Applications of WKB analysis in bottom-up holographic QCD models

Based on 2505.14324, 2501.01755 and work in progress

Miguel Angel Martin Contreras
University of South China

July 18th, 2025

Outline

1. WKB in Quarkonium Physics
2. Bottom-up in a nutshell
3. WKB I: Rydberg-Klein-Rees Formula
   - Illustrative example: D3/D7
   - Example: Heavy Multiquark states
4. WKB II: Segre-Fermi formula
   - Quark mass in the SoftWall (KKSS) model.
5. Conclusions

WKB in Quarkonium Physics

Motivation

What information is encoded in the mass spectrum?

Phys.Rept. 56 (1979) 167-235

QUANTUM MECHANICS WITH APPLICATIONS TO QUARKONIUM

C. QUIGG

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510, U.S.A.

and

Jonathan L. ROSNER

School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455, U.S.A.

PHYSICS REPORTS (Review Section of Physics Letters) 200, No. 4 (1991) 127–240. North-Holland

BOUND STATES OF QUARKS

Wolfgang LUCHA

Institut für Hochenergiephysik, Österreichische Akademie der Wissenschaften, Nikolsdorfergasse 18, A-1050 Vienna, Austria

Franz F. SCHÖBERL

Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Vienna, Austria

and

Dieter GROMES

Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, D-6900 Heidelberg, Fed. Rep. Germany

Editor: R. Petronzio Received August 1989

Big lesson: Defining properly the potential we can obtain a well-behaved mass spectrum

Spectrum

$$M_n^2 = a(n+b)^\nu$$

- Consequence of Confinement: bound states of quarks and gluons.
- Carries information about χ-sym. breaking.
- Testing tool for inner hadron dynamics.
- What other information can we extract?

Motivation

In NRQCD (quantum mechanics) , we learn:

Spectrum
$M_n^2 = a(n+b)^\nu$

WKB →

Potential
$V(r)$ at large r

We can do the same in bottom-up AdS/QCD:

Spectrum
$M_n^2 = a(n+b)^\nu$

→

Potential
$V(z)$ at large z

→

Bottom-up
dilaton
$\Phi(z)$

Bottom-up in a nutshell

Bottom-up methodology

1. Define an AdS-like geometric background: $dS^2 = e^{2\,A(z)}\eta_{mn}\,dx^m\,dx^n$, $\eta_{mn} = \text{diag}(-1, \vec{1}, 1)$.
2. Bulk action for hadrons: $I = \int d^5x\,\sqrt{-g}\,e^{-\Phi(z)}\,\mathcal{L}_{\text{hadrons}}\,[\psi, \nabla\,\psi, M_5\,\ldots]$, where $\Phi(z)$ is the dilaton (static or dynamic).
3. Obtain Sturm-Liouville bulk equations (after choosing the proper gauge fixing): $\partial_z\left[e^{-B(z)}\,\partial_z\psi(z)\right] + (-q^2)\,e^{-B(z)}\psi(z) - M_5^2\,e^{2\,A(z)}\,e^{-B(z)}\psi(z) = 0$, with $B(z) = \Phi(z) + (3 - 2\,S)A(z)$, where S is the hadron spin.
4. Transform to the Schrödinger-like equation $-\phi''(z) + V(z)\,\phi(z) = M_n^2\,\phi(z)$, using $\psi(z) = e^{\frac{1}{2}B(z)}\phi(z)$, where $M_n^2 = -q^2$, and

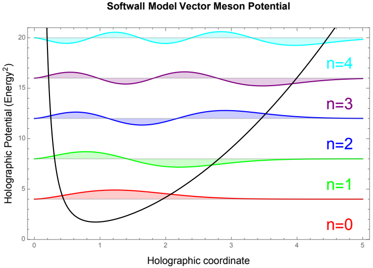
$$V(z) = \frac{B'(z)^2}{4} - \frac{B''(z)}{2} + M_5^2\,e^{2\,A(z)},$$

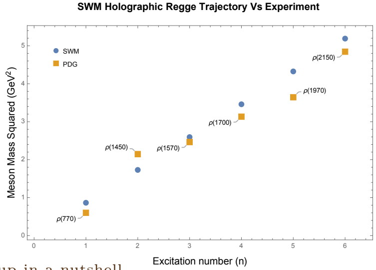
   defines the confining holographic potential.
5. Compute the Mass spectrum and the eigenfunction spectrum.

