An *explicit* parameterization of minimal **seesaw model**

Speaker: Zi-Qiang Chen (HIAS-UCAS)

Speech Date: Mar. 3. 28, 2025

[2504,xxxxx]

Zi-Qiang Chen, Xi-He Hu and Ye-Ling Zhou

Introduced from neutrinos:

	NH (1σ)	IH (1σ)
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.49^{+0.19}_{-0.19}$	$7.49^{+0.19}_{-0.19}$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.534^{+0.025}_{-0.023}$	$-2.510^{+0.024}_{-0.025}$

NuFIT 6.0 (2024)

Known:

- Massive: 1 may be relativistic
- Three flavor, PMNS mixing

Minimal Seesaw: $m_{\text{lightest}}=0$ [1909.09610]

NH: $m_1 = 0\,,\quad m_2 = \sqrt{\Delta m^2_{21}}\,,\quad m_3 = \sqrt{\Delta m^2_{31}}\,;$

IH: $m_1 = \sqrt{|\Delta m^2_{32}| - \Delta m^2_{21}}\,,\quad m_2 = \sqrt{|\Delta m^2_{32}|}\,,\quad m_3 = 0$

Unknown: Sterile-Heavy spices

- As HNL & LLP in collider
- As non-unitarity UV orig.

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_\nu} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_\nu} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} P_\nu$$

$$P_\nu \equiv \mathrm{Diag}\{e^{i\rho}, e^{i\sigma}, 1\}$$

=>Motivation: Quantitively constrains the SS through Parameterization

- Majorana or Dirac, Ordering, Responsible for matter-antimatter asymmetry…

P. S. Bhupal Dev [2503.21212]

Minimal-SS (MSM, Type-I):

$$-\mathcal{L}_{\rm lepton} = \overline{\ell_{\rm L}} Y_l H E_{\rm R} + \left\{\overline{\ell_{\rm L}} Y_\nu \widetilde{H} N_{\rm R} + \underline{\frac{1}{2}\overline{(N_{\rm R})^c} M_{\rm R} N_{\rm R}}\right\} + \text{h.c.}$$

$$\frac{\mathcal{L}_{\rm d=5}}{\Lambda_{\rm SS}} = \frac{1}{2}\overline{\ell_{\rm L}}\widetilde{H} Y_\nu M_{\rm R}^{-1} Y_\nu^T \widetilde{H}^T (\ell_{\rm L})^c + \text{h.c.}$$

EWSB

$$M_\nu \simeq -M_{\rm D} M_{\rm R}^{-1} M_{\rm D}^T$$

(Approximated, without Uni. V from UV?)

By UV Lag.: $\{\cdots\}$

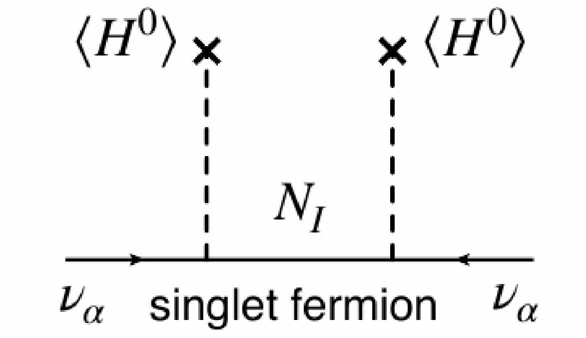
$$= \overline{\nu_{\rm L}} M_{\rm D} N_{\rm R} + \frac{1}{2}\overline{(N_{\rm R})^c} M_{\rm R} N_{\rm R} + \text{h.c.}$$

$$= \frac{1}{2}\overline{\begin{bmatrix}\nu_{\rm L} & (N_{\rm R})^c\end{bmatrix}}\begin{pmatrix} 0 & M_{\rm D} \\ M_{\rm D}^T & M_{\rm R}\end{pmatrix}\begin{bmatrix}(\nu_{\rm L})^c \\ N_{\rm R}\end{bmatrix} + \text{h.c.}$$

$$\boxed{\begin{aligned} 0 &= U D_\nu U^{\rm T} + R D_N R^{\rm T} \quad \text{(Exact)} \\ M_D &= (AU_0) D_\nu (U_0 S)^{\rm T} + R D_N B^{\rm T}\end{aligned}}$$

v-N mixing
Solve Out

<u>Tree Dominant</u>



Well-known seesaw formular:

$$\begin{pmatrix} 0 & M_{\rm D} \\ M_{\rm D}^T & M_{\rm R}\end{pmatrix} = \mathcal{U}\begin{pmatrix} D_\nu & 0 \\ 0 & D_N\end{pmatrix}\mathcal{U}^T$$

$$\mathcal{U} = \begin{pmatrix} I & 0 \\ 0 & U_0'\end{pmatrix}\underline{\begin{pmatrix} A & R \\ S & B\end{pmatrix}\begin{pmatrix} U_0 & 0 \\ 0 & I\end{pmatrix}}$$

