Universal Time Evolution of Holographic Complexity

Shan-Ming Ruan
Peking University

arXiv:2502.01266, arXiv:2507.XXXX

with Masamichi Miyaji, Shono Shibuya, Kazuyoshi Yano

Holographic Applications: From Quantum Realms to the Big Bang
2025-07-18@UCAS

- 01. Two paradoxes of black holes
- 02. Generating functions of holographic complexity
- 03. Black Hole interior and holographic complexity
- 04. Universal time evolution

Is the black hole interior finite?

01. Black Hole Information Paradox

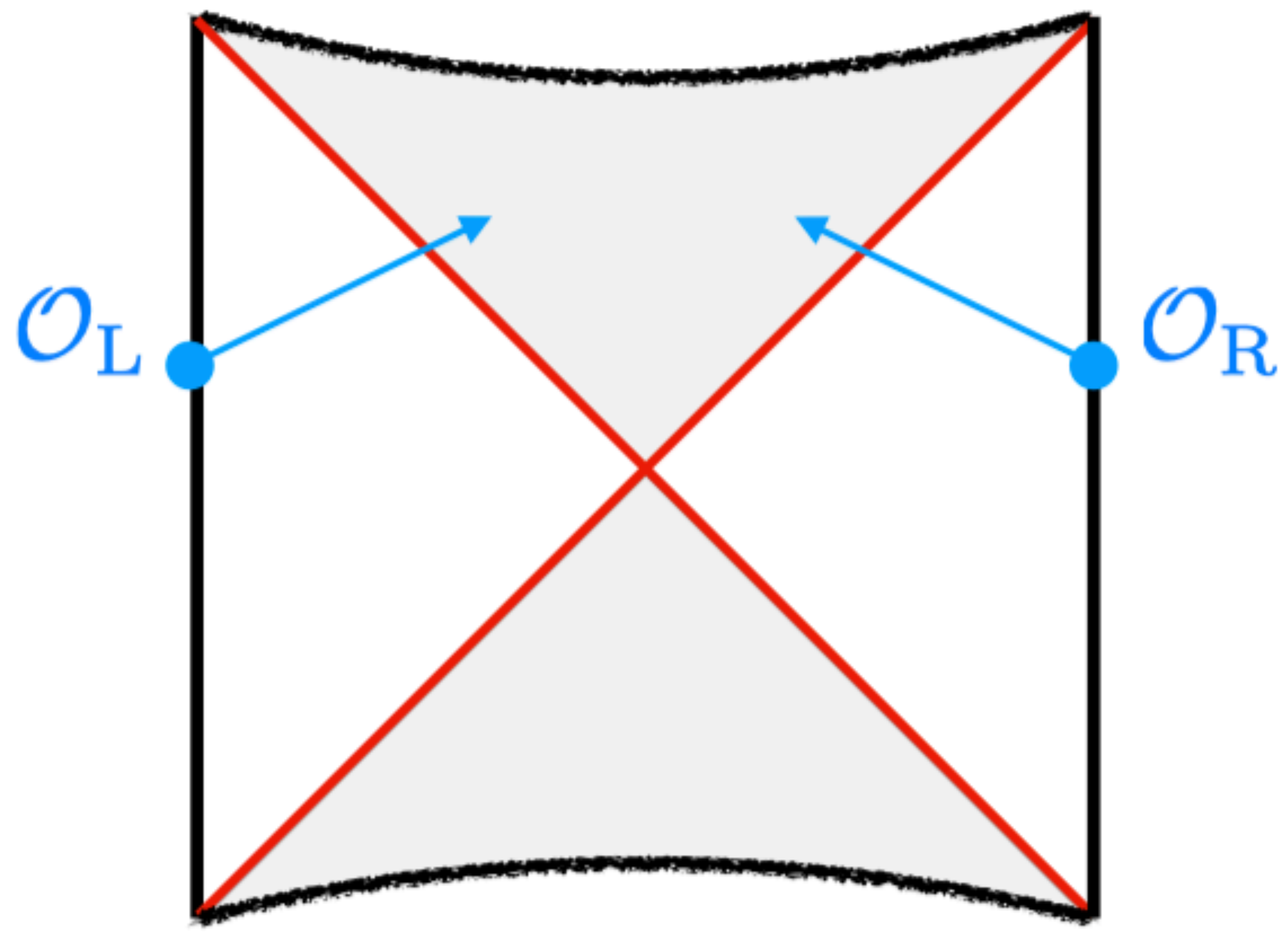
01. Maldacena's Black Hole Information Paradox

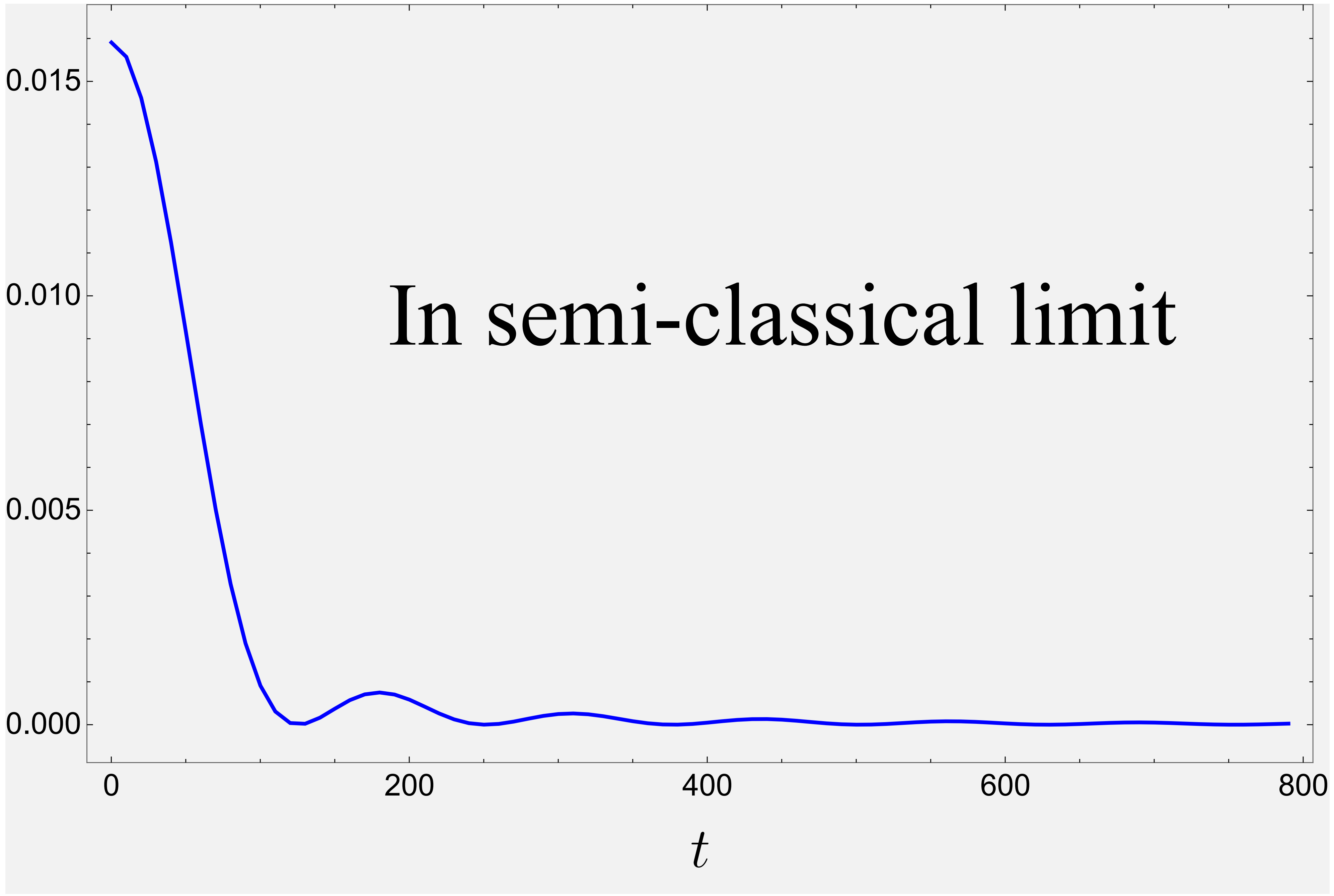
Maldacena, 2001
[hep-th/0106112]

Eternal AdS Black Hole Spacetime ⟷ (AdS/CFT) Thermofield double (TFD) state

Thermal correlation functions

$$\langle \mathrm{TFD}_\beta | \mathcal{O}(t_\mathrm{L}) \mathcal{O}(t_\mathrm{R}) | \mathrm{TFD}_\beta \rangle = \frac{1}{Z} \sum_{i,j} e^{-\frac{\beta}{2}(E_i+E_j)} e^{-iT(E_i-E_j)} \left| \langle E_i | \mathcal{O} | E_j \rangle \right|^2 ,$$



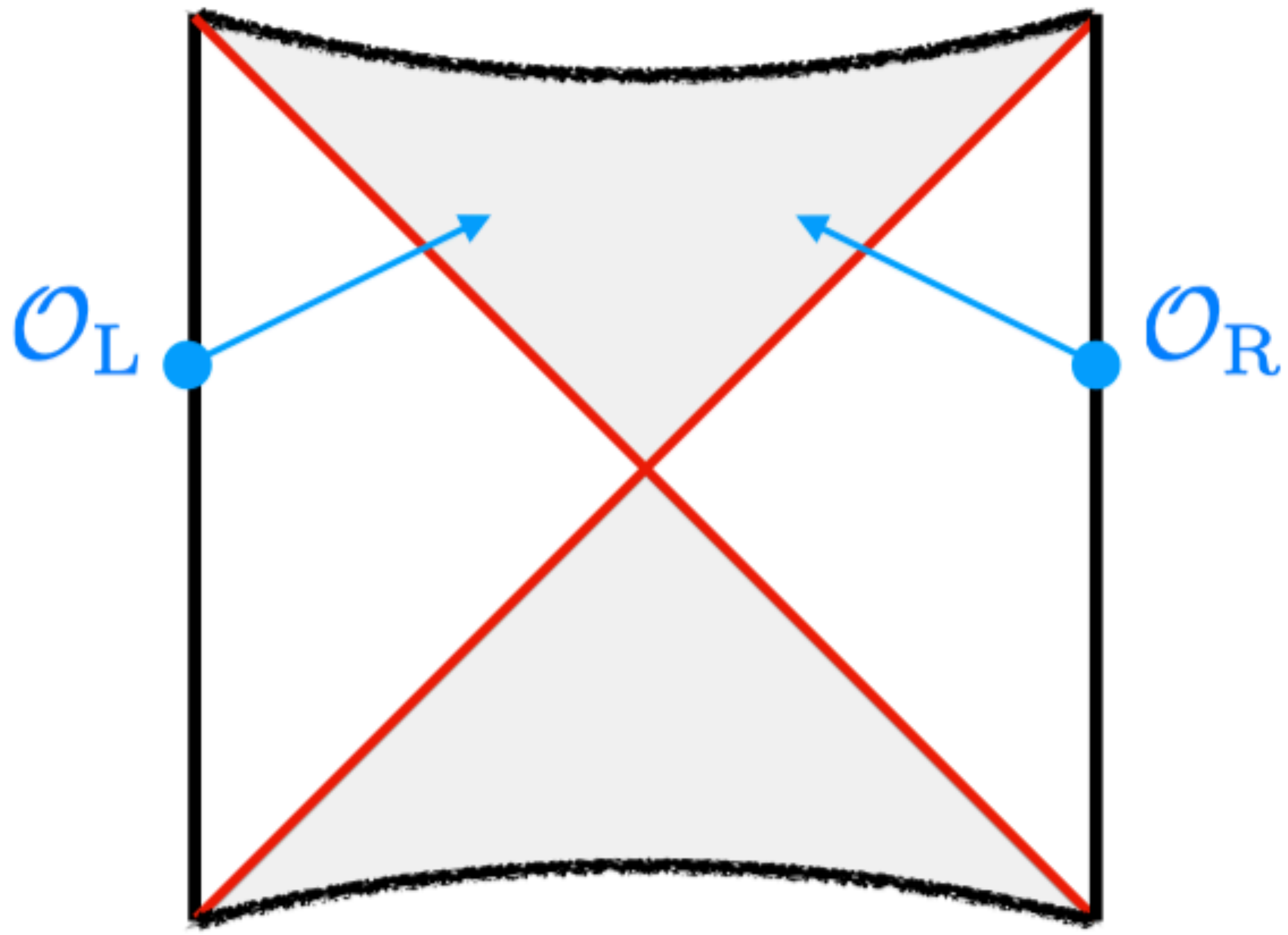


01. Maldacena's Black Hole Information Paradox

Maldacena, 2001
[hep-th/0106112]

Thermal correlation functions

$$\langle \mathrm{TFD}_\beta | \mathcal{O}(t_\mathrm{L}) \mathcal{O}(t_\mathrm{R}) | \mathrm{TFD}_\beta \rangle = \frac{1}{Z} \sum_{i,j} e^{-\frac{\beta}{2}(E_i+E_j)} e^{-iT(E_i-E_j)} \left| \langle E_i | \mathcal{O} | E_j \rangle \right|^2 ,$$



Holographic correlators decay forever
(particles fall into the black hole)

↑

Maldacena's Black Hole Information Paradox

↓

Unitary quantum system with a finite and discrete spectrum:
correlation functions cannot decay indefinitely

01. Maldacena's Black Hole Information Paradox

Thermal correlation functions

$$\langle \mathrm{TFD}_\beta | \mathcal{O}(t_\mathrm{L}) \mathcal{O}(t_\mathrm{R}) | \mathrm{TFD}_\beta \rangle = \frac{1}{Z} \sum_{i,j} e^{-\frac{\beta}{2}(E_i + E_j)} e^{-iT(E_i - E_j)} \left| \langle E_i | \mathcal{O} | E_j \rangle \right|^2 ,$$

Spectral Form Factor (SFF)

$$Z(\beta + iT) Z(\beta - iT) = \sum_{n,m} e^{-(\beta + iT)E_n} e^{-(\beta - iT)E_m} = \sum_{m,n} e^{-\beta(E_m + E_n) + iT(E_m - E_n)}$$

01. Maldacena's Black Hole Information Paradox

Typical time evolution

Correlation functions/Spectral Form Factor (SFF)

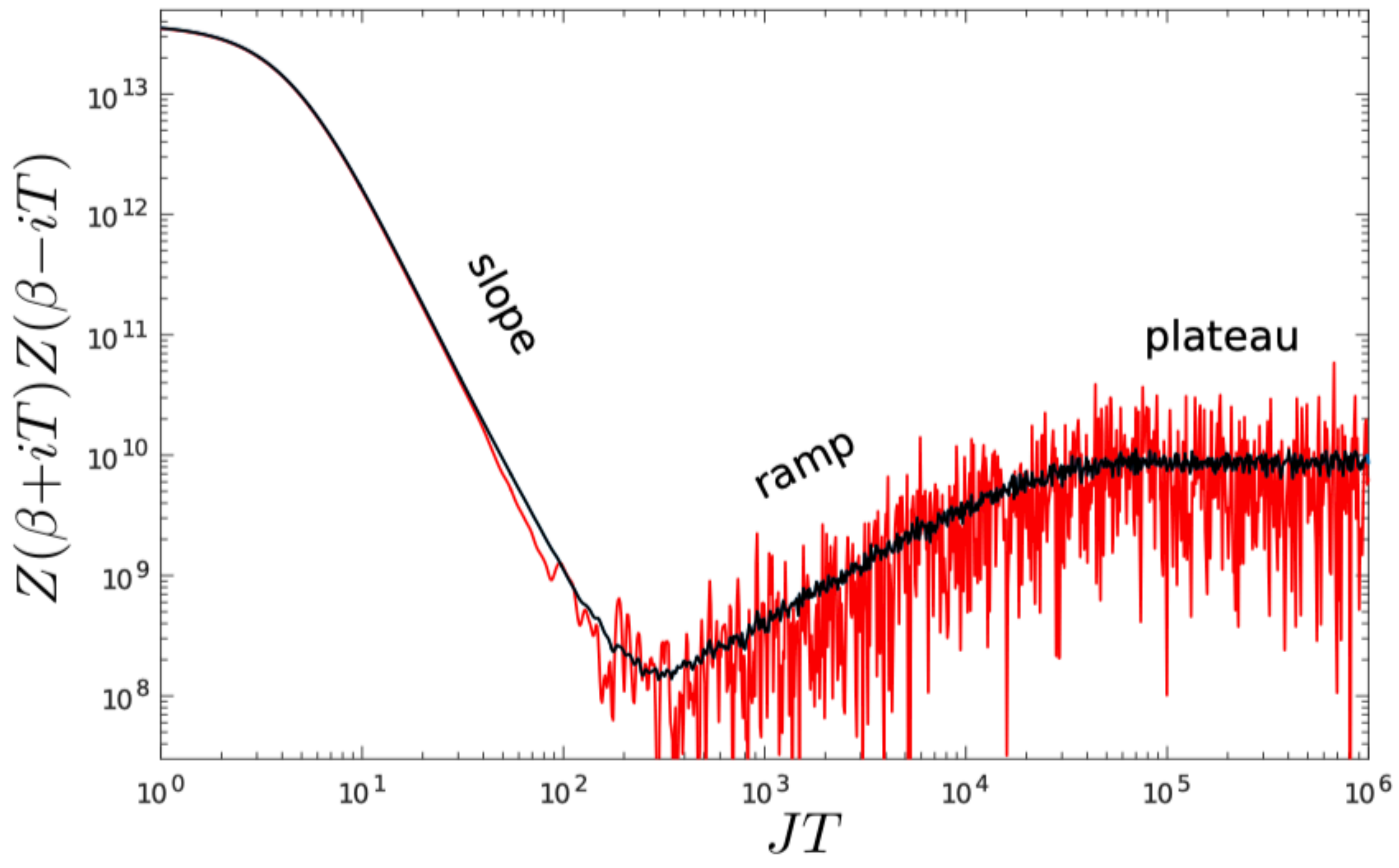


Figure 1: A log-log plot of the spectral form factor in SYK for $q = 4$, $N = 34$, $\beta J = 5$ [22]. A single sample (red, erratic) is plotted together with an average of 90 samples (black, smoother). The ramp is approximately linear $\propto T$ in standard variables, not just in the log-log variables.

Saad, Shenker, Stanford, 2018
arXiv: 1806.06840

01. Maldacena's Black Hole Information Paradox

↓

Correlators decay forever

↓ Solution

slope-ramp-plateau structure

We need to include quantum corrections!