Note : A Similar procedure applies for LF-holographic QCD.

Example: Vector Softwall model

- Quadratic dilaton (Karch et al. 2005): $\Phi(z) = \kappa^2\, z^2$ for light hadrons.
- Poincaré Patch: $A(z) = \log \frac{R}{z}$. It can be extended to dynamic models (see the EMD model).
- Bulk action $I_H = -\frac{1}{4\, g_5^2} \int d^5x\, \sqrt{-g}\, e^{-\kappa^2 z^2}\, F_{mn}\, F^{mn}$, with $F_{mn} = 2\, \partial_{[m}\, A_{n]}$.
- Holographic potential $V(z) = \frac{4}{3\, z^2} + \kappa^2\, z^2$.
- Mass spectrum for ρ mesons $I^G(J^{PC}) = 1^+(1^{--})$: $M_n^2 = 4\,\kappa^2(n+1)$ with $\kappa = 0.388$ GeV, and eigenunctions $\phi(z) = \sqrt{\frac{2\,\kappa^4\, n!}{(n+1)!}}\, e^{-\frac{1}{2}\kappa^2\, z^2}\, z^{\frac{3}{2}}\, L_n^1\left(\kappa^2\, z^2\right)$.





WKB I: Rydberg-Klein-Rees Formula

Rydberg-Klein-Rees formula (See C Quigg, 1979)

Given a phenomenological structure for $V(z)$, it is possible to infer $\Phi(z)$. Consider the holographic potential for vector bulk fields:

$$V(z) = \frac{1}{4}\Phi'(z)^2 - \frac{1}{2}\Phi''(z) + \frac{1}{2\,z}\Phi'(z) + \frac{4\,M_5^2\,R^2 + 3}{4\,z^2}$$

The high z region is dominated by $\Phi(z)$ derivatives.

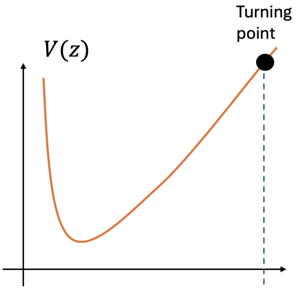
- From a eigenvalue spectrum $M_n^2(n)$, calculate the potential at high z (turning point), $V^*(z)$:

$$z(V) = 2\int_0^V \frac{d\,M^2}{\frac{\partial\,M^2}{\partial\,n}\sqrt{V^* - M^2}}.$$

- Compute the dilaton and deformation as

$$V^*[\Phi', \Phi''] = \frac{1}{4}\Phi'(z)^2 - \frac{1}{2}\Phi''(z) - \frac{\beta}{2\,z}\Phi'(z).$$

- Build the full potential by adding the low-z part.



Illustrative Example: "Bottom-up" D3/D7

1. From the spectrum, we solve the high-z potential $V*$:

$$M_{n,l}^2 = \frac{4\,L^2}{R^4}\,(n+l+1)\,(n+l+2) \rightarrow \quad z(V^*) = 2\int_0^{V^*} \frac{d\,M_{n,l}^2}{\frac{\partial\,M_{n,l}^2}{\partial\,n}\left(V*-M_{n,l}^2\right)^{1/2}} = \frac{2}{\sqrt{a}}\,\tan^{-1}\left(2\sqrt{\frac{V*}{a}}\right)$$

Thus

$$V^*(z) = \frac{a}{2}\tan^2\left(\frac{\sqrt{a}\,z}{2}\right).$$

with $a = 4\,L^2/R^4$.

2. The bottom-up effective potential has the structure

$$V_{\mathrm{D3/D7}}(z) = \frac{(2\,l+3)\,(2\,l+1)}{4\,z^2} + \frac{a}{4}\tan^2\left(\frac{\sqrt{a}\,z}{2}\right).$$

3. *Wall* locus:

$$z_{\mathrm{cutoff}} = \frac{\pi}{\sqrt{a}}\,\gamma = \frac{\pi\,R^2}{2\,L}\gamma, \quad \gamma \in \mathbb{N}.$$

Dilaton Reconstruction

Once we have the potential, we can calculate the associated dilaton as:

$$V^*(z) = \frac{1}{4}\Phi'(z)^2 - \frac{1}{2}\Phi''(z) - \frac{\beta}{2\,z}\Phi'(z).$$

We choose as BCs: $\Phi(z \to 0) = 0$ and $\Phi(z \to \infty) = \sqrt{V_{\mathrm{WKB}}(z^*)}$, with z^* a turning point at infinity.

