Chaos in the matrix models for holographic QCD

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Holographic applications: from Quantum Realms to the Big Bang

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Outline

- **Introduction: motivations, chaos, coupled oscillators and gauge theory**
- **Matrix models for mesons and baryons as coupled oscillators (the D4/D6/D6-bar approach)**
- **The classical and quantum chaos**
- **Summary**

1. Introduction: motivations, chaos, coupled oscillators and gauge theory

The motivation

Chaos: A bridge connecting the gravity theory and quantum theory…
Coupled oscillator: simple model in classical and quantum theories…

Chaos relates to the topics:

1. Nonlinear dynamics

S. Strogatz, “Nonlinear Dynamics and Chaos: With Applications to Physics, Biology Chemistry, and Engineering,” Westview Press.
K. Hashimoto, K. Murata, R.Yoshii, “Out-of-time-order correlators in quantum mechanics”, JHEP 10 (2017) 138, arXiv:1703.09435.

2. Quantum mechanics, quantum information

P. Hosur, X. Qi, D. Roberts, B. Yoshida, “Chaos in quantum channels”, JHEP 02 (2016) 004, arXiv:1511.04021.

3.Gauge theory

S. Matinyan, G. Savvidy, N. Savvidy, “Stochasticity of Classical Yang-Mills Mechanics and Its Elimination by Higgs Mechanism. (In Russian),” JETP Lett.34, 590-593 (1981)

4. SYK (Sachdev-Ye-Kitaev) model

J. Maldacena, D. Stanford, “Remarks on the Sachdev-Ye-Kitaev model”, Phys.Rev.D 94 (2016) 10, 106002, arXiv:1604.07818.

5. Gauge-gravity duality, matrix models, quantum gravity…

J. Maldacena, S Shenke, D. Stanford, “A bound on chaos”, JHEP 08 (2016) 106, arXiv:1503.01409.
K. Hashimoto, K. Murata, K.Yoshida, “Chaos in chiral condensates in gauge theories”, Phys.Rev.Lett. 117 (2016) 23, 231602, arXiv:1605.08124.
O. Fukushima, K. Yoshida, “Chaotic instability in the BFSS matrix model”, JHEP 09 (2022) 039, arXiv:2204.06391.
J. Polchinski, “Chaos in the black hole S-matrix”, arXiv:1505.08108.

Measurements for the classical chaos

1. Parcare section

A surface in phase space (defined by canonical coordinates and momenta) that transversally intersects the system's flow. All possible orbits cross this section repeatedly.

Orbits or trajectories on the Parcae section are discrete points:

Regular → Non-chaotic

Radom → Chaotic

2. Lyapunov exponent

The Lyapunov exponent quantifies the sensitivity of a classical system's evolution to initial conditions in time.

$$e^{Lt} = \left[\frac{\delta x\,(t)}{\delta x\,(0)}\right]^2$$

L : The Lyapunov coefficient; *t* : Time
x(t) the classical solution of a system
x(0) the initial condition of the classical solution *x(t)*

L>0, chaotic
L<0, regular
L=0, critical

Quantum chaos and Out-of-Time-Order Correlator (OTOC)

The OTOC is a quantum analog of the classical Lyapunov exponent:

$$e^{Lt} = \left[\frac{\delta x(t)}{\delta x(0)}\right]^2 = \{x(t), p(0)\}^2_{\text{P.B.}}$$

The quantum version: replace the Poisson bracket with the quantum commutator $\{,\}_{\text{P.B.}} \to -i[,]$ then take the average.

The microcanonical OTOC $c_n(t)$

$$c_n(t) = -\left\langle n \left| [x(t), p(0)]^2 \right| n \right\rangle$$

$|n\rangle$: the eigenstate of the Hamiltonian;
x(t), p(0) : operators (canonical coordinates and momentum) in the Heisenberg representation

The thermal OTOC $C_T(t)$

Taking the statistical average, Z is the partition function

$$e^{L_T t} = C_T(t) = \left\langle [x(t), p(0)]^2 \right\rangle_T = \frac{1}{Z}\sum_n c_n e^{-\frac{E_n}{T}},\ Z = e^{-\frac{E_n}{T}}$$

Generic form

$$C_T = -\left\langle [W(t), V(0)]^2 \right\rangle_T$$

Gauge theory with spontaneously broken symmetry as coupled oscillators

The SU(2) gauge theory with Higgs scalar

$$L = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \left(D_\mu\phi\right)^\dagger\left(D_\mu\phi\right) - V\left(\phi\right),$$

$$V\left(\phi\right) = -\chi^2\left|\phi\right|^2 + \frac{\lambda}{2}\left|\phi\right|^4, \chi^2 > 0$$

With the VEV (vacuum expectation value) for the Higgs scaler $\langle\phi\rangle = \left(\chi^2/\lambda\right)^{1/2}\xi$

$$L = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - m_A^2\xi^\dagger A_\mu A^\mu\xi + \ldots m_A = \left(\frac{\chi^2}{\lambda}\right)^{1/2}$$

Impose the spacially homogeneous ansatz

$$A_1 = \frac{1}{\sqrt{2}}x\left(t\right)\sigma^1, A_2 = \frac{1}{\sqrt{2}}y\left(t\right)\sigma^2,$$

The SU(2) gauge theory reduces to 2d coupled oscillator with homogeneous mass term

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m^2\left(x^2 + y^2\right) + gx^2y^2,$$

$$p_{x,y} = \dot{x}, \dot{y}, m = m_A$$

M. E. Peskin, D.V. Schroeder, “An introduction to quantum field theory”.

Gauge theory with spontaneously broken symmetry as coupled oscillators

The SU(2) gauge theory with Higgs scalar in the vector representation

$$L = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}f^{abc}f^{dec}A^a_\mu A^d_\mu \langle\phi^e\rangle \left\langle\phi^b\right\rangle + \dots$$

With the VEV for the Higgs scaler $\langle\phi^a\rangle = m_A\delta^{3a}$

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}m_A^2\left[\left(A^1_\mu\right)^2 + \left(A^2_\mu\right)^2\right] + \dots$$

Impose the spacially homogeneous ansatz

$$A^1_1 = x\,(t), A^2_2 = y\,(t), A^3_3 = w\,(t),$$

The SU(2) gauge theory reduces to 3d coupled oscillator with inhomogeneous mass term

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_w^2\right) + \frac{1}{2}m^2\left(x^2 + y^2\right) + g\left(x^2y^2 + x^2w^2 + y^2w^2\right)$$

M. E. Peskin, D.V. Schroeder, “An introduction to quantum field theory”.

Exercise: The OTOC for 1D harmonic oscillator

The Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \,.$$

The equations of motion in the Heisenberg representation

$$\frac{\partial x(t)}{\partial t} = -i\,[x, H] = \frac{p}{m},$$
$$\frac{\partial p(t)}{\partial t} = -i\,\left[p, H\right] = -m\omega^2 x,$$

