Bulk Spacetime Encoding via Boundary Ambiguities

arXiv:2506.12890

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July 17, 2025

Holographic applications: from Quantum Realms to the Big Bang

Overview

1. **Backgrounds**
   - Bulk metric reconstruction
   - Pole-skipping points
2. **Bulk metric reconstruction by pole-skipping points**
   - Flipping Near-Horizon Analysis
   - Reconstruction of g_{vv_n} and $g_{vr_{n-1}}$
3. **μ-polynomial constraints**
   - specific form
   - Valid region

Bulk metric reconstruction

- Reconstruct the bulk spacetime from boundary QFT data.
- Spacetime emerges from entanglement—reconstruct the bulk geometry from boundary entanglement entropy.

  [S. Ryu and T. Takayanagi, (2006)]; M. V. Raamsdonk, (2010); J. Maldacena and L. Susskind, (2013)]

  [S. Bilson, (2011); B. Czech, X. Dong, and J. Sully, (2014); S. R. Roy and D. Sarkar, (2018)]
- Employing machine learning techniques to explore the emergence of bulk spacetime [K. Hashimoto, et al., (2018); X. Dong and L. Zhou, (2018)]
- The bulk spacetime can be reconstructed from special boundary locations, known as "pole-skipping points", where the boundary Green's function becomes ambiguous;
- The reconstruction is fully analytical and only involves solving a set of linear equations.

Pole-skipping points

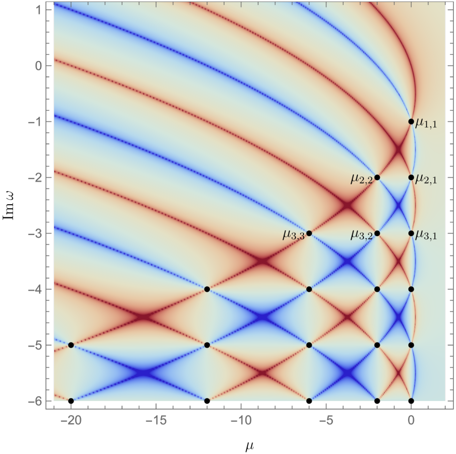

Figure: heatmaps showing the values of $\log|\mathcal{G}(\mathrm{Im}\omega, \mu)|$ with $\mu = k^2$.

- At pole-skipping points, the Green's function takes the form '$0/0$'.

[M. Blake, H. Lee, and H. Liu, (2018); S. Grozdanov, K. Schalm, and V. Scopelliti, (2018); M. Blake, et al. (2018)]

Pole-skipping points

- In theories with perturbation modes of spin l, pole-skipping points occur at complex Matsubara frequencies $\omega_n = i(l-n)2\pi T$.

  [D. Wang and Z.-Y. Wang, (2022); S. Ning, D. Wang, and Z.-Y. Wang, (2023)]

- The EOM of bulk perturbations exhibits two linearly independent ingoing solutions; The $(\omega_n, \mu_{n,q})$ can be determined from the 'near-horizon analysis'.

  [M. Blake, et al. (2018); M. Blake, R. A. Davison, and D. Vegh, (2020)]

- For a given n, different $\mu_{n,q}$ with $q \in \{1, \ldots, n\}$, correspond to the roots of a degree-n polynomial equation.

Pole-skipping points

Near-horizon analysis

- Assuming a probe massless scalar field $\phi(r)$ obeys the Klein-Gordon equation: $\nabla^2\phi(r) = 0$.
- Its fourier mode φ admits: $\varphi(r) = (r-1)^\alpha \sum_{p=0}^{\infty} \varphi_n (r-1)^n$. By substituting this into the Klein-Gordon equation and expanding up to order n at complex Matsubara frequency $\omega_n = -in2\pi T$, one obtains an $n \times n$ matrix $\mathcal{M}^{(n)}(\boldsymbol{\mu})$.
- The determinant of this matrix, Det$(\mathcal{M}^{(n)}(\boldsymbol{\mu})) = 0$, leads to a degree-$n$ polynomial equation in μ:

$$V_{n,n}\mu^n + V_{n,n-1}\mu^{n-1} + \cdots + V_{n,1}\mu + V_{n,0} = 0, \tag{1}$$

- The specific form of $V_{n,m}$ depends on the details of the theory.