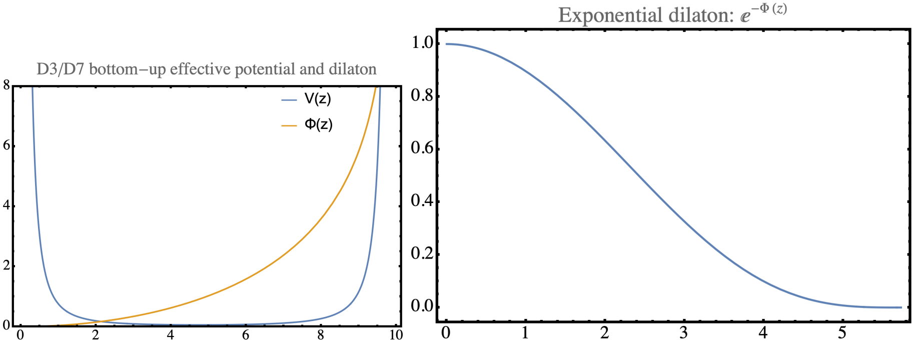


Figure: Reconstructed dilaton

WKB Bottom-up model for Charmonium

- How to infer the confining potential $V_{q\bar{q}}$ and the dilaton $\Phi(z)$ from a given Regge trajectory.

$$\begin{aligned} V_{Q\bar{Q}}(z) &= \frac{\beta(\beta-2)+4\,M_5^2\,R^2}{4\,z^2}+V_\Phi\left[\Phi',\Phi''\right] \\ M_5^2\,R^2 &= (3+L-J)\,(L+J-1) \\ \beta &= -3+2\,J, \end{aligned}$$

- Main idea: linearity in radial Regge trajectories ceases when considering heavy quarks (J. K. Chen, 2018), i.e., $M_n^2 = a(n+b)^\nu$.
- From RKR formula (MAMC et al, 2024), we obtain a static dilaton $\Phi(z) = (\kappa\,z)^{2-\alpha}$ with:

$$\kappa = \left[\frac{a^{1/\nu}}{2\,\pi^{1/2}}\,\frac{\nu\,\Gamma\left(\frac{\nu+2}{2\,\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)}\right]^{\frac{\nu}{2}}(2-\nu)^{\frac{2-\nu}{2}} \quad \text{and } \alpha = \frac{2\,(\nu-1)}{2-\nu}.$$

- We consider the vector charmonium n^3S_1 to fit the radial trajectory.

Heavy Quarkonia summary in MeV $I^G(J^{PC}) = 0^+(1^{--})$						
$Q\bar{Q}$ **state**	$1^3\,S_1$	$2^3\,S_1$	$3^3\,S_1$	$4^3\,S_1$	$5^3\,S_1$	$6^3\,S_1$
$c\bar{c}$	3096.900 ± 0.006,	3686.097 ± 0.011,	4040 ± 4,	4415 ± 5	Not Seen	Not Seen
	Non-Linear Regge Trajectory:		$a(\text{GeV}^2) = 7.88 \pm 2.62$,	$b = 0.39 \pm 0.67$,	$\nu = 0.61 \pm 0.17$,	$R = 0.999$