<u>6 × 6 Parameterized:</u>

[1110.0083] Xing

<u>MSM 5 × 5</u>

$$\begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix} = O_{23}O_{13}O_{12}, \quad \begin{pmatrix} I & 0 \\ 0 & U_0' \end{pmatrix} = O_{45}, \quad \begin{pmatrix} A & R \\ S & B \end{pmatrix} = O_{35}O_{25}O_{15}O_{34}O_{24}O_{14}$$

=>I

- Solution to R (m_{lightest}=0):

11 DoFs For MSM

$$m_2,\ m_3(m_1),\ \theta_{12},\ \theta_{13},\ \theta_{23},\ \delta_\nu,\ \sigma,\ M_1,\ M_2,\ \theta_{14},\ \delta_{14}$$

$$0 = UD_\nu U^{\mathrm{T}} + RD_N R^{\mathrm{T}}$$ (Exact)

$$\boxed{\sin\theta_{ik} = \frac{|p_{ik}|}{\sqrt{1+|p_{ik}|^2}}\,, \quad \delta_{ik} = -\arg p_{ik}}$$

EMSM

$$M_{\mathrm{D}} = (AU_0)D_\nu(U_0S)^T + RD_NB^T$$

tiny v-N mixing angle $D_\nu \ll D_N$

$$\tilde{R} \approx R \approx \begin{pmatrix} \hat{s}^*_{14} & \hat{s}^*_{15} \\ \hat{s}^*_{24} & \hat{s}^*_{25} \\ \hat{s}^*_{34} & \hat{s}^*_{35} \end{pmatrix}$$

$$\begin{cases} M_\nu \simeq -(RD_N)D_N^{-1}(RD_N)^{\mathrm{T}} \simeq -M_DM_N^{-1}M_D^T \\ M_D \simeq RD_N \end{cases}$$

Return to approximated case

$$\tilde{R} = \begin{pmatrix} \frac{\hat{s}^*_{14}}{c_{14}} & \frac{\hat{s}^*_{15}}{c_{14}c_{15}} \\ \frac{\hat{s}^*_{24}}{c_{14}c_{24}} & \frac{\hat{s}_{14}\hat{s}^*_{24}\hat{s}^*_{15}}{c_{14}c_{15}c_{24}} + \frac{\hat{s}^*_{25}}{c_{24}c_{15}c_{25}} \\ \frac{\hat{s}^*_{34}}{c_{14}c_{24}c_{34}} & \frac{\hat{s}_{14}\hat{s}^*_{34}\hat{s}^*_{15}}{c_{14}c_{24}c_{15}c_{34}} + \frac{\hat{s}_{24}\hat{s}^*_{34}\hat{s}^*_{25}}{c_{24}c_{34}c_{15}c_{25}} + \frac{\hat{s}^*_{35}}{c_{34}c_{15}c_{25}c_{35}} \end{pmatrix}$$

$$\begin{cases} -M_\nu \equiv -U_0D_\nu U_0^T = \tilde{R}D_N\tilde{R}^T \\ (U^{-1}R)_{11} = U_0^\dagger\tilde{R} = 0 \end{cases}$$

$$\tilde{R} = A^{-1}R$$

$$(\tilde{R}D_N\tilde{R}^T)_{11} = -m_{11}\,,$$
$$(\tilde{R}D_N\tilde{R}^T)_{22} = -m_{22}\,,$$
$$(\tilde{R}D_N\tilde{R}^T)_{33} = -m_{33}\,,$$
$$w_1\tilde{R}_{11} + w_2\tilde{R}_{21} + w_3\tilde{R}_{31} = 0\,,$$
$$w_1\tilde{R}_{12} + w_2\tilde{R}_{22} + w_3\tilde{R}_{32} = 0\,,$$

Comparing with CI & NR (approximated)