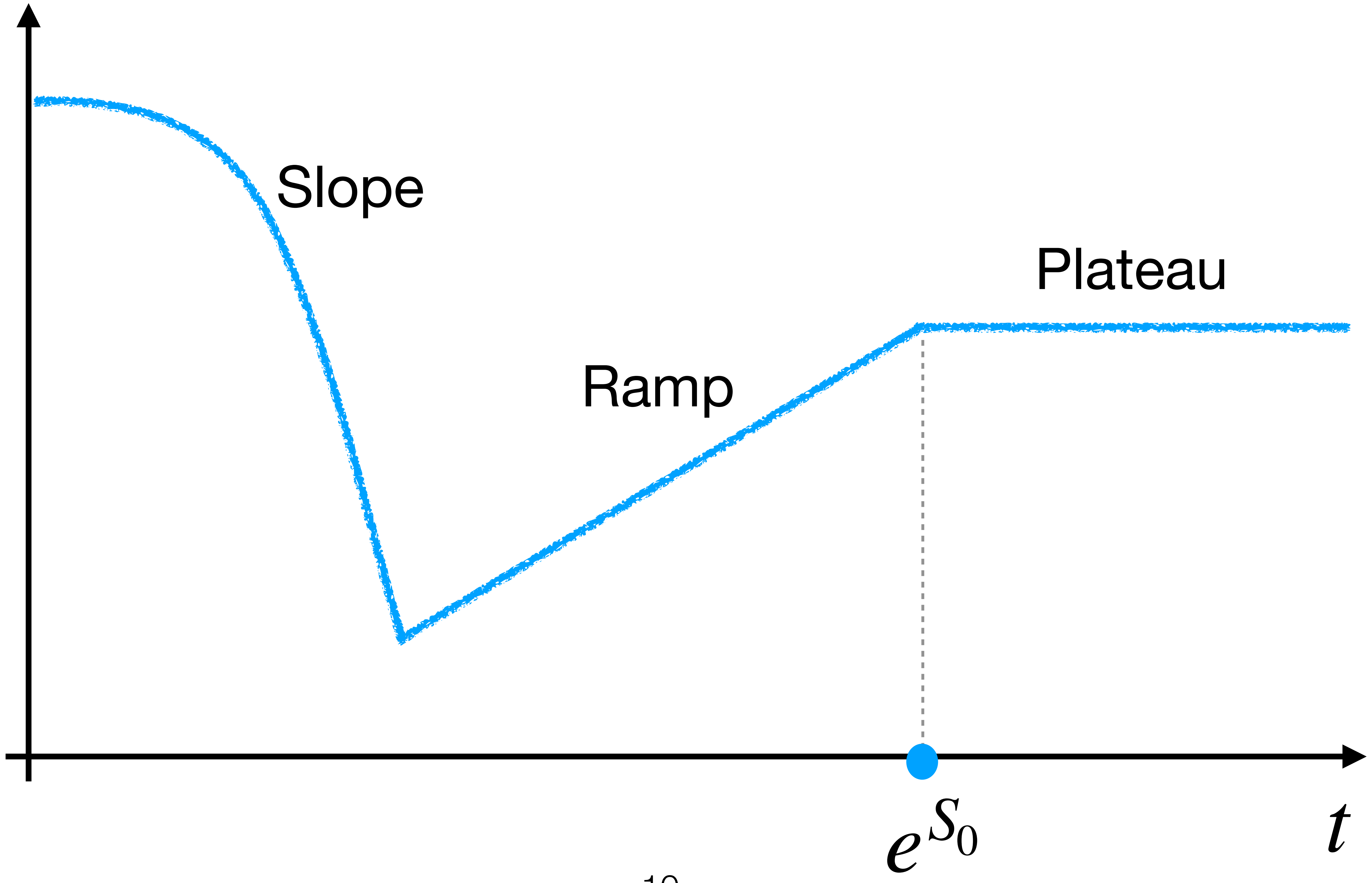
Random matrix universality
Spectral statistics

Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka 1611.04650
Saad, Shenker, Stanford, arXiv: 1806.06840, 1903.11115.
Blommaert, Kruthoff, Yao, 2208.13795
Saad, Stanford, Yang, Yao, 2210.11565.
Okuyama, Sakai, 2004.07555.
....

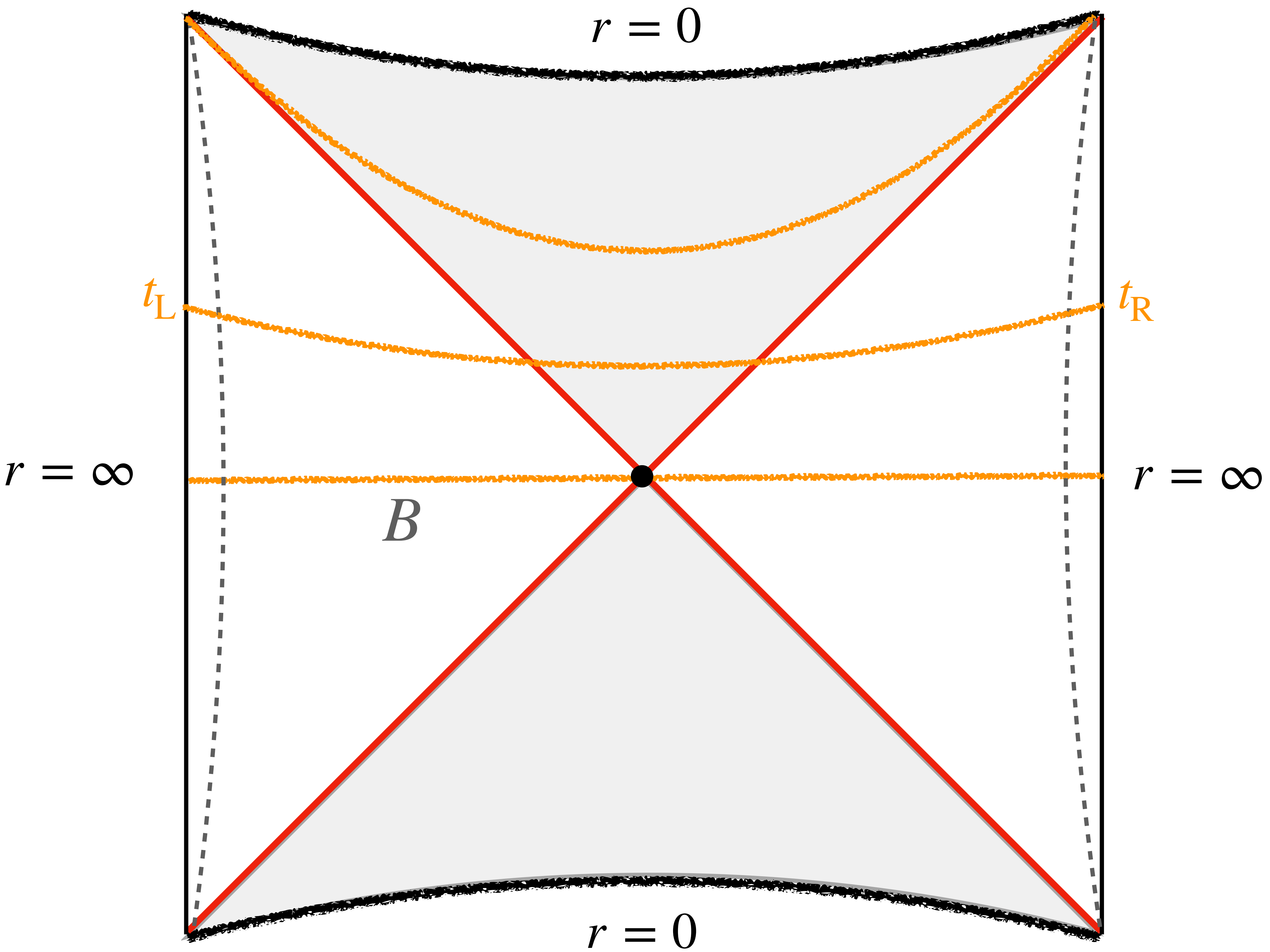
01. Maldacena's Black Hole Information Paradox

Typical time evolution

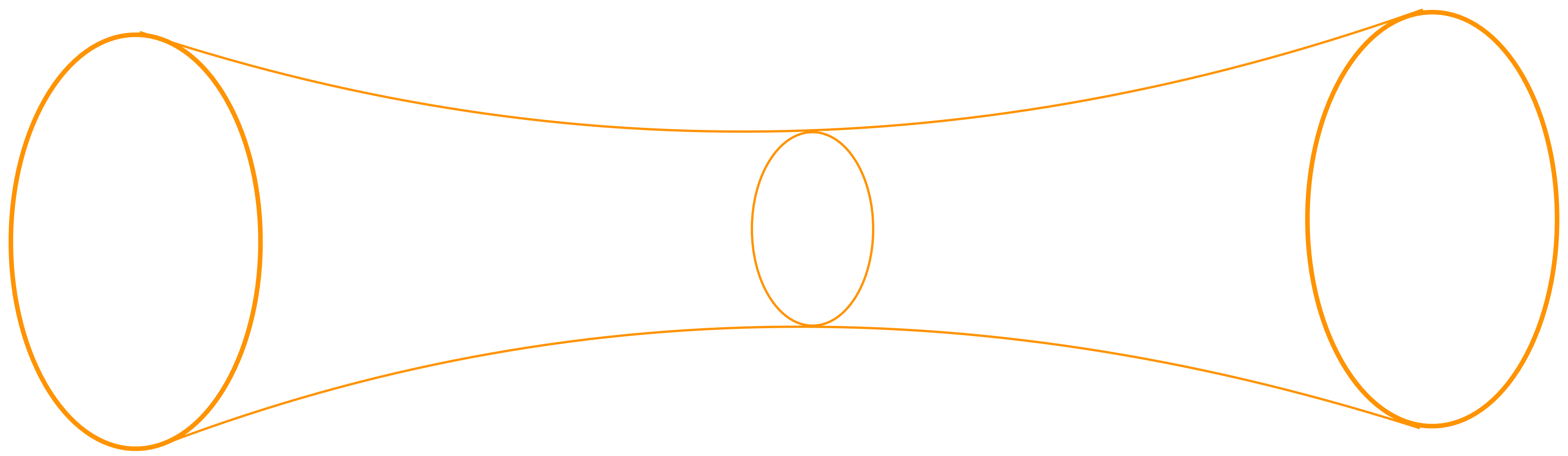
Correlation functions/Spectral Form Factor (SFF)