Bottom-up Charmonium spectroscopy

Summary of Charmonium states

$\kappa_c = 2.174$ GeV $\qquad \alpha_c = 0.561$

State	J^{PC}	$n^{2S+1}L_J$	$M_{\mathbf{Exp}}$ **(MeV)**	$M_{\mathbf{Th}}$ **(MeV)**	$\Delta M(\%)$
η_c	0^{-+}	$1^1 S_0$	2984 ± 0.4	3633.9	21.8
		$2^1 S_0$	3637.7 ± 0.9	4051.8	11.4
ψ	1^{--}	$1^3 S_1$	3096.097 ± 0.006	3042.75	1.74
		$2^3 S_1$	3686.097 ± 0.011	3619.8	1.80
		$3^3 S_1$	4040 ± 4	4041.0	0.1
		$4^3 S_1$	4415 ± 5	4380.7	0.8
		$1^3 D_1$	3773.7 ± 0.7	3909.9	3.61
		$2^3 D_1$	4191 ± 5	4264.6	1.8
	2^{--}	$1^3 D_2$	3823.51 ± 0.34	3212.7	15.98
	3^{--}	$1^3 D_3$	3842.71 ± 0.20	3282.7	14.57
χ_c	0^{++}	$1^3 P_0$	3414.71 ± 0.30	3852.5	12.82
		$2^3 P_0$	3862^{+50}_{-35}	4228.9	9.51
		$3^3 P_0$	3922.1 ± 1.8	4508.8	15.77
	1^{++}	$1^3 P_1$	3510.67 ± 0.05	3387.3	3.51
	2^{++}	$1^3 P_2$	3556.17 ± 0.07	2807.8	21.04
		$2^3 P_2$	3922.5 ± 1.0	3430.9	12.53
h_c	1^{+-}	$1^1 P_1$	3525.37 ± 0.14	3387.3	3.91

RMS error (17 states, 2 parameters)= 12.18%.

Tetraquarks from a bottom-up perspective

- Starting point: confining potential for charmonium (MAMC and A. Vega, 2021.)

$$V_{Q\bar{Q}}(z) = \frac{\beta(\beta-2) - 4\,M_5^2\,R^2}{4\,z^2} + V^*[\Phi', \Phi''], \text{ with: } \Phi(z) = (\kappa\,z)^{2-\alpha}, \text{ Parameters: } \{\kappa(m_q),\, \alpha(m_q)\}$$

Consider tetraquarks as an interacting diquark-antidiquark cluster.

- Inner structure effects: $\mathcal{O}_m^{4Q} = \bar{Q}_1\,\Gamma_r\,D^{2i}\,Q_2 \times \bar{Q}_3\,\Gamma_p\,D^{2j}\,Q_4$, $i, j \in \mathbb{Z}$, with $m = r + p$, flavor and color indices are omitted,and D_μ is the (boundary) gauge covariant derivative.
- $\dim \mathcal{O}_m^{4Q} = 6 + 2(i + j) + L$, therefore $M_5^2\,R^2 = [6 + L + 2(i + j) - J]\,[6 + L + 2(i + j) - 4]$.
- From the Bethe-Salpeter equation, we learn that Regge trajectories for heavy diquarks (X. Feng, et al., 2023) can be written as

$$M_n^2 \sim (3\,\pi)^{2/3}\,(m_1 + m_2)\left(\frac{\sigma_c^2}{2\,\mu}\right)^{1/3} n^{2/3}, \text{ for large } n,$$

with $m_1 + m_2$ is the diquark mass, μ the reduced mass, and σ_c is the string tension for charmonium-like states described by the Cornell potential (W. Lucha, 1991).

RKB potential for heavy tetraquarks

- WKB reconstructed potential:

$$V_{\mathrm{WKB}}(z) = 2\, g_{eff}\, (m_1 + m_2)^{3/2} \left(\frac{\sigma_c^2}{2\,\mu} \right)^{1/2} z$$

- Tetraquark potential

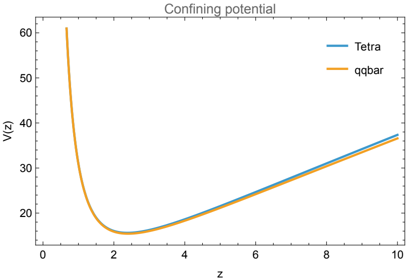
$$V_{4Q}(z) = V_{q\bar{q}}(z) + V_{\mathrm{WKB}}(z),$$