Analytical solution for the operators

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m}\sin\omega t,$$
$$p(t) = \omega p(0)\cos\omega t - m\omega x(0)\sin\omega t,$$

The microcanonical OTOC

$$c_n(t) = -\left\langle \left[x(t), p(0)\right] \right\rangle^2 = \cos^2\omega t,$$

The thermal OTOC

$$C_T(t) = \frac{1}{Z}\sum_n c_n(t)\, e^{-\frac{E_n}{T}} = \cos^2\omega t$$

2. Matrix models for mesons and baryons as coupled oscillators (the D4/D6/D6-bar approach)

The D4/D6/D6-bar model

Bulk supergravity

$$S_{\mathrm{IIA}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(\mathscr{R} + 4\partial_M\phi\partial^M\phi\right) - \frac{g_s^2}{2}\left|F_4\right|^2\right],$$

The D4-brane soliton solution (Bubble)

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}\left[\eta_{\mu\nu}dx^\mu dx^\nu + f(U)\,d\tau^2\right] + \left(\frac{R}{U}\right)^{3/2}\frac{dU^2}{f(U)} + R^{3/2}U^{1/2}d\Omega_4,$$

$$e^\phi = \left(\frac{U}{R}\right)^{3/4},\ F_4 = dC_3 = \frac{N_c}{\Omega_4}\epsilon_4,\ f(U) = 1 - \frac{U_{KK}^3}{U^3}.$$

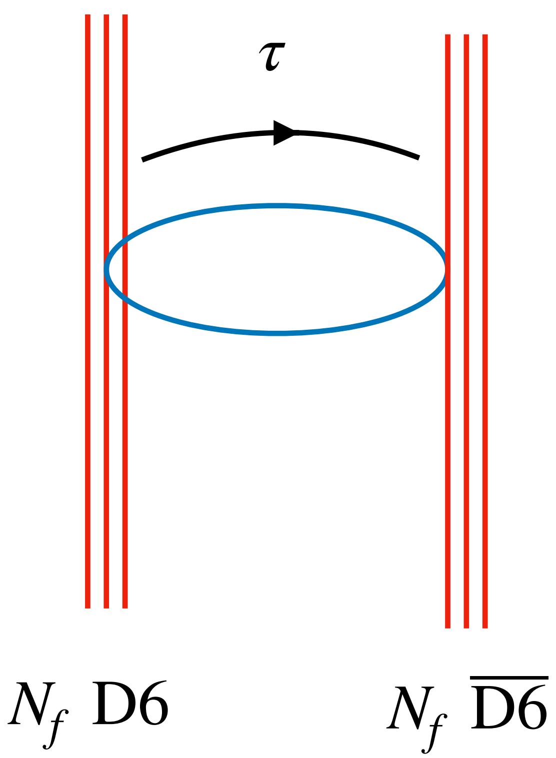
τ is compactified on a circle S^1 with size $\delta\tau$

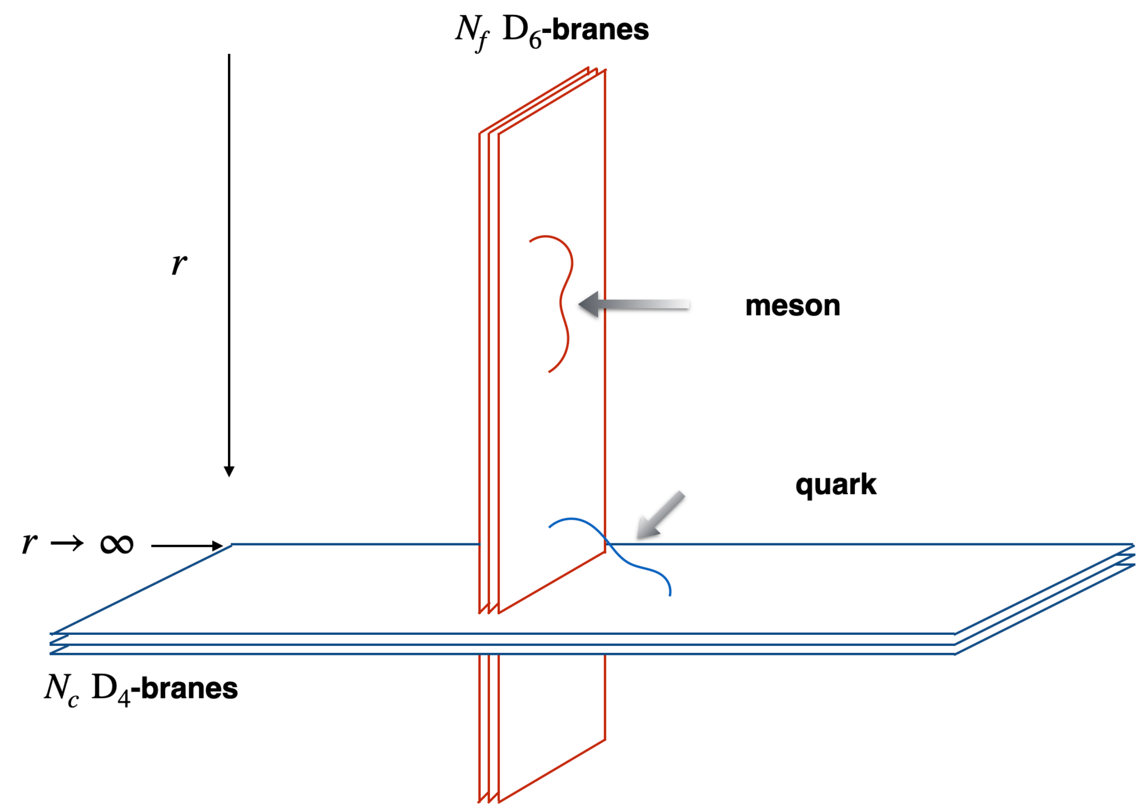
$$\tau \sim \tau + \delta\tau, M_{KK} = \frac{2\pi}{\delta\tau} = \frac{3U_{KK}^{1/2}}{2R^{3/2}}$$

E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories", Adv. Theor. Math. Phys. 2 (1998), 505-532, arXiv:hep-th/9803131.

The D4/D6/D6-bar model

The intersectional embedding of the D6/D6-bar





	0	1	2	3	4 (τ)	5 (U)	6	7	8	9
D4-branes	-	-	-	-	-					
D6/$\overline{\text{D6}}$-branes	-	-	-	-		-	-	-		
Baryon vertex	-						-	-	-	-

Table 1: The D-brane configuration in the D4/D6/$\overline{\text{D6}}$ approach.

M. Kruczenski, D.Mateos, R. Myers, D. Winters, “Towards a holographic dual of large-Nc QCD”, JHEP 05 (2004) 041, arXiv:hep-th/0311270.