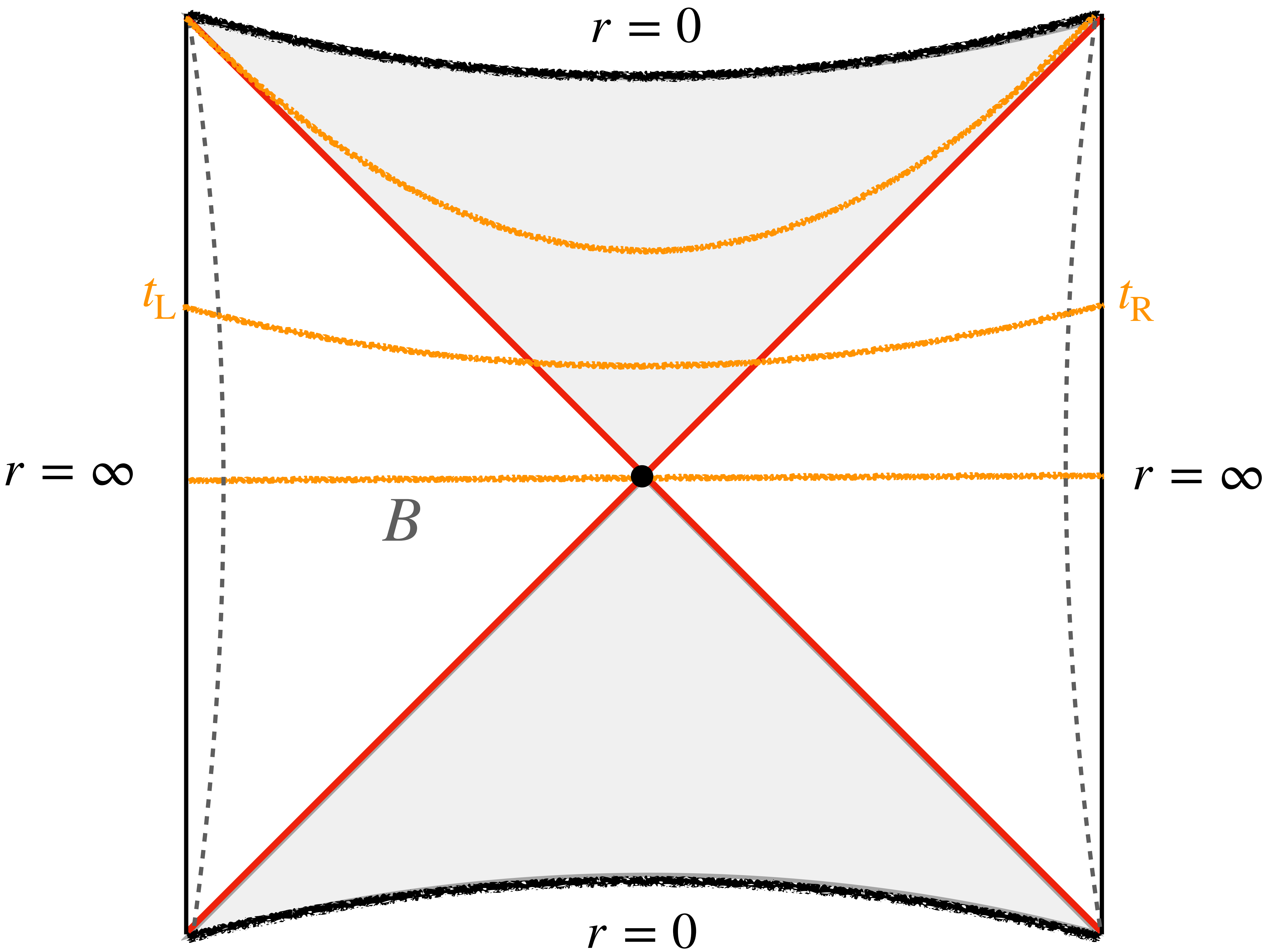
01. Wormhole Size Paradox



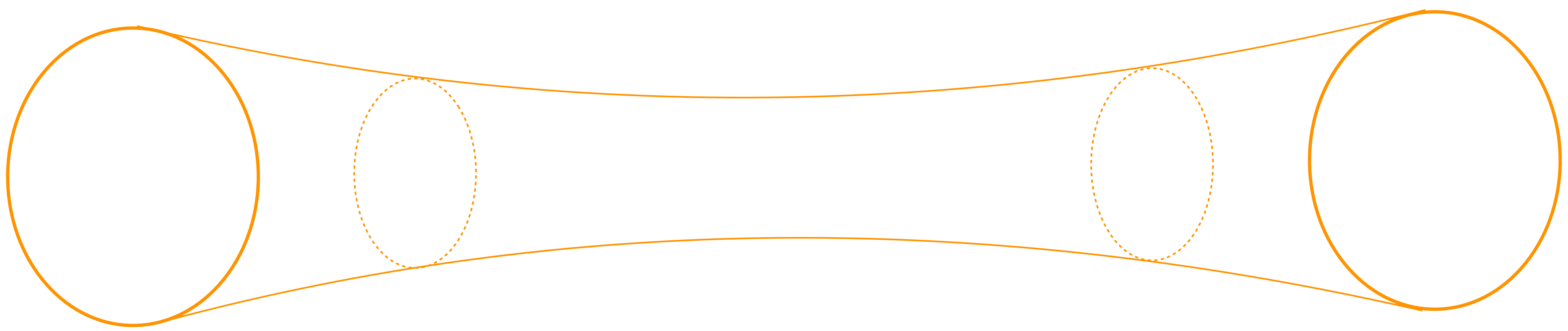
Wormhole/ Einstein-Rosen Bridge



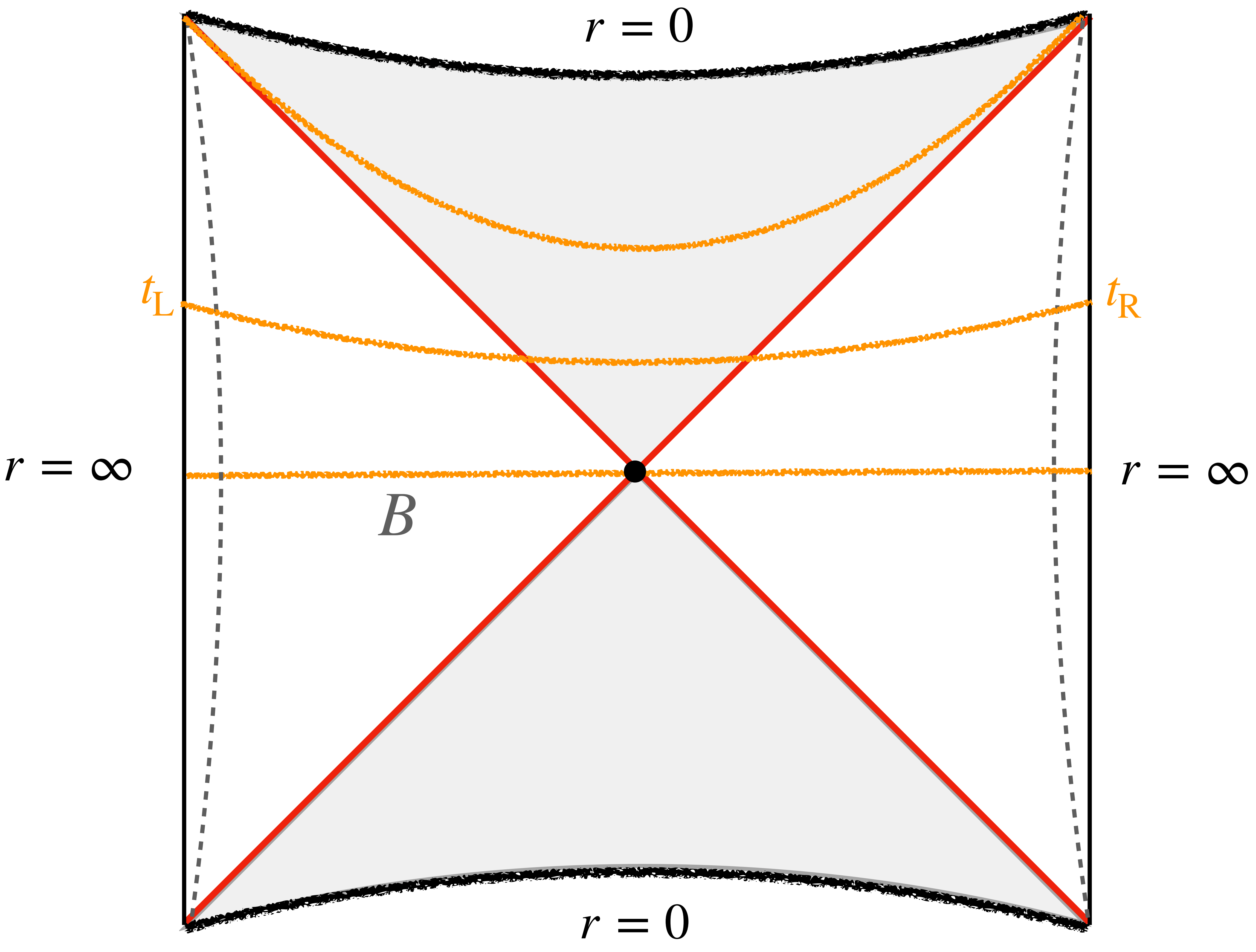
01. Wormhole Size Paradox



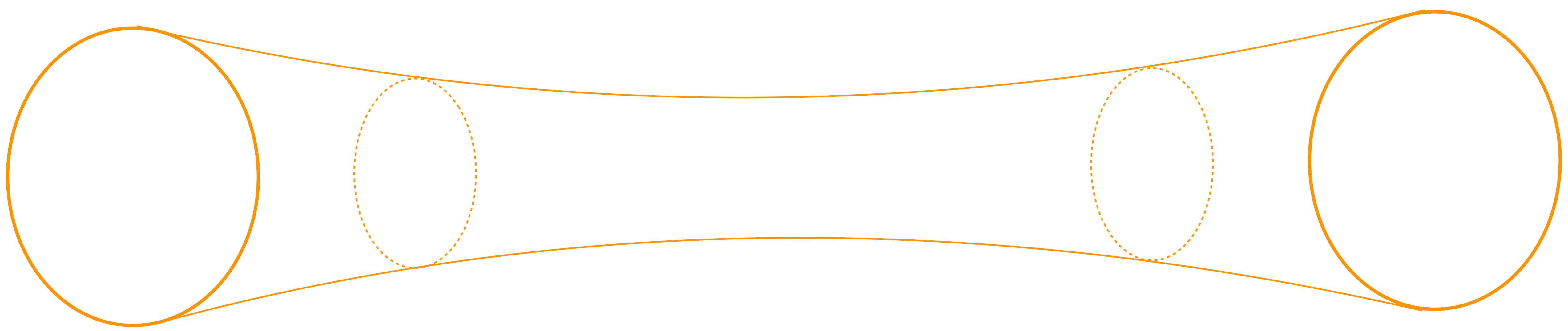
Linear growth of the wormhole size



01. Wormhole Size Paradox



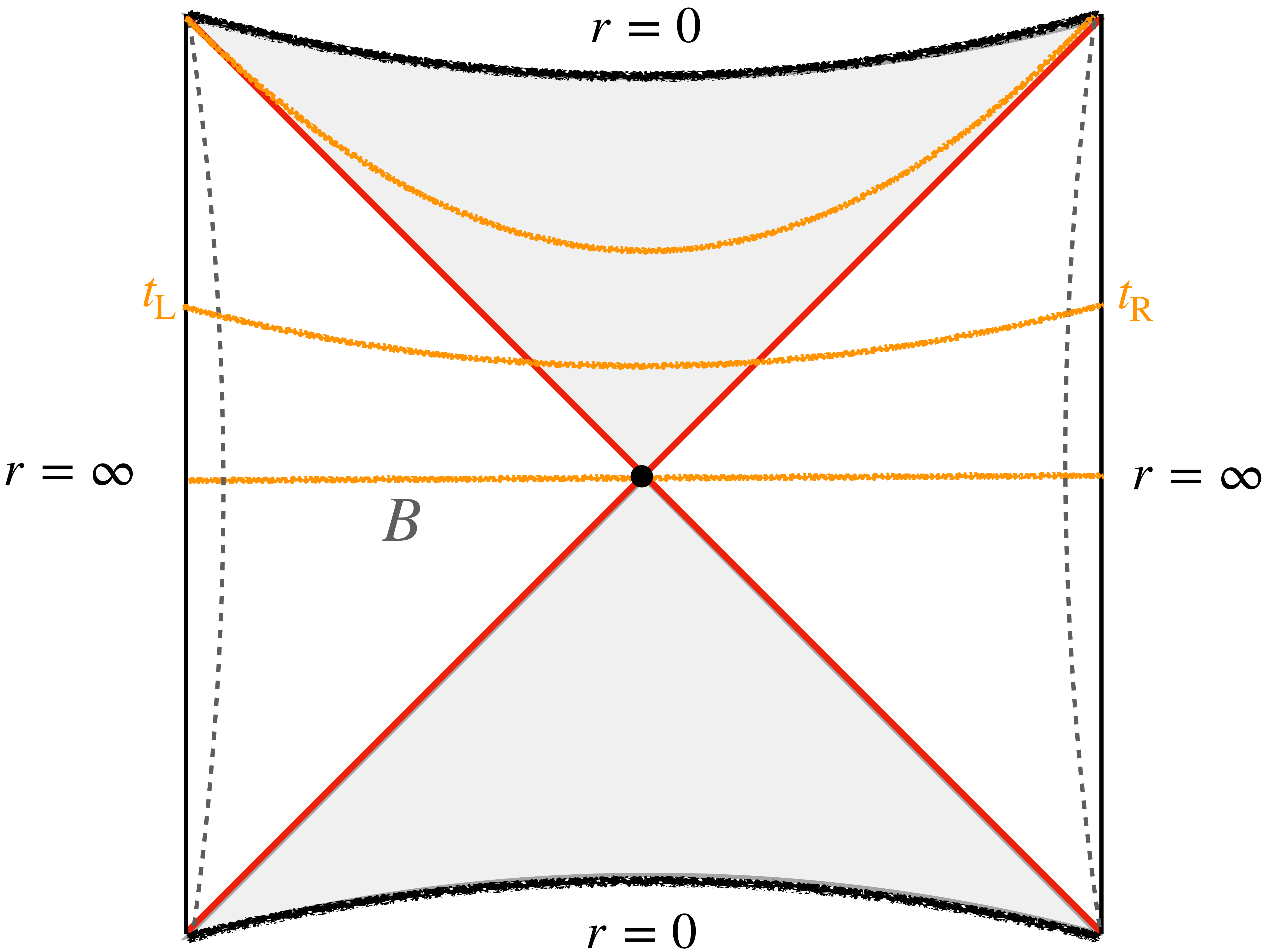
Linear growth of the wormhole size



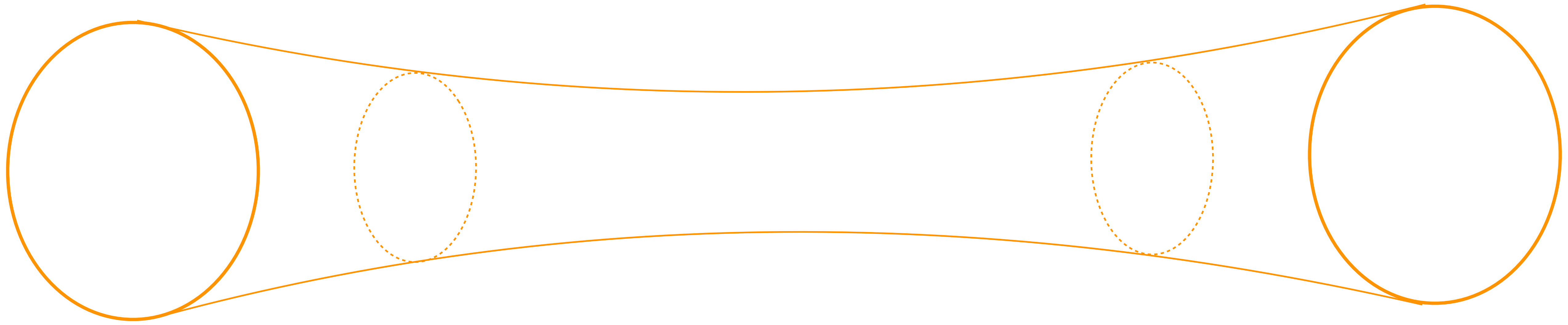
Boundary time $t \to \infty$

Size of BH interior $V \to \infty$

01. Wormhole Size Paradox



Linear growth of the wormhole size



size $\sim t_L + t_R$

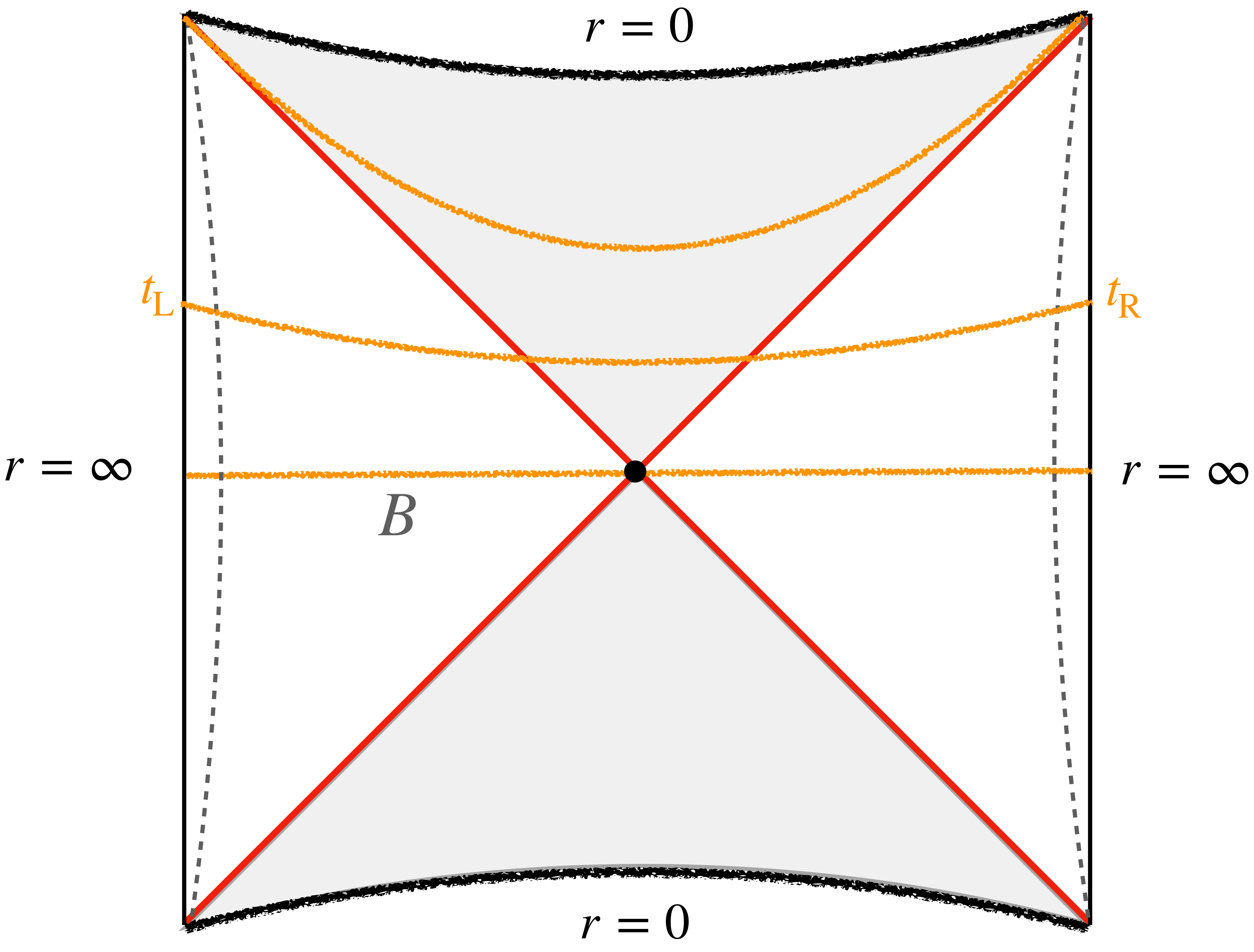
Geometries behind the horizon,
i.e., black hole interior

Entanglement entropy is not enough!

Susskind [1411.0690]

01. Holographic Complexity

Susskind, 1403.5695
Susskind, Stanford, 1406.2678



Complexity = Volume Conjecture:

$$\mathcal{C}_{\mathrm{V}} = \max\left[\frac{\mathcal{V}}{G_N \ell_{\mathrm{bulk}}}\right]$$

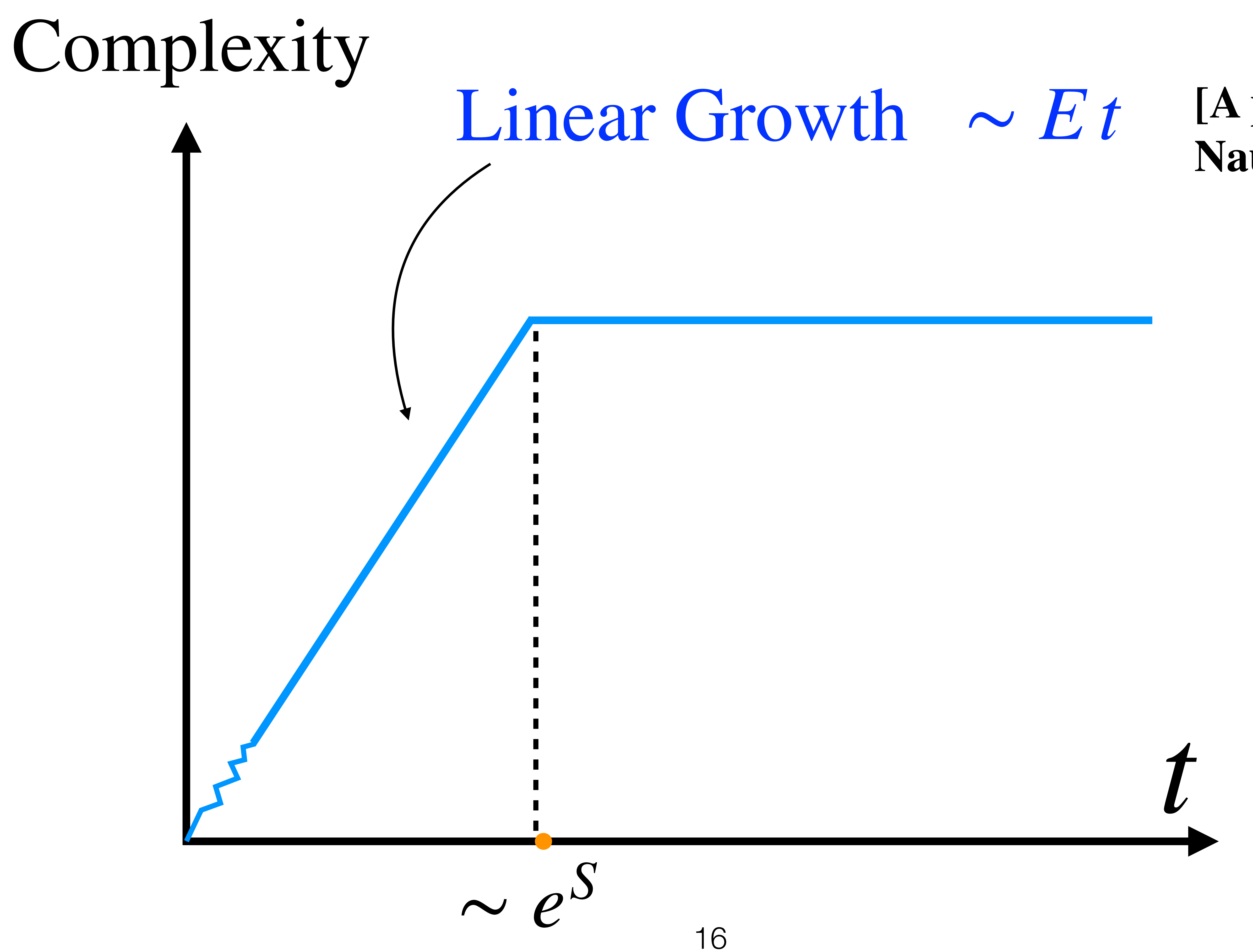
$$C_{\mathrm{V}} \sim M\,|\,t_{\mathrm{L}} + t_{\mathrm{R}}\,|$$

Complexity = Action
Complexity = Anything

2. Quantum Circuit Complexity

$$|\psi_{\mathrm{T}}(t)\rangle = e^{-iHt}|\psi\rangle$$

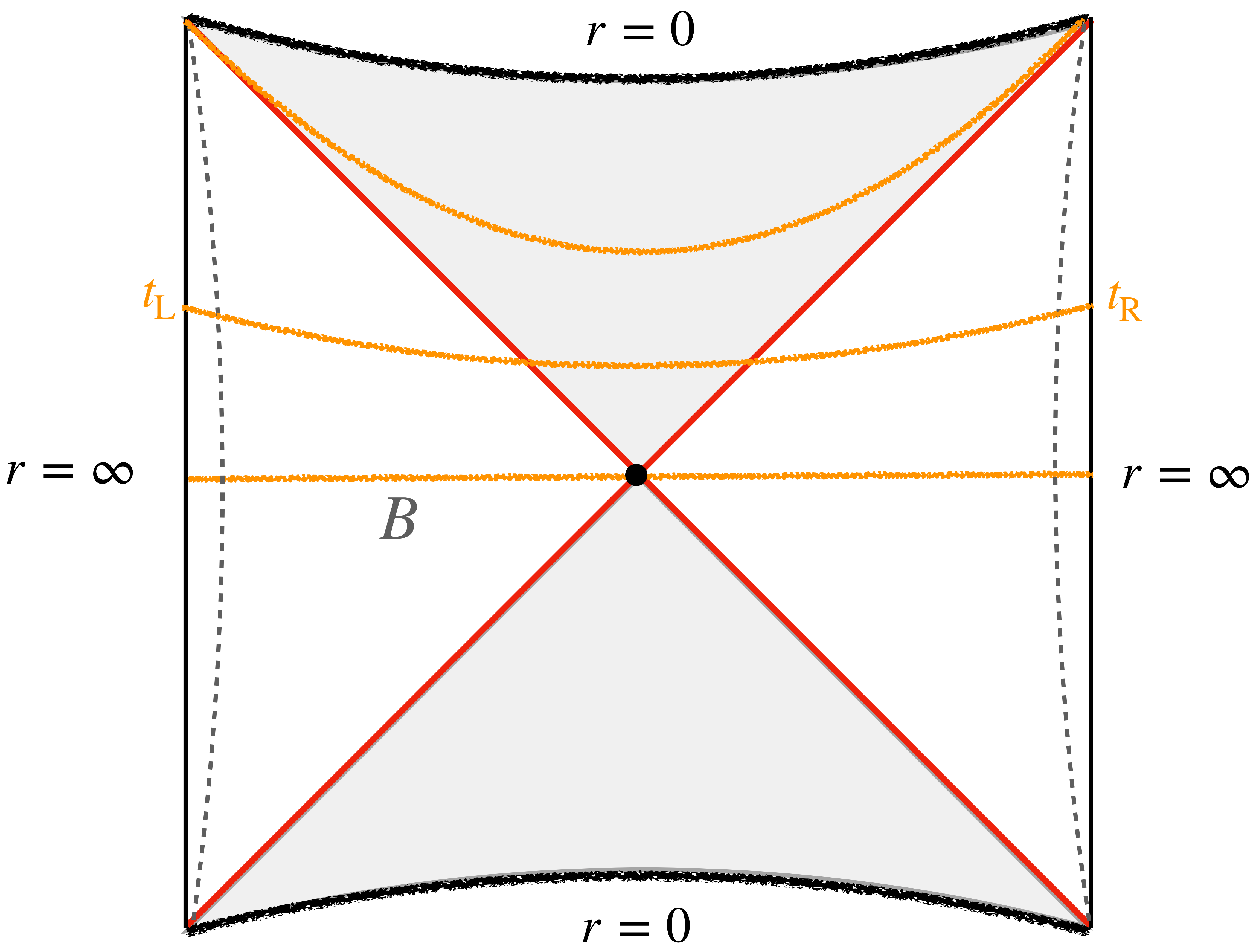
Susskind, Brown (2017)



[A proof for random circuits of qubits]
Nautre Physics 18, 528-532 (2022)

01. Wormhole Size Paradox

Linear growth of wormhole size = linear growth of complexity



Wormhole Size Paradox

In semi-classical
Wormhole size/complexity grows forever

?

