Resolving the Replica Problem with Supersymmetric SYK Models

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Based on the work in cooperation with Chenhao Zhang. (arXiv:2507.XXXXX)

Holographic applications: from Quantum Realms to the Big Bang

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Non-Renormalizability of 4D Gravity

- **Einstein-Hilbert action in 4D**:

$$S_{\text{EH}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R$$

- **Problem: Non-renormalizable UV behavior**
  - Perturbative expansion generates infinite counterterms
  - Coupling constant G has negative mass dimension $[G] = -2$
  - No predictive power at high energies
- **Motivation for low-dimensional toy models**
  - Simpler dynamics, controlled quantum gravity
  - SYK model as a dual to 2D JT gravity

[1] ’t Hooft and Veltman, **20** (1974) 69

Two-Dimensional Gravity and JT Model

- **Jackiw-Teitelboim (JT) gravity**:

$$S_{\mathrm{JT}} = \frac{1}{2}\int d^2x\sqrt{-g}\,\phi(R+2)$$

- **Features**:
  - Topological gravity coupled to dilaton ϕ
  - AdS_2 solutions with boundary dynamics
  - UV-finite and analytically solvable
- **Connection to SYK**:
  - Low-energy limit of SYK matches JT boundary dynamics
  - Schwarzian action emerges in both setups

[2] Almheiri and Polchinski, JHEP **11** (2015) 014

Quantum No-Cloning and Replica Wormholes

- **Quantum No-Cloning Theorem**:

  Unknown quantum states cannot be perfectly copied.

  - Preparing n identical wormhole replicas requires full knowledge of the quantum state.
  - Without prior knowledge, replica construction violates no-cloning principles.
- **Information Recovery and Replicas**:
  - Fewer replicas ($n \to 1$) minimize operational complexity and resource requirements.
  - Smaller n risks incomplete information extraction from gravitational path integrals.
- **Unexplored Physics of n-Dependence**:
  - Finite-n quantum effects may enrich the physics of non-perturbative effects (e.g., topological wormholes, replica asymmetry).
  - The transition from wormhole saddle to Hawking saddle in gravitational path integrals may exhibit n-dependent directionality.

Wormholes in n-replica SYK

- **Competing Saddles in Entanglement Entropy**

  Hawking Saddle: Neglects quantum gravity effects $\Rightarrow$ S_{rad} increases monotonically

  Replica Wormholes: Include non-perturbative contributions $\Rightarrow$ Unitary Page curve

- **Supersymmetric Enhancement**

$$\mathcal{N} = 1 \text{ Supersymmetric SYK Model}$$
$$\Downarrow$$
$$\text{Analytically tractable replica wormhole solutions}$$

[3] Penington et al., JHEP **03** (2020) 205

Special SYK with Replicas

Two-Copy SYK System

- Consider two independent SYK models: left (L) and right (R)

$$H_L = \sum_{1\le i_1<\cdots<i_q\le N_L} J^{(L)}_{i_1\cdots i_q}\,\psi_{i_1,L}\cdots\psi_{i_q,L}$$
$$H_R = \sum_{1\le i_1<\cdots<i_q\le N_R} J^{(R)}_{i_1\cdots i_q}\,\psi_{i_1,R}\cdots\psi_{i_q,R}$$

- Index structure:
  - $i \in \{1,\ldots,N_a\}$ —*flavor index*
  - $a \in \{L, R\}$ —*physical system label*
  - $\alpha \in \{1,\ldots,n\}$ —*replica index*

[4] G. Penington, S. H. Shenker, D. Stanford, Z. Yang, JHEP **03** (2022) 205

Inter-Replica Interactions as Wormholes

- **Non-local interaction between replicas α and α':**

$$V^{\alpha\alpha'} = \sum_{\substack{1\le i_1^{(1)}<\cdots<i_{\bar{q}}^{(1)}\le N \\ 1\le i_1^{(2)}<\cdots<i_{\bar{q}}^{(2)}\le N}} \bar{J}_{i_1^{(1)}\dots i_{\bar{q}}^{(1)};\, i_1^{(2)}\dots i_{\bar{q}}^{(2)}}\, \psi^{\alpha}_{i_1^{(1)}}\cdots\psi^{\alpha}_{i_{\bar{q}}^{(1)}}\, \psi^{\alpha'}_{i_1^{(2)}}\cdots\psi^{\alpha'}_{i_{\bar{q}}^{(2)}}$$

- **After disorder averaging:** generates an effective bi-local interaction term

$$\mathcal{V} = \frac{1}{2}\frac{\bar{J}^2}{q}\int_C d\tau_1\, d\tau_2 \sum_{\substack{\alpha,\alpha' \\ \gamma,\gamma'}} \left[G_L^{\alpha\alpha'}(\tau_1,\tau_2)\right]^{\bar{q}} g^{\alpha\gamma}(\tau_1)\, g^{\alpha'\gamma'}(\tau_2) \left[G_R^{\gamma\gamma'}(\tau_1,\tau_2)\right]^{\bar{q}}$$

- **Physical interpretation:** such couplings mediate *replica wormholes* —correlations across spacetime boundaries in gravity dual.

