An Analytic Computation of Planar Three-Loop Five-Point Feynman Integrals

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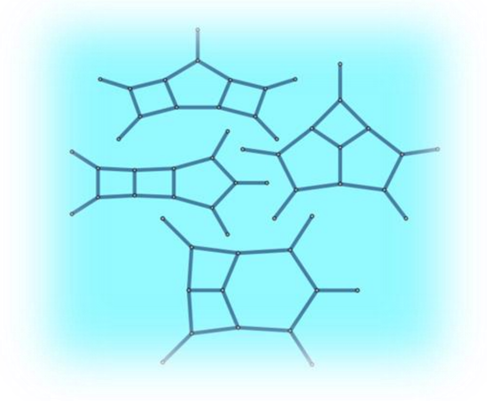

粒子物理精细计算研讨会
河北 保定

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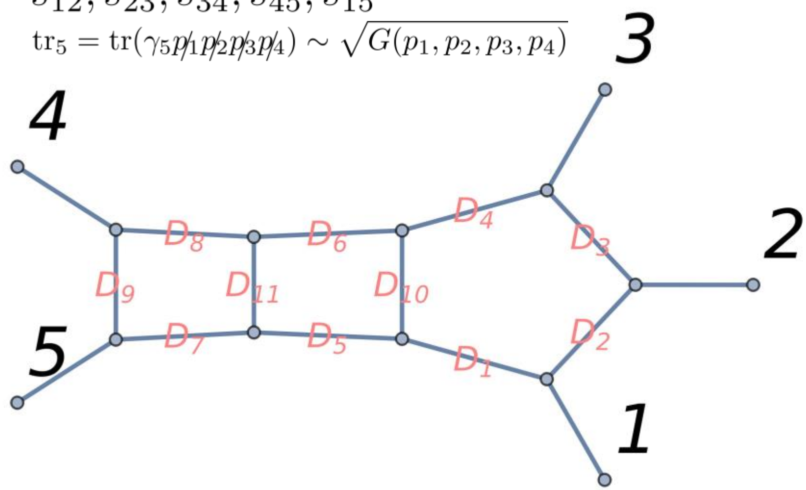
An Analytic Computation of Three-Loop Five-Point Feynman Integrals

👉[arXiv:2411.18697] with Liu, Matijašić, Miczajka, Xu, Zhang

- IBP Reduction
- UT Basis
- Differential Equation
- Boundary Values
- Pentagon Functions
- Auxiliary Matrix
- Outlook

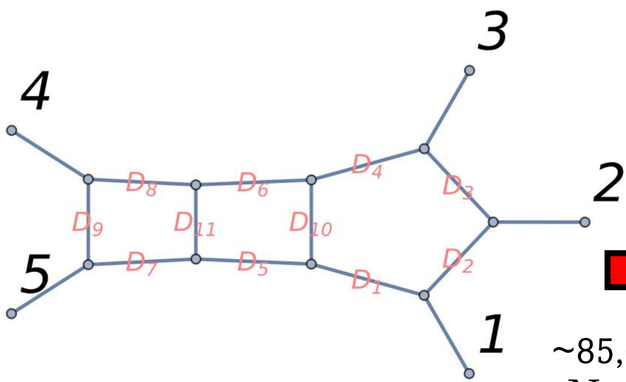
$s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

$\mathrm{tr}_5 = \mathrm{tr}(\gamma_5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) \sim \sqrt{G(p_1, p_2, p_3, p_4)}$



Many thanks to my incredible collaborators!