The matrix model for mesons as coupled oscillators

The induced metric on the D6-branes

$$ds^2_{\mathrm{D6}/\overline{\mathrm{D6}}} = \left(\frac{U}{R}\right)^{3/2} \eta_{\mu\nu} dx^\mu dx^\nu + K\left(d\xi^2 + \xi^2 d\Omega_2^2\right).$$

$$U(\rho) = A(\rho)\, U_{KK},$$

$$A(\rho) = \left(\rho^{3/2} + \frac{1}{4\rho^{3/2}}\right)^{2/3},$$

$$K(\rho) = \frac{R^{3/2} U_{KK}^{1/2} A^{1/2}}{\rho^2},$$

The matrix action for the mesonic excitation on the D6-branes (i.e. fluxes of $X_{8,9}$)

$$S_{\mathrm{D6}} = -T_{\mathrm{D6}} \int d^7x e^{-\phi} \sqrt{-g}\,\mathrm{Tr}\left\{\frac{1}{4} g^{ab} \partial_a w^m \partial_b w^n g_{mn} - \frac{1}{4\left(2\pi\alpha'\right)^2} g_{mr} g_{ns} \left[w^m, w^n\right]\left[w^r, w^s\right]\right\}$$

where $w^8 = \delta X_8, w^9 = \delta X_9$ are generators of $U(N_f)$

The matrix model for mesons coupled oscillators

Impose the spacially homogeneous ansatz

$$w^8 = \mathcal{N} P\left(\xi\right) \frac{x\left(t\right)}{2\pi\alpha'}\sigma^1, w^9 = \mathcal{N} P\left(\xi\right) \frac{y\left(t\right)}{2\pi\alpha'}\sigma^2, P\left(\xi\right) = A^{-1}\rho^{-1/2},$$

The matrix action can be simplified to be a 2D coupled oscillator

$$S_{\mathrm{D6}} = V_3 \int dt \left[\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2} m^2 \left(x^2 + y^2\right) - g x^2 y^2\right].$$

with the normalization and the parameters

$$\frac{M_{KK}^2 N_c \lambda^2 \mathcal{N}^2}{2^4 3^4} \int d\xi \frac{A^2 \xi^2 P^2}{\rho^4} = \frac{1}{4},$$

$$\frac{M_{KK}^4 N_c \lambda^2 \mathcal{N}^2}{2 \times 3^6} \int d\xi \frac{A^3 \xi^2}{\rho^2} P'^2 = m^2 = \frac{4}{27} M_{KK}^2 \left[7 + 11\ {}_2F_1\left(1,1,\frac{1}{3},-1\right)\right] \simeq M_{KK}^2,$$

$$\frac{M_{KK}^4 N_c \lambda^4 \mathcal{N}^4}{2^2 3^8 \pi^2} \int d\xi \frac{A^4 \xi^2}{\rho^6} P^4 = g = \frac{16 \times 2^{2/3}}{5\pi^2 N_c} \simeq 0.55 N_c^{-1}.$$

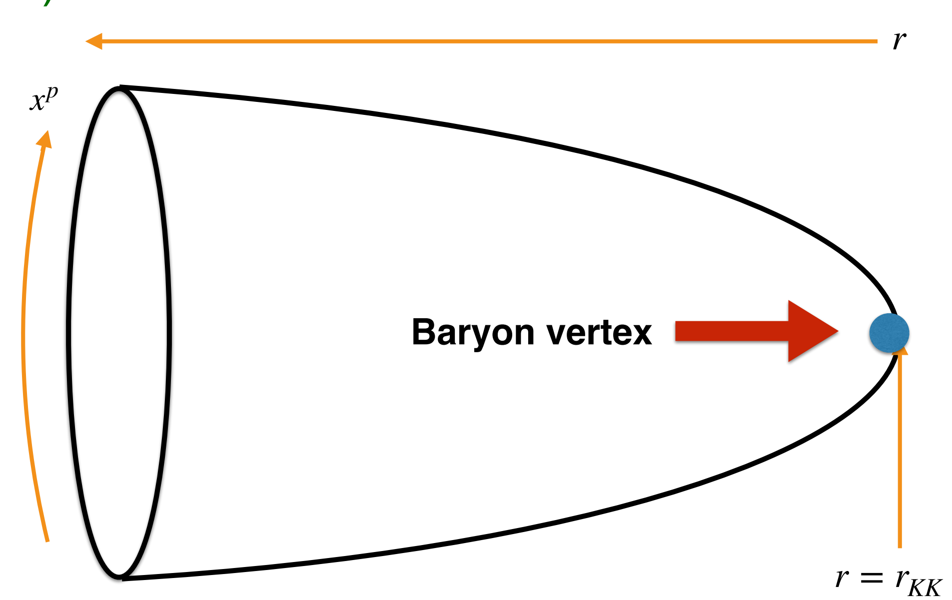
The matrix model for baryons as coupled oscillators

- **Baryon in the Dp-brane background: wrapped D*8-p* -brane on S^{8-p} with N_c string endpoints (the baryon vertex)**
  E.Witten, JHEP 9807 (1998) 006
  D. J. Gross, H. Ooguri, Phys.Rev. D58 (1998) 106002

  - D3/D7 model: Baryon vertex is a D5-brane wrapped on S^5

- **Baryon in the D4-brane background:**
  - D4-brane wrapped on S^4 (the baryon vertex)
  T. Sakai, S. Sugimoto, Prog.Theor. Phys. 113 (2005) 843-882
  H. Hata, T. Sakai, S. Sugimoto, S. Yamato, Prog.Theor. Phys. 117 (2007) 1157

  - equivalently: instantons on the D8-branes
  D.Tong, hep-th/0509216



The matrix model for baryons as coupled oscillators

The action for the baryon vertex

$$S_{\mathrm{D4}} = S_{\mathrm{DBI}} + S_{\mathrm{CS}}$$

$$S_{\mathrm{DBI}} = -T_{\mathrm{D4}} \int d^5 x e^{-\phi} \mathrm{Tr} \sqrt{-\det\left[g_{ab} + 2\pi\alpha' F_{ab}\right]}$$

$$S_{\mathrm{CS}} = -g_s T_{\mathrm{D4}} \left(2\pi\alpha'\right) \mathrm{Tr} \int F \wedge C_3,$$

Expand the action up to quadratic terms and impose the T-duality rules,

$$S = \frac{2}{27\pi} \lambda M_{KK} N_c \int dt \mathrm{Tr} \left\{ \frac{1}{2} D_0 X^M D_0 X^M - \frac{2}{3} M_{KK}^2 \left(X^4\right)^2 + \frac{2}{3^6 \pi^2} \lambda^2 M_{KK}^4 \left[X^M, X^N\right]^2 \right\}$$
$$+ N_c \int dt \mathrm{Tr} A_0,$$

X is the U(k) matrix, where k is the number of the baryon with the covariant derivative $D_0 X^M = \partial_0 X^M - i\left[A_0, X^M\right], M, N = 1,2,3,4$

K. Hashimoto, N. Iizuka, P. Yi, “A Matrix Model for Baryons and Nuclear Forces”, JHEP 10 (2010) 003, arXiv:1003.4988.
S. Aoki, K. Hashimoto, N. Iizuka, “Matrix Theory for Baryons: An Overview of Holographic QCD for Nuclear Physics”, Rept.Prog.Phys. 76 (2013) 104301, arXiv:1203.5386.
S. Li, T. Jia, “Matrix model and Holographic Baryons in the D0-D4 background”, Phys.Rev.D 92 (2015) 4, 046007, arXiv:1506.00068.

