Tree-level soft emission for two pairs of quarks

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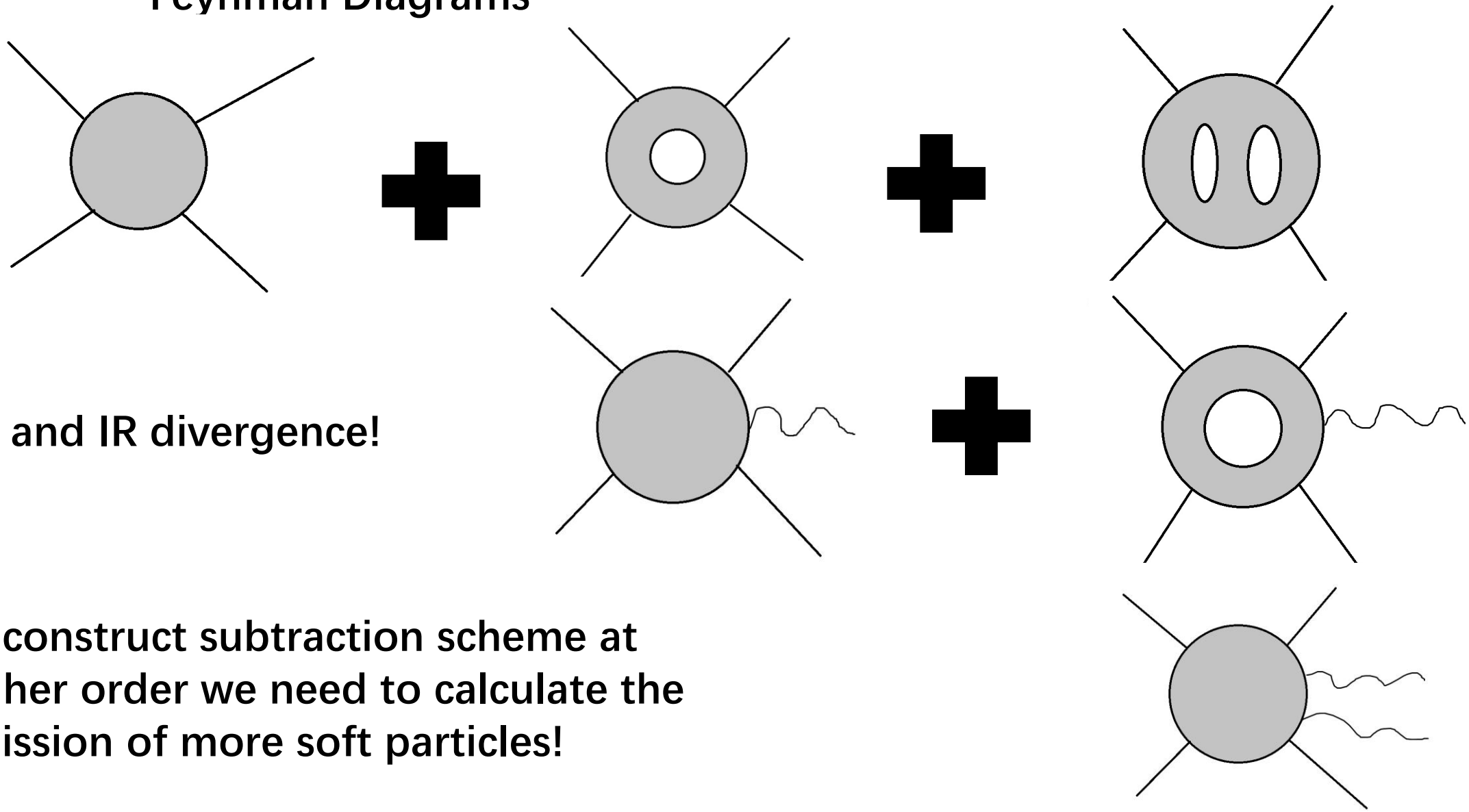
Quantum Field Theory + Perturbative expansion ->
Feynman Diagrams