Using Algebraic Geometry for Feynman Integral Reduction



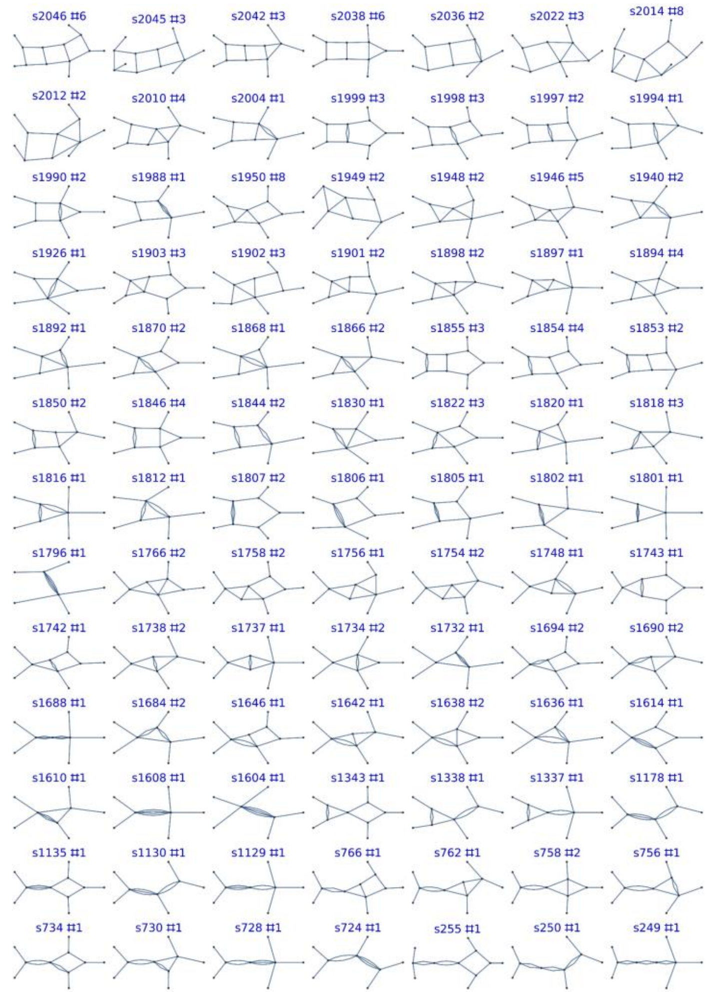
Picture created by Azurite 1.1.0

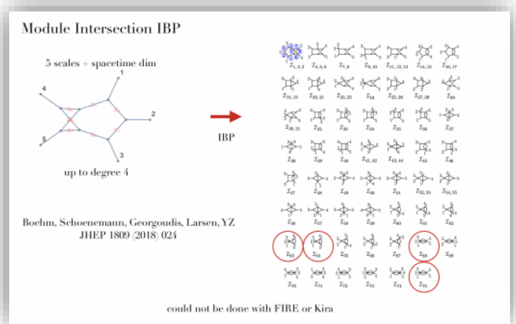
~85,000 IBPs

NeatIBP

Master Integral

#316





Similar Stories at Two Loop

Picture from Yang Zhang

Zihao Wu’s talk

Could not be done with FIRE or Kira

4D-Leading Singularities Analysis

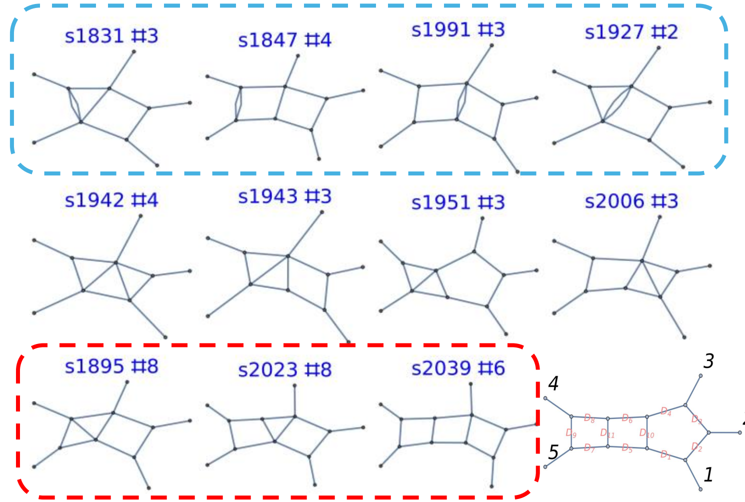
Dlogbasis

INITIAL

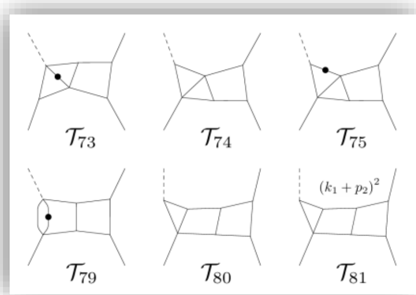
Constructing

UT basis

Integrating dotted bubble Loop



Genuine Five-Point Topologies (mod $\mathbb{Z}_2$)

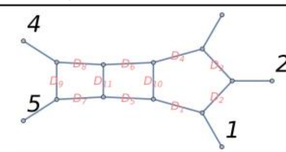


3L4P with 1 off-shell leg

[Di Vita, and Mastrolia, and Schubert, and Yundin '14]
[Henn, Lim, Bobadila'23] and so on..