For a finite quantum system:
Quantum complexity is bounded (plateau)

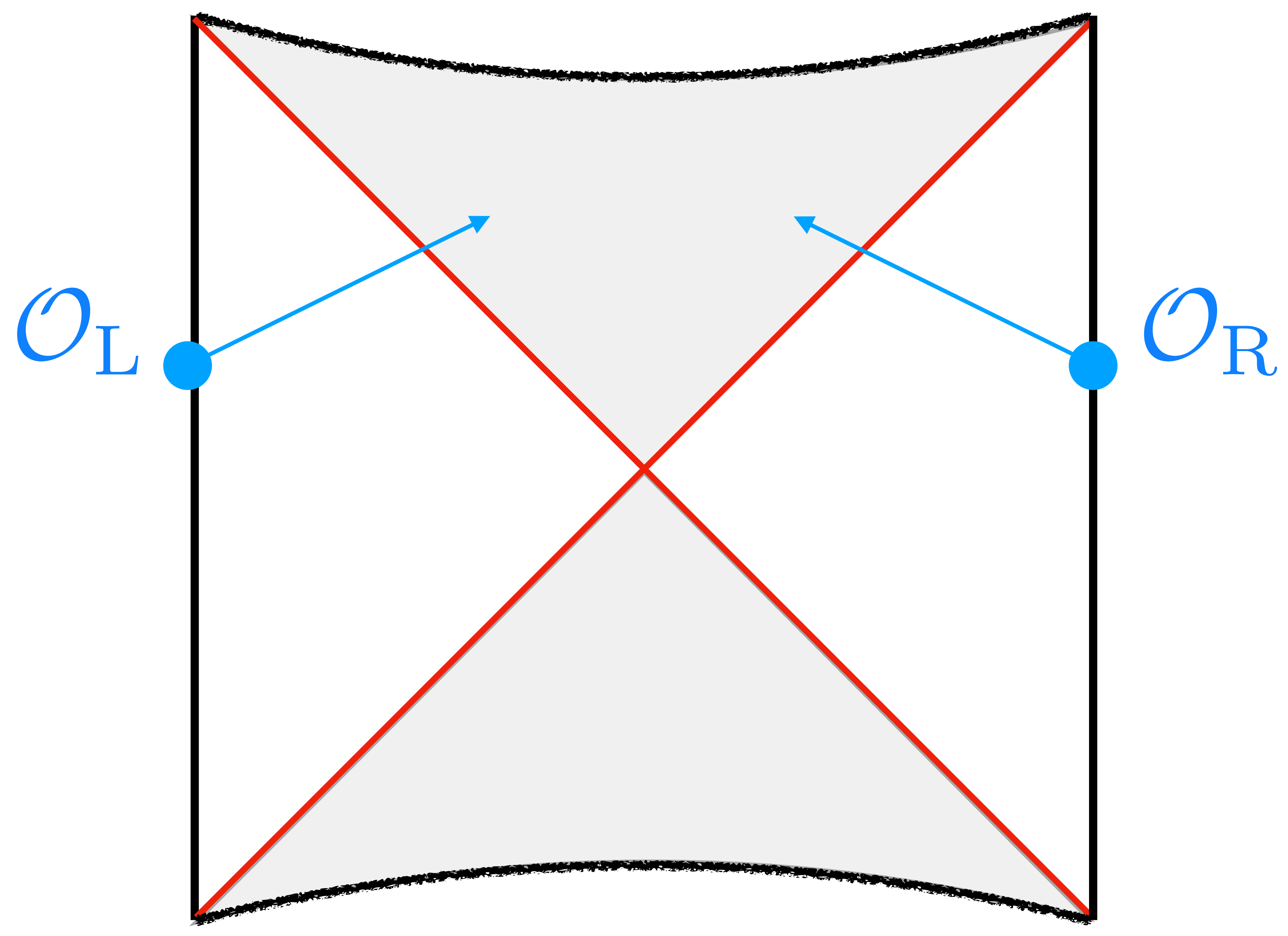
01. Two Paradoxes

Black hole information Paradox

Correlator decays forever

Wormhole Size Paradox

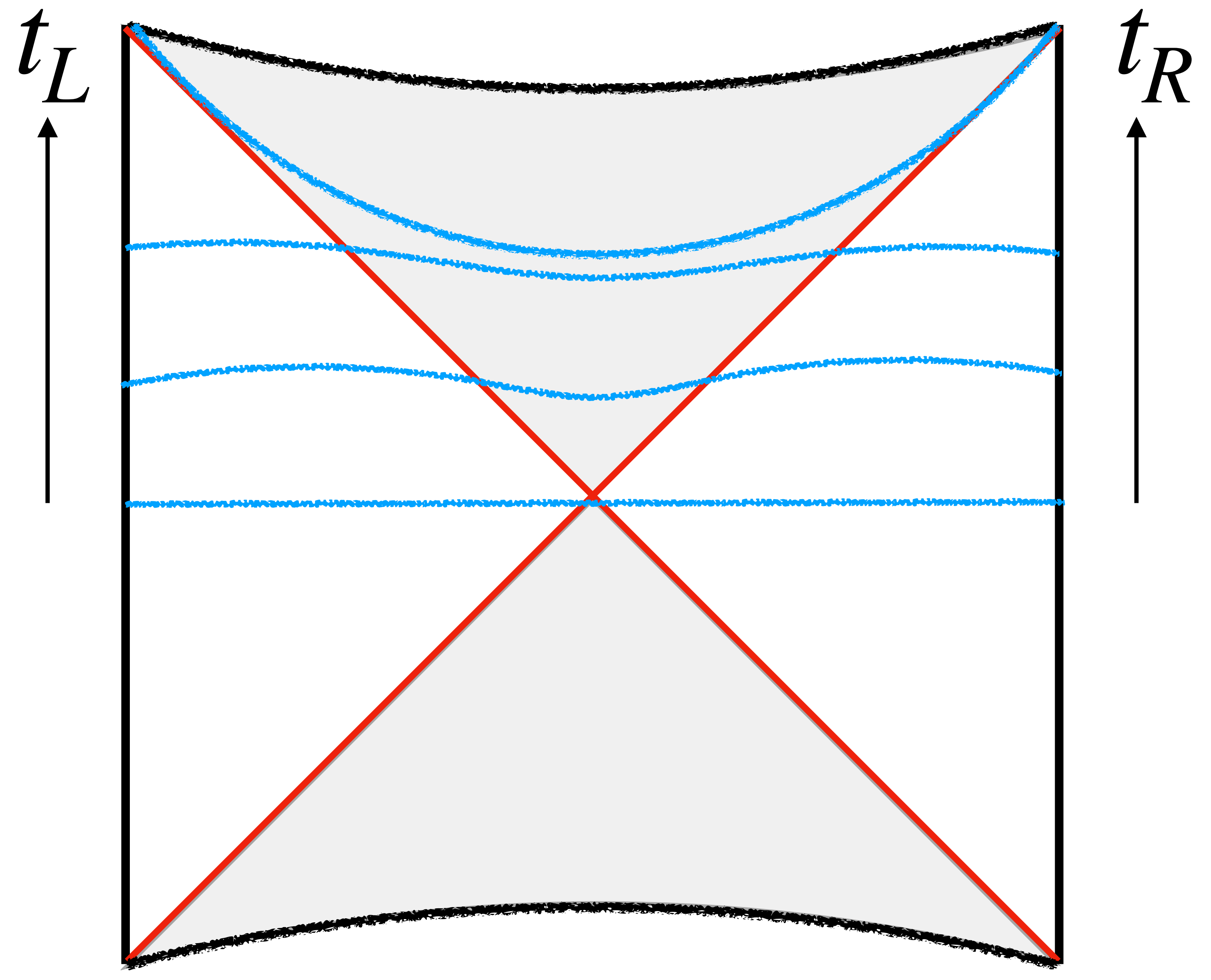
Wormhole/complexity grows forever



Semi-classical limit

geodesic approximation

$$\langle \mathcal{O}_\mathrm{L} \mathcal{O}_\mathrm{R} \rangle \sim \sum e^{-\alpha L_{\mathrm{geodesic}}}$$



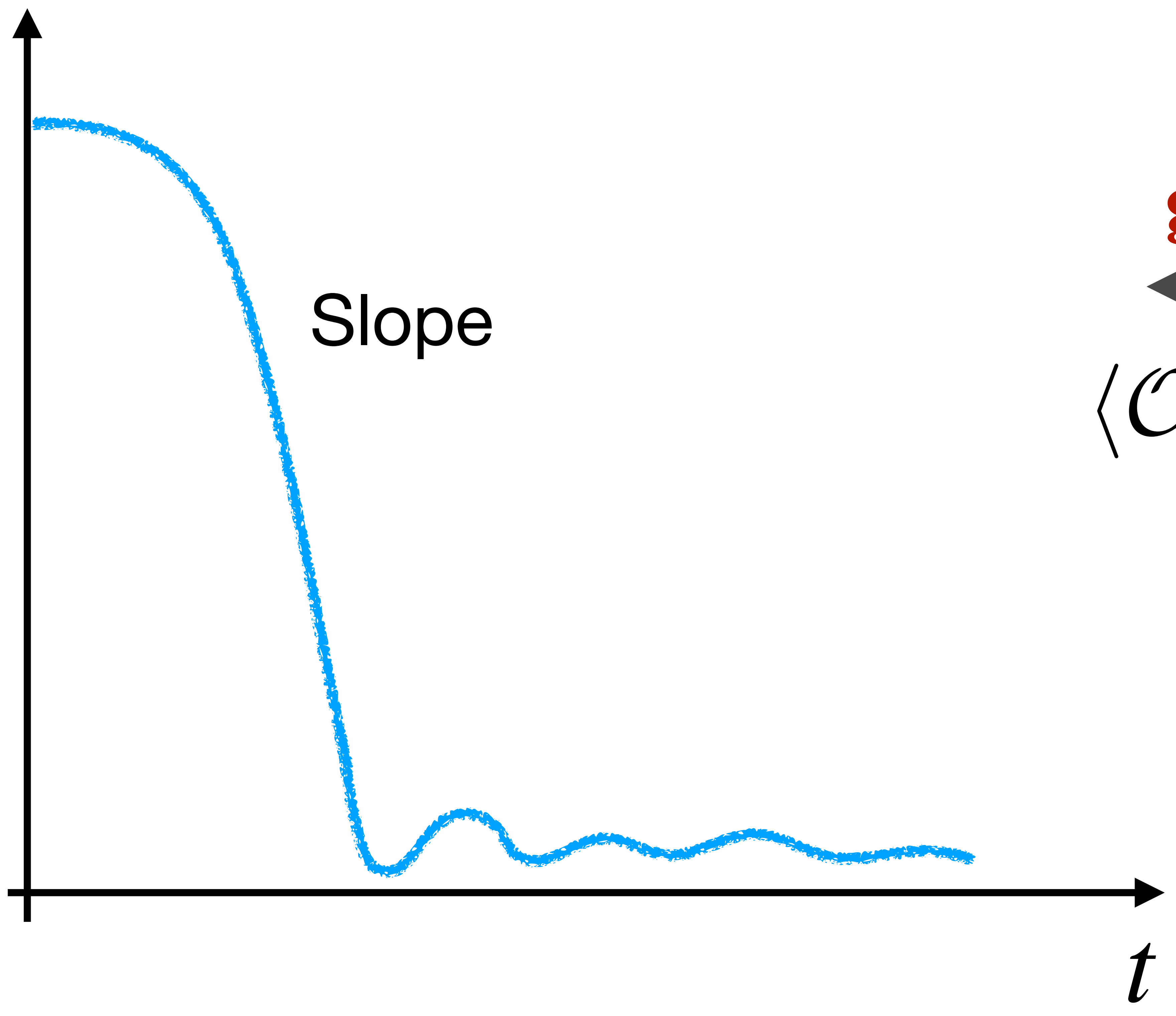
01. Two Paradoxes

Black hole information Paradox

Correlator decays forever

Wormhole Size Paradox

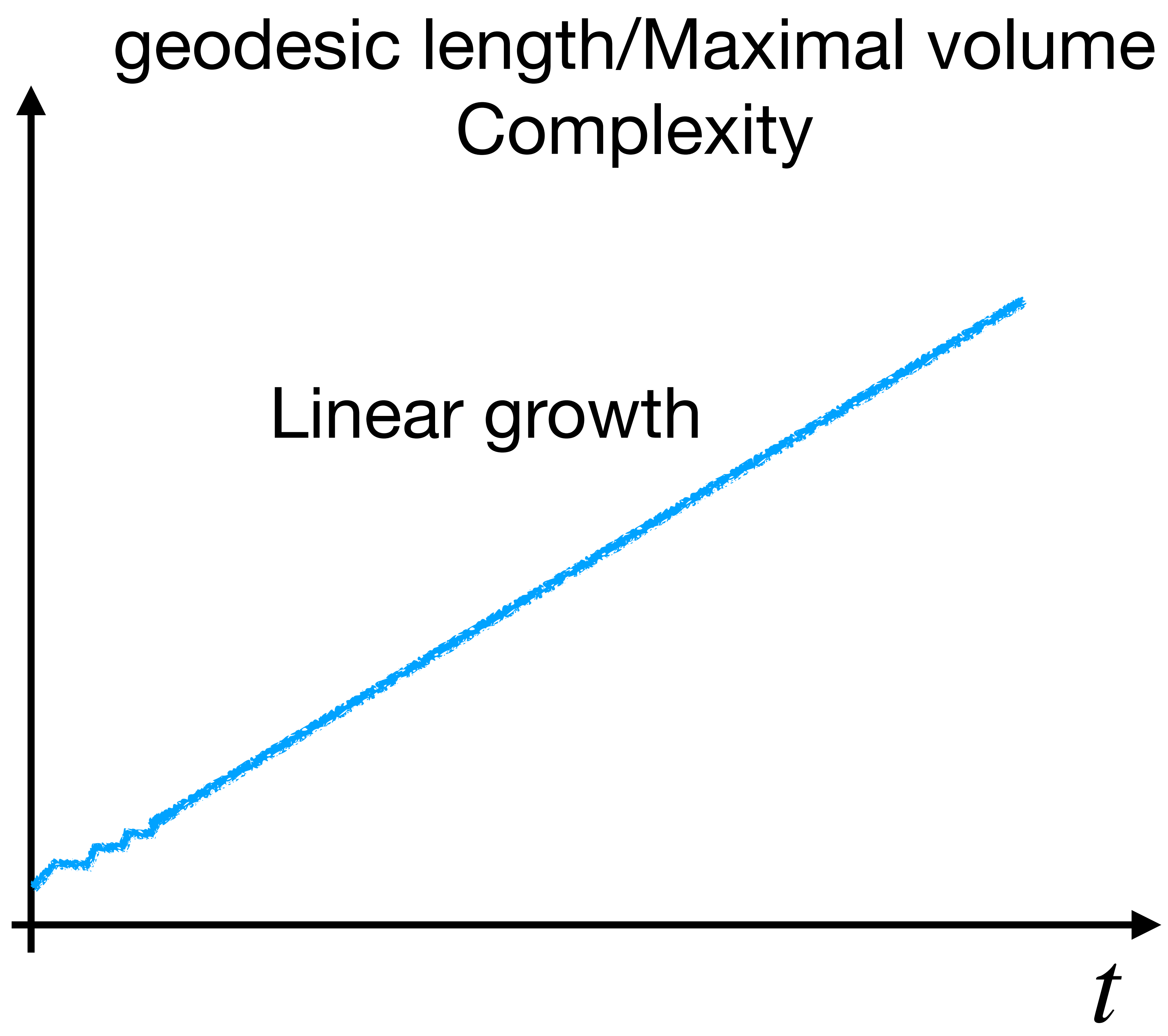
Wormhole/complexity grows forever



Semi-classical limit

geodesic approximation

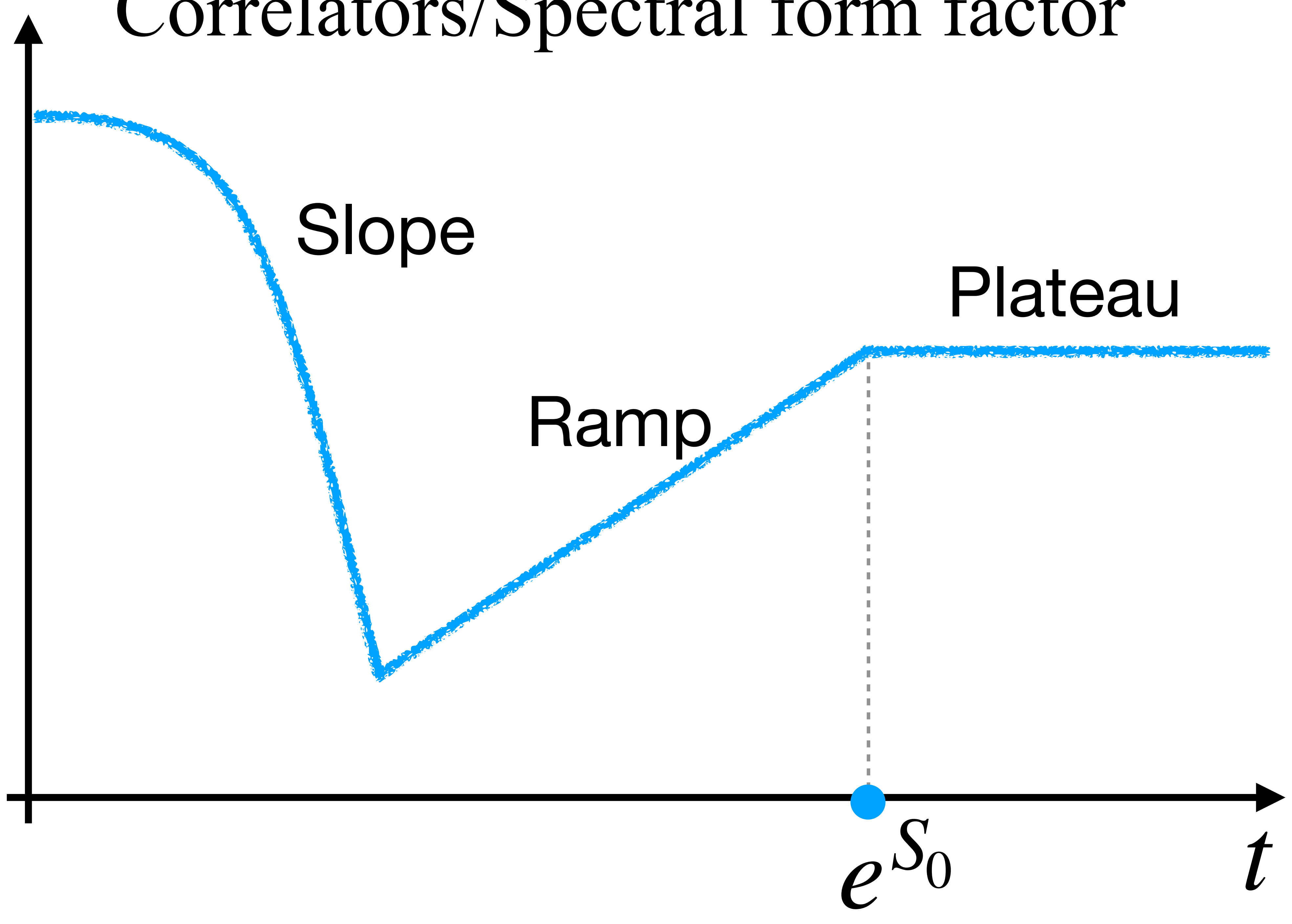
$$\langle \mathcal{O}_{\mathrm{L}} \mathcal{O}_{\mathrm{R}} \rangle \sim \sum e^{-\alpha L_{\text{geodesic}}}$$



01. Two Paradoxes

Black hole information Paradox

Quantum corrections + RMT universality



Slope-ramp-plateau structure

Wormhole Size Paradox

Wormhole/complexity grows forever

Solution ?