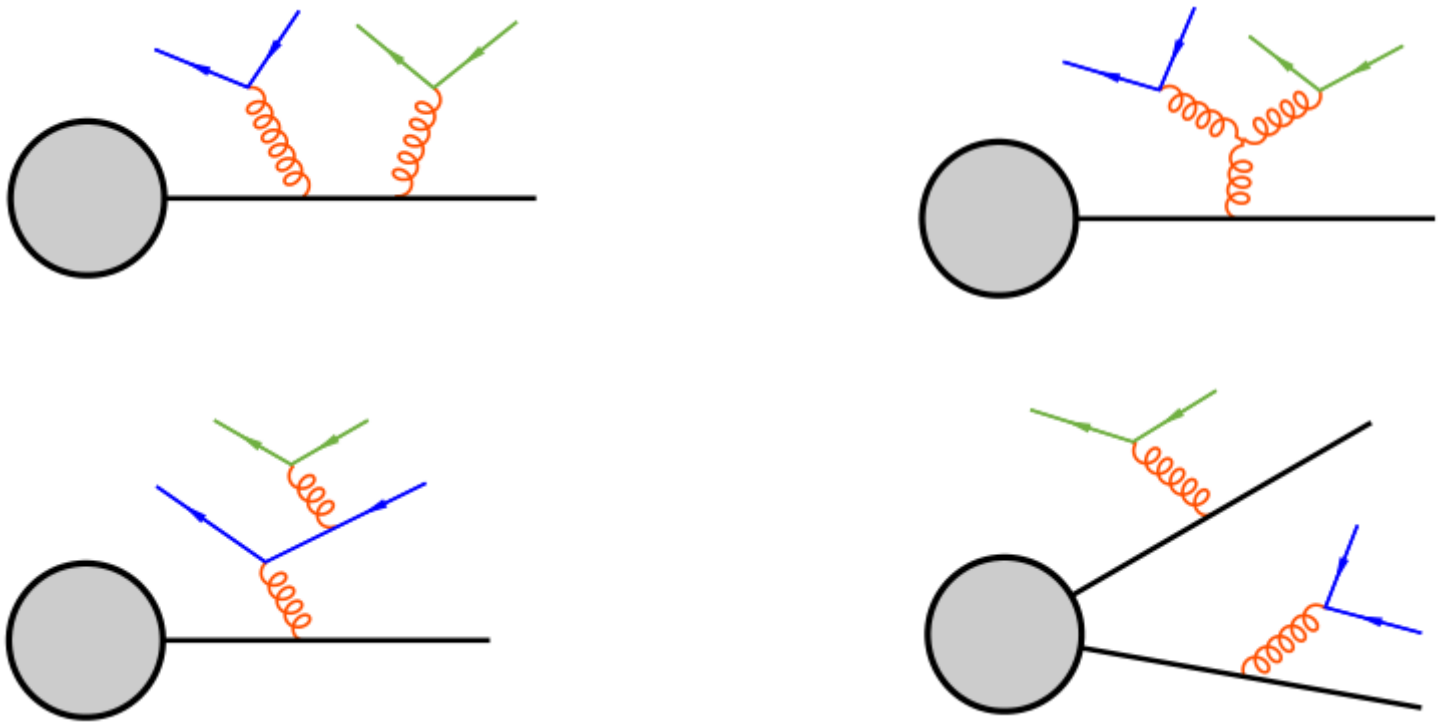
UV and IR divergence!

To construct subtraction scheme at higher order we need to calculate the emission of more soft particles!

Previous Work

- Single soft gluon: tree, one-loop, two-loop.
- Two gluons and quark-antiquark pair: tree and one-loop.
- ggg, $gq\bar{q}$, tree level