1. Comparing with Casas-Ibarra's:

- EMSM's :

$$M_{\rm D} = (AU_0)D_\nu(U_0S)^T + RD_NB^T$$
$$\simeq \boldsymbol{RD_N}$$

- Casas-Ibarra's:

$$\boldsymbol{M_D \simeq i\, U\sqrt{D_\nu}\,\Omega\,\sqrt{D_N}}$$

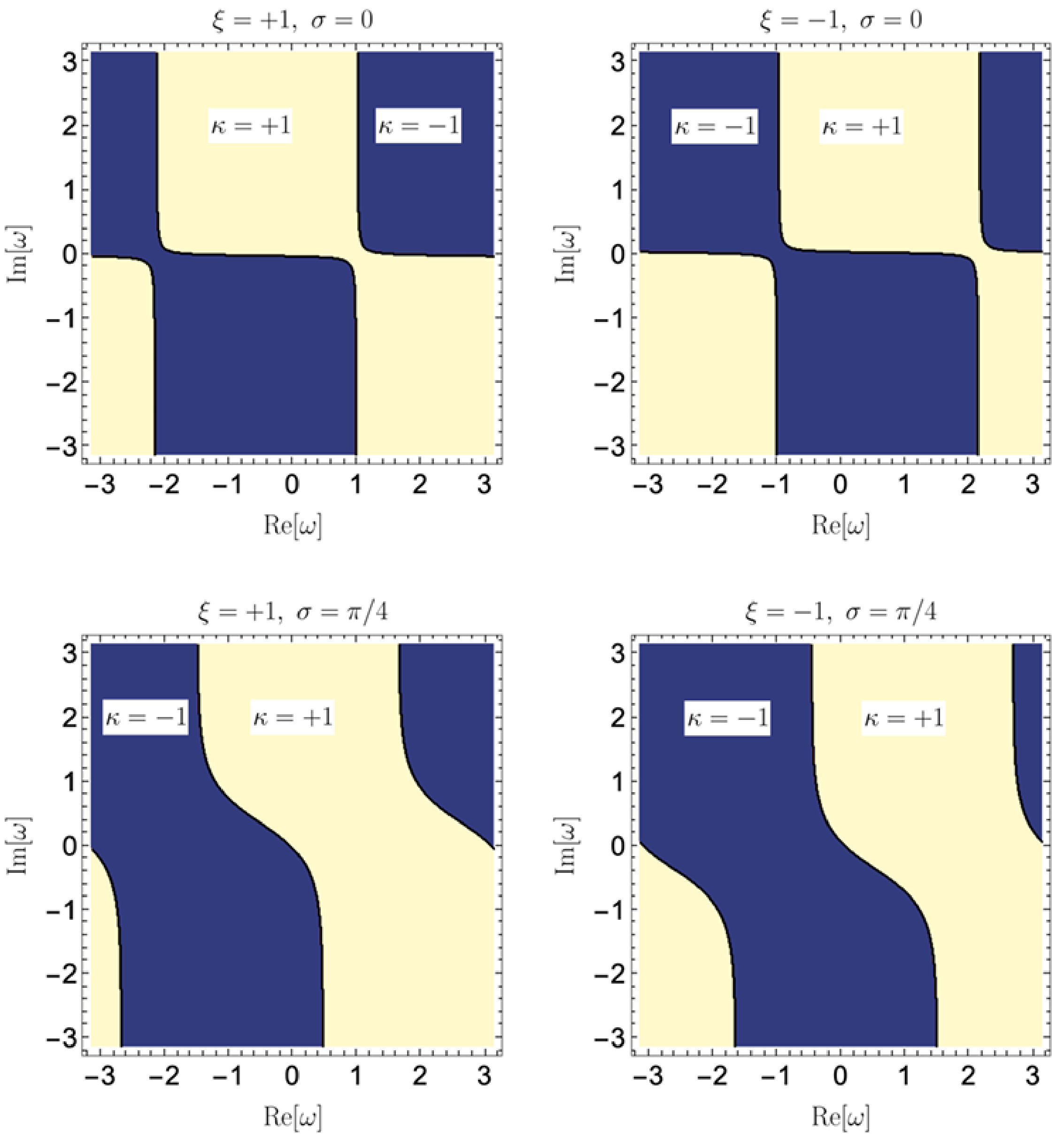
$$(M_{\rm D}^{\rm EMSM})_{11} = (M_{\rm D}^{\rm CI})_{11}$$