The matrix model for baryons as coupled oscillators

Impose the spacially homogeneous ansatz

$$X^1 = \frac{1}{\mathcal{N}}\frac{\sigma^1}{\sqrt{2}}x(t), X^2 = \frac{1}{\mathcal{N}}\frac{\sigma^2}{\sqrt{2}}y(t), X^3 = \frac{1}{\mathcal{N}}\frac{h(t)}{\sqrt{2}}\mathbf{1}_{2\times 2}, X^4 = \frac{1}{\mathcal{N}}\frac{\sigma^3}{\sqrt{2}}w(t), A_0 = \frac{\hat{A}_0(t)}{\sqrt{2}}\mathbf{1}_{2\times 2},$$

The baryonic matrix model can be simplified to be a 3D coupled oscillator with inhomogeneous mass term

$$S_{\mathrm{ocs}} = \int dt \left[\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{w}^2\right) - \frac{1}{2}m^2 w^2 - g\left(x^2 y^2 + x^2 w^2 + y^2 w^2\right)\right],$$

$$m = \frac{2\sqrt{3}}{3}M_{KK}, g = \frac{4M_{KK}^3}{27\pi N_c}, \mathcal{N} = \left(\frac{2}{27\pi}\lambda M_{KK} N_c\right)^{1/2}$$

3. The classical and quantum chaos

The models and the methods

The mesonic matrix model

$$S_{\rm D6} = V_3 \int dt \left[\frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 \right) - \frac{1}{2} m^2 \left(x^2 + y^2 \right) - g x^2 y^2 \right].$$

The baryonic matrix model

$$S_{\rm ocs} = \int dt \left[\frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 + \dot{w}^2 \right) - \frac{1}{2} m^2 w^2 - g \left(x^2 y^2 + x^2 w^2 + y^2 w^2 \right) \right],$$

Classical chaos

Parcare section:

1. solve the classical equations of motion with various total energies numerically
2. find all the position of t satisfying x(t)=0 as the Parcare section in the phase space.

Lyapunov exponent:

1. solve the classical equations of motion with various total energies numerically
2. Taking the average of the Lyapunov coefficient at late-time $t \to \infty$

Analysis of the classical chaos

1. Parcare section

The mesonic matrix model

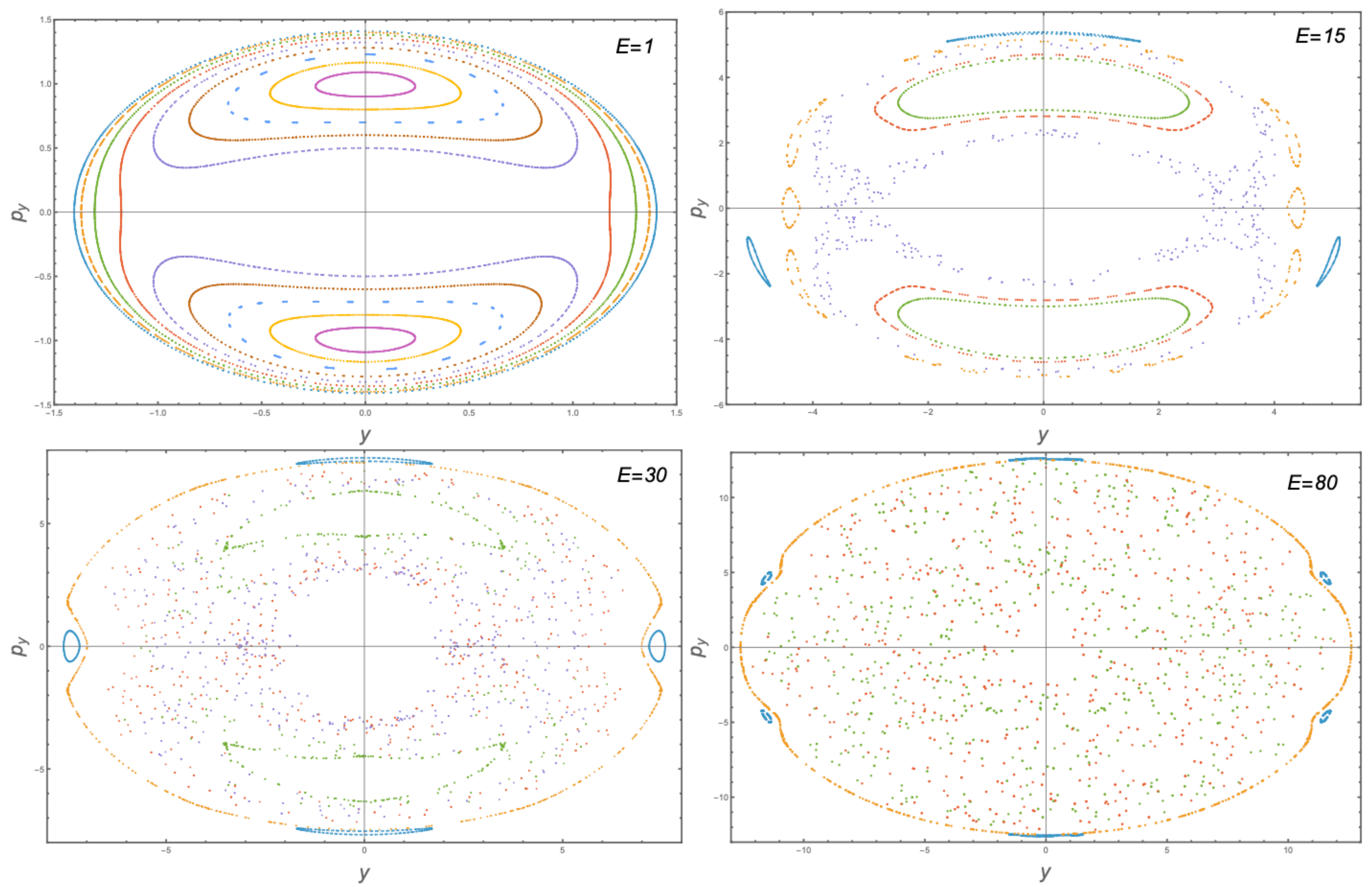


Figure 1: The classical trajectory on the Poincare section of the mesonic matrix model at $x(t) = 0$ with various energy $E = 1, 15, 30, 80$. The horizontal axis is $y(t)$, the vertical axis is $p_y = \dot{y}(t)$.