Tree-level soft emission for two pairs of quarks



Color space formalism

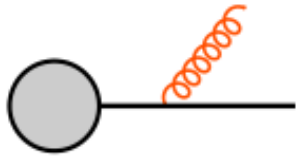
- Represent amplitude as a vector in the color $\otimes$spin space
- $|M_{f_1 f_2 \dots f_n}(p_1, p_2, \dots p_n) > = |c_1 c_2 \dots c_n > \otimes |\lambda_1 \lambda_2 \dots \lambda_n > M_{f_1 f_2 \dots f_n}^{c_1 c_2 \dots c_n; \lambda_1 \lambda_2 \dots \lambda_n}(p_1, p_2, \dots p_n)$
- The squared matrix element:
- $\left|M_{f_1 \dots f_n}\right|^2 =< M_{f_1 \dots f_n} | M_{f_1 \dots f_n} > = \sum_{c_1 \dots c_n, \lambda_1 \dots \lambda_n} \left(M_{f_1 \dots f_n}^{c_1 \dots c_n; \lambda_1 \dots \lambda_n}\right)^* M_{f_1 \dots f_n}^{c_1 \dots c_n; \lambda_1 \dots \lambda_n}$
- the color charge carried by a parton can be represented as an abstract operator that acts on the color space
- $T_j^a |c_1 c_2 \dots c_n > = |c_1 c_2 \dots c_{j-1} c_j' c_{j+1} \dots c_n > T_{c_j' c_j}^a$
- $\left[T_j^a, T_l^b\right] = i\, f^{abc} T_j^c \delta_{jl}$

Color space formalism

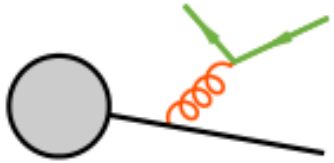
- Color is conserved, the amplitude $|M>$ is a color-singlet
- $\sum_i T_i^a |M> = 0$

- Factorization formula can be expressed as follow: when the momentum $q_1, q_2 \ldots$ goes soft, for the leading singular terms we have
- $|M(p_1, \ldots p_n; q_1 \ldots q_m)> \rightarrow (\mu^\epsilon g_s)^m J(q_1, \ldots q_m; p_1 \ldots p_n)|M(p_1, \ldots p_n)>$

Eikonal Feynman rules

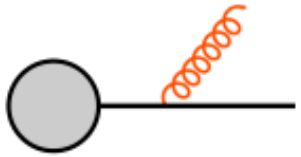


$$= J_i^{a;\mu}(p_i, q) = S_i^{\mu}(q) T_i^a = \frac{p_i^{\mu}}{p_i \cdot q} T_i^a$$



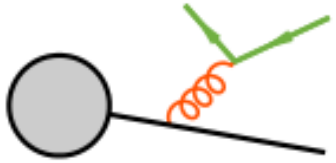
$$= \frac{-1}{q_{12}^2} S_i^{\mu}(q_{12})\, \bar{u}_{\lambda}(q_1) \gamma_{\mu} v_{\lambda'}(q_2) T_i^a t_{j\bar{\imath}}^a$$

Eikonal Feynman rules

squared 

$$\sum_{i,j} d_{\mu\nu}(q,r) S_i^\mu(q) S_j^\nu(q)\,(T_i \cdot T_j) \cong \sum_{i,j} -\frac{p_i \cdot p_j}{(q \cdot p_i)(q \cdot p_j)}\, T_i \cdot T_j\,,$$

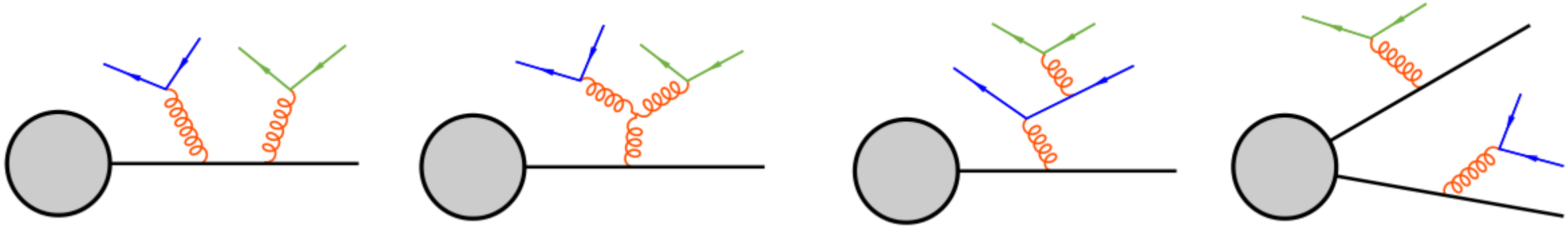
$$d_{\mu\nu}(q,r) = \sum_{\lambda=1}^{D-2} \epsilon_\mu^\lambda(q,r)\left[\epsilon_\nu^\lambda(q,r)\right]^* = -g_{\mu\nu} + \frac{q_\mu r_\nu + r_\mu q_\nu}{q \cdot r}$$