Gram Determinant

$$e^{3\varepsilon\gamma_E}\int\prod_{j=1}^{3}\frac{\mathrm{d}^{4-2\varepsilon}\ell_j}{(i\pi)^{2-\varepsilon}}\,\frac{\frac{s_{45}^2}{\epsilon_5}G\begin{pmatrix}\ell_1,k_1,k_2,k_3,k_4\\ \ell_3,k_1,k_2,k_3,k_4\end{pmatrix}}{\text{[diagram]}}$$



Constructing Canonical Differential Equation

UT basis

NEATIBP

$$\frac{\partial}{\partial s_{ij}} = \beta_{ij}^{k\mu} \frac{\partial}{\partial p_k^\mu}$$

Fine-Tuned
Differential Operator

[Magerya '22]
[Abreu,Page,Zeng'18]

Semi-Analytic

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(\/)‾‾‾‾\
\''/_(__~~~~
 ``    ratracer
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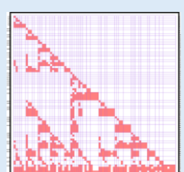
reconstruction

Letter	v notation	momentum notation	cylic
W_1	v_1	$2p_1 \cdot p_2$	+ cyclic (4)
W_6	$v_3 + v_4$	$2p_4 \cdot (p_3 + p_5)$	+ cyclic (4)
W_{11}	$v_1 - v_4$	$2p_3 \cdot (p_4 + p_5)$	+ cyclic (4)
W_{16}	$v_4 - v_1 - v_2$	$2p_1 \cdot p_3$	+ cyclic (4)
W_{21}	$v_3 + v_4 - v_1 - v_2$	$2p_3 \cdot (p_1 + p_4)$	+ cyclic (4)
W_{26}	$\frac{v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 - \sqrt{\Delta}}{v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 + \sqrt{\Delta}}$	$\frac{\mathrm{tr}[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2]}{\mathrm{tr}[(1+\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2]}$	+ cyclic (4)
W_{31}	$\sqrt{\Delta}$	$\mathrm{tr}[\gamma_5\not{p}_1\not{p}_2\not{p}_3\not{p}_4]$	

Five-particle letter alphabet
[Gehrmann, Henn, Lo Pres5 '18]

Full-Analytic

$$d\mathbf{I} = \varepsilon \left[\quad \right] \mathbf{I}$$



316×316

$$d\mathbf{I} = \varepsilon d\tilde{\mathbf{A}} \cdot \mathbf{I}$$

$$d\tilde{\mathbf{A}} = \sum a_i \mathrm{dlog} W_i$$

Solving
Analytic Boundary Values
from Physics Consistency

Trivial Integral

Spurious Pole Cancellation $\Rightarrow$

$$\mathbf{I}^{(n)}(x_0) = I_1^{(n)}(x_0) + \mathbb{Q}\left(\int_\gamma \mathrm{d}\tilde{\mathbf{A}} \cdot \mathbf{I}^{(n-1)}(x)\right)$$

$$\mathrm{Res}\left(\frac{\mathrm{d}\tilde{\mathbf{A}}}{\mathrm{d}x}\right)\Bigg|_{y_i} \cdot \mathbf{I}^{(n)}(y_i) = 0$$

$$y_1 = \{-2, -1, -1, -1, -1\}$$

$$y_6 = \{1/4, -1, -1, -1, -1\}$$

Line Integration

Symmetric Point in Euclidean Region

$$x_0 = y_0 = \{-1, -1, -1, -1, -1\}$$

Solve the boundary values up to weight-six

Written in special values of GPL functions

[Gehrmann, Henn, Lo Pres5 ‘18] e.t.c

- Pentagon Functions

Weight-1: $\log(-s_{12}), \log(-s_{23}), \log(-s_{34}), \log(-s_{45}), \log(-s_{15})$

Weight-2: $\mathrm{Li}_2\left(1-\frac{s_{45}}{s_{12}}\right), \mathrm{Li}_2\left(1-\frac{s_{15}}{s_{23}}\right), \mathrm{Li}_2\left(1-\frac{s_{12}}{s_{34}}\right), \mathrm{Li}_2\left(1-\frac{s_{23}}{s_{45}}\right), \mathrm{Li}_2\left(1-\frac{s_{34}}{s_{15}}\right)$