Extending to n Replicas: Multi-Replica Interactions

Interacting Fermions Across n Replicas

- Consider non-local interaction terms involving fermions from n distinct replicas:

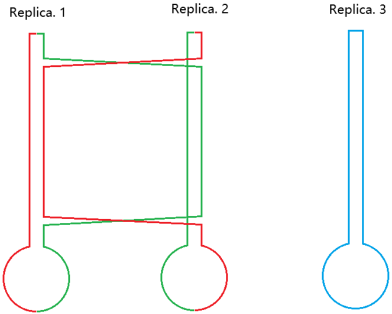
$$V^{\alpha_1\cdots\alpha_n} = \sum_{\substack{1\le i_1^{(1)}<\cdots<i_{\bar{q}}^{(1)}\le N \\ \vdots \\ 1\le i_1^{(n)}<\cdots<i_{\bar{q}}^{(n)}\le N}} J^{\alpha_1\cdots\alpha_n}_{i_1^{(1)}\cdots i_{\bar{q}}^{(1)};\cdots;i_1^{(n)}\cdots i_{\bar{q}}^{(n)}} \prod_{a=1}^{n} \psi^{\alpha_a}_{i_1^{(a)}}\cdots\psi^{\alpha_a}_{i_{\bar{q}}^{(a)}}$$

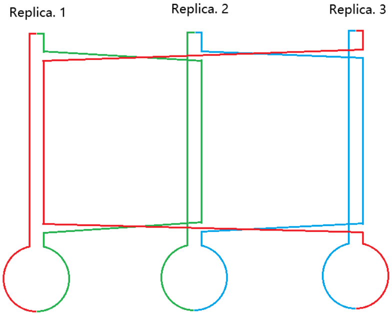
- These interactions entangle all n replicas, potentially generating connected multi-replica geometries in the dual gravity.
- **Interpretation:** these couplings lead to *replica wormholes* that connect n geometries —central to resolving the entropy puzzle via the gravitational path integral.

Interaction for the 3-th replica and the contours

$$V = \int_{C_1} \left(V^{11} + V^{22} + V^{33} \right) + \int_{C_2} \left(V^{12} + V^{21} + V^{13} + V^{31} + V^{23} + V^{32} \right)$$

$$+ \int_{C_2} \left(V^{123} + V^{213} + V^{132} + V^{312} + V^{231} + V^{321} \right) = \int_C \sum_{\alpha\beta\gamma} V^{\alpha\beta\gamma} g^{\alpha\beta\gamma} (\tau).$$





Effective Action

- **Effective Action for n Replicas:**

$$I_n = -\log\left(\partial_\tau\, \delta_{aa'}^{\alpha\alpha'} - \Sigma_{aa'}^{\alpha\alpha'}\right) + \frac{1}{2}\int_C d\tau_1\, d\tau_2 \Bigg[\Sigma_{aa'}^{\alpha\alpha'}(\tau_1,\tau_2)\, G_{aa'}^{\alpha\alpha'}(\tau_1,\tau_2)$$

$$- \frac{J^2}{q}\left[G_{aa'}^{\alpha\alpha'}(\tau_1,\tau_2)\right]^q\Bigg] - \mathcal{V},$$

$$\bar{J}^2(n) = \frac{2\bar{J}^2(2)}{n!}, \qquad \frac{\bar{q}(n)}{n} = \frac{\bar{q}(2)}{2}$$

$$\Rightarrow \quad I_n\left(\bar{q}(n), \bar{J}(n)\right) = I_2\left(\bar{q}(2), \bar{J}(2)\right)$$

- **Subtlety:** For $n \geq 3$, constructing consistent off-shell wormhole configurations becomes nontrivial.

Challenges in n-Replica SYK Calculations

Key Obstacles to Full Understanding

- **Limited Solvability:**
  Analytical methods break down for $n \geq 3$ due to increased complexity of replica wormholes.