Flipping Near-Horizon Analysis

Background spacetime

A static planar symmetric black hole metric in EF coordinate:
$ds^2 = -g_{vv}(r)dv^2 + g_{vr}(r)dvdr + r^2 d\vec{x}^2$, with
$g_{vv}(r) = g_{vv_1}(r-1) + g_{vv_2}(r-1)^2 + \dots$
$g_{vr}(r) = g_{vr_0} + g_{vr_1}(r-1) + g_{vr_2}(r-1)^2 + \dots$

- To apply our reconstruction method, we swap the roles of $(g_{vv_n}, g_{vr_{n-1}})$ and $(\omega_n, \mu_{n,q})$, treating the latter as input and the former as unknowns.
- In Eq. (1), $V_{n,m}$ dependent on g_{vv_n} and $g_{vr_{n-1}}$, while μ assumes n values $\mu_{n,q}$.
- Instead of treating n individual pole-skipping points separately, it is more intuitive to exploit the S_n symmetry in Eq. 1 and work with the related elementary symmetric polynomials, e.g.,
$E_1(\mu) \equiv \mu_{1,1}, \quad E_2(\mu) \equiv \mu_{2,1} + \mu_{2,2}$
$E_2(\mu^2) \equiv \mu_{2,1}\mu_{2,2}, \quad E_3(\mu^2) \equiv \mu_{3,1}\mu_{3,2} + \mu_{3,2}\mu_{3,3} + \mu_{3,3}\mu_{3,1}$

Reconstruction of g_{vv_n} and $g_{vr_{n-1}}$

- Those elementary symmetric polynomials links to the g_{vv_n} and $g_{vr_{n-1}}$ by Vieta formula:
  $E_n(\mu^m) - \frac{v_{n,n-m}}{v_{n,n}} = 0$ where we define $v_{n,m} = (-1)^{n-m} V_{n,m}$
- For $n = 1$, combining $E_1(\mu) + \frac{dg_{vv_1}}{2g_{vr_0}^2} = 0$ and $i\frac{\omega_1}{2\pi} = \frac{g_{vv_1}}{4\pi g_{vr_0}}$ yielding:

$$g_{vv_1} = \frac{2d\omega_1^2}{E_1(\mu)}, \quad g_{vr_0} = -\frac{id\omega_1}{E_1(\mu)}. \tag{2}$$

- For $n = 2$, after substituting the solution at $n = 1$, the two equations $E_2(\mu^1) = \frac{v_{2,1}}{v_{2,2}}$ and $E_2(\mu^2) = \frac{v_{2,0}}{v_{2,2}}$ solves for g_{vv_2} and g_{vr_1}

$$\begin{aligned} g_{vv_2} &= \frac{d^2\omega_1^2 E_2(\mu^2)}{4E_1(\mu)^3} + \frac{2d^2\omega_1^2}{E_1(\mu)} - \frac{d^2\omega_1^2 E_2(\mu)}{E_1(\mu)^2} + \frac{3d\omega_1^2}{E_1(\mu)}, \\ g_{vr_1} &= \frac{id^2\omega_1 E_2(\mu)}{2E_1(\mu)^2} - \frac{id^2\omega_1}{E_1(\mu)} - \frac{id^2\omega_1 E_2(\mu^2)}{4E_1(\mu)^3} - \frac{id\omega_1}{E_1(\mu)}. \end{aligned} \tag{3}$$

Reconstruction of $g_{vr_{n-1}}$ and g_{vv_n}

- For $n > 2$, there are n equations but only two variables: g_{vv_n} and $g_{vr_{n-1}}$ to solve.

Unique solutions

Only $E_n(\mu^{n-1}) - \frac{v_{n,1}}{v_{n,n}} = 0$ and $E_n(\mu^n) - \frac{v_{n,0}}{v_{n,n}} = 0$ are **linear** in terms of g_{vv_n} and $g_{vr_{n-1}}$.