Analysis of the classical chaos

The baryonic matrix model

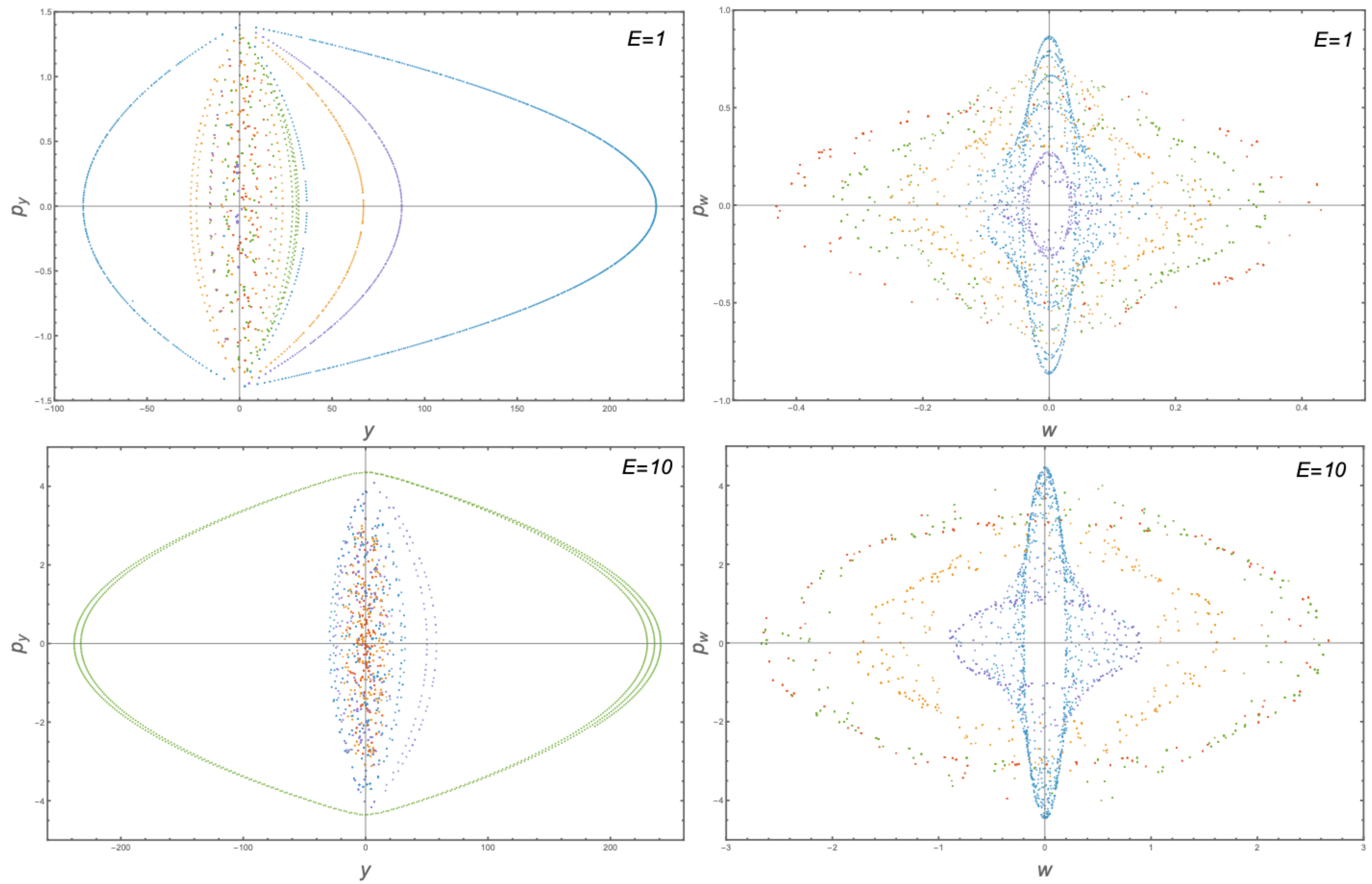


Figure 2: The classical trajectory on the Poincare section of the baryonic matrix model at $x(t) = 0$ with various energy $E = 1, 10$. The horizontal axis is $y(t)$ or $w(t)$, the vertical axis is $p_y = \dot{y}(t)$ or $p_w = \dot{w}(t)$.