Weight-3: $\mathrm{Li}_3\left(1-\frac{s_{45}}{s_{12}}\right), \mathrm{Li}_3\left(1-\frac{s_{15}}{s_{23}}\right), \mathrm{Li}_3\left(1-\frac{s_{12}}{s_{34}}\right), \mathrm{Li}_3\left(1-\frac{s_{23}}{s_{45}}\right), \mathrm{Li}_3\left(1-\frac{s_{34}}{s_{15}}\right)$

Same as Two-Loop Pentagon Box

• • •

How to do Calculus?

$$\int \frac{\log(1-x)\log x}{(x-2)\sqrt{1+4x}}\mathrm{d}x = ?$$

How to put a $\sqrt{\cdots}$ into Li_3

- Function Type ?
- Admissible Argument ?

$$\mathrm{SB}(\mathrm{Li}_n(z)) = -(1-z)\otimes \underbrace{z\otimes \ldots \otimes z}_{n-1}$$

Only $\log, \mathrm{Li}_2, \mathrm{Li}_3$ are needed

Weight-3: $\mathrm{Li}_3\left(..\sqrt{..}\right) + \mathrm{Li}_2\left(..\sqrt{..}\right)\log(..) + \frac{\pi^2}{6}\log(..\sqrt{..}) + \zeta_3$

- What about Higher Transcendental Weight?

One-fold integration with Auxiliary Matrix.

Weight-4:
$$\mathbf{I}^{(4)}(x) = \mathbf{I}^{(4)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{\mathbf{A}}(t)}{\mathrm{d}t}\mathbf{I}^{(3)}(t)\mathrm{d}t$$

IBP

Well-Known at Two Loop

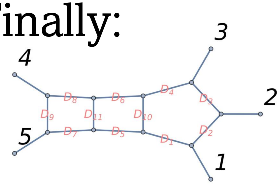
Weight-5:
$$\mathbf{I}^{(5)}(x) = \mathbf{I}^{(5)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{\mathbf{A}}(t)}{\mathrm{d}t}\mathbf{I}^{(4)}(x_0)\mathrm{d}t + \int_0^1 (\tilde{\mathbf{A}}(1) - \tilde{\mathbf{A}}(t))\frac{\mathrm{d}\tilde{\mathbf{A}}(t)}{\mathrm{d}t}\mathbf{I}^{(3)}(t)\mathrm{d}t.$$

Weight-6: IBP

$$\mathrm{d}\tilde{\mathbf{B}} = (\mathrm{d}\tilde{\mathbf{A}})\tilde{\mathbf{A}} \qquad \tilde{\mathbf{B}} \sim \mathrm{Li}_2(..\sqrt{..}) + \log(..\sqrt{..})^2$$

$$\begin{aligned}\mathbf{I}^{(6)}(x) =& \mathbf{I}^{(6)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{\mathbf{A}}}{\mathrm{d}t}\mathbf{I}^{(5)}(x_0)\mathrm{d}t + \int_0^1 \left(\tilde{\mathbf{A}}(1) - \tilde{\mathbf{A}}(t)\right)\frac{\mathrm{d}\tilde{\mathbf{A}}}{\mathrm{d}t}\mathbf{I}^{(4)}(x_0)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{\mathbf{A}}(t) - \tilde{\mathbf{A}}(1)\right)\tilde{\mathbf{A}}(t)\frac{\mathrm{d}\tilde{\mathbf{A}}}{\mathrm{d}t}\mathbf{I}^{(3)}(t)\mathrm{d}t + \int_0^1 \left(\tilde{\mathbf{B}}(1) - \tilde{\mathbf{B}}(t)\right)\frac{\mathrm{d}\tilde{\mathbf{A}}}{\mathrm{d}t}\mathbf{I}^{(3)}(t)\mathrm{d}t.\end{aligned}$$

Finally:



$$\sim \frac{\mathbb{Q}}{\varepsilon^6} + \frac{\log}{\varepsilon^5} + \frac{\mathrm{Li}_2}{\varepsilon^4} + \frac{\mathrm{Li}_3}{\varepsilon^3} + \frac{\int \mathrm{Li}_3}{\varepsilon^2} + \frac{\int \mathrm{Li}_3 \log}{\varepsilon^1} + \int \mathrm{Li}_3\mathrm{Li}_2$$

- Why analytic integrals?

