Applications of numerical relativity in holography

Yu Tian

School of Physical Sciences, University of Chinese Academy of Sciences

Holographic applications: from Quantum Realms to the Big Bang, UCAS, Beijing

July 18, 2025

Outline

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Numerical relativity: introduction

- Motivation: Einstein equations cannot be solved analytically in less symmetric cases. In particular, most of gravitational dynamics needs to be done numerically.

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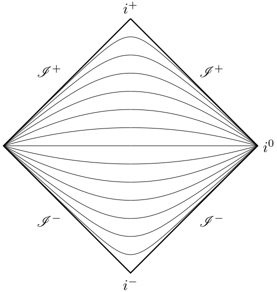
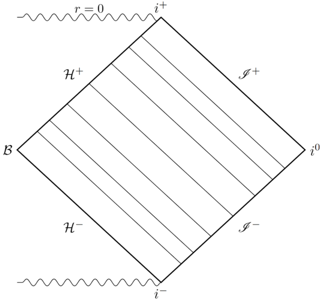
- Difficulties: coordinate gauges and constraints, singularities, instabilities, ...
- Formalisms: space-like time slicing ($3+1$ formalisms) versus null (light-front) time slicing (characteristic formalisms)





Numerical relativity: formalisms

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- Bondi-Sachs
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Gravitational physics is so complicated that no formalism is well suited for most problems.

Numerical relativity: applications

- Dynamics of real black holes/relativistic compact objects and gravitational waves
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- Dynamics of real black holes/relativistic compact objects and gravitational waves
- Dynamics of theoretical gravitational models (higher dimensions, asymptotically non-flat cases, singularities, cosmological censorship, no-hair theorem, …)
- AdS/CFT duality (holography)
  - Fundamental problems of the duality and quantum gravity
  - Describing quantum fields or quantum many-body systems (applied holography)

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Levels of problems in numerical holography (with backreaction)

More generally, there are three kinds (levels) of problems for numerical relativity in holography.

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- Equilibrium (stationary solution)
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- Near equilibrium (linear perturbation)
  Linear stability (related to phase transition), linear response (susceptibility, transport coefficients), $\cdots$
- Far from equilibrium (dynamical evolution)
  Nonlinear transport, dynamical phase transition, general nonlinear dynamics, $\cdots$

Review of previous works

- Equilibrium: Stationary solution of coupled gravity-matter equations of motion
  - Analytical solution
  - Numerical solution with maximum symmetry
  - Numerical solution with less symmetry: Deturk method

    [O.J.C. Dias, J.E. Santos & B. Way, arXiv:1510.02804]

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    [O.J.C. Dias, J.E. Santos & B. Way, arXiv:1510.02804]
- Near equilibrium: Linear perturbation on top of bulk black holes (in-going BC at the horizon)
  - Linear responses
  - Quasi-normal modes (QNM) and linear stability

    Many different perspectives and formalisms for the numerical computation!

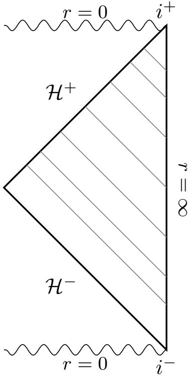
Review of previous works

- Far from equilibrium: Evolution of the coupled gravity-matter dynamics by numerical relativity
  - Homogeneous setups (setups with maximum spatial symmetry)
  - Inhomogeneous setups
    - Chesler-Yaffe formalism (characteristic)

      [Chesler & Yaffe, arXiv:1309.1439]

      (Wilke van der Schee's talk the day before yesterday)
    - BSSN or generalized harmonic gauge formalism (3+1)

      (See, e.g. L. Rossi, arXiv:2205.15329 for a review.)