- **Off-Shell Ambiguity:**
  The path integral fails to capture essential off-shell contributions needed for entropy computations.

- **Degenerate Saddles:**
  Multiple replica wormhole solutions complicate interpretation and dilute predictive power.

These issues obstruct a complete microscopic derivation of the Page curve in the n-replica SYK framework.

The $\mathcal{N} = 1$ Supersymmetric SYK Model

Supersymmetry Overview:

- $\mathcal{N} = 1$ SUSY introduces a fermionic generator Q satisfying: $Q^2 = \mathcal{H}$
- The Hamiltonian emerges as the square of the supercharge

Supercharge (for general q-fermion interactions):

$$Q = i^{\frac{q-1}{2}} \sum_{i_1,\ldots,i_q} C_{i_1\ldots i_q} \psi^{i_1}\psi^{i_2}\cdots\psi^{i_q}$$

Hamiltonian (e.g., for $q = 3$):

$$\mathcal{H} = Q^2 = E_0 + \sum_{i<j<k<l} J_{ijkl}\,\psi^i\psi^j\psi^k\psi^l$$

Supersymmetric Structure:

$$\{Q, \psi^i\} = Q\psi^i = i\sum_{j<k} C_{ijk}\psi^j\psi^k \equiv b^i$$

Coupled $\mathcal{N} = 1$ SSYK Model

Total Hamiltonian:

$$H_{\text{total}} = H_L + H_R + H_{\text{int}}$$

Interaction Term:

$$H_{\text{int}} = i\mu \int d\theta \left(\Psi_L \Psi_R - \Psi_R \Psi_L\right)$$

Expanded in Components:

$$H_{\text{int}} = i\mu \left(\psi_L b_R - b_L \psi_R - \psi_R b_L + b_R \psi_L\right)$$

- μ controls the strength of the supersymmetric coupling between left and right systems.
- Interaction preserves $\mathcal{N} = 1$ SUSY and mixes fermionic and bosonic degrees of freedom across replicas.

[5] C.H. Zhang, W.H. Cai, *Off-diagonal coupling of supersymmetric SYK model*, JHEP **2025**, 133 (2025).

W. Fu, D. Gaiotto, J. Maldacena, S. Sachdev, *Phys. Rev. D* **95**, 026009 (2017).

Fractal Extension of the $\mathcal{N} = 1$ SSYK Model (I)

Motivation: Explore hierarchical entanglement structures via iterated supersymmetric couplings.

Recursive Fermion Expansion:

$$\psi_n^i \to \psi_n^i + \psi_m^i \to \psi_n^i + \psi_m^i + \psi_l^i + \cdots$$

- Fermion degrees of freedom span multiple nested subsystems.
- Each level introduces additional symmetry or coupling.

Fractal Supercharge Structure:

$$Q_n \to Q_{nm} \to Q_{nm\ldots l}$$
$$\{Q_{nm\ldots l}, \psi_n^i\} \equiv b_{nm\ldots l}^i$$

- Higher-order supercharges act across multiple layers.
- SUSY algebra produces increasingly structured bosonic partners.

Fractal Extension of the $\mathcal{N} = 1$ SSYK Model (II)

Total Supersymmetric Action:

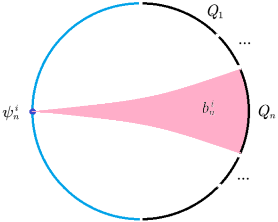
$$\mathcal{L}_{\text{total}} = \mathcal{L}_n + \mathcal{L}_{nm} + \cdots + \mathcal{L}_{nm\ldots l}$$

- Each term $\mathcal{L}_{\ldots}$ describes a level in the recursive hierarchy.
- The action accumulates contributions from all nested SYK sectors.

Physical Picture:

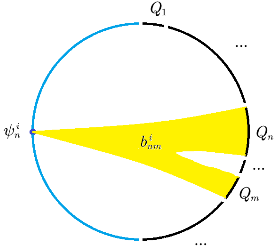
- The model builds a **fractal-like** structure in theory space.
- Captures richer replica entanglement patterns and wormhole connections.
- Offers a natural generalization for constructing higher-order non-local couplings.

Inner Structure of Superfields via Chord Diagrams



(a) Single-Layer Coupling

Supercharge terms C_{ijk} on the LHS act via bosonic partners b_n^i (pink chord region) on the RHS SYK model. *Interpretation:* Supersymmetry links fermions and bosons across subsystems.



