Gravity Dual of Network and Entanglement

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Outline

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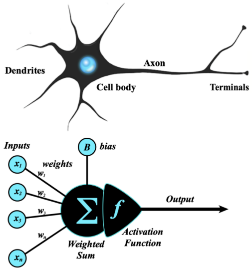
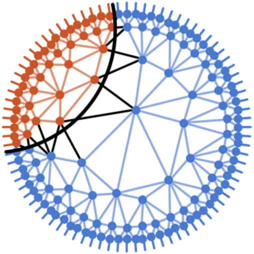
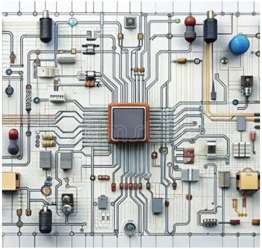
Motivations for Networks

Everything in the universe is interconnected (gravity/entanglement). Networks offer a strong framework for studying these connections.

- Neural networks are driving revolution in artificial intelligence.
- Tensor networks offer insights into entanglement and gravity.
- Circuits etal naturally exhibit network structures.



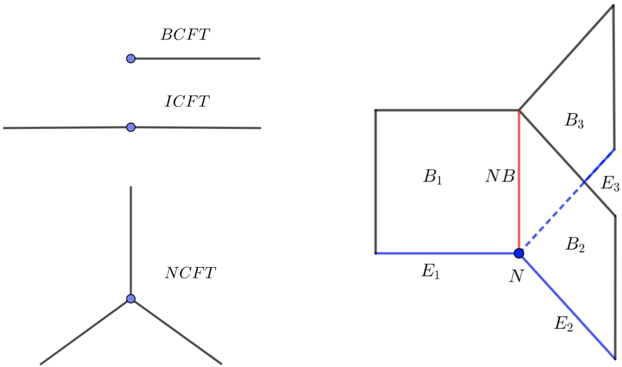




Motivations for Networks

We aim to study the CFT in networks(NCFT) and its gravity dual.

- NCFT can describe electron motion in nanoscale circuits.
- NCFT is a multi-branch generalization of BCFT and ICFT.
- AdS/NCFT is a natural realization of parallel universe.



Outline

NCFT vs BCFT

NCFT is the multi-branch generalization of BCFT.

- The symmetry group is reduced from $O(d+1,1)$ to $O(d,1)$ for both BCFT and NCFT.
- Boundary conditions for CFT

$$\text{BCFT}: \ J_n|_{\text{bdy}} = 0, \tag{1}$$

$$\text{NCFT}: \ \sum_m \overset{(m)}{J}_n|_{\text{node}} = 0, \tag{2}$$

- Boundary conditions for AdS/BCFT

$$\text{AdS/BCFT}: \ \left(K_{ij} - K h_{ij}\right)|_{\text{EOW brane}} = -T h_{ij}, \tag{3}$$

- Junction conditions for AdS/NCFT

$$\text{AdS/NCFT}: \ \sum_m \left(\overset{(m)}{K}_{ij} - \overset{(m)}{K} h_{ij}\right)|_{\text{Net-brane}} = -T h_{ij}, \tag{4}$$

Summary of main results

- We prove the junction condition on Net-brane leads to energy conservation on network node.
- We find the spectrum of KK modes on Net-brane is a combination of the spectra from the AdS/BCFT with NBC and DBC/CBC.
- We obtain the general form of two-point functions of NCFT.

$$\langle O(x)O(x')\rangle = \begin{cases} \dfrac{F_I(v_I)}{|\overset{(m)}{x} - \overset{(m)}{x}{}'|^{2\Delta}}, & \text{same edge}, \\ \dfrac{F_{II}(v_{II})}{|\overset{(m)}{x} - \overset{(n)}{x}{}'|^{2\Delta}}, & \text{mixed edge}, \end{cases} \tag{5}$$

- We propose the rules to calculate holographic entanglement entropy and define a positive network entropy.

$$S_{\text{Net}} = S_{\text{NCFT}} - S_{\text{BCFT}} \geq 0. \tag{6}$$

- Shortest path problem is dual to seeking the shortest geodesic in bulk.