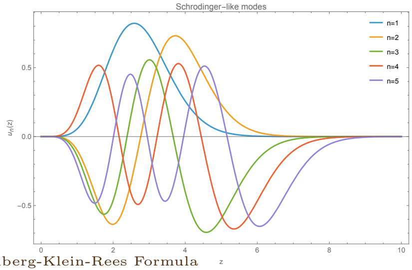
- Schrodinger-like equation:

$$-u_n''(z) + V_{4Q}(z)\, u_n(z) = M_n^2\, u_n(z)$$

- Parameters and constituent masses:

m_l	0.3375 GeV	σ_c	0.24 GeV^2
m_s	0.510 GeV	g_{eff}	$\frac{1}{2\,\pi}$
m_c	1.55 GeV	κ_c	2.174
-	-	α	0.561





Numerical Results for charm tetraquarks

Summary of Charmonium Tetraquarks $\sigma_c = 0.24$ GeV^2 $g_{eff} = \frac{1}{2\pi}$						
State	J^P **or** J^{PC}	$[q_1\,q_2][q_3\,q_4]$	$M_{\textbf{Exp}}$ **(MeV)**	$M_{\textbf{Th}}$ **(MeV)**	$\Delta\,M(\%)$	(i,j)
$T^*_{cs0}(2870)^0$	0^+	$[u\bar{s}][d\bar{c}]$	2866 ± 7	2704.9	5.62	$(0,0)$
$T^*_{cs1}(2900)$	1^-	$[d\bar{c}][u\bar{s}]$	2904 ± 4	2943.5	1.4	$(0,0)$
$T_{cc}(3875)^+$	1^+	$[d\bar{c}][u\bar{c}]$	3874.83 ± 0.11	3990.32	2.98	$(1,1)$
	0^+		3874.83 ± 0.11	3846.2	0.74	$(0,1)$
$T_{c\bar{c}1}(3900)$	1^{+-}	$[c\bar{c}][d\bar{u}]$	3887.1 ± 2.6	3918.3	0.80	$(0,0)$
$T_{c\bar{c}\bar{s}1}(4000)$	1^+	$[c\bar{c}][u\bar{s}]$	3980 to 4010	4133.9	3.47	(0,0)
$T_{c\bar{c}\bar{s}1}(4220)$			4220^{+50}_{-40}	4451.8	5.49	
$T_{c\bar{c}}(4050)^+$	$1^{?+}$	$[c\bar{c}][u\bar{d}]$	4051^{+24}_{-40}	4215.4	4.1	$(1,1)$
	$0^{?+}$			4511.3	11.3	$(0,0)$
$T_{c\bar{c}}(4055)^+$	$1^{?-}$	$[c\bar{c}][u\bar{d}]$	4054 ± 3.2	3998.3	1.37	$(1,1)$
	$0^{?-}$			4331.8	6.85	$(0,0)$
$T_{c\bar{c}1}(4200)$	1^{+-}	$[c\bar{c}][u\bar{d}]$	4196^{+35}_{-32}	4318.2	2.91	$(1,0)$
$T_{c\bar{c}0}(4240)$	0^{--}	$[c\bar{c}][u\bar{d}]$	4239^{+50}_{-21}	4253.3	0.34	$(0,0)$
$\chi_{c1}(3872)$	1^{++}	$[c\bar{c}][u\bar{d}]$	3871.64 ± 0.06	4137.7	6.89	$(0,0)$
$\chi_{c1}(4140)$			4146.5 ± 0.3	4456.1	7.48	
$\chi_{c1}(4274)$			4286^{+8}_{-9}	4731.3	10.4	
$\chi_{c1}(4685)$			4684^{+15}_{-17}	4975.2	6.22	

RMS error (17 states, 4 parameters)= 5.75%.

Figure: Experimental data is read from PDG 2024.

WKB II: Segre-Fermi formula

Fermi-Segre formula in QCD

Fermi-Segre formula (C. Quigg and Rosner, 1977) connects the hadron binding energy spectrum E_n with the hadron wave-function at the origin

$$|\Psi_n(0)|^2 = \frac{(2\mu)^{\frac{3}{2}}}{4\pi^2} E_n^{\frac{1}{2}} \frac{d}{dn} E_n.$$

with $M_n = 2\,m_Q + E_n$. From the Van Royen-Weisskopf formula, we connect with the decay constants f_n (in MeV): $f_n^2\,M_n^2 \propto |\Psi(0)|^2$. Thus, for two different radial states we have

$$\frac{|\Psi_{n_1}(0)|^2}{|\Psi_{n_2}(0)|^2} = \frac{M_{n_1} f_{n_1}^2}{M_{n_2} f_{n_2}^2} = \frac{(M_{n_1} - 2\,m_q)^{1/2}}{(M_{n_2} - 2\,m_q)^{1/2}} \frac{\left.\frac{d\,M_n}{d\,n}\right|_{n=n_1}}{\left.\frac{d\,M_n}{d\,n}\right|_{n=n_2}}.$$

Thus, from the mass spectrum, we can infer information related to quotients of the wave function at the origin, connected with other decay properties as branching ratios (see Lucha et al., 1991.)