Analysis of the classical chaos

2. Lyapunov exponent

The mesonic matrix model

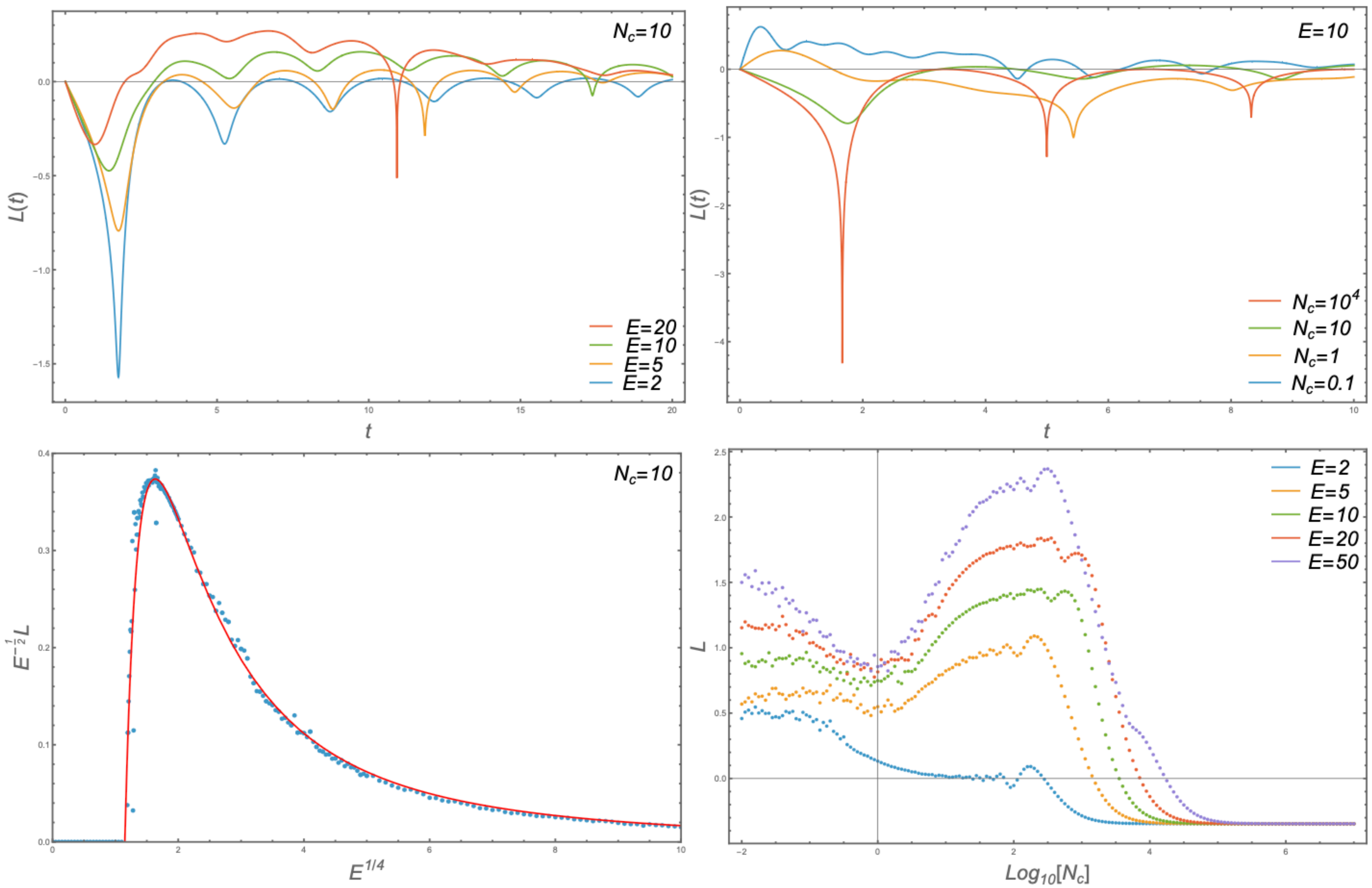


Figure 3: Lyapunov coefficient of the mesonic matrix model . **Upper:** Lyapunov coefficient as a function of time t. **Lower:** Lyapunov coefficient as a function of E and N_c.

Analysis of the classical chaos

2. Lyapunov exponent

The baryonic matrix model

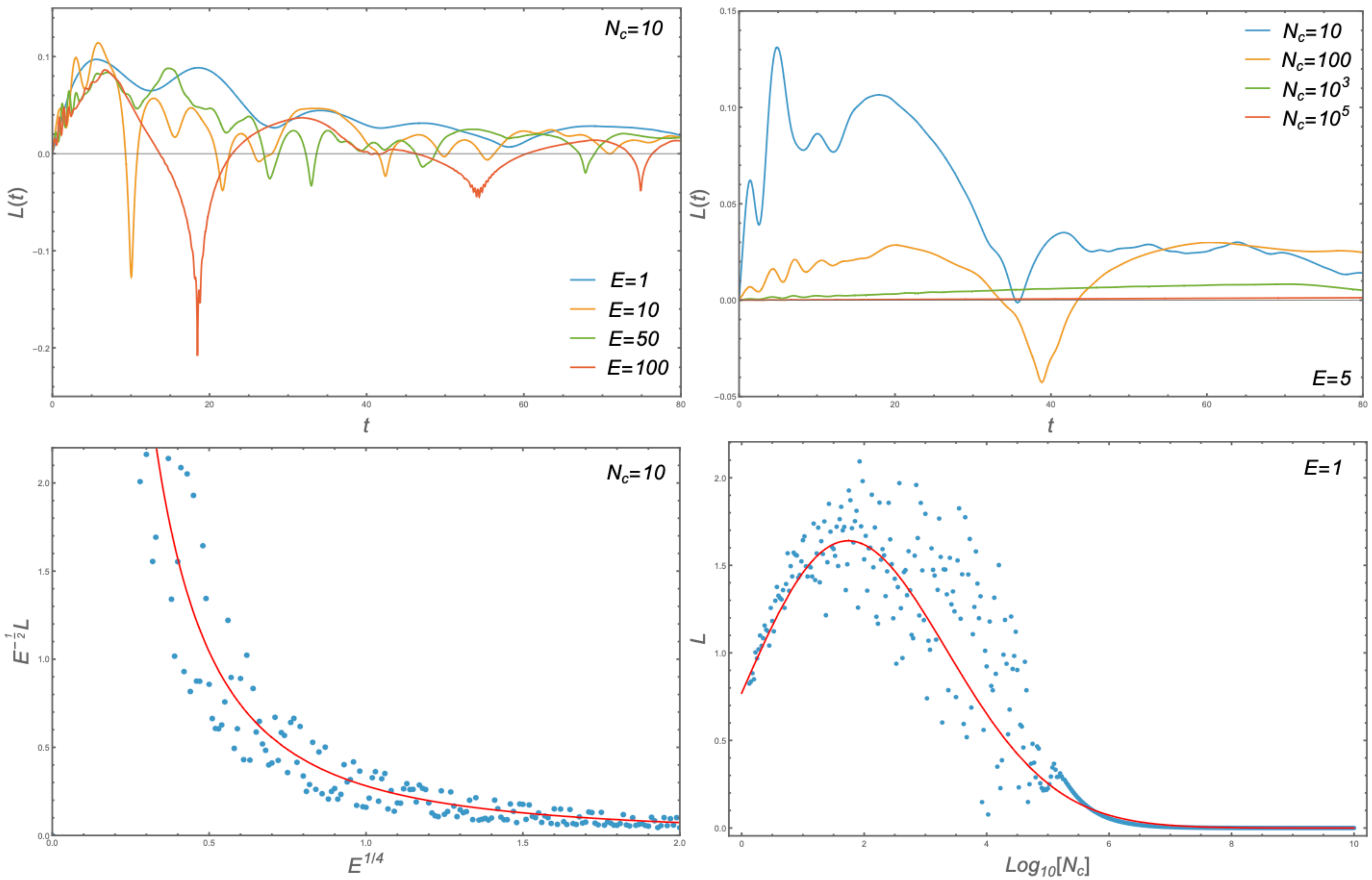


Figure 4: Lyapunov coefficient of the baryonic matrix model . **Upper:** Lyapunov coefficient as a function of time t. **Lower:** Lyapunov coefficient as a function of E and N_c.

Analysis of the quantum chaos

The numerical method for OTOC

The equations of motion in the Heisenberg representation is usually non-analytical, so a numerical method is necessary.

$$c_n(t) = -\left\langle n \left| \left[x(t), p\right]^2 \right| n \right\rangle,$$

$$c_n(t) = -\sum_m \left\langle n \left| \left[x(t), p\right] \, |m\rangle \langle m| \, \left[x(t), p\right] \right| n \right\rangle \equiv \sum_m b_{nm} b_{nm}^*,$$

$$b_{nm}(t) = -i \left\langle n \left| \left[x(t), p\right] \right| m \right\rangle .$$

the completeness condition $\sum_m |m\rangle \langle m| = 1$

Use the time-dependent unitary transformation $x(t) = e^{iHt} x e^{-iHt}$

Analysis of the quantum chaos

$$b_{nm}(t) = -i\sum_k \left(e^{iE_{nk}t}x_{nk}p_{km} - e^{iE_{km}t}p_{nk}x_{km}\right),$$

where $\left\langle n \left| x \right| m \right\rangle = x_{nm}, p_{nm} = \left\langle n \left| x \right| m \right\rangle$ are the matrix elements in the energy representation, $E_{nm} = E_n - E_m$. Since the Hamiltonian is proportional to p^2 $\left(H = \dfrac{p^2}{2m} + \ldots\right)$, one can find

$$\frac{1}{i}[x, H] = p$$

$$p_{nm} = -i\left\langle n \left| xH - Hx \right| m \right\rangle = iE_{nm}x_{nm}.$$

$$b_{nm}(t) = \sum_k x_{nk}x_{km}\left(e^{iE_{nk}t}E_{km} - e^{iE_{km}t}E_{nk}\right).$$