squared 

$$T_F \sum_{i,j} I_{i,j}(q_1,q_2)\, T_i \cdot T_j$$

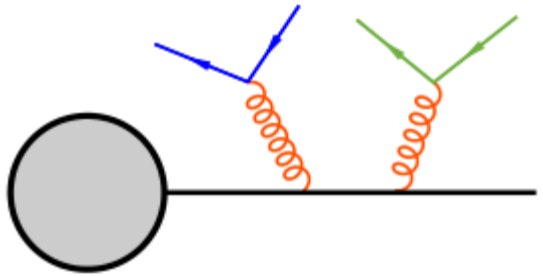
$$I_{i,j}(q_1,q_2) = \frac{1}{q_1 \cdot q_2} S_i^\mu(q_{12}) S_j^\nu(q_{12}) d_{\mu\nu}(q_1,q_2) = \frac{(p_i \cdot q_1)(p_j \cdot q_2) + (p_i \cdot q_2)(p_j \cdot q_1) - (q_1 \cdot q_2)(p_i \cdot p_j)}{(q_1 \cdot q_2)^2 (p_i \cdot q_{12})(p_j \cdot q_{12})}$$

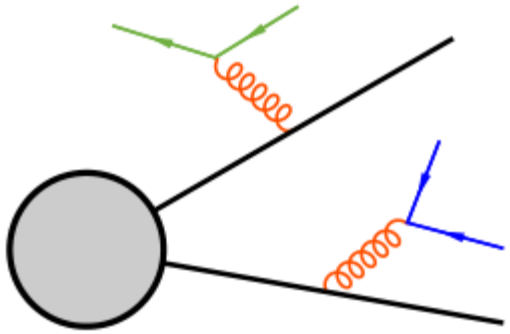
Take the softest diagrams



We need to consider these 4 types of diagrams, including some permutation of quark pairs.

Use eikonal Feynman rules to calculate the current



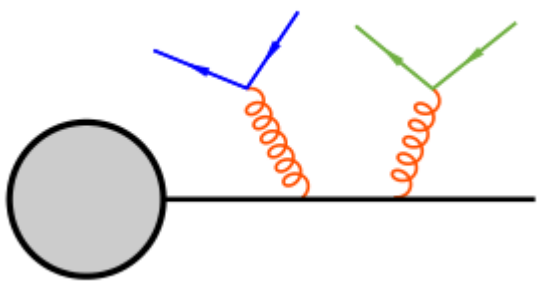


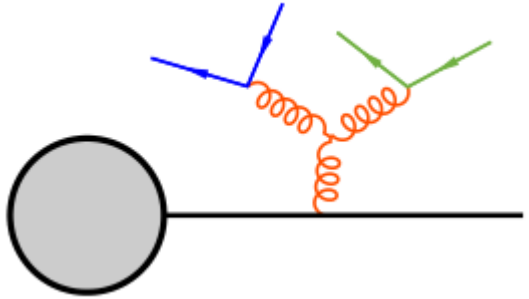
Symmetrize the color charges through $T_i^a T_i^b = \frac{1}{2}\{T_i^a, T_i^b\} + \frac{i}{2} f^{abc} T_i^c$

Combining the symmetric part of first kind of diagram and forth kind of diagram.

$$A_1 = \sum_{k,l} \frac{[\bar{u}_1 \gamma_\mu v_2][\bar{u}_3 \gamma_\nu v_4]}{q_{12}^2 q_{34}^2} \times \frac{1}{2}\{J_k^{a;\mu}(q_{12}), J_l^{b;\nu}(q_{34})\} t_{i\bar{i}}^a t_{j\bar{j}}^b$$