Illustrative example: Heavy quark mass in the softwall model

- Suppose a linear Regge trajectory for heavy quarks $M_n = 2\,\kappa(n+1)^{1/2} = 2\,m_Q + E_n$.
- Compute decay constants: $f_n^2 = \frac{1}{g_5^2\,M_n^2}\,\lim_{\varepsilon\to 0}\,e^{-2\,B(\varepsilon)}\,|\psi_n(\varepsilon,q)|^2 = \frac{2\,\kappa^2}{g_5^2}$.
- Use Segre-Fermi formula:

$$\frac{|\Psi_{n_1}(0)|^2}{|\Psi_{n_2}(0)|^2} = \frac{M_{n_1}}{M_{n_2}} = \frac{\left[2\,\kappa(n_1+1)^{1/2} - 2\,m_Q\right]^{1/2}}{\left[2\kappa(n_2+1)^{1/2} - 2\,m_Q\right]^{1/2}}\,\frac{(n_1+1)^{-1/2}}{(n_2+1)^{-1/2}}.$$

  We obtain:

$$\frac{\left[2\,\kappa(n_1+1)^{1/2} - 2\,m_Q\right]^{1/2}}{\left[2\kappa(n_2+1)^{1/2} - 2\,m_Q\right]^{1/2}} = \frac{n_1+1}{n_2+1}.$$

- Fix for the ground ($n_1 = 0$) and the first excited ($n_2 = 1$) states, and solve for the quark mass, we obtain: $m_Q = \frac{4-\sqrt{2}}{3}\,\kappa$.
- For J/ψ, we obtain: $m_c = 1.33(14.1\%)$ GeV.
- For Υ, we obtain: $m_b = 4.07(13.9\%)$ GeV.
- For $\rho(770)$, we obtain: $m_l = 0.334$ GeV.

Conclusions

Summary and Conclusions

- Mass spectrum controls high-z behavior observed in the confining potential $\Rightarrow$ Dilaton reconstruction.
- Using WKB, it is possible to include hadronic inner structure effects.
- RMS for 17 $c\bar{c}$ states with two parameters is 12.18%
- RMS for 17 charm *exotic* states with four parameters is 5.75%.
- Things to do next:
  - Include Coulombian effects (1-Gluon Exchange) for heavy states (See Afonin & Solomonko, 2023)
  - Test other hadron structures, such as hadroquarkonium and molecular pictures.
  - Extend to Pentaquarks.

Backup slides

Hadrons in holography

In AdS/CFT:

- Gauge/gravity duality: hadrons are non-perturbative boundary objects dual to bulk fields.

$$\mathcal{O}\,|0\rangle = \alpha\,|p\rangle$$

with operator creating hadrons $\mathcal{O}$ defined as

$$\begin{aligned} \mathcal{O} &= f(q, \bar{q}, G_{\mu\nu}, D_\mu) \\ \mathbf{dim}\,\mathcal{O} &= \Delta + L + \gamma \end{aligned}$$

$L =$ Orb. Ang. momentum, $\gamma =$ anom. Dim.

- Field Operator duality ($L = \gamma = 0$):

$$\mathbf{dim}\,\mathcal{O} \Leftrightarrow \dim \psi(z,q) \equiv \Delta$$

Bulk field: p-form in AdS_5
$A_p(z,q) = A_p(q)\,\psi(z,q)$ with mass M_5.

$$\psi(z,q)\big|_{z\to 0} = \mathcal{C}\, z^{\Delta - p}$$

See Polchinski 2002.

- Hadronic identity: is defined by the bulk field mass $M_5 = M_5(\Delta, L, \gamma)$.