Holographic evolution in the Chesler-Yaffe formalism

- The general (in-going) Chesler-Yaffe gauge:

$$ds^2 = 2dt(dr - Adt - F_i dx^i) + \Sigma^2 h_{ij} dx^i dx^j$$

with $\det(h_{ij}) = 1$, (A, F_i, Σ, h_{ij}) functions of $(t, r, \vec{x})$ and r is an affine parameter of the null generators of a constant t surface

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- Residual gauge freedom:

$$\tilde{r} = r + \lambda(t, \vec{x})$$

  which can be used to fix the position of the apparent horizon to the boundary of the computational domain

Holographic evolution in the Chesler-Yaffe formalism

- The (vacuum) Einstein equations (not fully independent due to the Bianchi identity)

$$\Sigma'' = S_\Sigma(h) \qquad (X' := \partial_r X) \tag{2.1}$$

$$F_i'' = S_F^i(h, \Sigma) \tag{2.2}$$

$$A'' = S_A(h, \Sigma, F, d_+\Sigma, d_+h) \tag{2.3}$$

$$(d_+\Sigma)' = S_{d_+\Sigma}(h, \Sigma, F) \tag{2.4}$$

$$(d_+h_{ij})' = S_{d_+h}^{ij}(h, \Sigma, F, d_+\Sigma) \tag{2.5}$$

$$(d_+F_i)' = S_{d_+F}^{i}(h, \Sigma, F, d_+\Sigma, d_+h, A) \tag{2.6}$$

$$d_+(d_+\Sigma) = S_{d_+^2 F}(h, \Sigma, F, d_+\Sigma, d_+h, A) \tag{2.7}$$

with $d_+ := \partial_t + A\partial_r$ the null direction other than ∂_r

Holographic evolution in the Chesler-Yaffe formalism

Evolution scheme (basic):

1. h_{ij} given on the initial time slice t_0
2. Σ obtained from (2.1)
3. F_i obtained from (2.2)
4. $d_+\Sigma$ obtained from (2.4)
5. d_+h obtained from (2.5)
6. A obtained from (2.3)
7. Evolution of h_{ij} obtained by $\partial_t h_{ij} = d_+ h_{ij} - A h'_{ij}$, marching h_{ij} to the next time slice and repeating the process from the start

Holographic evolution in the Chesler-Yaffe formalism

Physical boundary conditions at the AdS conformal boundary $r \to \infty$:

Initial physical conditions

- Boundary energy density for $d_+\Sigma$ on the initial time slice t_0
- Boundary momentum density for F_i on the initial time slice t_0
- Boundary stress tensor for h_{ij} on the initial time slice t_0

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Subsequent physical conditions

- Evolution of the above quantities by the Ward-Takahashi identities

Holographic evolution in the Chesler-Yaffe formalism

Gauge fixing boundary conditions for F_i and gauge fixing conditions for $\lambda(t,\vec{x})$:

Initial gauge fixing conditions

- λ given on the initial time slice t_0

Holographic evolution in the Chesler-Yaffe formalism

Gauge fixing boundary conditions for F_i and gauge fixing conditions for $\lambda(t, \vec{x})$:

Initial gauge fixing conditions

- λ given on the initial time slice t_0

Then evolution of λ by

Subsequent gauge fixing conditions

- either a prescribed $\partial_t \lambda$ subsequently
- or a $\partial_t \lambda$ determined by the apparent horizon fixing condition (solving a 2nd order linear elliptic PDE for A on the horizon)

Outline

Dynamical spacetime under the Bondi-Sachs gauge

- Very brief history of the Bondi-Sachs gauge (and generally the characteristic formalism)

  [J. Winicour, Living Rev. Relativity 8 (2005) 10; A. Giannakopoulos, arXiv:2308.16001]

Dynamical spacetime under the Bondi-Sachs gauge

- Very brief history of the Bondi-Sachs gauge (and generally the characteristic formalism)

  [J. Winicour, Living Rev. Relativity 8 (2005) 10; A. Giannakopoulos, arXiv:2308.16001]
- A simple and unified framework of numerical relativity for holographic systems
  - Equilibrium configurations
  - Linear responses and QNM
  - Dynamical evolution

Dynamical spacetime under the Bondi-Sachs gauge

- The in-going Bondi-Sachs gauge (null foliation $v=$ const with null generator ∂_z) for general $(1+d)$D AdS black holes:

$$ds^2=\frac{L^2}{z^2}(-fe^{-\chi}dv^2-2e^{-\chi}dvdz+h_{ij}[dx^i-\xi^i dv][dx^j-\xi^j dv])$$

with $\det(h_{ij})=1$ for the planar case (Poincare patch, meaning $r=\frac{L}{z}$ the areal radius or luminosity distance with L the AdS radius) and (f,χ,h_{ij},ξ^i) functions of $(v,z,\vec{x})$
[Z. Ning, Q. Chen, YT, X. Wu & H. Zhang, arXiv:2307.14156]