Use eikonal Feynman rules to calculate the current



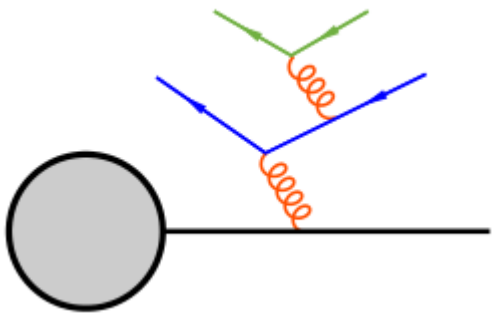


Combining the anti-symmetric part of first kind of diagram and second kind of diagram.

$$N = \frac{[\bar{u}_1\gamma_\mu v_2][\bar{u}3\gamma_\nu v_4]\, t^a_{i\bar{i}} t^b_{j\bar{j}}}{q_{12}^2 q_{34}^2} \sum_k i f^{abc} T^c_k S^\alpha_k(q_{1234})\left(\frac{1}{2}\delta^\mu_\alpha[S^\nu_k(q_{12}) - S^\nu_k(q_{34})] - \frac{g_{\alpha\sigma}}{q_{1234}^2} V^{\mu\nu\sigma}(q_{12}, q_{34})\right)$$

$$V^{\mu\nu\alpha}(q_{12}, q_{34}) = g^{\mu\nu}(q_{12} - q_{34})^\alpha + g^{\nu\alpha}(q_{12} + 2q_{34})^\mu - g^{\mu\alpha}(q_{34} + 2q_{12})^\nu$$

Use eikonal Feynman rules to calculate the current



The contribution from the third kind of diagram

$$A_2 = \frac{[\bar{u}_3\gamma^\nu v_4]t^b_{j\bar{j}}}{q^2_{1234}q^2_{34}}\sum_k S^\mu_k(q_{1234})\bar{u}_1\left[\frac{(t^b t^a)_{i\bar{i}}}{q^2_{134}}\gamma_\nu\gamma_\sigma q^\sigma_{134}\gamma_\mu - \frac{(t^a t^b)_{i\bar{i}}}{q^2_{234}}\gamma_\mu\gamma_\sigma q^\sigma_{234}\gamma_\nu\right]v_2 + (q\bar{q} <-> Q\bar{Q})$$

Use eikonal Feynman rules to calculate the current

- $A_1 = \sum_{k,l} \frac{[\bar{u}_1 \gamma_\mu v_2][\bar{u}_3 \gamma_\nu v_4]}{q_{12}^2 q_{34}^2} \times \frac{1}{2} \{ J_k^{a;\mu}(q_{12}), J_l^{b;\nu}(q_{34}) \} t_{i\bar{i}}^a t_{j\bar{j}}^b$
- $A_2 = \frac{[\bar{u}_3 \gamma^\nu v_4] t_{j\bar{j}}^b}{q_{1234}^2 q_{34}^2} \sum_k S_k^\mu(q_{1234}) \bar{u}_1 \left[\frac{(t^b t^a)_{i\bar{i}}}{q_{134}^2} \gamma_\nu \gamma_\sigma \mathrm{q}_{134}^\sigma \gamma_\mu - \frac{(t^a t^b)_{i\bar{i}}}{q_{234}^2} \gamma_\mu \gamma_\sigma q_{234}^\sigma \gamma_\nu \right] v_2 + (q\bar{q} <-> Q\bar{Q})$
- $N = \frac{[\bar{u}_1 \gamma_\mu v_2][\bar{u}_3 \gamma_\nu v_4] t_{i\bar{i}}^a t_{j\bar{j}}^b}{q_{12}^2 q_{34}^2} \sum_k i\, f^{abc} T_k^c S_k^\alpha(q_{1234}) (\frac{1}{2} \delta_\alpha^\mu [S_k^\nu(q_{12}) - S_k^\nu(q_{34})] - \frac{g_{\alpha\sigma}}{q_{1234}^2} V^{\mu\nu\sigma}(q_{12}, q_{34}))$
- $V^{\mu\nu\alpha}(q_{12}, q_{34}) = g^{\mu\nu}(q_{12} - q_{34})^\alpha + g^{\nu\alpha}(q_{12} + 2q_{34})^\mu - g^{\mu\alpha}(q_{34} + 2q_{12})^\nu$

$$J_{q\bar{q}Q\bar{Q}}^{i\bar{i}j\bar{j};\lambda_1 \ldots \lambda_4}(q_1, q_2, q_3, q_4) = A_1 + A_2 + N$$