- Analytic Structure
- Mathematical Physics
- More Theoretical study
- ….

pySecDec

NVIDIA A100 GPU

$\times 4$

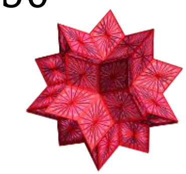
~38 hours

One top Integral with 6 digits

V.S.

$$\frac{\mathbb{Q}}{\varepsilon^6}+\frac{\log}{\varepsilon^5}+\frac{\mathrm{Li}_2}{\varepsilon^4}+\frac{\mathrm{Li}_3}{\varepsilon^3}+\frac{\int \mathrm{Li}_3}{\varepsilon^2}+\frac{\int \mathrm{Li}_3 \log}{\varepsilon^1}+\int \mathrm{Li}_3\mathrm{Li}_2$$

Intel Xeon Gold 6330



$\times 30$

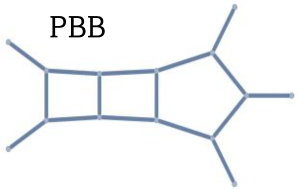
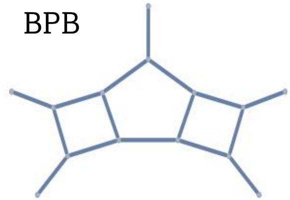
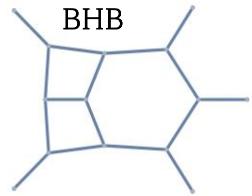
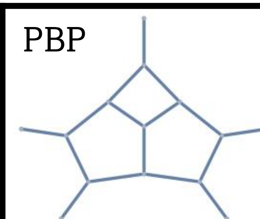
< 40 mins

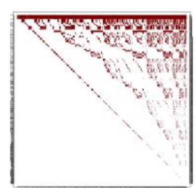
316 Integrals with 32 digits

https://github.com/YongqunXu/3L5P_PBB

What's Next?

Toward the complete UT basis for Planar Three-Loop Five-Point Feynman Integral

Family	PBB 	BPB 	BHB 	PBP 
# of Master Integrals	316	367	**431**	**734**
UT basis	316/316	367/367	~427/431	???
Differential Equation	✓	✓	✓	✓
# IBPs for building DE	~85,000	~180,000	Spanning Cuts	Spanning Cuts



734×734

Difficulties

Powered by NeatIBP 1.1.0.3

👉 Zihao Wu's talk

Toward the complete function space for Planar Three-Loop Five-Point Feynman Integral

Letter Alphabets are DNA of Feynman Integral

Letter	v notation	momentum notation	cylic
W_1	v_1	$2p_1 \cdot p_2$	$+$ cyclic (4)
W_6	$v_3 + v_4$	$2p_4 \cdot (p_3 + p_5)$	$+$ cyclic (4)
W_{11}	$v_1 - v_4$	$2p_3 \cdot (p_4 + p_5)$	$+$ cyclic (4)
W_{16}	$v_4 - v_1 - v_2$	$2p_1 \cdot p_3$	$+$ cyclic (4)
W_{21}	$v_3 + v_4 - v_1 - v_2$	$2p_3 \cdot (p_1 + p_4)$	$+$ cyclic (4)
W_{26}	$\frac{v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 - \sqrt{\Delta}}{v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 + \sqrt{\Delta}}$	$\frac{\mathrm{tr}[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2]}{\mathrm{tr}[(1+\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2]}$	$+$ cyclic (4)
W_{31}	$\sqrt{\Delta}$	$\mathrm{tr}[\gamma_5\not{p}_1\not{p}_2\not{p}_3\not{p}_4]$	

Five-particle letter alphabet at Two-Loop
[Gehrmann, Henn, Lo Pres5 '18]

$$\widetilde{W}_i := \tau^{i-1}\left(\Delta_2^{(1)}\right), \quad \widetilde{W}_{5+i} := \tau^{i-1}\left(\widetilde{W}_6\right),$$