02. Generating functions of holographic complexity

The key lesson from the two paradoxes

Time evolution of the size of black hole interior is governed by the generating function

Example: geodesic length in JT $\lim_{\alpha\to 0}\left(-\partial_\alpha \left\langle e^{-\alpha\ell}\right\rangle\right) = \langle\hat{\ell}\rangle$ **Iliesiu, Mezei, Sárosi, [2107.06286]**

Generating function of complexity

$$\langle e^{-\alpha C}\rangle$$

Slope-ramp-plateau structure

$\alpha \to 0$

Holographic complexity measures

$$\langle \hat{C} \rangle$$

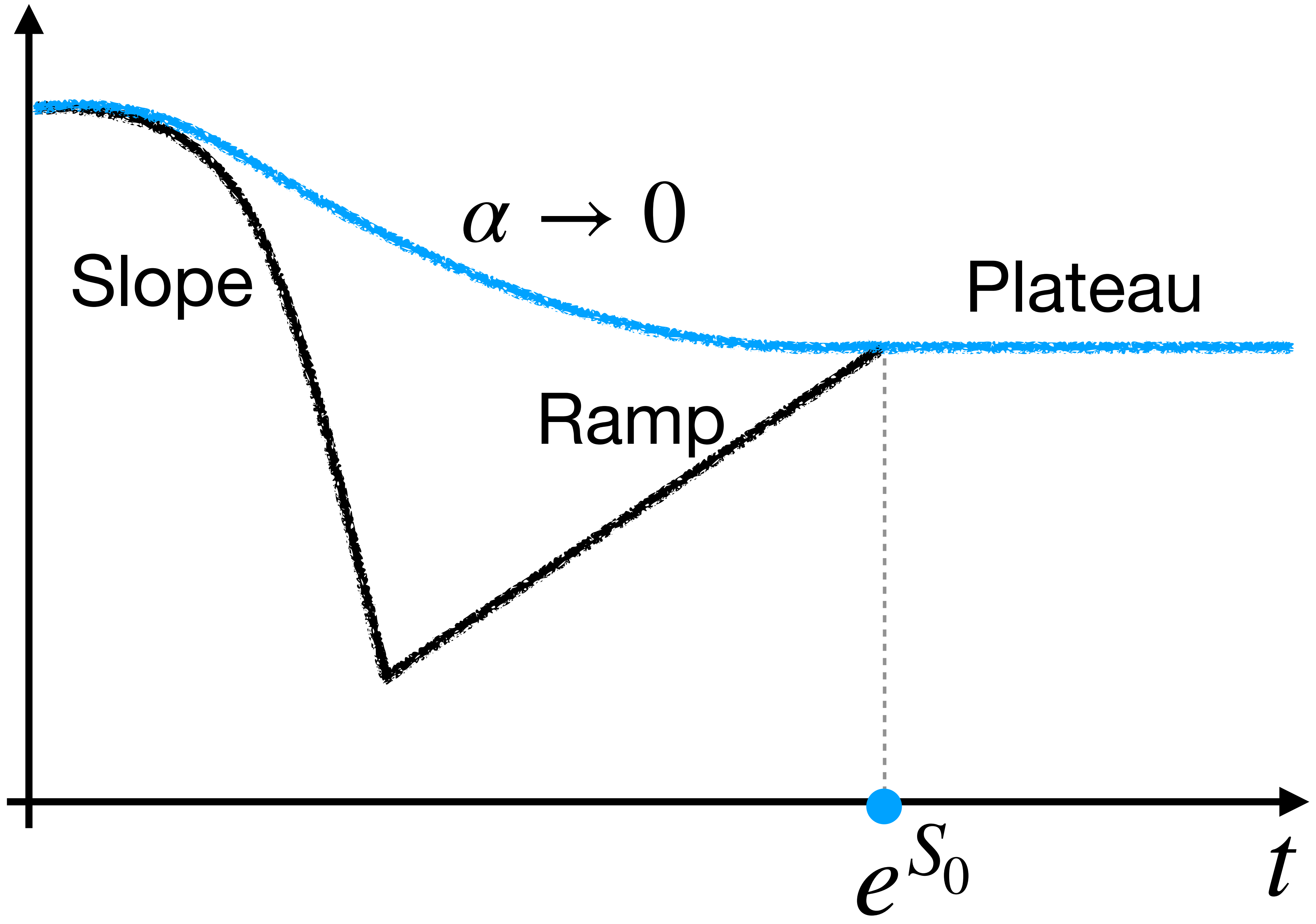
Linear-plateau structure

02. Generating functions of holographic complexity

Quantum corrections + RMT universality

Generating functions of complexity

$\langle e^{-\alpha C} \rangle$



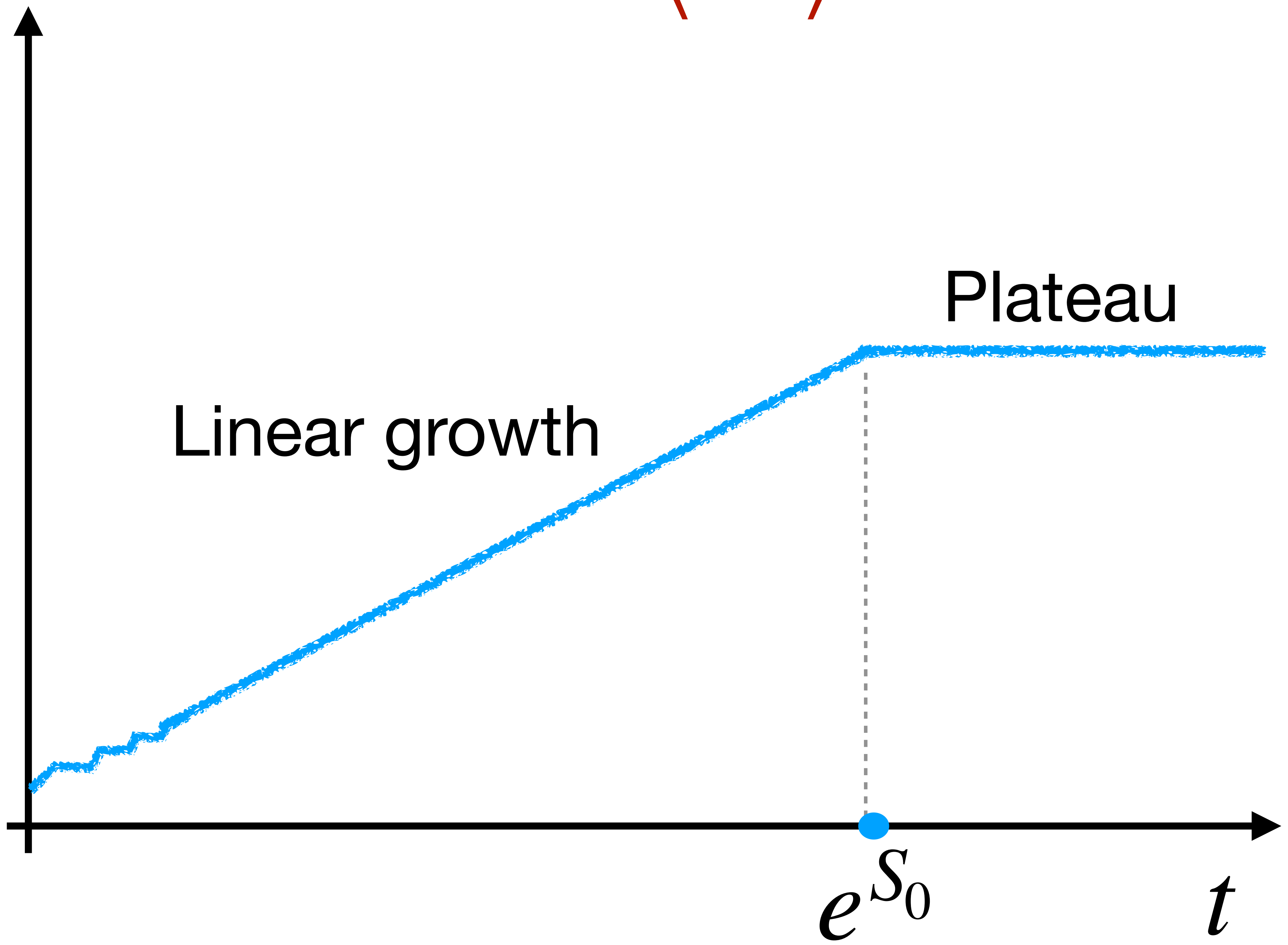
Slope-ramp-plateau structure

$\alpha \to 0$

Quantum corrections + RMT universality

Quantum complexity/wormhole size

$\langle \hat{C} \rangle$



linear-plateau structure

02. Generating functions of holographic complexity

(rigged) Hilbert space

Energy eigenstates $|E_i\rangle$ $\langle E \mid E' \rangle = \frac{\delta(E - E')}{e^{S_0} D(E)},$

Density of states $D(E)$

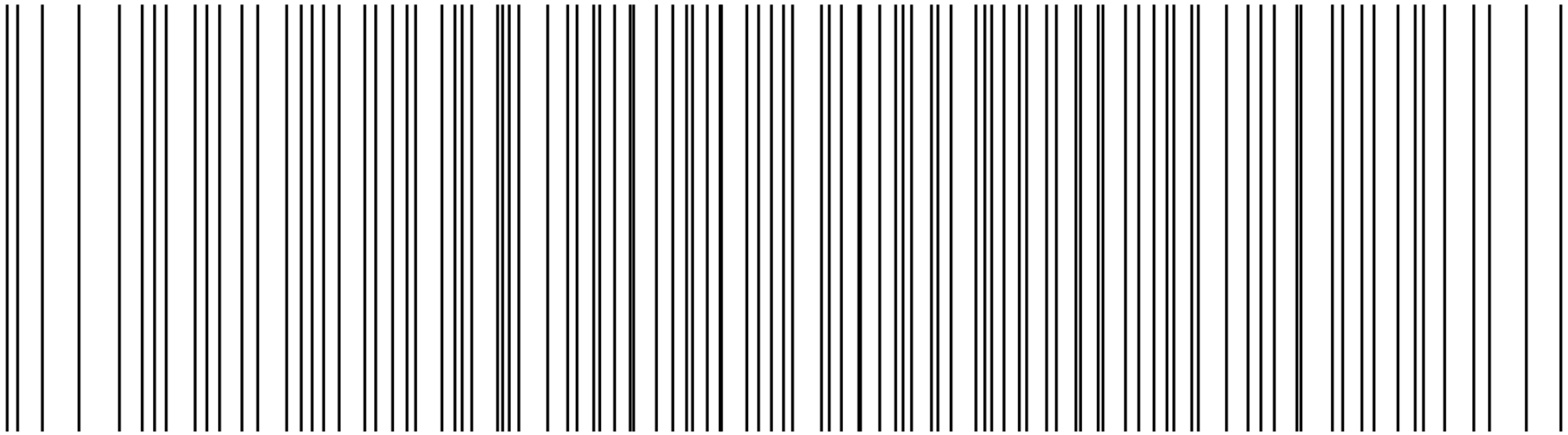
microcanonical Hilbert subspace

$$E \in \left[E_0 - \frac{\Delta E}{2}, E_0 + \frac{\Delta E}{2}\right], \quad \text{with} \quad \Delta E \ll E_0 .$$

Heisenberg time
(inverse of the mean energy level spacing)

$$T_{\mathrm{H}} := 2\pi e^{S_0} D(E_0)$$

Random Matrix Theory
GUE, N=100

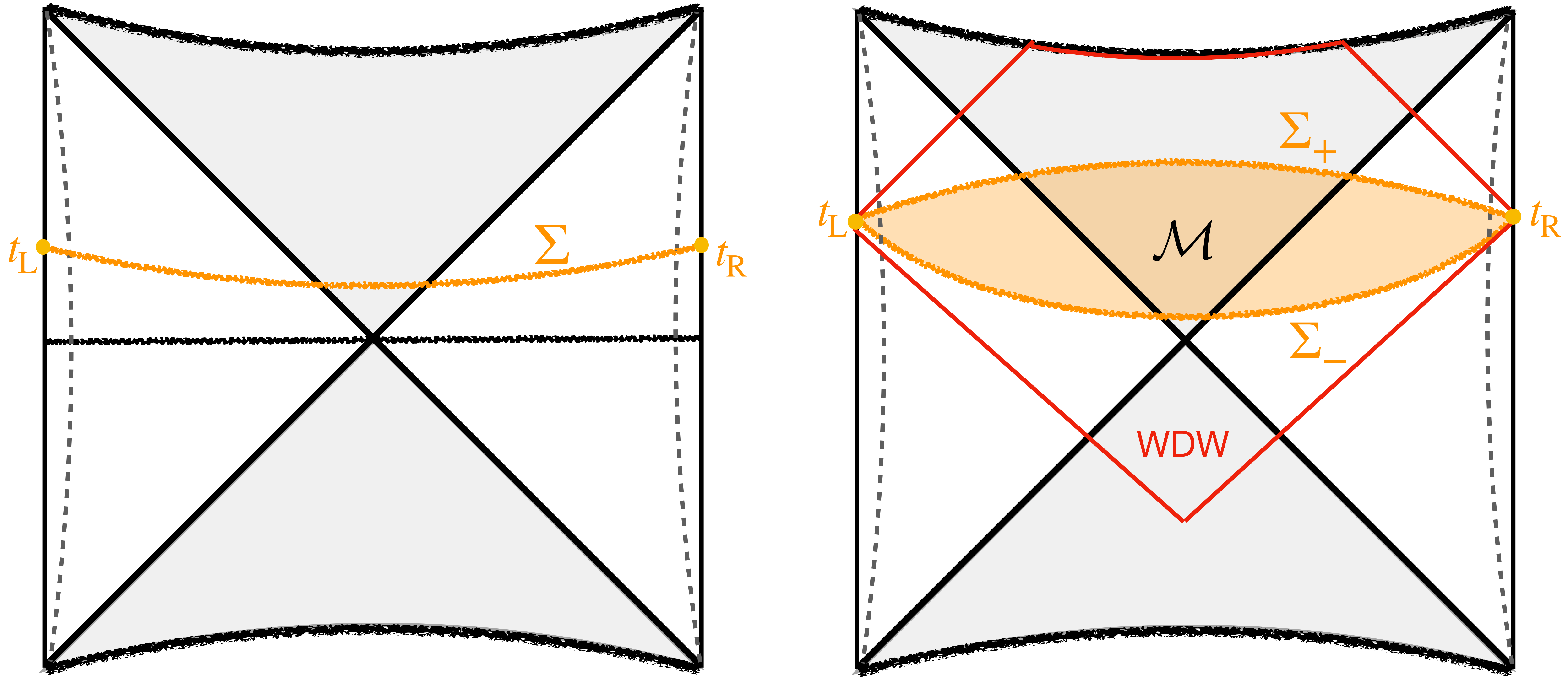


03. Black hole interior and holographic complexity

03. Black hole interior and holographic complexity

Belin, Myers, Ruan, Sárosi, Speranza
[2111.02429] [2210.09647]

Complexity=Anything Proposal

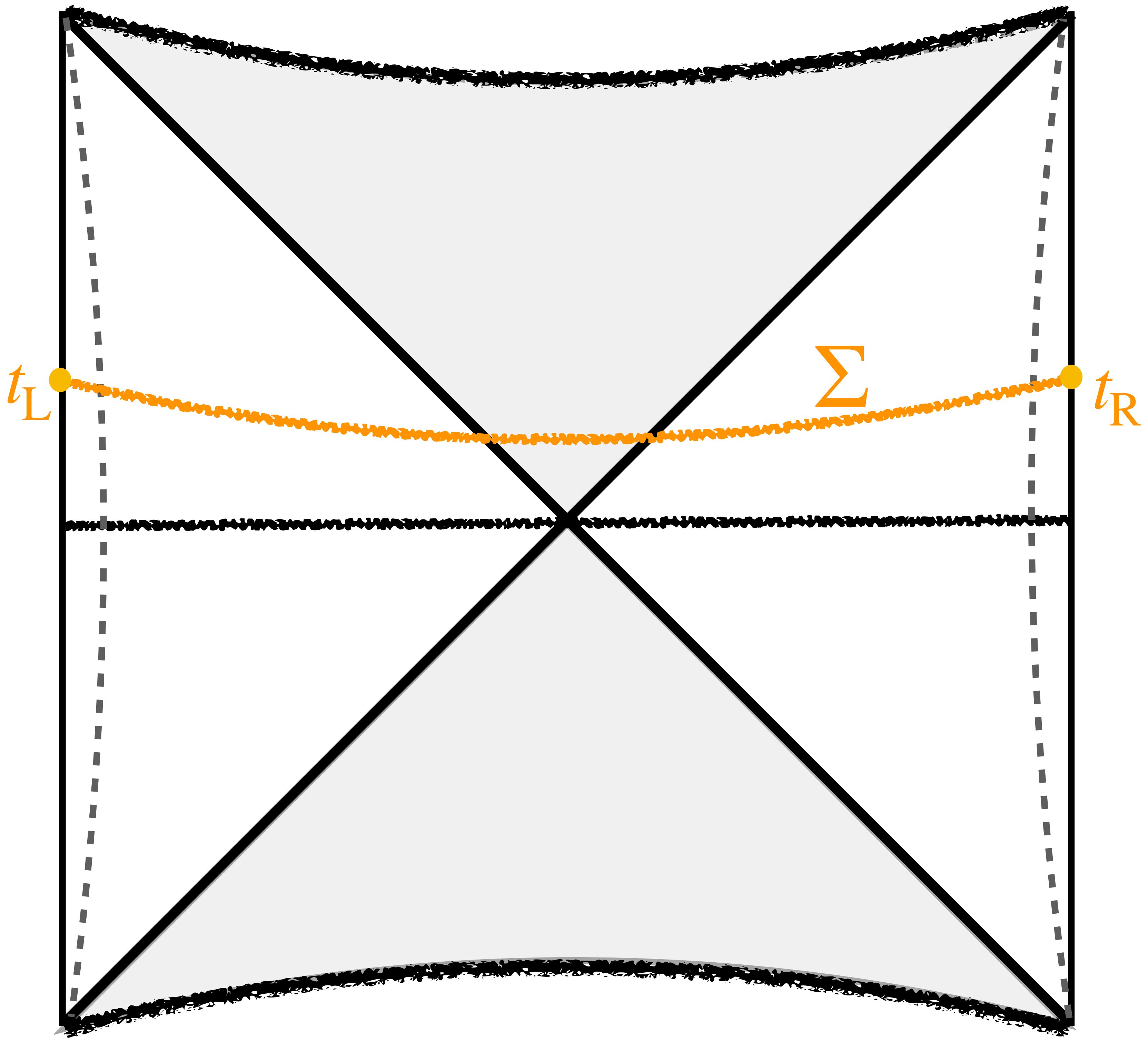


03. Black hole interior and holographic complexity

To appear *with* Masamichi Miyaji, Shono Shibuya, Kazuyoshi Yano

Infinite generalized volume $\widehat{e^{-\alpha\mathcal{C}}} := \int e^{-\alpha\mathcal{C}} |\mathcal{C}\rangle\langle\mathcal{C}| \, d\mathcal{C} \,.$

Generating functions $\langle e^{-\alpha\mathcal{C}}\rangle \equiv \langle \mathrm{TFD}(t)|\widehat{e^{-\alpha\mathcal{C}}}|\,\mathrm{TFD}(t)\rangle = \frac{e^{2S_0}}{Z}\int dE_i dE_j e^{-iE_{ij}t}\langle D(E_i)D(E_j)\rangle \times \langle E_i|\widehat{e^{-\alpha\mathcal{C}}}|E_j\rangle$



RMT Universality (e.g., GUE)