Outline

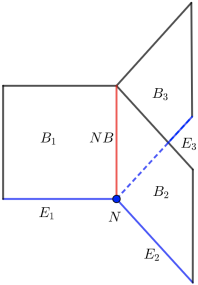
Geometry and Junction Condition

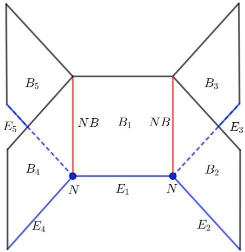
- Action

$$I = \sum_{m}^{p} \int_{B_m} d^{d+1}x \sqrt{|g|}(R - 2\Lambda) + 2\int_{NB} d^d y \sqrt{|h|}(-T + \sum_{m}^{p} \overset{(m)}{K}),$$

- Junction condition

$$\delta I|_{NB} = \int_{NB} d^d y \sqrt{|h|} \left[T h_{ij} + \sum_{m} \left(\overset{(m)}{K}_{ij} - \overset{(m)}{K} h_{ij} \right) \right] \delta h^{ij} = 0.$$





Energy Conservation at Node I

Junction condition on Net-brane leads to energy conservation at node.

- Codazzi's equation

$$D^i(K_{ij} - Kh_{ij}) = \frac{1}{2}R_{nj} = \frac{1}{4}T_{nj}^{\text{matter}} = 0 \tag{7}$$

- Conserved stress tensors on edge E and Net-brane NB

$$D^i(K_{ij} - Kh_{ij})|_{E_m} = 0,$$

$$\sum_m D^i(\overset{(m)}{K}_{ij} - \overset{(m)}{K} h_{ij})|_{NB} = 0.$$

- Analog Maxwell's theory: $\nabla \cdot \mathbf{D} = 0 \to D_{2n} = D_{1n}$
- We expect

$$\sum_m (\overset{(m)}{K_E}{}_{an} - \overset{(m)}{K_E} h_{an})|_N = \sum_m (\overset{(m)}{K_{NB}}{}_{na} - \overset{(m)}{K_{NB}} h_{na})|_N = -T h_{na}|_N = 0$$

Energy Conservation at Node II

Junction condition on Net-brane leads to energy conservation at node.

- Problem I: no exact Killing vector, no exact conserved current

$$J^i = (K^{ij} - Kh^{ij})\xi_j \tag{8}$$

- Problem II: $K^{ij} - Kh^{ij}|_E$ is not the NCFT stress tensor
- Solution I: local Killing vector is sufficient

$$\xi_j = O(y), \quad \nabla_{(i}\xi_{j)} = 0 + O(y) \tag{9}$$

- Solution II: $K^{ij} - Kh^{ij}|_E$ include information of NCFT stress tensor

$$K_{na} \sim -\frac{\epsilon^{d-2}}{2} T^{\mathrm{CFT}}_{na}. \tag{10}$$

- We prove

$$\sum_m \overset{(m)}{T}{}^{\mathrm{CFT}}_{na}|_N = 0. \tag{11}$$

Typical solutions

By gluing Poincaré AdS/ black holes, we get gravity duals of general networks.

- Poincaré AdS

$$ds^2 = \frac{dz^2 - dt^2 + d\overset{(m)}{x}{}^2 + \delta_{ab}dy^a dy^b}{z^2}, \quad NB : \overset{(m)}{x} = -\sinh(\rho)z$$

Poincaré AdS is not the vacuum solution for general network.

- Black hole

$$ds^2 = \frac{\frac{dz^2}{f(z)} - f(z)dt^2 + d\overset{(m)}{x}{}^2 + \delta_{ab}dy^a dy^b}{z^2}, \quad NB : \overset{(m)}{x} = 0$$

- Black string

$$ds^2 = d\overset{(m)}{r}{}^2 + \cosh^2\overset{(m)}{r}\,\frac{\frac{dw^2}{h(w)} - h(w)dt^2 + \delta_{ab}dy^a dy^b}{w^2}, \quad NB : \overset{(m)}{r} = \rho$$

Perturbative solution I: conservation law at node

- Perturbative Poincaré AdS

$$ds^2 = \frac{dz^2 - dt^2 + d\overset{(m)}{x}{}^2 + \frac{2}{d} f_m(z, \overset{(m)}{x}) dt d\overset{(m)}{x} + \delta_{ab} dy^a dy^b}{z^2}, \quad (12)$$

- Solution

$$f_m(z, \overset{(m)}{x}) = X_m(\overset{(m)}{x}) + c_m z^d,$$

$$\overset{(m)}{T}{}^{\text{CFT}}_{xt} = c_m$$

- Junction condition

$$\cosh(\rho) \sum_m c_m = 0.$$

- Conservation of energy flux at node

$$\sum_m \overset{(m)}{T}{}^{\text{CFT}}_{xt}|_N = \sum_m c_m = 0.$$

Perturbative solution II: gravitational KK modes

Gravitational KK modes combines those of AdS/BCFT with NBC and DBC, corresponding to isolated and transparent modes.