- In this perspective, a hadron is *a bag of constituents* characterized by $M_5(\Delta)$ (No inner structure *ab initio*).
- Hadrons in AdS/QCD are normalizable modes labeled by their bulk mass.

Bottom-up approach

AdS-like Background

$$dS^2 = \frac{R^2}{z^2}\left[dz^2 + \eta_{\mu\nu}\, dx^\mu\, dx^\nu\right]$$

with R defined as the AdS radius and $\eta_{\mu\nu} = \text{diag}(-1, \vec{1})$.

General Action with minimal coupling for $p-$forms

$$I_{\text{H}} = \int d^5x\,\sqrt{-g}\, e^{-\Phi(z)}\left[\nabla_m\, \phi^{m_1\dots\, m_p}\, \nabla^m\, \phi_{m_1\dots\, m_p} + M_5^2\, \phi^{m_1\dots\, m_p}\, \phi_{m_1\dots\, m_p}\right],$$

with $\Phi(z)$ defined as a static dilaton field (confinement). The bulk field $\phi^{m_1\dots\, m_p}$ carries the information about the hadron living at the boundary.
In general, for bulk $p-$forms (dual to hadrons at the boundary), the equations of motion have the structure:

$$\partial_z\left[e^{-B(z)}\,\partial_z\,\psi(z)\right] + \left(-q^2\right)\, e^{-B(z)}\,\psi(z,q) - \frac{M_5^2\, R^2}{z^2}\, e^{-B(z)}\psi(z,q) = 0$$

Where in Fourier space, the $p-$form is written as: $\phi_{\mu_1\dots\mu_p}\,(z,q) = \tilde{\phi}_{\mu_1\dots\mu_p}\,(q)\;\psi\,(z,q)$.

Bottom-up approach

with the following definitions:

$$\begin{aligned} B(z) &= \Phi(z) + \beta \log\left(\frac{R}{z}\right) \\ M_5^2\, R^2 &= (\Delta + L - p)\,(\Delta + L + p - 4) \\ \beta &= -(3 - 2\,p),\ \text{Defines hadronic spin: } p \equiv J \\ J &= L + S \\ M_n^2 &= -q^2,\ \text{on-shell condition.} \end{aligned}$$

Boguliubov transformation: $\psi(z) = e^{\frac{1}{2}B(z)}\, u(z)$.
From the action, we construct the Schrödinge-like form as (Karch et al., 2005):

$$-u''(z) + V(z)\, u(z) = M_n^2\, u(z),$$

where $V(z)$ is the holographic potential written in terms $\Phi(z)$, and M_n^2 is the hadronic spectrum (Regge trajectory).

Tetraquarks spectra calculation

- We organize the quark content in diquark-antidiquark clusters:
  - *Open charm:* $[c\,\bar{Q}_2][Q_1\,\bar{c}]$.
  - *Closed charm:* $[c\,\bar{c}][Q_1\,\bar{Q}_2]$.
- One diquark cluster defines the parameters for $V_{q\bar{q}}(z)$ following M.A.M.C and A. Vega, 2023. We use *calibration curves* from the isoscalar mesons to compute $\{\kappa, \alpha\}$:

$$\kappa(\bar{m}) = 15.2085 - 14.808\,e^{-0.0524\,\bar{m}^2}$$
$$\alpha(\bar{m}) = 0.8454 - 0.8455\,e^{-0.4233\,\bar{m}^2}$$

- The other cluster defines $V_{\text{WKB}}(z)$.
- We solve the Schrödinger-like bulk equations to find M_n^2 for each tetraquark candidate.

$$V_{4Q}(z) = V_{q\bar{q}}(z) + V_{\text{WKB}}(z)$$
$$V_{q\bar{q}}(z) = \frac{\beta(\beta-2) - 4\,M_5^2\,R^2}{4\,z^2} + V^*[\Phi', \Phi'']$$
$$\Phi(z) = (\kappa\,z)^{2-\alpha}$$
$$V_{\text{WKB}}(z) = 2\,g_{eff}\,(m_1+m_2)^{3/2}\left(\frac{\sigma_c^2}{2\,\mu}\right)^{1/2} z$$
$$M_5^2\,R^2 = [6 + L + 2(i+j) - J]\,[2 + L + 2(i+j)]$$