(b) Multi-Layer Extension

Fermion doubling $(\psi_n^i + \bar{\psi}_n^i)$ enables construction of higher supercharges Q_{nm}, generating new bosons b_{nm}^i. *Interpretation:* Iterative replica couplings form nested chord structures.

Superfield dynamics emerge naturally from recursive chord interactions in replicated SYK sectors.

Interaction to make nontrivial solution

Interactions by orders

$$H_{int} = i\mu^n \int d\theta \Psi^i_{L,n}\Psi^i_{R,n} + i\mu^{nm} \int d\theta \Psi^i_{L,nm}\Psi^i_{R,nm} + \ldots + i\mu^{nm\ldots l} \int d\theta \Psi^i_{L,nm\ldots l}\Psi^i_{R,nm\ldots l}$$

$$\begin{aligned}
&= i\mu^n \left(\psi_{L,n} b_{R,n} - b_{L,n}\psi_{R,n} - \psi_{R,n} b_{L,n} + b_{R,n}\psi_{L,n}\right) \\
&+ i\mu^{nm} \left(\psi_{L,n} b_{R,nm} - b_{L,nm}\psi_{R,n} - \psi_{R,n} b_{L,nm} + b_{R,nm}\psi_{L,n}\right) \\
&+ \ldots + i\mu^{nm\ldots l} \left(\psi_{L,nm\ldots l} b_{R,nm\ldots l} - b_{L,nm}\psi_{R,nm\ldots l} - \psi_{R,n} b_{L,nm\ldots l} + b_{R,nm\ldots l}\psi_{L,n}\right),
\end{aligned}$$

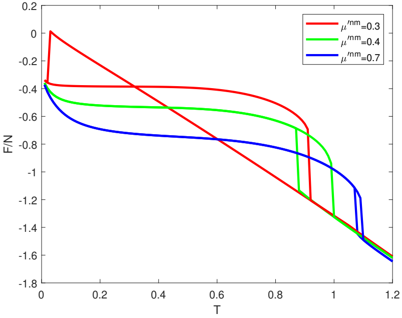
Physical constrain in thermal limit

$$C^n_{i_1 i_2 \ldots i_q} \neq 0, \;\; C^{nm}_{i_1 i_2 \ldots i_q} \to 0, \;\; C^{nm\ldots l}_{i_1 i_2 \ldots i_q} \to 0.$$

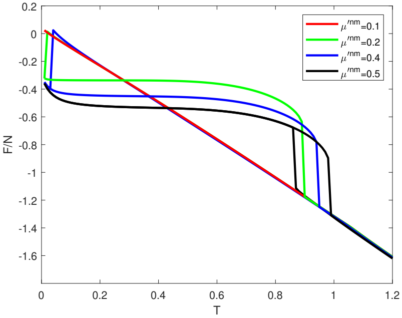
Introduction of reduction parameter

$$C^2_n \mu^{nm} + \ldots + C^n_n \mu^{nm\ldots l} \to \mu'^{nm}$$

Thermal Phase Diagram



(a) No primary coupling ($\mu^{(n)} = 0$): Increasing inter-replica coupling $\mu'^{(nm)}$ softens the phase transition.



(b) With primary coupling ($\mu^{(n)} = 0.3$): Two distinct transitions emerge at intermediate $\mu'^{(nm)}$.

Critical Point: $T_c = 1.15$ at $\mu_c^{(n)} = 0.3$, $\mu_c'^{(nm)} = 0.84$

Effective Action in Lorentzian Time

Wick Rotation via Wightman Correlators:

$$\mathcal{G}^{>}_{AB}(t_1, t_2) = -i \lim_{\epsilon \to 0^-} \mathcal{G}_{AB}(it_1 + \epsilon, it_2 - \epsilon)$$

$$\mathcal{G}^{<}_{AB}(t_1, t_2) = -i \lim_{\epsilon \to 0^+} \mathcal{G}_{AB}(it_1 - \epsilon, it_2 + \epsilon)$$