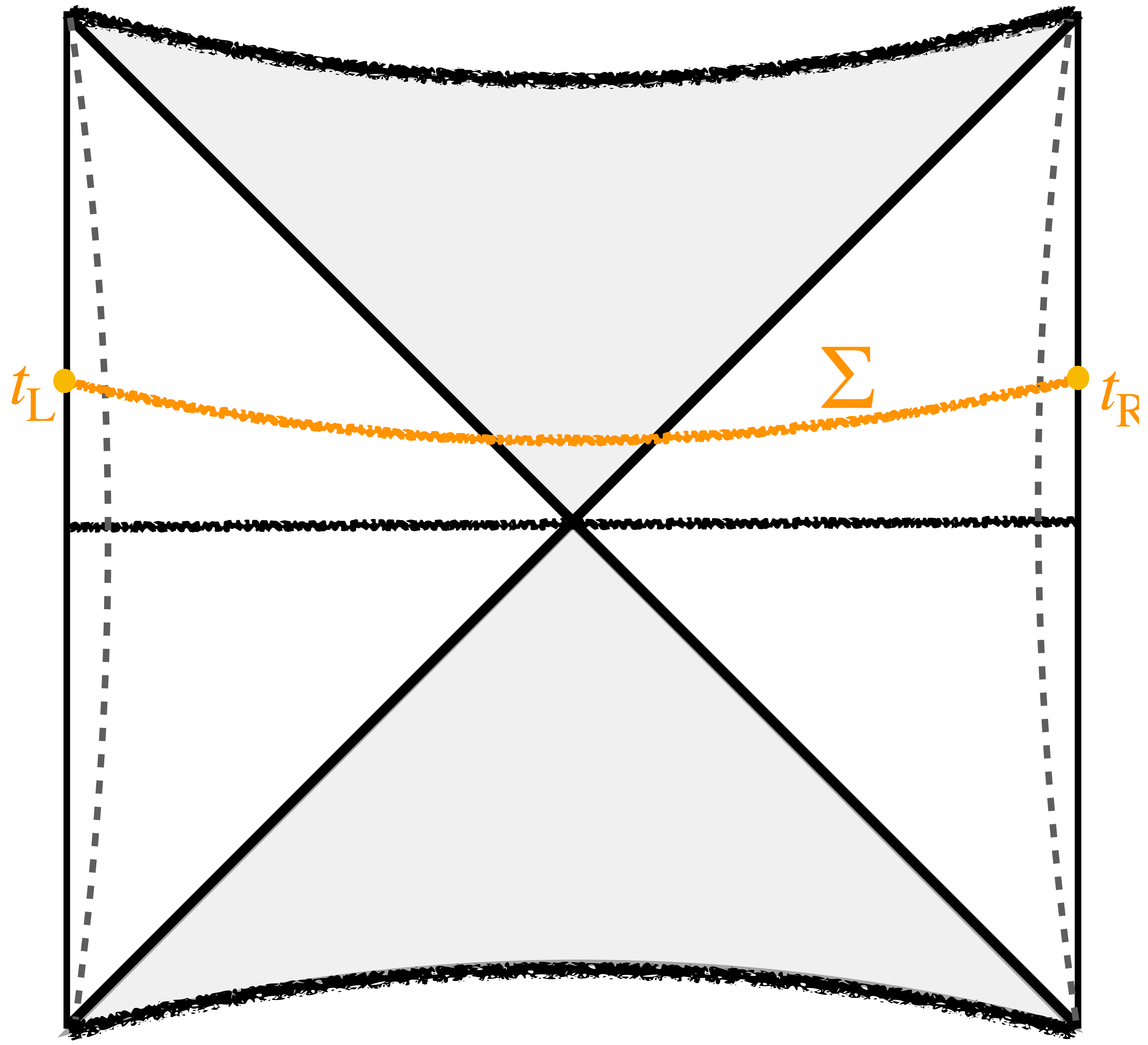
$$\begin{aligned}\langle D(E_i)D(E_j)\rangle &= D(E_i)D(E_j) + \langle D(E_i)D(E_j)\rangle_c \\ &\approx D(E_i)D(E_j) + e^{-S_0}\delta(E_i - E_j)D(E_i) - \left(\frac{\sin(\pi e^{S_0}D(\bar{E})(E_i - E_j))}{e^{S_0}\pi(E_i - E_j)}\right)^2 ,\end{aligned}$$

03. Black hole interior and holographic complexity

To appear *with* Masamichi Miyaji, Shono Shibuya, Kazuyoshi Yano

Infinite generalized volume $$\widehat{e^{-\alpha \mathcal{C}}} := \int e^{-\alpha \mathcal{C}} |\mathcal{C}\rangle\langle\mathcal{C}|\, d\mathcal{C} \,.$$

Generating functions $$\langle e^{-\alpha\mathcal{C}}\rangle \equiv \langle \mathrm{TFD}(t)|\widehat{e^{-\alpha\mathcal{C}}}|\,\mathrm{TFD}(t)\rangle = \frac{e^{2S_0}}{Z}\int dE_i dE_j e^{-iE_{ij}t}\underline{\langle D(E_i)D(E_j)\rangle} \times \underline{\langle E_i|\widehat{e^{-\alpha\mathcal{C}}}|E_j\rangle}$$



(ensemble-averaged) spectral two-point function

$$\langle D(E_i)D(E_j)\rangle = e^{-2S_0}\langle\rho(E_i)\rho(E_j)\rangle = e^{-2S_0}\left(\underline{R_2(E_i,E_j)} + \delta(E_i - E_j)\rho(E_i)\right) \,.$$

joint eigenvalue distribution

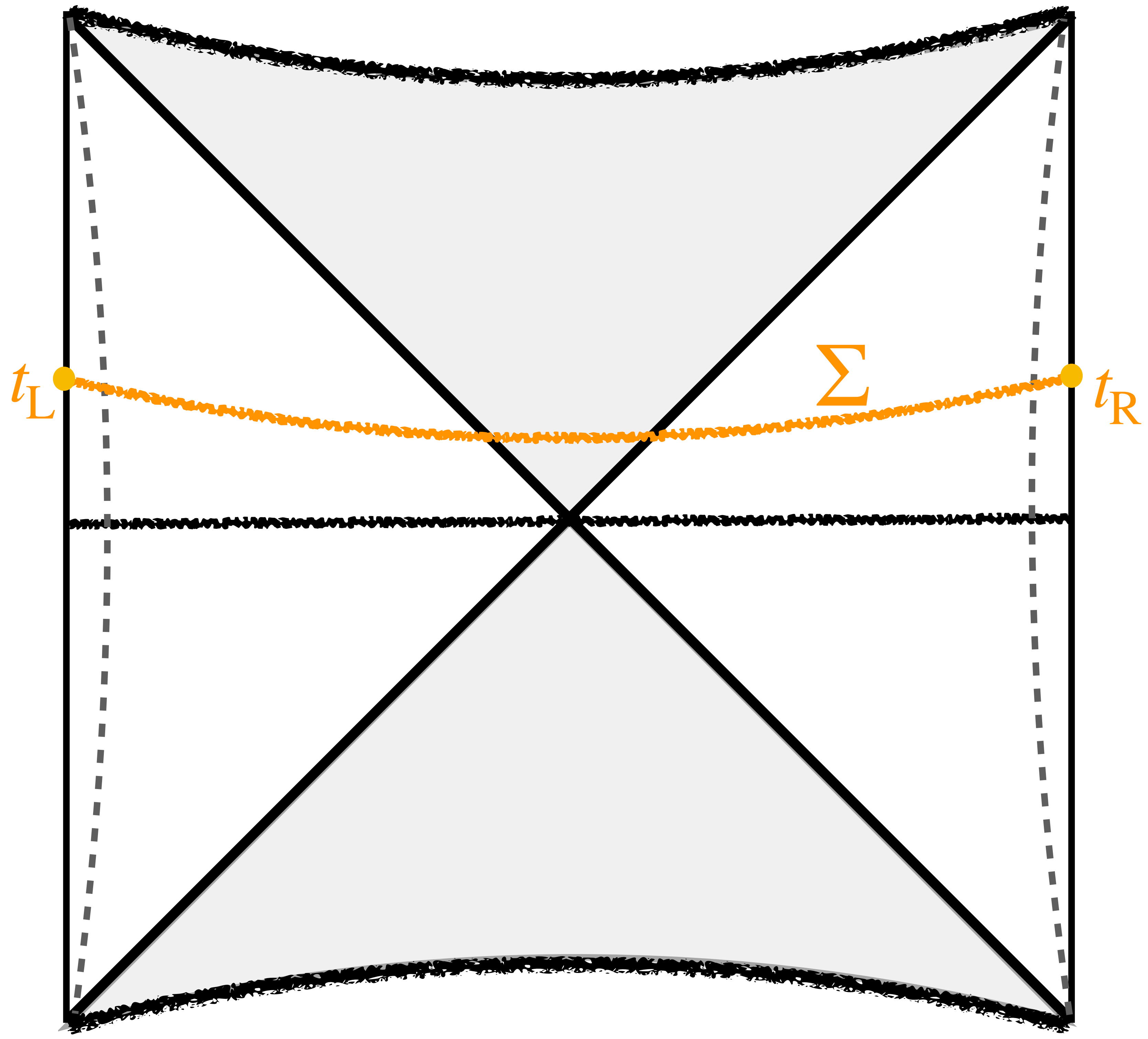
Matrix Element of Operator?

03. Black hole interior and holographic complexity

To appear *with* Masamichi Miyaji, Shono Shibuya, Kazuyoshi Yano

Infinite generalized volume $\widehat{e^{-\alpha\mathcal{C}}} := \int e^{-\alpha\mathcal{C}} |\mathcal{C}\rangle\langle\mathcal{C}|\, d\mathcal{C}\,.$

Generating functions $\langle e^{-\alpha\mathcal{C}}\rangle \equiv \langle \mathrm{TFD}(t)|\widehat{e^{-\alpha\mathcal{C}}}|\,\mathrm{TFD}(t)\rangle = \frac{e^{2S_0}}{Z}\int dE_i dE_j e^{-iE_{ij}t}\langle D(E_i)D(E_j)\rangle \times \langle E_i|\widehat{e^{-\alpha\mathcal{C}}}|E_j\rangle$



Trick: Analytical continuation to the complex energy plane

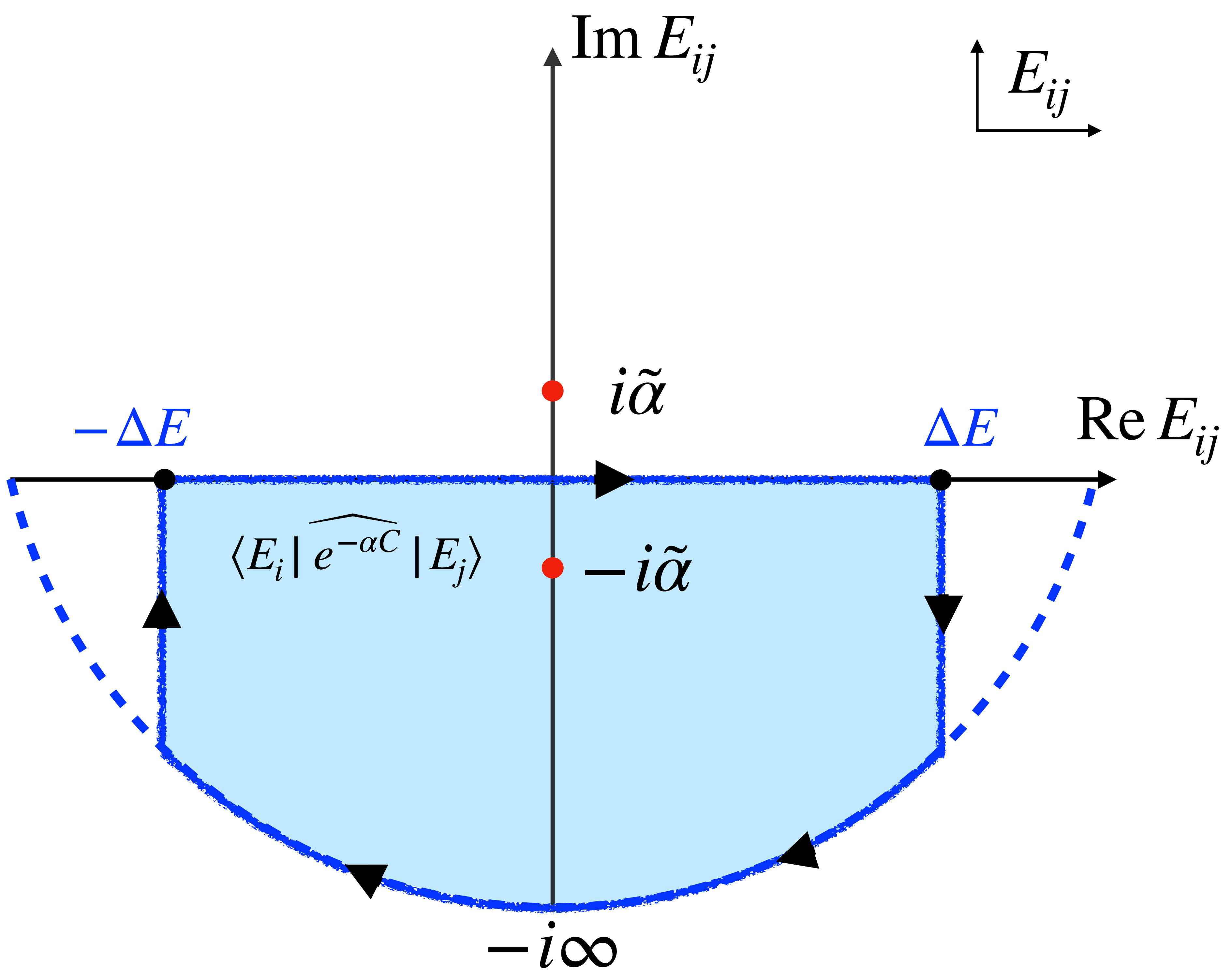
$$\int dE_i dE_j \longrightarrow \int d\bar{E}\int dE_{ij}$$

03. Black hole interior and holographic complexity

To appear *with* Masamichi Miyaji, Shono Shibuya, Kazuyoshi Yano

Infinite generalized volume $\widehat{e^{-\alpha\mathcal{C}}} := \int e^{-\alpha\mathcal{C}} |\mathcal{C}\rangle\langle\mathcal{C}|\, d\mathcal{C}\,.$

Generating functions $\langle e^{-\alpha\mathcal{C}}\rangle \equiv \langle \mathrm{TFD}(t)|\widehat{e^{-\alpha\mathcal{C}}}|\,\mathrm{TFD}(t)\rangle = \frac{e^{2S_0}}{Z}\int dE_i dE_j e^{-iE_{ij}t}\langle D(E_i)D(E_j)\rangle \times \langle E_i|\widehat{e^{-\alpha\mathcal{C}}}|E_j\rangle$



Trick: Analytical continuation to the complex energy plane

$$\int dE_i dE_j \longrightarrow \int d\bar{E}\int dE_{ij}$$

Time evolution ⟵ $e^{-iE_{ij}t}$

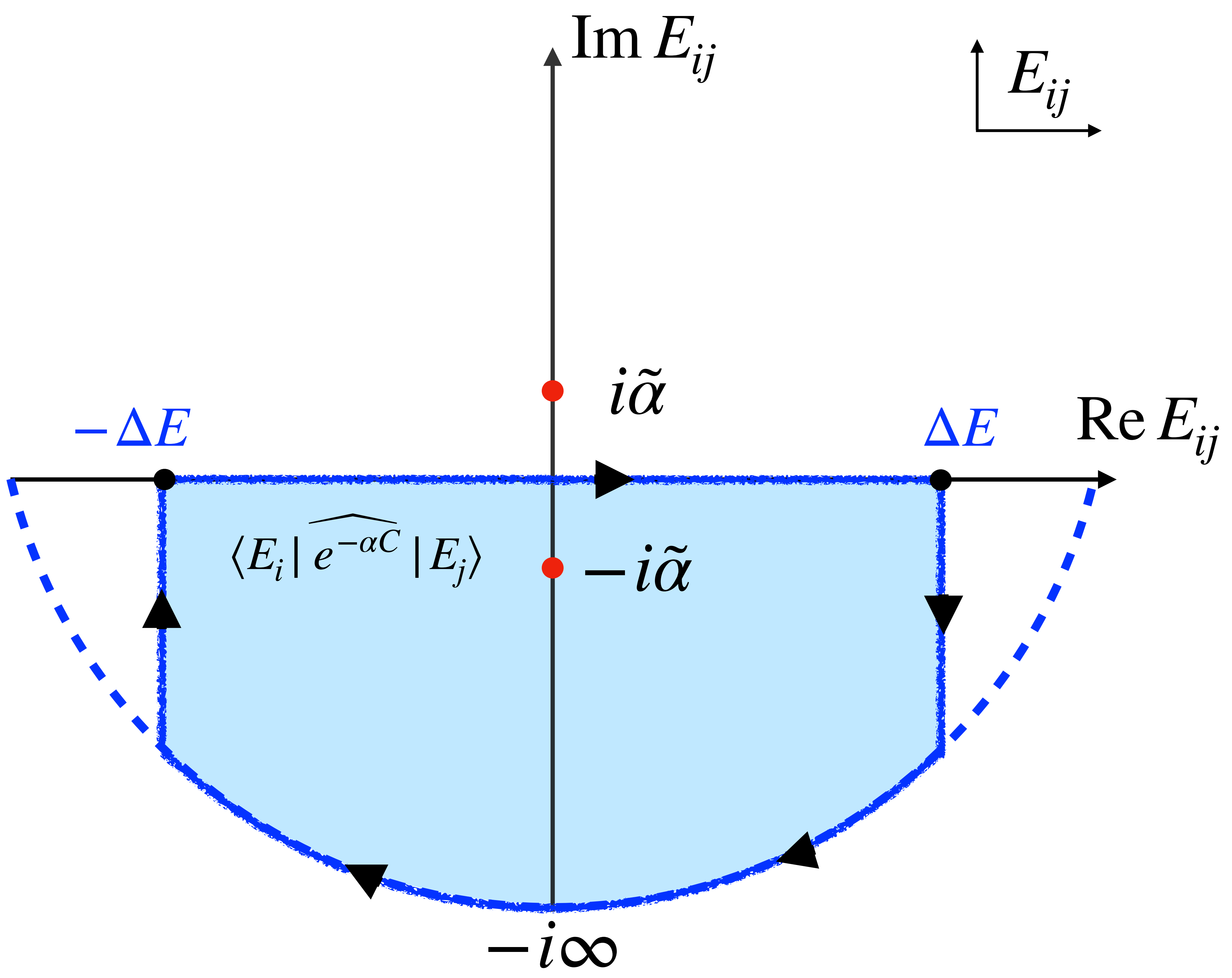
Time evolution ⟷ Pole structure

03. Black hole interior and holographic complexity

To appear *with* Masamichi Miyaji, Shono Shibuya, Kazuyoshi Yano

Infinite generalized volume $\widehat{e^{-\alpha \mathcal{C}}} := \int e^{-\alpha \mathcal{C}} |\mathcal{C}\rangle\langle\mathcal{C}|\, d\mathcal{C}\,.$

Generating functions $\langle e^{-\alpha \mathcal{C}} \rangle \equiv \langle \mathrm{TFD}(t) | \widehat{e^{-\alpha \mathcal{C}}} | \mathrm{TFD}(t) \rangle = \frac{e^{2S_0}}{Z} \int dE_i dE_j e^{-iE_{ij}t} \langle D(E_i) D(E_j) \rangle \times \langle E_i | \widehat{e^{-\alpha \mathcal{C}}} | E_j \rangle$