Analysis of the quantum chaos

The inputs for the numerical calculation: eigenvalues and eigenfunctions

mesonic matrix model

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{1}{2}m^2\left(x^2+y^2\right)+gx^2y^2\right]\psi_n=E_n\psi_n,$$

baryonic matrix model

$$\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial w^2}\right)+\frac{1}{2}m^2w^2+g\left(x^2y^2+x^2w^2+y^2w^2\right)\right]\psi_n=E_n\psi_n$$

Analysis of the quantum chaos

OTOCs, eigenvalues, quantum Lyapunov coefficient in the mesonic matrix model

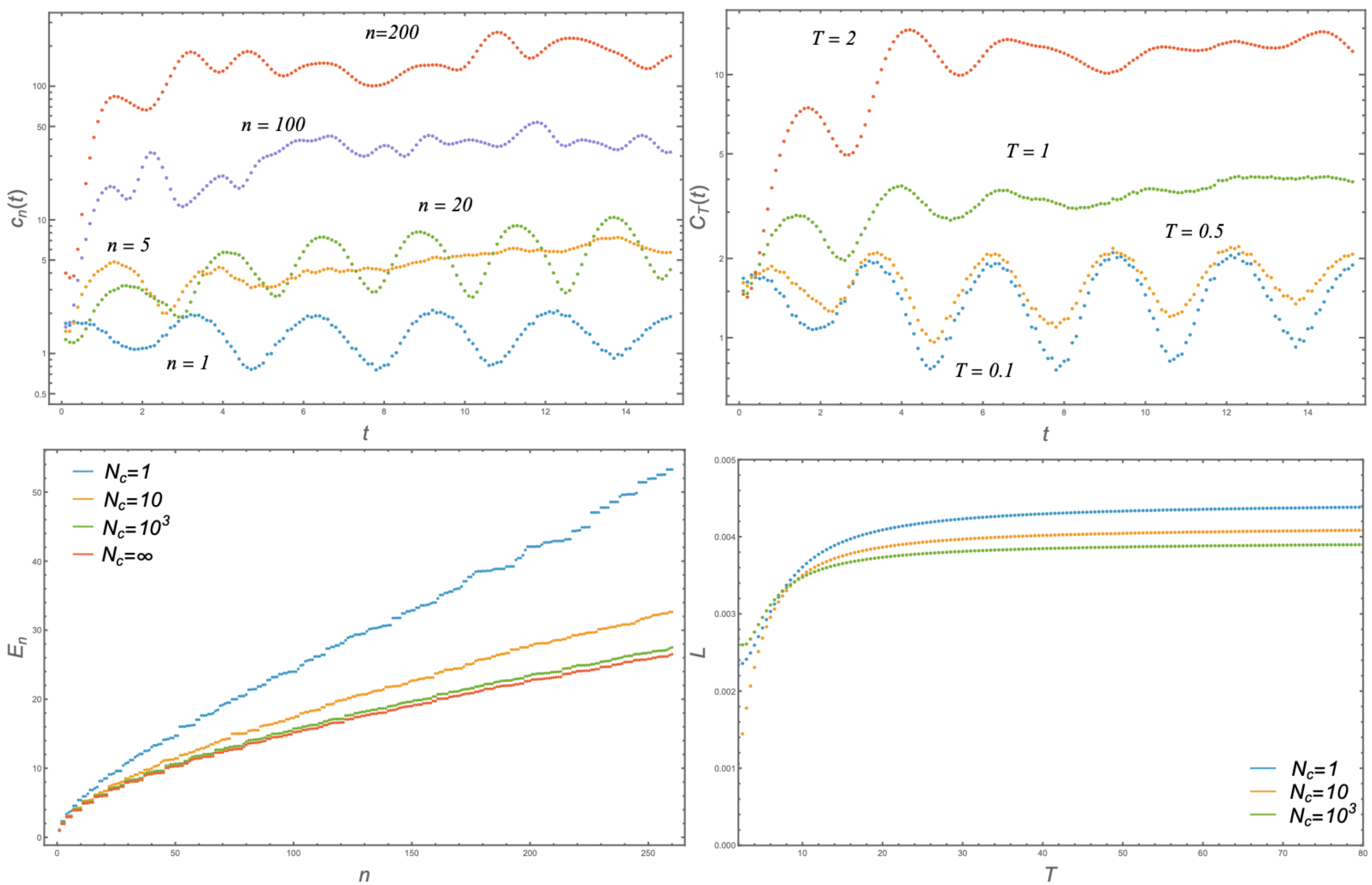


Figure 5: The quantum properties of the mesonic matrix model including the microcanonical and thermal OTOCs, average of the quantum Lyapunov coefficient and eigen energies. **Upper:** The microcanonical OTOC $c_n(t)$ and thermal OTOC $C_T(t)$ as functions of time t at various temperatures with $N_c = 10$. **Lower:** Eigen energies E_n as a function of quantum number n and the average of quantum Lyapunov coefficient L as a function of temperature T with various N_c.