Equations of motion under the Bondi-Sachs gauge

- The (vacuum) Einstein equations (not fully independent due to the Bianchi identity)

$$
\begin{aligned}
\chi' &= S_\chi(h') \qquad (X' := \partial_z X) && (3.1)\\
\xi^{i\prime\prime} &= S^i_\xi(h', \chi') && (3.2)\\
f' &= S_f(h, \chi, \xi') && (3.3)\\
\dot{h}'_{ij} &= S^{ij}_h(h'', \chi, \xi', f') \qquad (\dot{X} := \partial_v X) && (3.4)\\
\dot{\xi}^{i\prime} &= T^i_\xi(\dot{h}', \dot{\chi}', \xi', f') && (3.5)\\
\dot{f} &= T_f(\dot{h}', \dot{\chi}', \dot{\xi}', f') && (3.6)
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- Independent equations can be taken as (3.1,3.2,3.3,3.4), while constraint equations (3.5) and (3.6) act as boundary conditions (momentum and energy conservation).

Discussion about the Bondi-Sachs gauge

- Dynamical fields: h_{ij} (propagating degrees of freedom)
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- Dynamical fields: h_{ij} (propagating degrees of freedom)
- Semi-dynamical fields: (χ, ξ^i, f)
- Explicitly reflecting the counting of degrees of freedom in the Einstein gravity
- Very nice nested structure, convenient for numerical evolution

Discussion about the Bondi-Sachs gauge

- In 4D ($d = 3$), there are 6 fields in total and 2 dynamical fields in h_{ij}. Particularly, if the system is translation invariant in one of the spatial directions (say, y), it turns out that ξ^y and the off-diagonal component h_{xy} can be consistently switched off, which leads to the simplified metric ansatz

$$ds^2 = \frac{L^2}{z^2}(-fe^{-\chi}dv^2 - 2e^{-\chi}dvdz + e^A[dx - \xi dv]^2 + e^{-A}dy^2)$$

with $\xi := \xi^x$ and $e^A := h_{xx} = h_{yy}^{-1}$.

Discussion about the Bondi-Sachs gauge

- For the spherical case (global AdS), one just uses the coordinate transformation $x = -\cos\theta$ to obtain the metric

$$ds^2 = \frac{L^2}{z^2}(-fe^{-\chi}dv^2 - 2e^{-\chi}dvdz + e^{\tilde{A}}[d\theta - \tilde{\xi}dv]^2 + e^{-\tilde{A}}\sin^2\theta d\varphi^2)$$

with y renamed as φ, $e^{\tilde{A}} := \frac{e^A}{\sin^2\theta}$ and $\tilde{\xi} := \xi\sin\theta$, which is axi-symmetric with the corresponding Killing vector ∂_φ but has no angular momentum.

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- The angular momentum can be turned on by turning on ξ^φ and $h_{\theta\varphi}$ (corresponding to ξ^y and h_{xy} in the planar case).
- The corresponding Einstein equations can be accordingly transformed from their planar counterparts.

Typical evolution scheme (brief)

- Initial data (unconstrained) on the $v = 0$ slice: dynamical fields h_{ij}
- Obtaining χ from its constraint equation (3.1) with a gauge fixing BC
- Obtaining ξ^i from its constraint equation (3.2) with a gauge fixing BC and a physical BC from evolution (3.5) of the boundary value of ξ'
- Obtaining f from its constraint equation (3.3) with a physical BC from evolution (3.6) of the boundary value of f
- Evolving h'_{ij} with (3.4) to the next v slice and obtaining h_{ij} with its physical BC (source) at the conformal boundary $z = 0$
- Repeating the above procedure with h_{ij} on the next v slice as the initial data

[Z. Ning, Q. Chen, YT, X. Wu and H. Zhang, arXiv:2307.14156]

Typical numerical implementation (4D)

- Fourier pseudo-spectral expansion in x and y
  Fourier pseudo-spectral in θ and φ
- Chebyshev pseudo-spectral expansion in z
- For evolution: the 4th-order Runge-Kutta marching in v

Examples of applications

- Spontaneous deformation of spherical AdS black holes

  [Z. Ning, Q. Chen, YT, X. Wu and H. Zhang, arXiv:2307.14156]