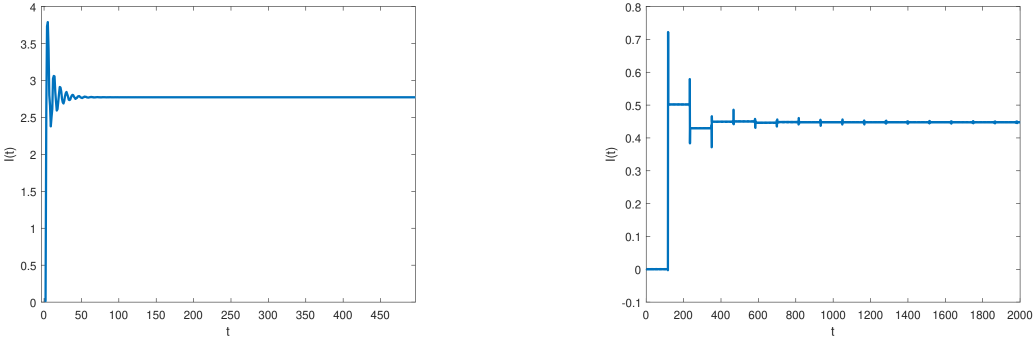
Time Evolution of Effective Action:

$$I(t) = I(0) + \int dt\, d\theta\, (2H_0 + H_{\text{int}})$$

$$I(t) = I(0) + 2\int dt \left[\left(G^{>}_{\psi\psi}(t) \right)^{q-1} G^{>}_{bb}(t) + \left(G^{>}_{\psi\psi}(t) \right)^{q-2} G^{>}_{b\psi}(t)\, G^{>}_{\psi b}(t) \right] + i\mu \int dt \left[G^{>}_{\psi b}(t) - G^{>}_{b\psi}(t) \right]$$

- **Lorentzian effective action** tracks real-time dynamics of the coupled SSYK system.
- Wightman functions encode quantum fluctuations out of equilibrium.

Effective Action Evolution in Lorentzian Time

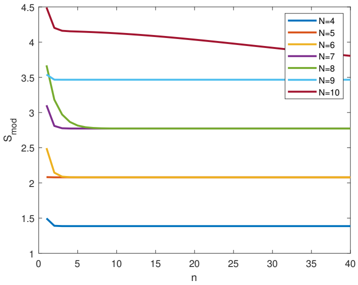


(a) Black hole phase: $\mu^{(n)} = 0.5,\ T = 2.5$

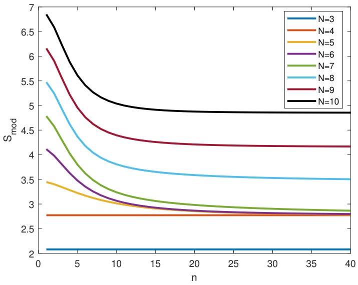
(b) Wormhole phase: $\mu^{(n)} = 0.5,\ T = 0.5$

Effective action $I_{\text{eff}}(t)$ dynamics in Lorentzian time

Finite-N Modular Entropy via Full Diagonalization

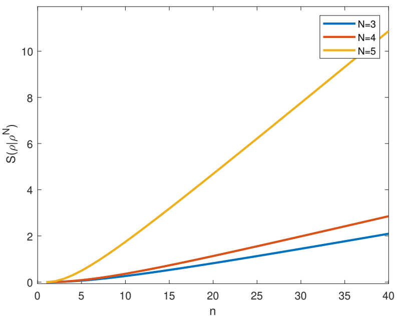


(a) Non-SUSY: Stepwise convergence of $S_{\text{mod}}(n)$ as n increases.

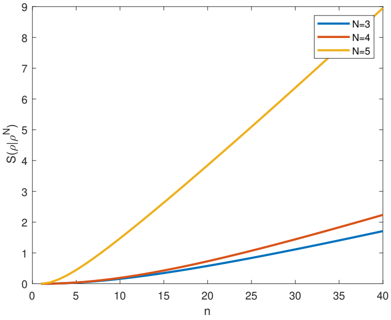


(b) $\mathcal{N} = 1$ SUSY: Smooth convergence with distinct asymptotes ($H = Q^2$ constraint)

Finite-N Relative Entropy via Full Diagonalization



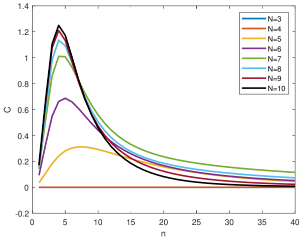
(c) Full coupling: $\mu^{(n)} = 0.1$ for all replicas ($N_{\text{ferm}} > 4$)



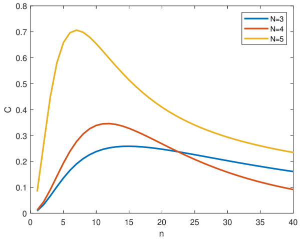
(d) Partial coupling: $\mu^{(n)} = 0$ for half replicas ($N_{\text{ferm}} > 4$)