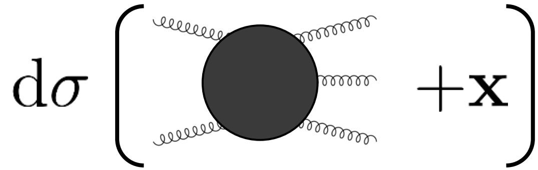
$$\widetilde{W}_{10+i} := \tau^{i-1}\left(\widetilde{W}_{11}\right), \quad \widetilde{W}_{15+i} := \tau^{i-1}\left(\Delta_4^{(1)}\right), \quad i = 1, \ldots, 5$$

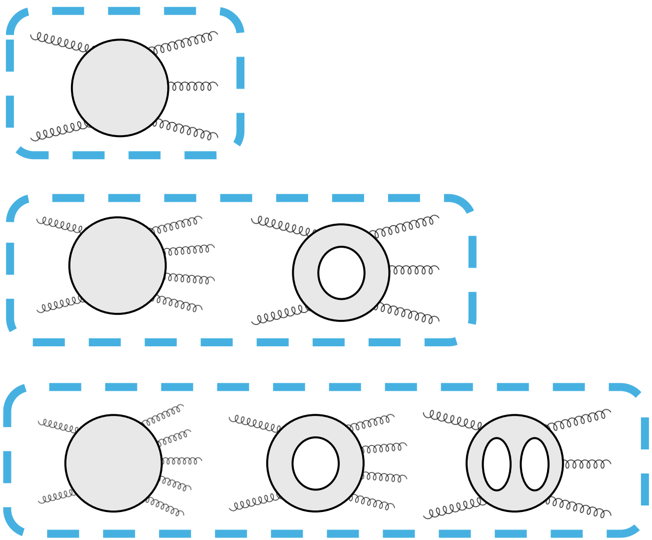
Novel Letters appear at Three-Loop
[Chicherin, Henn, Trnka, Zhang '24]

$$\widetilde{W}_6 := s_{12}s_{15}^2 - s_{12}s_{15}s_{23} - s_{12}s_{23}s_{34} + s_{23}s_{34}^2 + s_{15}s_{34}s_{45},$$

$$\widetilde{W}_{11} := s_{12}^2 s_{15} + s_{12}s_{15}s_{23} + s_{12}s_{23}s_{34} + s_{23}^2 s_{34} + s_{15}^2 s_{45} - 2s_{15}s_{34}s_{45} + s_{34}^2 s_{45}.$$

Toward the Five-particle cross sections at NNNLO

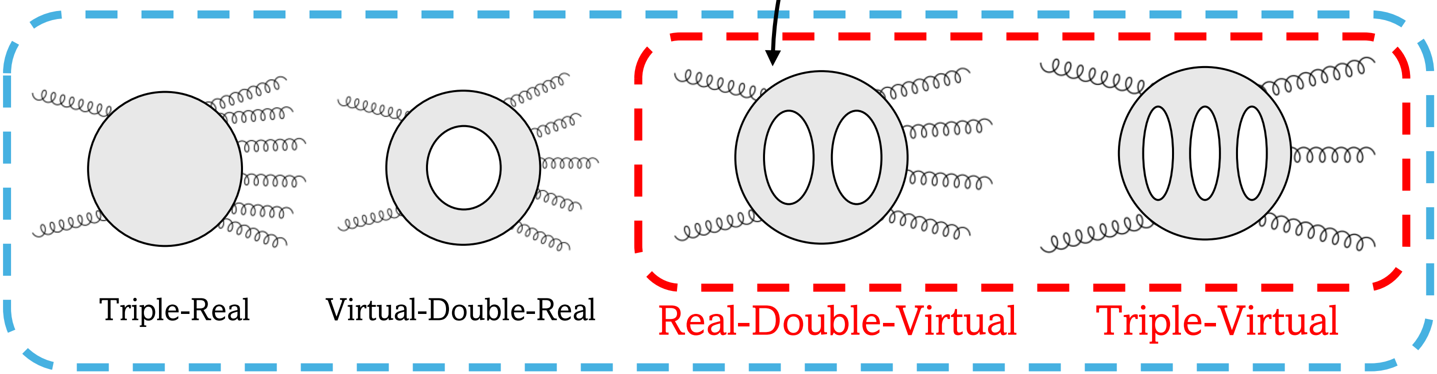




Necessary integrals for each perturbative order

👉 Yang Zhang's talk

Two-loop six-point computation



$\mathcal{O}(\alpha_s^7)$

Thank You!

Thank You!