- Perturbative metric

$$ds^2 = dr^2 + \cosh^2(r)\left(\bar{h}^{(0)}_{ij}(y) + \epsilon \overset{(m)}{H}(r)\bar{h}^{(1)}_{ij}(y)\right) dy^i dy^j$$

- EOM on Net-brane

$$\left(\bar{\Box} + 2 - M^2\right)\bar{h}^{(1)}_{ij}(y) = 0,$$

$$\cosh^2(r)\overset{(m)}{H}{}''(r) + d\sinh(r)\cosh(r)\overset{(m)}{H}{}'(r) + M^2\overset{(m)}{H}(r) = 0,$$

- Solutions of KK modes

$$\overset{(m)}{H}(r) = \overset{(m)}{c}H(r) = \overset{(m)}{c}\begin{cases} \mathrm{sech}^{\frac{d}{2}}(r)\; P^{\frac{d}{2}}_{l_M}(-\tanh r), & \text{even } d, \\ \mathrm{sech}^{\frac{d}{2}}(r)\; Q^{\frac{d}{2}}_{l_M}(-\tanh r), & \text{odd } d. \end{cases}$$

Perturbative solution II: gravitational KK modes

- Junction condition and continuity condition

$$\sum_{m=1}^{p} \overset{(m)}{H}{}'(\rho) = 0 \ \rightarrow \ \sum_{m=1}^{p} \overset{(m)}{c} H'(\rho) = 0,$$

$$\overset{(i)}{H}(\rho) = \overset{(j)}{H}(\rho) \ \rightarrow \quad \overset{(i)}{c} H(\rho) = \overset{(j)}{c} H(\rho).$$

- One class of modes obeying Neumann boundary condition

$$\text{NBC}: \ H'(\rho) = 0, \ \overset{(i)}{c} = \overset{(j)}{c}, \tag{13}$$

NBC corresponds to isolated mode $\overset{(m)}{J}{}_n|_N = 0$

- $(p-1)$ classes of modes satisfying Dirichlet boundary condition

$$\text{DBC}: \ H(\rho) = 0, \ \sum_{m=1}^{p} \overset{(m)}{c} = 0. \tag{14}$$

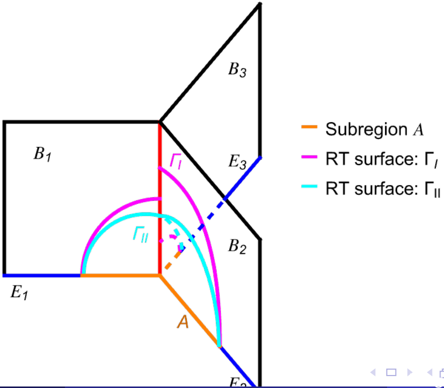
DBC corresponds to transparent mode $\overset{(m)}{J}{}_n|_N \neq 0$

Outline

Proposal of HEE

While proposal I leads to smaller HEE, proposal II is the correct one.

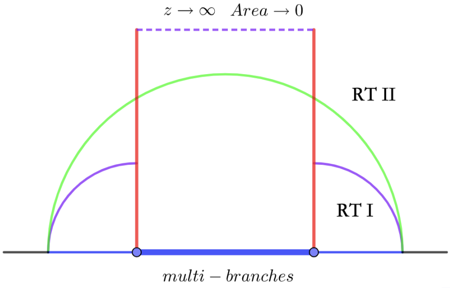
- Disconnected proposal I: inspired by AdS/BCFT, RT surfaces are perpendicular to Net-brane
- Connected proposal II: RT surfaces must be interconnected through the same intersection on Net-brane



Why connected RT surface?

RT surfaces intersect at the same point on Net-brane for connected subsystems within the network

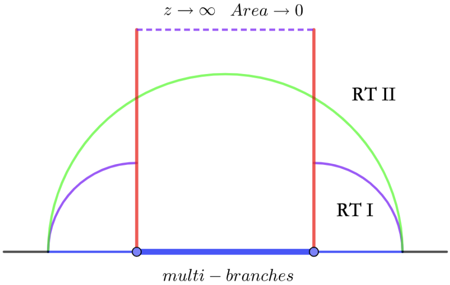
- Naturally, connected subsystem corresponds to connected RT surface.
- Both proposal I and proposal II obeys strong subadditivity of entanglement entropy.
- Only proposal II agrees with monotonicity of entanglement entropy when increasing numbers or lengths of internal edges.



Monotonicity of entanglement entropy I

EE increases with numbers of internal edges.

- Label network with p internal edges by Net(p), the HEE by $S(p)$
- Remove one internal edge while keeping invariant the RT surfaces of other edges, $S(p) > S_0(p-1)$
- RT surfaces change between Net(p) and Net($p-1$), $S_0(p-1) > S(p-1)$.