Key note: Strong subadditivity manifests via relative entropy monotonicity, confirming holographic subsystem inequalities

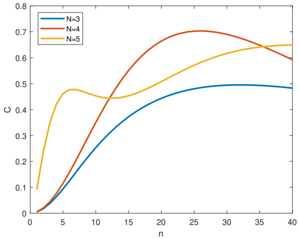
Entanglement Capacity: Finite-N Results



(a) No coupling:
Peak at $n = 5$



(b) Full coupling $\mu^{(n)} = 0.1$:
Peak shifts left



(c) Half coupling:
Peak splitting at $n = 5$

$$C = n^2(\langle H^2\rangle_n - \langle H\rangle_n^2)$$

Fractal Symmetry and Supersymmetry Breaking in $\mathcal{N} = 1$ SYK

Fractal Structure of Supercharges:

$$Q_{nm\ldots l} = i^{\frac{q-1}{2}} \sum_{i_1 \cdots i_q} C^{nm\ldots l}_{i_1 \cdots i_q} \psi^{i_1}_{nm\ldots l} \cdots \psi^{i_q}_{nm\ldots l}$$

Supersymmetry Breaking via Fermion Overlap:

$$b = \{Q, \psi\} = \{Q_{12}, \psi_1 + \psi_2\} = b_1 + b_2 + \text{cross terms}$$

- Cross terms arise from fermion mixing between layers.
- SUSY is explicitly broken when cross terms are non-vanishing.

Interaction Hamiltonian Decomposition:

$$H_{\text{int}} = H_{\text{int},1} + H_{\text{int},2} + \text{cross terms}$$

- Cross-layer couplings induce SUSY-breaking interactions; Reflects spontaneous breaking of fractal symmetry at large N.

The Law of Parallelism

Core Principle: Parallel Structure Across All Levels

In this framework, the $\mathcal{N} = 1$ supersymmetric SYK model maintains:

- Identical local dynamics at **every site**
- Recurrent structure at **every recursive level**

As a result:

Each component follows the same form of effective action, mirroring a parallel and self-similar structure.

The Law of Parallelism: Superconformal Structure Across Replicas

Extended Superconformal Symmetry with Replica Indices:

$$\mathcal{G}^{(n)}(\tau_1,\theta_1;\tau_2,\theta_2) = \left(D_{\theta_1}\theta_1'\right)^{1/q}\left(D_{\theta_2}\theta_2'\right)^{1/q}\mathcal{G}^{(n)}(\tau_1',\theta_1';\tau_2',\theta_2')$$

$$\mathcal{G}^{(nm\ldots l)}(\tau_1,\theta_1;\tau_2,\theta_2) = \left(D_{\theta_1}\theta_1'\right)^{1/q}\left(D_{\theta_2}\theta_2'\right)^{1/q}\mathcal{G}^{(nm\ldots l)}(\tau_1',\theta_1';\tau_2',\theta_2')$$

Hierarchical Schwarzian-like Effective Action:

$$\begin{aligned} S_A = \int d\tau\, d\theta \Big[&- N\alpha_S^{(n)}\, S\left(\tanh\frac{h_A^{(n)}}{2}, \theta_A^{(n)\prime}; \tanh\frac{\tau_A^{(n)}}{2}, \theta_A^{(n)}\right) \\ &- N\alpha_S^{(nm\ldots l)}\, S\left(\tanh\frac{h_A^{(nm\ldots l)}}{2}, \theta_A^{(nm\ldots l)\prime}; \tanh\frac{\tau_A^{(nm\ldots l)}}{2}, \theta_A^{(nm\ldots l)}\right) \\ &- \cdots \Big] \end{aligned}$$

- Each level of the hierarchy respects the same superconformal transformation rules.
- The effective action naturally extends in a recursive, parallel form.

The Law of Perpendicularity

Principle: Coordinate Invariance in Supersymmetric Theories

- The form of any physical action is determined by how it transforms under coordinate changes.
- **If we understand how one part transforms**, the rest follows from symmetry.

Implication for the Super-Schwarzian Action:

- Its structure is **invariant under super-reparameterizations** (coordinate changes in τ, θ).
- Transformations may introduce **boundary terms**, but do not alter the bulk dynamics.
- Guarantees consistency across all layers and replicas in supergravity or SSYK constructions.