  - Bondi-Sachs as a unified framework for static solutions, linear perturbations, and nonlinear dynamics

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  - Bondi-Sachs as a unified framework for static solutions, linear perturbations, and nonlinear dynamics
- Static configurations with less symmetry (working for non-AdS asymptotics as well)
- Fluid/gravity duality and turbulence

Fluid/gravity duality and turbulence

- Hydrodynamics as a universal effective field theory of interacting quantum many-body systems at low energy and long wavelength
- Fluid/gravity duality as the low-energy, long-wavelength expansion of AdS/CFT

  [S. Bhattacharyya, V.E. Hubeny, S. Minwalla & M. Rangamani, arXiv:0712.2456]

Fluid/gravity duality and turbulence

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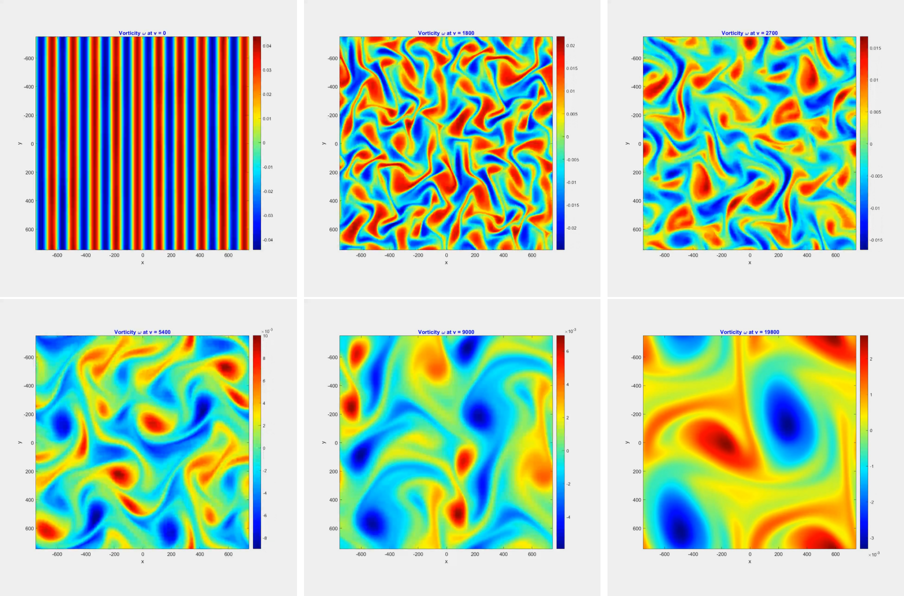
  [S. Bhattacharyya, V.E. Hubeny, S. Minwalla & M. Rangamani, arXiv:0712.2456]
- Holographic turbulence from a shear flow and fractal horizons
  [A. Adams, P.M. Chesler & H. Liu, arXiv:1307.7267]
- Implementation of holographic turbulence in the Bondi-Sachs gauge (with no symmetry), suitable for GPU acceleration


Vorticity ω at v = 0
Vorticity ω at v = 1800
Vorticity ω at v = 2700
Vorticity ω at v = 5400
Vorticity ω at v = 9000
Vorticity ω at v = 19800
x
y

Bondi-Sachs gauge vs Chesler-Yaffe gauge

- Metrics (both with $\det(h_{ij}) = 1$)
  - Bondi-Sachs:

$$ds^2 = \frac{L^2}{z^2}(-fe^{-\chi}dv^2 - 2e^{-\chi}dvdz + h_{ij}[dx^i - \xi^i dv][dx^j - \xi^j dv])$$

  - Chesler-Yaffe:

$$ds^2 = 2dt(dr - Adt - F_i dx^i) + \Sigma^2 h_{ij} dx^i dx^j$$

- Similarities: Null foliation, nice nested structure of the Einstein equations
- Differences: Areal radius $\frac{1}{z}$ vs affine parameter r, extra gauge freedom and one more nontrivial equation of motion in Chesler-Yaffe, and different philosophies of evolution schemes
- Bondi-Sachs goes deeper and deeper into the black hole during the evolution.

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Summary and outlook

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- The nested structure in characteristic formalisms is fragile (may be ruined by matter fields), so sometimes $3 + 1$ is more suitable.
- Further exploration of optimized/adapted formalisms of numerical relativity for various problems is still important.

Thank You!