Monotonicity of entanglement entropy II

EE increases with the length of internal edges.

- Focus on Poincaré AdS, increase edge length by $dL \to 0$, we get

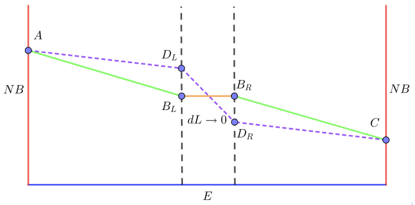
$$\text{Area}(D_L D_R) > \text{Area}(D_L B_L \cup D_R B_R)$$

- Near the local regions of $D_L B_L$ and $D_R B_R$, we have

$$\text{Area}(AD_L \cup D_L B_L) > \text{Area}(AB_L), \quad \text{Area}(CD_R \cup D_R B_R) > \text{Area}(CB_R)$$

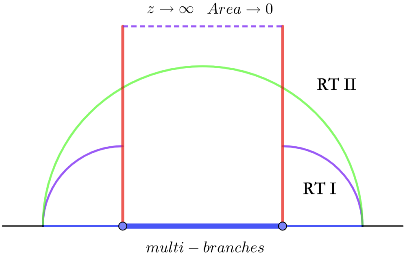
- Finally, we have at the linear order of $dL \to 0$

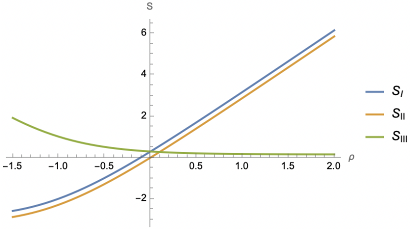
$$\text{Area}(AD_L \cup D_L D_R \cup D_R C) > \text{Area}(AB_L \cup B_R C).$$



We propose several natural definitions of network entropy.

- Proposal I: $S_{\mathrm{I}} = S_{\mathrm{NCFT}} - S_{\mathrm{CFT}}$, obeying g-theorem $S_{\mathrm{I}}|_{\mathrm{IR}} \leq S_{\mathrm{I}}|_{\mathrm{UV}}$
- Proposal II: $S_{\mathrm{II}} = S(\rho) - S(0)$, obeying g-theorem $S_{\mathrm{II}}|_{\mathrm{IR}} \leq S_{\mathrm{II}}|_{\mathrm{UV}}$
- Proposal III: $S_{\mathrm{III}} = S_{\mathrm{NCFT}} - S_{\mathrm{BCFT}} > 0$, entanglement excluding isolated modes from BCFT.





Outline

Shortest path problem I

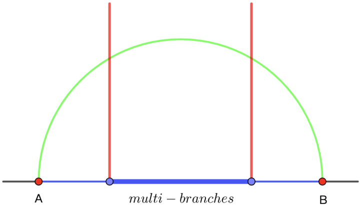
The shortest path problem seeks to determine shortest path between two points within networks. equivalent to finding the shortest distance in bulk.

- Bulk: multi-Euclidean AdS_2 glued by tensionless Net-branes

$$d\overset{(m)}{s}{}^2 = \frac{dz^2 + d\overset{(m)}{x}{}^2}{z^2}, \ 0 \leq \overset{(m)}{x} \leq L_m,$$

- Every loop-free path from A to B is dual to a single AdS_2

$$L_{AB} = 2\log(\frac{l_{AB}}{\epsilon})$$



Shortest path problem II

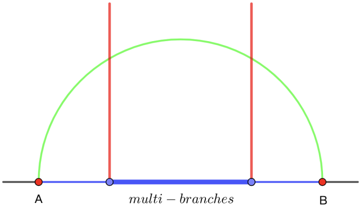
The shortest path problem is equivalent to calculating the holographic two-point correlators of massive operators.

- Every loop-free path from A to B is dual to a single AdS_2

$$L_{AB} = 2\log(\frac{l_{AB}}{\epsilon})$$

- Two-point function of operators dual to massive particles is determined by the proper distance in bulk

$$\langle O(A)O(B)\rangle \sim e^{-ML_{AB}}, \tag{15}$$



Summary and Outlook

Summary:

- We define NCFT (network CFT) and propose its gravity dual.
- Junction condition in bulk leads to energy conservation on node.
- KK modes on Net-brane combines that of AdS/BCFT with NBC and DBC.
- We propose rules to calculate HEE in AdS/NCFT and define a positive network entropy.
- We discuss the holographic dual of shortest-path problems.

Outlook:

- Multi-branches of braneworld, holographic circuit/chip/AI?
- EE for free NCFT
- Phase transitions of HEE in AdS/NCFT
- New insight into famous network problem?

Thanks!

Welcome to new world of holographic network!