- By computing g_{vv_n} and $g_{vr_{n-1}}$ to sufficiently large n, one can approximate the metric functions $g_{vv}(r)$ and $g_{vr}(r)$ with arbitrary accuracy within the convergence radius, determined by the nearest singularities in the complexified r-plane.

- The reconstruction can be generalized to the massive Klein-Gordon equation and extended to reconstruct metrics with Lifshitz scaling and hyperscaling violation.

μ-polynomial constraints
at $n = 3, 4$

- How about the rest $n-2$ equations: $E_n(\mu^m) - \frac{v_{n,n-m}}{v_{n,n}} = 0$ with m being 1 to $n-2$?
- For $n = 3$, $E_3(\mu) - \frac{v_{3,2}}{v_{3,3}} = 0$ takes the form:

$$E_3(\mu) + \frac{3dg_{vv_1}}{2g_{vr_0}^2} + \frac{8g_{vv_1}}{g_{vr_0}^2} + \frac{4g_{vr_1}g_{vv_1}}{g_{vr_0}^3} - \frac{8g_{vv_2}}{g_{vr_0}^2} = 0.$$

$$\Downarrow$$

- $$P_3(\mu) \equiv E_3(\mu) - 4E_2(\mu) + 5E_1(\mu) = 0$$

$$\Updownarrow$$

$$\mu_{3,1} + \mu_{3,2} + \mu_{3,3} - 4(\mu_{2,1} + \mu_{2,2}) + 5\mu_{1,1} = 0$$

- For $n = 4$,

$$\begin{aligned} P_4(\mu) &\equiv E_4(\mu) - 10E_2(\mu) + 16E_1(\mu) = 0 \\ P_4(\mu^2) &\equiv E_4(\mu^2) - 6E_3(\mu^2) + 14E_2(\mu^2) - 9E_2(\mu)^2 + 40E_2(\mu)E_1(\mu) \\ &\quad - 46E_1(\mu)^2 = 0 \end{aligned} \tag{4}$$

μ-polynomial constraints
at arbitrary n

- In master equation of the form $(\nabla^2 + V(r))\Phi(r) = 0$, for arbitrary n, the following $n-2$ redundant equations:

  [H. Kodama and A. Ishibashi, (2003); H. Kodama and A. Ishibashi, (2004)]

$$E_n(\mu^m) - \frac{v_{n,\,n-m}}{v_{n,\,n}} = 0, \quad \text{for } m = 1, 2, \ldots, n-2.$$
$$\Downarrow$$
$$P_n(\mu^m) = 0, \quad \text{for } m = 1, 2, \ldots, n-2.$$

- i.e. a set of $n-2$ universal polynomial constraints on pole-skipping μ alone, free of any other quantities.
- For each $n > 2$, these μ-polynomial constraints reduce the number of independent $\mu_{n,q}$ from n to 2, consistent with the number of the bulk variables g_{vv_n} and $g_{vr_{n-1}}$.

Precondition for μ-polynomial constraints

Det$(\mathcal{M}^{(n)}(\boldsymbol{\mu})) = 0$ derived from $(\nabla^2 + V(r))\Phi(r) = 0$ is a degree-n polynomial in μ, taking the form of Eq. (1).

Some comments

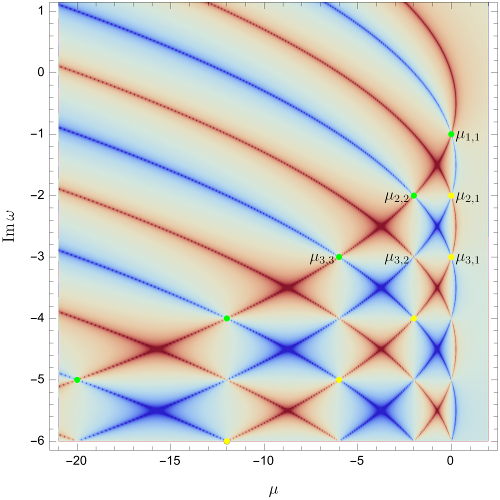

Figure: $2n-1$ pole-skipping points $\rightarrow$ bulk metric (with KG equation) $\rightarrow$ Green's function (numerically)

Thank you!