The law of perpendicularity

The chain rule under finite reparameterizations

$$S[\tau'',\theta'';\tau,\theta]=\left(D\theta'\right)^3 S[\tau'',\theta'';\tau',\theta']+S[\tau',\theta';\tau,\theta]$$

Rewrite the effective action

$$\begin{aligned}
I^{(n)} &= -\sum_n \alpha_S^{(n)} \int_{\partial\widetilde{M}_n} d\tau\, d\theta\, S\left[\tau^{(n)}\prime, \theta^{(n)}\prime; \tau, \theta\right] - \ldots \\
&\quad - \sum_{nm\ldots l} \alpha_S^{(nm\ldots l)} \int_{\partial\widetilde{M}_{nm\ldots l}} d\tau\, d\theta\, S\left[\tau^{(nm\ldots l)}\prime, \theta^{(nm\ldots l)}\prime; \tau, \theta\right] \\
&= -\sum_n \sum_{nm\ldots l} \left(\alpha_S^{(n)\prime} + \alpha_S^{(nm)\prime} + \ldots + \alpha_S^{(nm\ldots l)\prime}\right) \int_{\partial\widetilde{M}_n} d\tau\, d\theta\, S\left[\tau^{(n)}\prime, \theta^{(n)}\prime; \tau, \theta\right] \\
&\quad + \left(D_\theta \theta''\right)^3 S\left[\tau^{(n)}\prime, \theta^{(n)}\prime; \tau^{(nm\ldots l)}\prime, \theta^{(nm\ldots l)}\prime\right] + \ldots
\end{aligned}$$

The Law of Finite Displacement

Key Principle: Super-Schwarzian coordinate transformations are not arbitrary —they are constrained to finite-dimensional submanifolds that preserve the underlying symmetries of the supersymmetric SYK model.

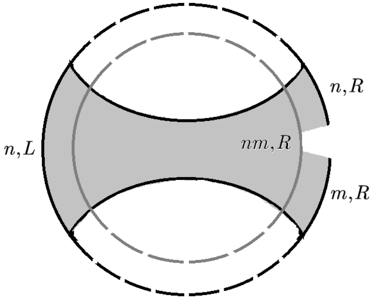
Implication: Hierarchical Influence in Fractal Geometry

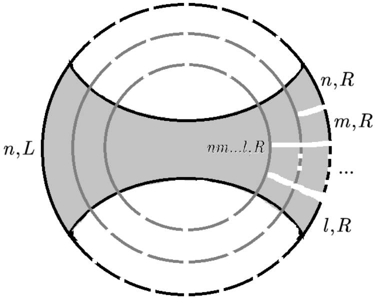
- Higher-order (replica-extended) surfaces contribute dynamically to lower-order sectors.
- However, **lower-order surfaces cannot influence higher-order ones**.
- This asymmetry encodes a directional flow of information in the hierarchy of supersymmetric actions.

Analogy: Like gravitational backreaction from a bulk geometry to a brane —but not vice versa.

Express the constrained theory

We use diagrams of discrete replicas related by path integrals over gray area boundaries. It illustrates Hawking and wormhole saddles, super-reparameterization of dilatons in the nm-sector, and interactions across sectors. Supersymmetry breaking prevents full mapping between some regions.





Research on deformed N=1 SJT gravity

The N=1 SJT gravity with replica ansatz

$$S=-\frac{1}{16\pi G}\sum_{n}\sum_{nm}\sum_{nm\ldots l}\left[\left[i\int d^2zd^2\theta\left(E^{(n)}\Phi^{(n)}+E^{(nm)}\Phi^{(nm)}+\ldots E^{(nm\ldots l)}\Phi^{(nm\ldots l)}\right)(R_{+-}-2)\right]\right.$$
$$+2\int_{\partial M^{(n)}}du^{(n)}d\theta^{(n)}\Phi^{(n)}K^{(n)}+2\int_{\partial M^{(nm)}}du^{(nm)}d\theta^{(nm)}\Phi^{(nm)}K^{(nm)}$$
$$\left.+\ldots+2\int_{\partial M^{(nm\ldots l)}}du^{(nm..l)}d\theta^{(nm..l)}\Phi^{(nm..l)}K^{(nm..l)}\right],$$