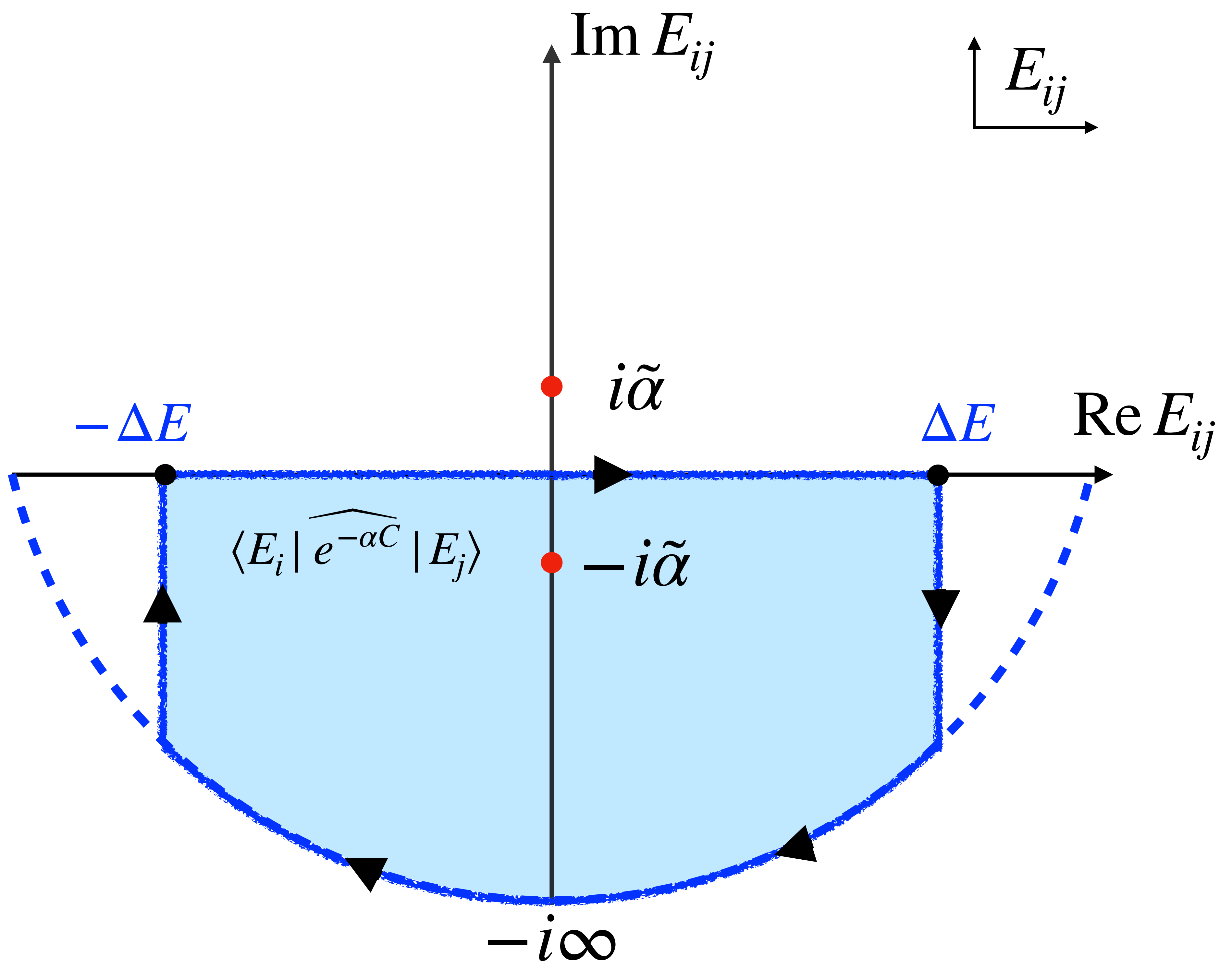
Time evolution ⟷ Pole structure

$$\langle E_i | \widehat{e^{-\alpha \mathcal{C}}} | E_j \rangle \sim \frac{1}{(\tilde{\alpha} + iE_{ij})(\tilde{\alpha} - iE_{ij})}$$

sufficient and necessary condition for the linear growth!

03. Black hole interior and holographic complexity

Classical linear growth and Pole Structure



$$\langle \hat{C} \rangle \Big|_{\text{classical}} = \lim_{\alpha \to 0} \left(-\partial_\alpha G_{\text{classical}}\, (\alpha, t) \right) \sim Mt$$

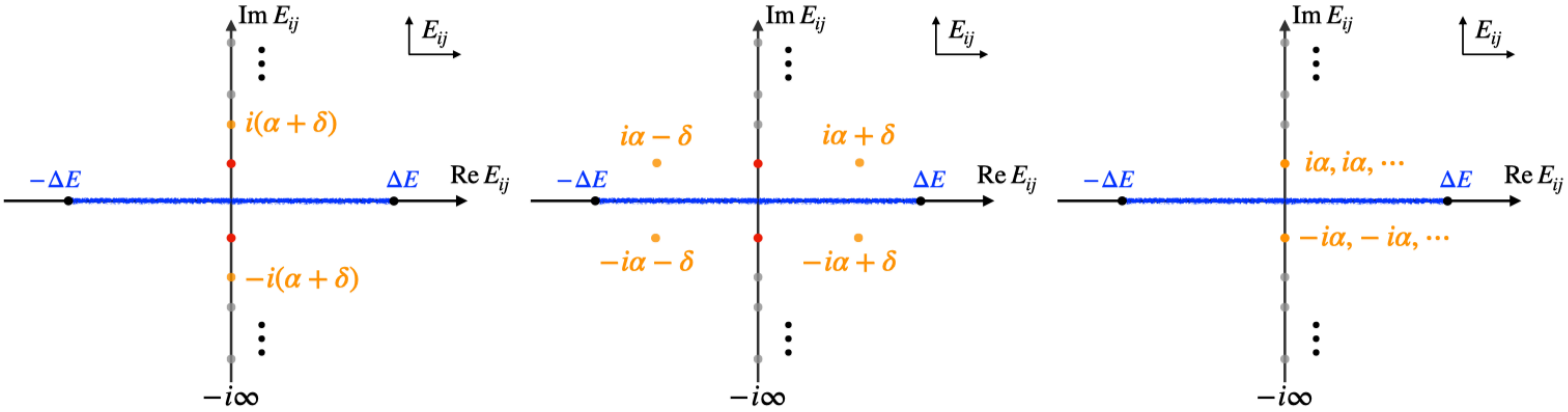
Necessary and Sufficient condition

$$\langle E_i | \widehat{e^{-\alpha \mathcal{C}}} | E_j \rangle \sim \frac{\tilde{\alpha}}{(\tilde{\alpha} + iE_{ij})(\tilde{\alpha} - iE_{ij})} + \mathcal{O}(\alpha^2).$$

$$\tilde{\alpha} = M\alpha + \mathcal{O}(\alpha^2)$$

03. Black hole interior and holographic complexity

Time Evolution in Classical Spacetime?



exponential decay | oscillatory behavior | quadratic growth

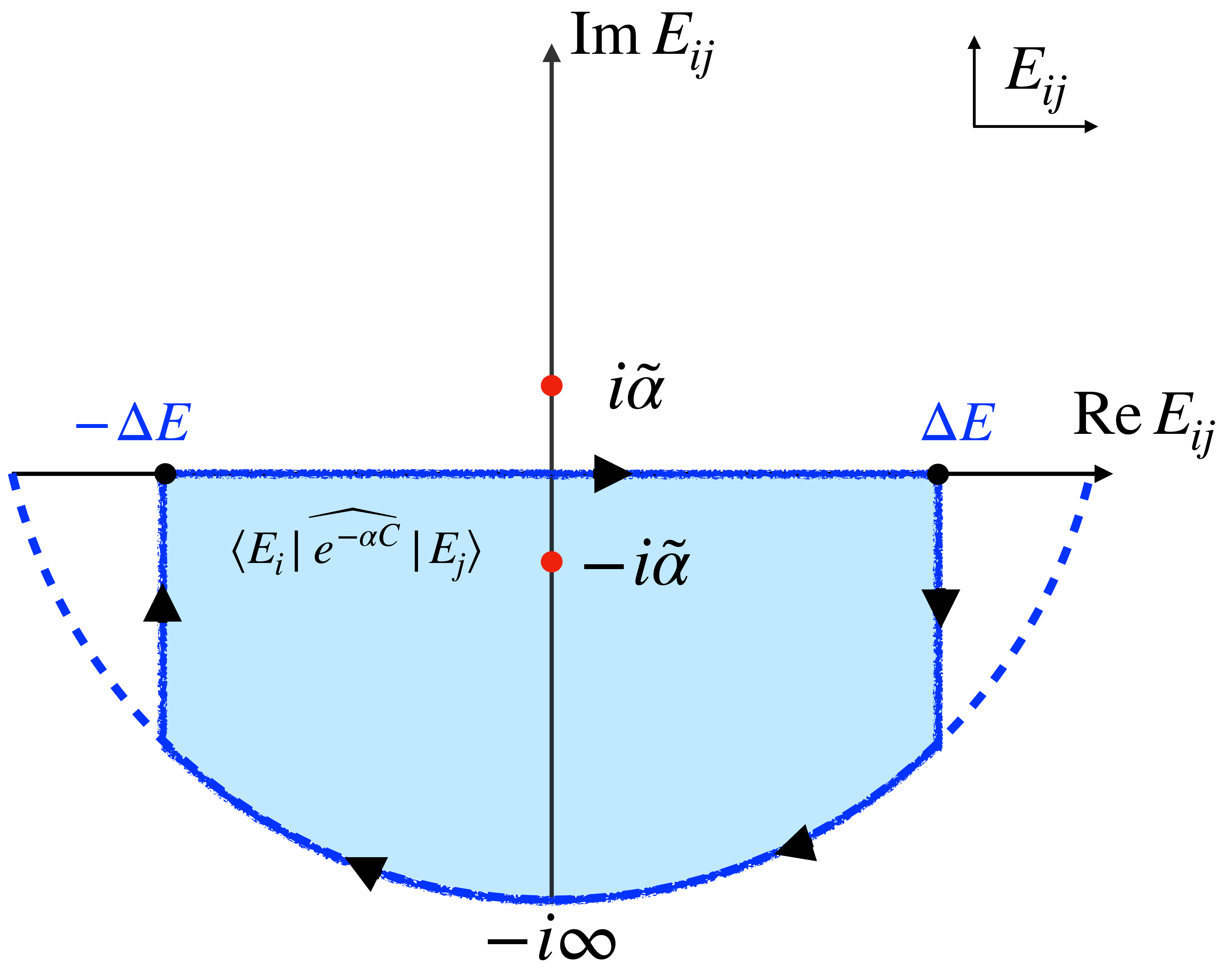
Quantum Corrections?

(ensemble-averaged) spectral two-point function

$$\langle D(E_i)D(E_j)\rangle = e^{-2S_0}\langle\rho(E_i)\rho(E_j)\rangle = e^{-2S_0}\left(R_2(E_i,E_j) + \delta(E_i - E_j)\rho(E_i)\right) \, .$$

04. Toward the universal time evolution

Generating functions $\langle e^{-\alpha\mathcal{C}}\rangle \equiv \langle \mathrm{TFD}(t)|\widehat{e^{-\alpha\mathcal{C}}}|\,\mathrm{TFD}(t)\rangle = \frac{e^{2S_0}}{Z}\int dE_i dE_j e^{-iE_{ij}t}\langle D(E_i)D(E_j)\rangle \times \langle E_i|\widehat{e^{-\alpha\mathcal{C}}}|E_j\rangle$



(ensemble-averaged) spectral two-point function

$$\langle D(E_i)D(E_j)\rangle = e^{-2S_0}\langle\rho(E_i)\rho(E_j)\rangle = e^{-2S_0}\left(\underline{R_2(E_i,E_j)} + \delta(E_i-E_j)\rho(E_i)\right).$$

Includes quantum corrections

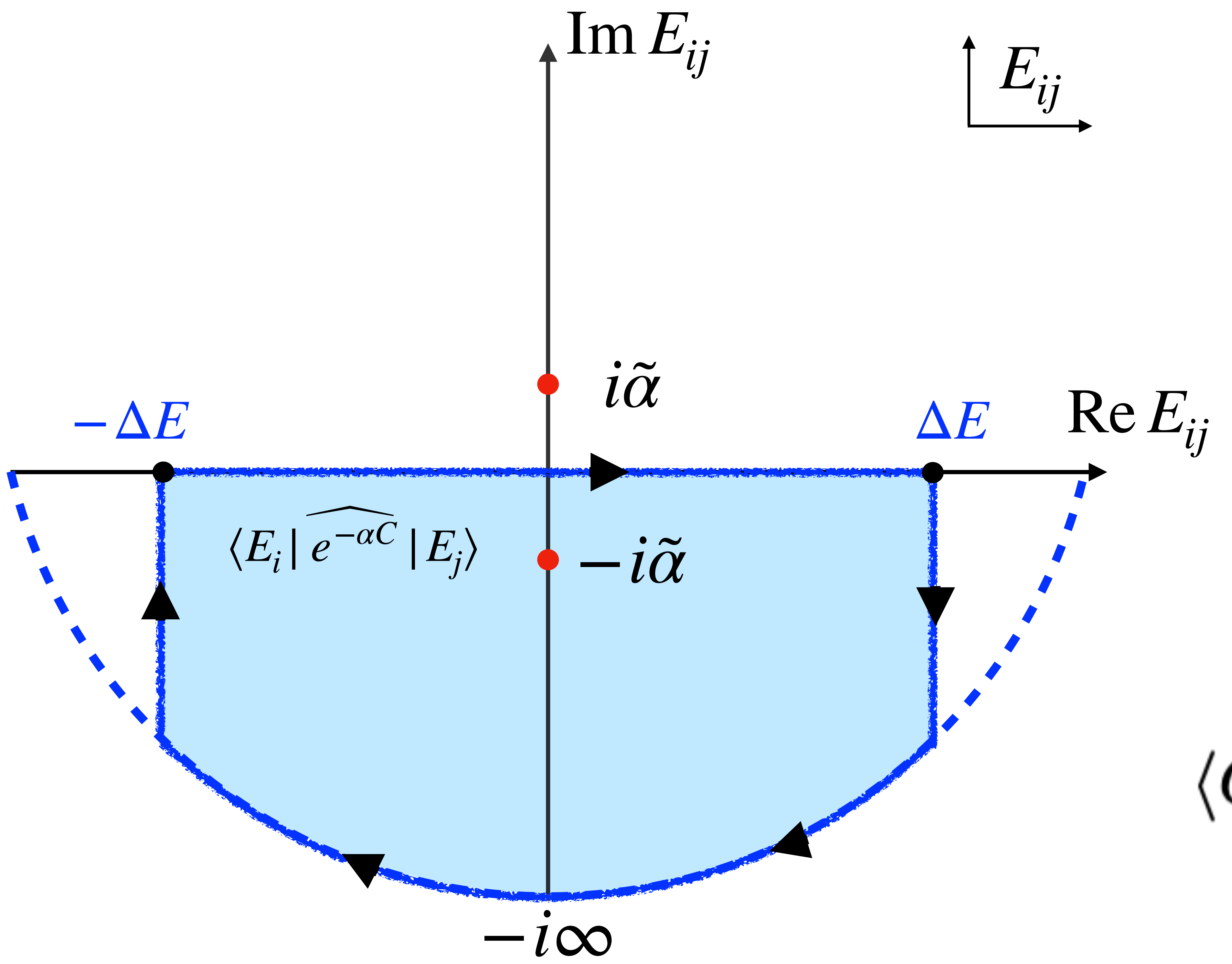
$$R_2(E_{ij},\bar{E}) = K(E_i,E_i)\,K(E_j,E_j) - K(E_i,E_j)\,K(E_j,E_i) = \rho_i\rho_j - (K_{ij})^2 .$$

Time evolution ⟷ Pole structure

late times: $G_{\mathrm{classical}}(\alpha,t) + G_{\mathrm{quantum}}(\alpha,t) \sim \mathrm{Con} + e^{-\tilde{\alpha}t}\times\left(R_2(E_{ij},\bar{E})\right)\big|_{E_{ij}=-i\tilde{\alpha}}\,.$

04. Toward the universal time evolution

Generating functions $\langle e^{-\alpha\mathcal{C}}\rangle \equiv \langle \mathrm{TFD}(t)|\widehat{e^{-\alpha\mathcal{C}}}|\,\mathrm{TFD}(t)\rangle = \frac{e^{2S_0}}{Z}\int dE_i dE_j e^{-iE_{ij}t}\langle D(E_i)D(E_j)\rangle \times \langle E_i|\widehat{e^{-\alpha\mathcal{C}}}|E_j\rangle$



(ensemble-averaged) spectral two-point function

$$\langle D(E_i)D(E_j)\rangle = e^{-2S_0}\langle\rho(E_i)\rho(E_j)\rangle = e^{-2S_0}\left(R_2(E_i,E_j)+\delta(E_i-E_j)\rho(E_i)\right).$$

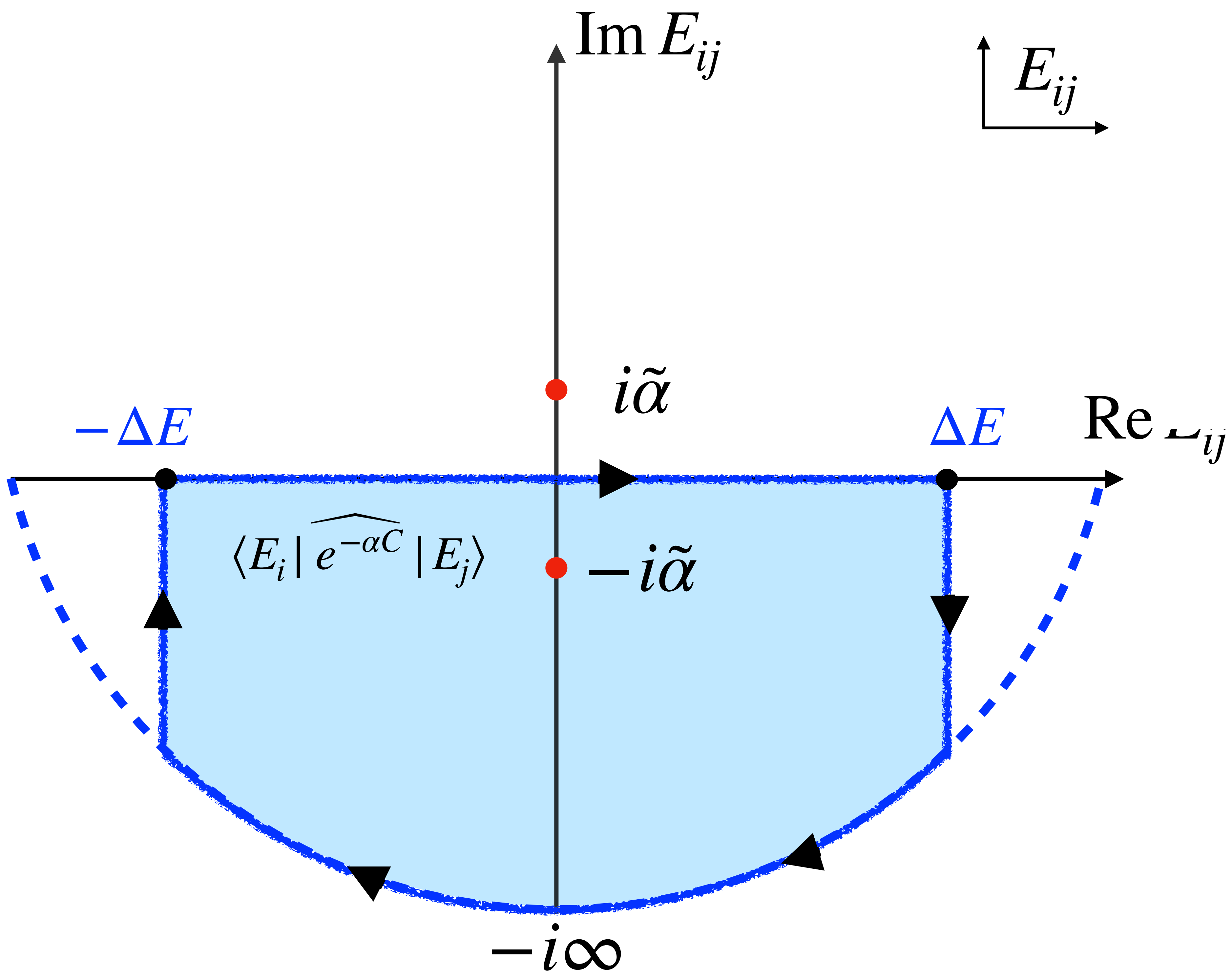
the late-time evolution of complexity measures

$$\langle\hat{C}\rangle = \langle\hat{C}\rangle|_{\text{classical}} + \langle\hat{C}\rangle|_{\text{quantum}} \approx Mt\times\left(R_2(E_{ij},\bar{E})\right)\Big|_{E_{ij}=-i\alpha=0}.$$

$$\lim_{\alpha\to 0} \overset{E_{ij}=-i\tilde{\alpha}}{=} \lim_{E_{ij}\to 0}$$

04. Toward the universal time evolution

Generating functions $\langle e^{-\alpha\mathcal{C}}\rangle \equiv \langle \mathrm{TFD}(t)|\widehat{e^{-\alpha\mathcal{C}}}|\,\mathrm{TFD}(t)\rangle = \frac{e^{2S_0}}{Z}\int dE_i dE_j e^{-iE_{ij}t}\langle D(E_i)D(E_j)\rangle \times \langle E_i|\widehat{e^{-\alpha\mathcal{C}}}|E_j\rangle$



the late-time evolution of complexity measures

$$\langle\hat{C}\rangle = \langle\hat{C}\rangle|_{\text{classical}} + \langle\hat{C}\rangle|_{\text{quantum}} \approx Mt \times \left(R_2(E_{ij},\bar{E})\right)\Big|_{E_{ij}=-i\alpha=0}\,.$$