Analysis of the quantum chaos

OTOCs, eigenvalues, quantum Lyapunov coefficient in the baryonic matrix model

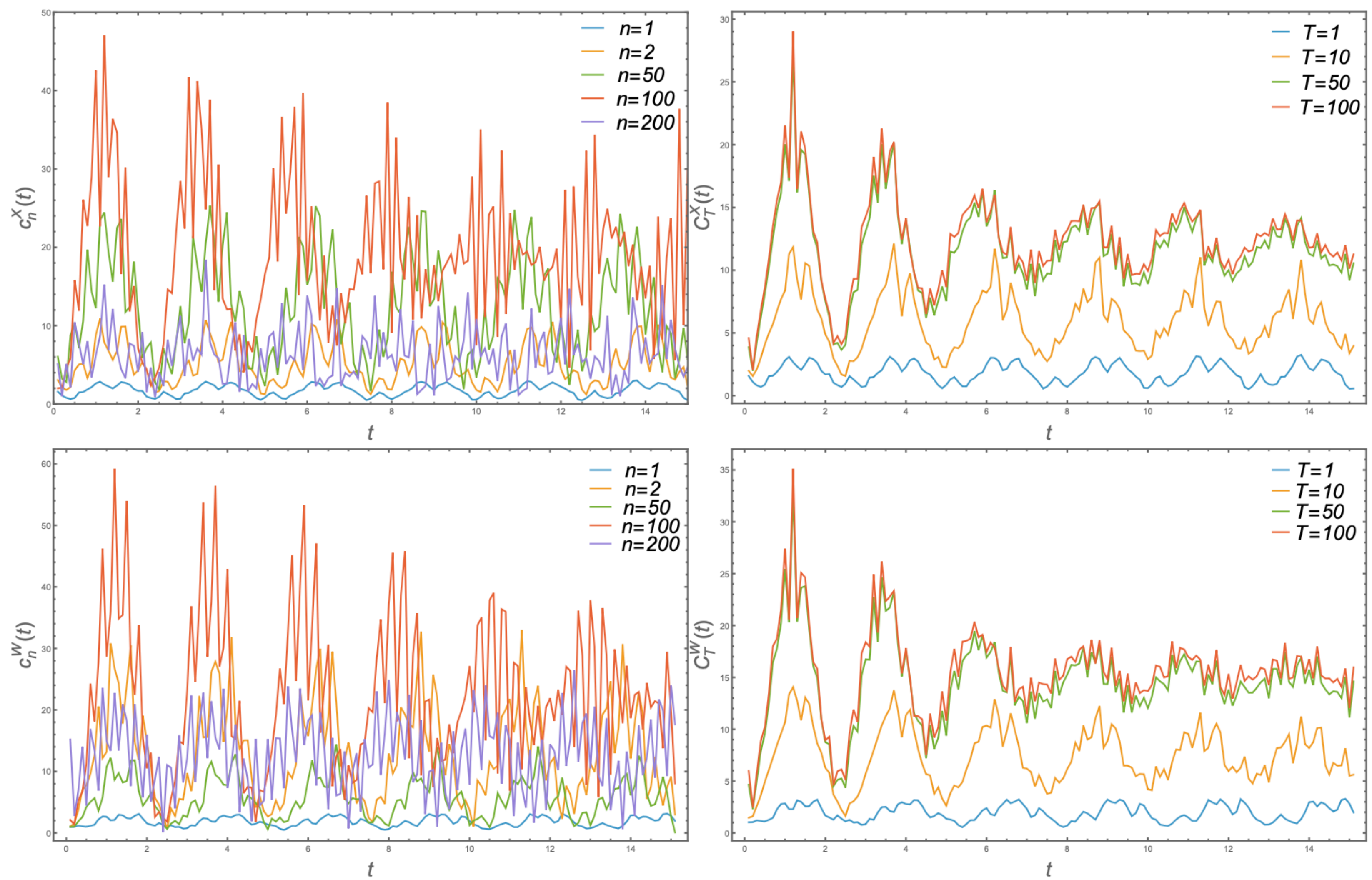


Figure 6: The microcanonical and thermal OTOCs of the baryonic matrix model at $N_c = 10$. The index X, W refers respectively to the OTOCs form the commutator of $[x(t), p_x]^2$ and $[w(t), p_w]^2$. **Upper:** The microcanonical OTOC $c_n^X(t)$ and thermal OTOC $C_T^X(t)$ as functions of time t with various temperatures. **Lower:** The microcanonical OTOC $c_n^W(t)$ and thermal OTOC $C_T^W(t)$ functions of time t with various temperatures.

Analysis of the quantum chaos

OTOCs, eigenvalues, quantum Lyapunov coefficient in the baryonic matrix model

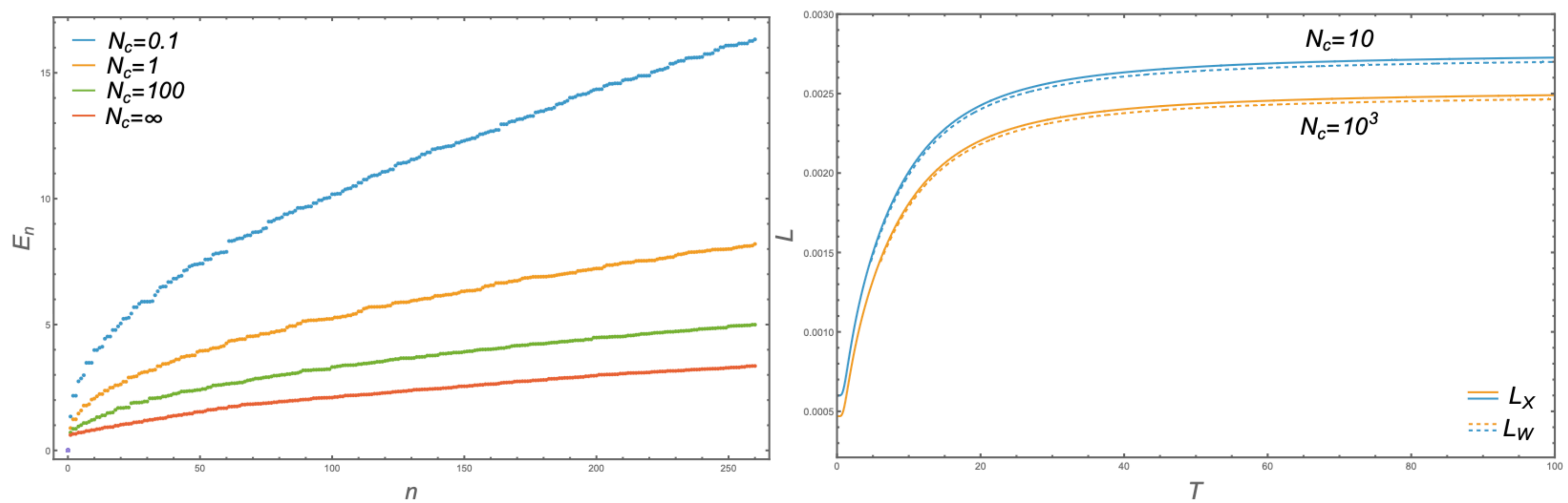


Figure 7: Energy spectrum E_n and quantum Lyapunov coefficient L of the baryonic matrix model with various N_c. The index X, W refers respectively to the Lyapunov coefficient from the OTOC $C_T^X(t)$ and $C_T^W(t)$. **Left:** Energy spectrum E_n as a function of the quantum number n. **Right:** Lyapunov coefficient L as a function of temperature T.

Summary

We study the classical and quantum chaos in the matrix models for mesons and baryons from the D4/D6/D6 bar approach

Analysis of the classical chaos:

The mesonic matrix model displays the phase transition from regular phase to chaotic phase when the total energy increases. The baryonic matrix model does not display the phase transition exactly.

The Lyapunov coefficient decreases when the color number increases as it is expected.

K. Hashimoto, K. Murata, K.Yoshida, "Chaos in chiral condensates in gauge theories", Phys.Rev.Lett. 117 (2016) 23, 231602, arXiv:1605.08124.

Analysis of the quantum chaos:

the thermal OTOC oscillates periodically in time direction at low temperature while it trends to be saturated at high temperature.

the quantum Lyapunov coefficient begins to saturate after a critical energy (as a critical temperature) and this behavior covers qualitatively the analysis of the classical Lyapunov coefficient

The quantum Lyapunov coefficient decreases when the color number increases