Effective Schwarzian action on single surface

$$I^{(n)}=-\sum_{n}\phi_r^n\int_{\partial\widetilde{M}_n}d\tau\, d\theta\, S\left[\tau^{(n)}{}',\theta^{(n)}{}';\tau,\theta\right]-\ldots-\sum_{nm\ldots l}\phi_r^{nm\ldots l}\int_{\partial\widetilde{M}_{nm\ldots l}}d\tau\, d\theta\, S\left[\tau^{(nm\ldots l)}{}',\theta^{(nm\ldots l)}{}';\tau\right.$$
$$=-\sum_{n}\sum_{nm\ldots l}\left(\phi{\prime}_r^n+\phi{\prime}_r^{nm}+\ldots+\phi{\prime}_r^{nm\ldots l}\right)\int_{\partial\widetilde{M}_n}d\tau\, d\theta\, S\left[\tau^{(n)}{}',\theta^{(n)}{}';\tau,\theta\right]$$
$$+\left(D_\theta\theta''\right)^3S\left[\tau^{(n)}{}',\theta^{(n)}{}';\tau^{(nm\ldots l)}{}',\theta^{(nm\ldots l)}{}'\right]+\ldots$$

The SYK constraint

Higher-ordered reparameterization

$$
\begin{aligned}
&Dz'^{(nm\ldots l)} = \theta'^{(nm\ldots l)} D\theta'^{(nm\ldots l)},\\
&z'^{(nm\ldots l)} = t'^{(nm\ldots l)} + i\epsilon \left(D\xi'^{(nm\ldots l)}\right)^2,\\
&Dt'^{(nm\ldots l)}\left(u'^{(nm\ldots l)}, \theta'^{(nm\ldots l)}\right) = \xi'\left(u'^{(nm\ldots l)}, \theta'^{(nm\ldots l)}\right) D\xi'^{(nm\ldots l)}\left(u'^{(nm\ldots l)}, \theta'^{(nm\ldots l)}\right).
\end{aligned}
$$

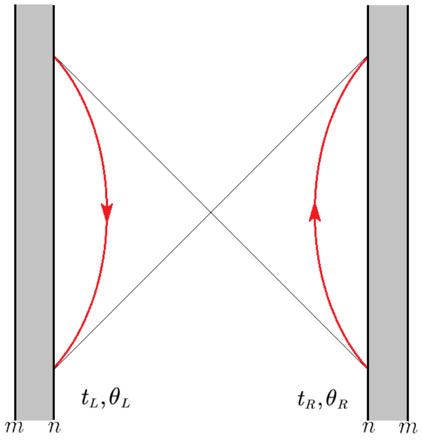
Low-ordered reparameterization

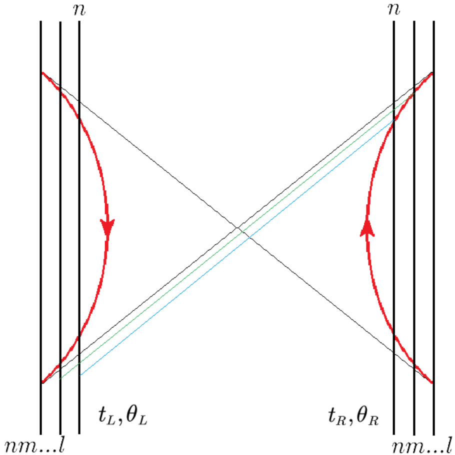
$$
\begin{aligned}
&Dz'^{(n)} = \theta'^{(n)} D\theta'^{(n)},\\
&z'^{(n)} = t'^{(n)} + i\epsilon \left(D\xi'^{(n)}\right)^2,\\
&Dt'^{(n)}\left(u'^{(n)}, \theta'^{(n)}\right) = \xi\left(u'^{(n)}, \theta'^{(n)}\right) D\xi'^{(n)}\left(u'^{(n)}, \theta'^{(n)}\right).
\end{aligned}
$$

The action on higher-ordered surfaces will influence the lower-ordered surfaces, but not vice versa.

SJT gravity with orders

In the extended $NAdS_2$ spacetime with JT gravity, the boundaries are reparameterized using a bosonic coordinate t and a Grassmann parameter θ.





Summary of Results

1. Multi-Ordered Trick

- Introduced a hierarchical framework for exact multi-replica wormhole solutions in $\mathcal{N} = 1$ supersymmetric SYK.
- Generalized from 2 to n replicas with systematically controlled off-diagonal interactions.

2. Fractal Symmetry and Supersymmetric Unification

- Leveraged recursive fractal structure to unify replica sectors through extended supercharges.
- Analyzed Lorentzian dynamics with emergent superconformal invariance and cross-replica reparametrizations.

3. Holographic Dual and Entanglement

- Matched the effective theory to $\mathcal{N} = 1$ super-Jackiw–Teitelboim gravity.
- Computed entanglement measures via exact diagonalization across replica sectors.

Thank You!