Square the amplitude: color

- $|A_1|^2 \to \{\{T_i^a, T_j^b\}, \{T_k^a, T_l^b\}\} + \{\{T_i^a, T_l^b\}, \{T_k^a, T_j^b\}\}$

- Czakon's identity
- $\{\{T_i^a, T_j^b\}, \{T_k^a, T_l^b\}\} + \{\{T_i^a, T_l^b\}, \{T_k^a, T_j^b\}\} = 8\{T_i \cdot T_k, T_j \cdot T_l\} + 2C_A[3\delta_{il}\delta_{jk}(T_i \cdot T_j) + 3\delta_{ij}\delta_{kl}(T_i \cdot T_k) - 2\delta_{ij}\delta_{jk}(T_i \cdot T_l) - 2\delta_{ij}\delta_{jl}(T_i \cdot T_k) - 2(\delta_{ik}\delta_{kl} + \delta_{jk}\delta_{kl})(T_i \cdot T_j)]$

- Double-dipole and dipole contribution.

Square the amplitude: color

- $A_1^\dagger N + N^\dagger A_1 \to f^{abc}\left[\left\{T_j^a, T_k^b\right\}, T_l^c\right]$

- Czakon's identity
- $if^{abc}\left[\left\{T_i^a, T_j^b\right\}, T_k^c\right] = 2C_A\left(T_i \cdot T_j\right)\left(\delta_{ik} - \delta_{jk}\right)$

- Dipole contribution

Square the amplitude: color

- $tr\left(t^a t^b t^c\right) = \frac{1}{4} d^{abc} + \frac{i}{2} T_F f^{abc}$

- $A_1^\dagger A_2 + A_2^\dagger A_1 \rightarrow f^{abc}\left[\left\{T_i^a, T_j^b\right\}, T_l^c\right]$ *and* $d^{abc}\left\{\left\{T_i^a, T_j^b\right\}, T_l^c\right\} \rightarrow \left(\mathrm{T_i} \cdot T_j\right)\left(\delta_{il} - \delta_{jl}\right)$ *and* $d^{abc} T_i^a T_j^b T_l^c$

- Dipole contribution and three-parton contribution!

Square the amplitude: color

- $|N|^2 \to f^{abc} f^{abe} T_i^c T_j^e \to C_A (T_i \cdot T_j)$
- $N^\dagger A_2 + A_2^\dagger N \to f^{abc} tr\left(t^a t^b t^e\right) T_i^c T_j^e = \frac{i}{2} T_F C_A (T_i \cdot T_j)$
- $|A_2|^2 \to$
- $tr\left(t^a t^a t^b t^c\right) T_i^b T_j^c = T_F C_F (T_i \cdot T_j)$
- $tr\left(t^a t^b t^a t^c\right) T_i^b T_j^c = T_F (C_F - \frac{1}{2} C_A)(T_i \cdot T_j)$
- $tr\left(t^a t^b t^c\right) tr\left(t^a t^b t^e\right) = T_F^2 (2C_F - \frac{3}{4} C_A - \frac{1}{4} C_A)(T_i \cdot T_j)$
- Dipole contribution!

Square the amplitude: color

- $|A_1|^2 \to \{T_i \cdot T_k, T_j \cdot T_l\}$ *and* $(T_i \cdot T_j)$, double-dipole part and dipole part
- $A_1^\dagger N + N^\dagger A_1 \to (T_i \cdot T_j)$, dipole contribution
- $A_1^\dagger A_2 + A_2^\dagger A_1 \to \left(T_i \cdot T_j\right)$ *and* $d^{abc} T_i^a T_j^b T_l^c$, dipole and **three-parton correlation**
- $|A_2 + N|^2 \to (T_i \cdot T_j)$, dipole contribution

Strongly-order limit

- Consider the region $E_1, E_2 \gg E_3, E_4$
- Double-dipole function remains exact.
- Three-parton function and dipole function are significantly simplified.
- New feature: terms that have C_F also contribute to the strongly-order limit