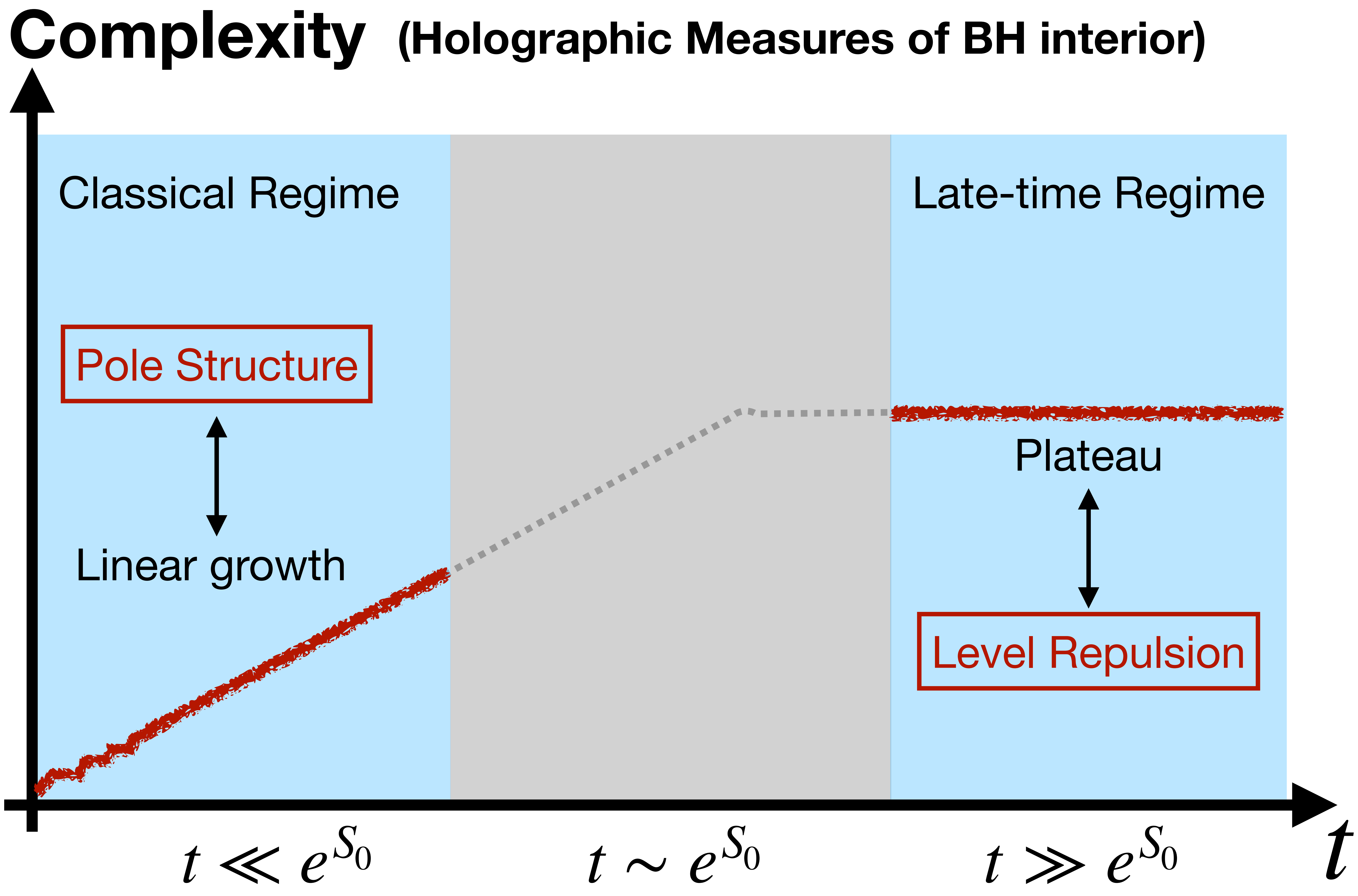
Late-time Plateau

(linear growth is cancelled by quantum corrections)

$$\lim_{E_{ij}\to 0} R_2(E_i,E_j) = R_2(E_i,E_i) = 0.$$

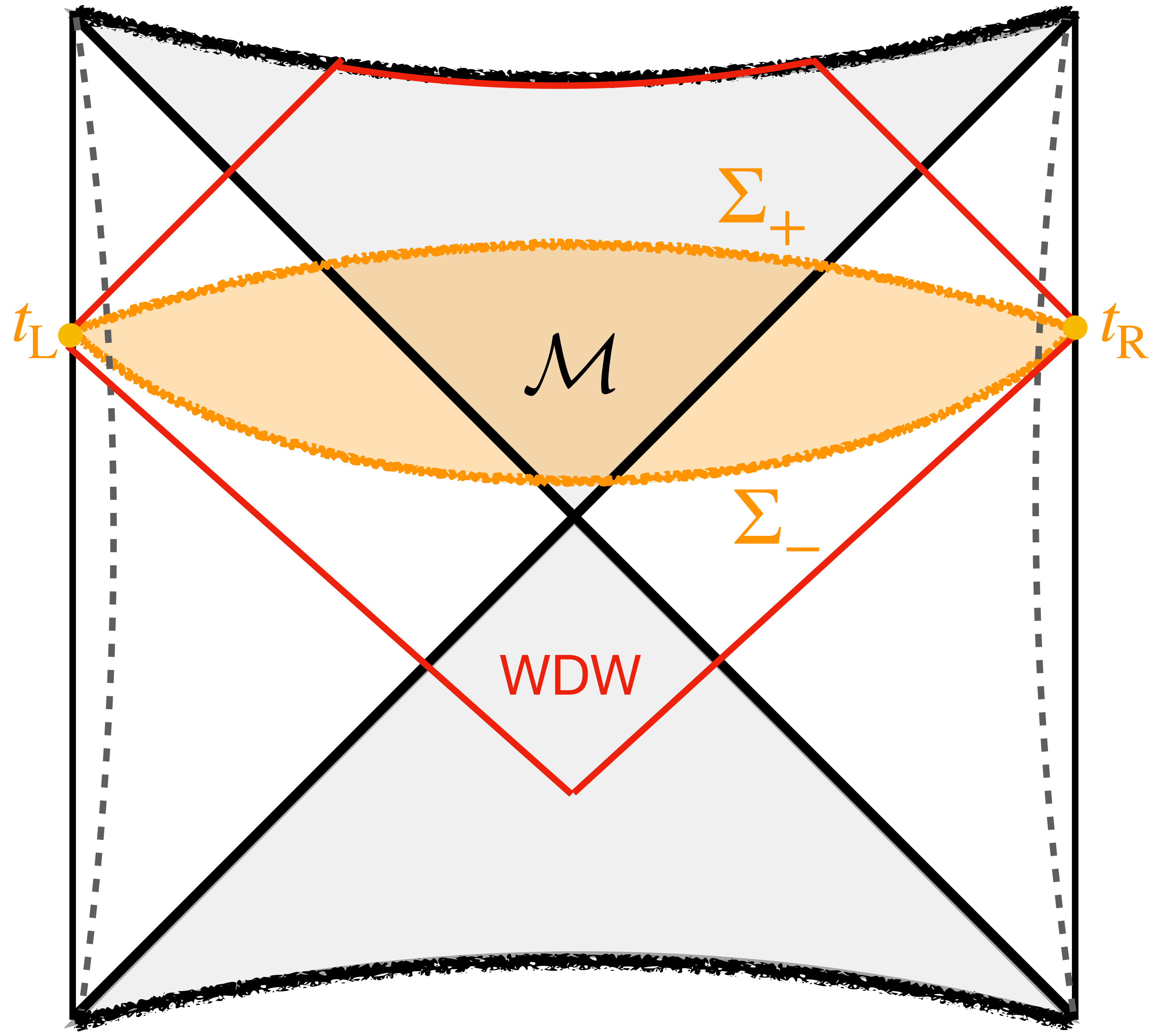
This is the *level repulsion* (hallmark of Quantum Chaos)!

04. Toward the universal time evolution



04. Toward the universal time evolution

How about codimension-zero holographic complexity (like CA)?



$$\langle e^{-\alpha_+ \mathcal{C}_+ - \alpha_- \mathcal{C}_-} \rangle = \frac{e^{3S_0}}{Z} \int dE_1 dE_2 dE_3 \, e^{-iE_{12}t} \langle D(E_1) D(E_2) D(E_3) \rangle \times \langle E_1 | \widehat{e^{-\alpha_+ \mathcal{C}_+}} | E_3 \rangle \langle E_3 | \widehat{e^{-\alpha_- \mathcal{C}_-}} | E_2 \rangle \,.$$

To appear *with* Masamichi Miyaji, Shono Shibuya, Kazuyoshi Yano

04. Toward the universal time evolution

How about codimension-zero holographic complexity (like CA)?

$$\langle e^{-\alpha_+ \mathcal{C}_+ - \alpha_- \mathcal{C}_-} \rangle = \frac{e^{3S_0}}{Z} \int dE_1 dE_2 dE_3\, e^{-iE_{12}t} \underline{\langle D(E_1) D(E_2) D(E_3) \rangle} \times \langle E_1 | \widehat{e^{-\alpha_+ \mathcal{C}_+}} | E_3 \rangle \langle E_3 | \widehat{e^{-\alpha_- \mathcal{C}_-}} | E_2 \rangle \,.$$

Spectral three-point function!

$$\begin{aligned} \langle D(E_1) D(E_2) D(E_3) \rangle &= D(E_1) D(E_2) D(E_3) + \langle D(E_1) D(E_2) D(E_3) \rangle_c \\ &+ \langle D(E_1) \rangle \langle D(E_2) D(E_3) \rangle_c + \langle D(E_2) \rangle \langle D(E_3) D(E_1) \rangle_c + \langle D(E_3) \rangle \langle D(E_2) D(E_1) \rangle_c . \end{aligned}$$

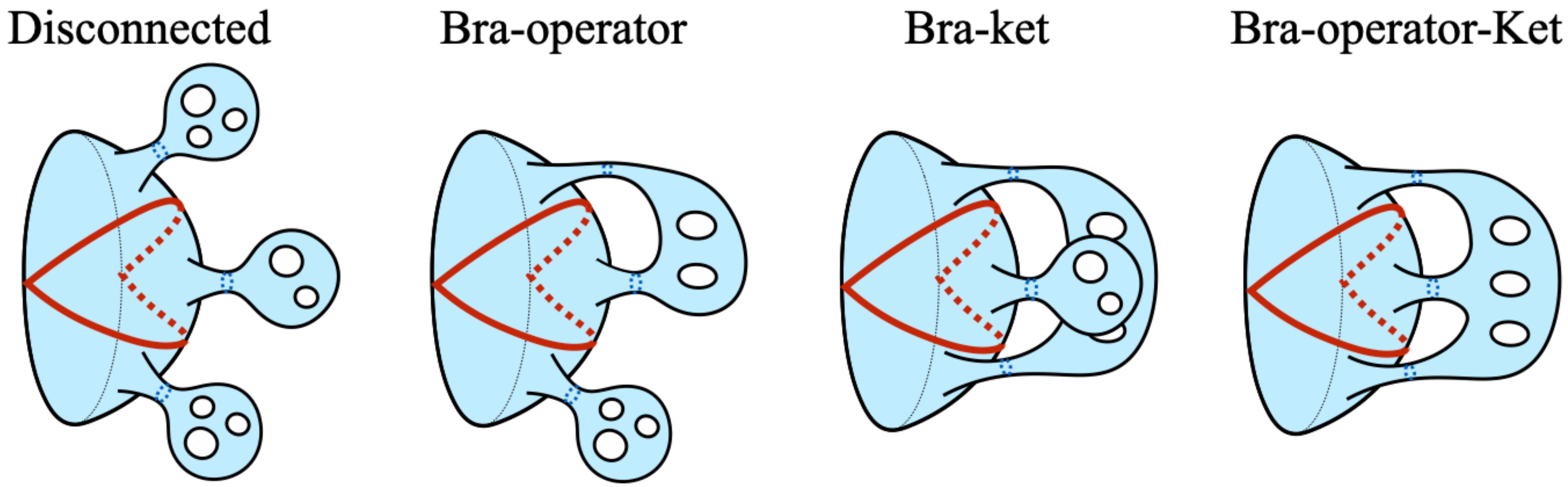
To appear *with* Masamichi Miyaji, Shono Shibuya, Kazuyoshi Yano

04. Toward the universal time evolution

How about codimension-zero holographic complexity (like CA)?

$$\langle e^{-\alpha_+\mathcal{C}_+-\alpha_-\mathcal{C}_-}\rangle = \frac{e^{3S_0}}{Z}\int dE_1 dE_2 dE_3\, e^{-iE_{12}t}\langle D(E_1)D(E_2)D(E_3)\rangle \times \langle E_1|\widehat{e^{-\alpha_+\mathcal{C}_+}}|E_3\rangle\,\langle E_3|\widehat{e^{-\alpha_-\mathcal{C}_-}}|E_2\rangle\,.$$

Spectral three-point function!



To appear *with* Masamichi Miyaji, Shono Shibuya, Kazuyoshi Yano

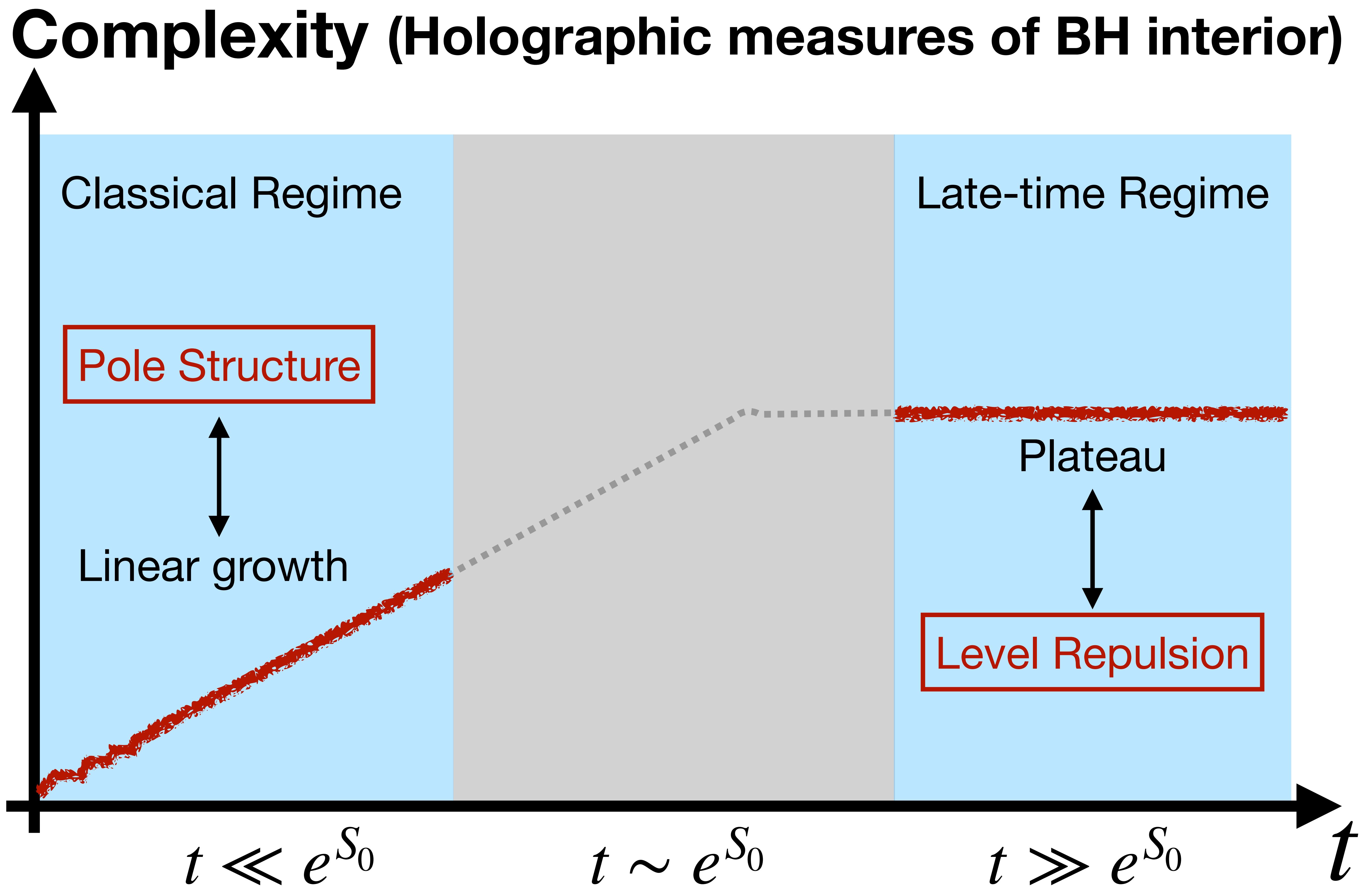
Is the black hole interior finite?

Is the black hole interior finite?

Black hole interior is finite ⟷ Dim of Hilbert space of quantum gravity is finite

04. Toward the universal time evolution

The size of black hole interior is finite ⟷ Dim of Hilbert space of quantum gravity is finite



Thanks for your attention!