- Correlated through:

$$\begin{cases} \theta_{14} = |(V_0)_{12}\sqrt{m_2}\cos\omega + \xi\ (V_0)_{13}\sqrt{m_3}\sin\omega|/\sqrt{M_1} \\ \delta_{14} = -\arg[i((V_0)_{12}\sqrt{m_2}\cos\omega + \xi\ (V_0)_{13}\sqrt{m_3}\sin\omega)] \end{cases}$$

$$\boxed{\kappa = \eta_0 \quad \text{and} \quad \xi = \eta_1}$$

$$\kappa \equiv \text{sign}\left\{\sqrt{(M_D^{\rm CI})^2_{12}}/(M_D^{\rm CI})_{12}\right\} = \pm 1$$



2. Comparing with the Nature Reconstruction: [hep-ph/0310278]

- EMSM's : $M_{\mathrm{D}} = (AU_0)D_\nu(U_0S)^T + RD_NB^T$

$$\simeq RD_N$$

- Natural Reconstruction's:

By solving: $M_\nu \simeq -M_{\mathrm{D}}M_{\mathrm{R}}^{-1}M_{\mathrm{D}}^T$

$$\left\{\begin{aligned}
(M_{\mathrm{D}}^{\mathrm{NR}})_{12} &\simeq \zeta_1\sqrt{-(M_\nu)_{11}M_2 - \frac{M_2}{M_1}(M_{\mathrm{D}}^{\mathrm{NR}})_{11}^2}\,, \\
(M_{\mathrm{D}}^{\mathrm{NR}})_{\alpha 1} &\simeq \frac{1}{(M_\nu)_{11}}\left\{(M_{\mathrm{D}}^{\mathrm{NR}})_{11}(M_\nu)_{1\alpha} + \zeta_\alpha(M_{\mathrm{D}}^{\mathrm{NR}})_{12}\sqrt{\frac{M_1}{M_2}}\sqrt{(M_\nu)_{11}(M_\nu)_{\alpha\alpha} - (M_\nu)_{1\alpha}^2}\right\} \\
(M_{\mathrm{D}}^{\mathrm{NR}})_{\alpha 2} &\simeq \frac{1}{(M_\nu)_{11}}\left\{(M_{\mathrm{D}}^{\mathrm{NR}})_{12}(M_\nu)_{1\alpha} - \zeta_\alpha(M_{\mathrm{D}}^{\mathrm{NR}})_{11}\sqrt{\frac{M_2}{M_1}}\sqrt{(M_\nu)_{11}(M_\nu)_{\alpha\alpha} - (M_\nu)_{1\alpha}^2}\right\}
\end{aligned}\right.$$

- Correlated through: $\zeta_1 = \eta_0$ and $\zeta_2 = -\zeta_3 = \eta_1$

Summary on DoFs:

The 11 DoFs for 7 different parameterization scheme are listed:

Parameterization	11 independent parameters in parameterizations of MSM
EMSM	$m_2,\ m_3(m_1),\ \theta_{12},\ \theta_{13},\ \theta_{23},\ \delta_\nu,\ \sigma,\ M_1,\ M_2,\ \boxed{\theta_{14},\ \delta_{14}}$
Casas–Ibarra	$m_2,\ m_3(m_1),\ \theta_{12},\ \theta_{13},\ \theta_{23},\ \delta_\nu,\ \sigma,\ M_1,\ M_2,\ \lvert\omega\rvert,\ \arg\omega$
Natural Reconstruction	$m_2,\ m_3(m_1),\ \theta_{12},\ \theta_{13},\ \theta_{23},\ \delta_\nu,\ \sigma,\ M_1,\ M_2,\ \lvert a_1\rvert,\ \arg a_1$
Blennow–Fernandez-Martinez	$m_2,\ m_3(m_1),\ \theta_{12},\ \theta_{13},\ \theta_{23},\ \delta_\nu,\ \sigma,\ M_1,\ M_2,\ \theta_y,\ \theta_z(\theta_x),$
Modified Casas–Ibarra	$m_2,\ m_3(m_1),\ \theta_{12},\ \theta_{13},\ \theta_{23},\ \delta_\nu,\ \sigma,\ \lvert P_{13}\rvert,\ \lvert P_{23}\rvert,\ \lvert P_{12}\rvert,\ \arg P_{12}$
Bi-Unitary	$m_2,\ m_3(m_1),\ M_1,\ M_2,\ \theta_1,\ \theta_2,\ \theta_3,\ \delta_L,\ \gamma_L,\ \theta_R,\ \gamma_R$
Vector Representation	$m_2,\ m_3(m_1),\ M_1,\ M_2,\ u_{e1},\ u_{\mu 1},\ \lvert u_{e2}\rvert,\ \lvert u_{\mu 2}\rvert,$ $\arg u_{e2},\ \arg u_{\mu 2},\ \arg u_{\tau 2}$

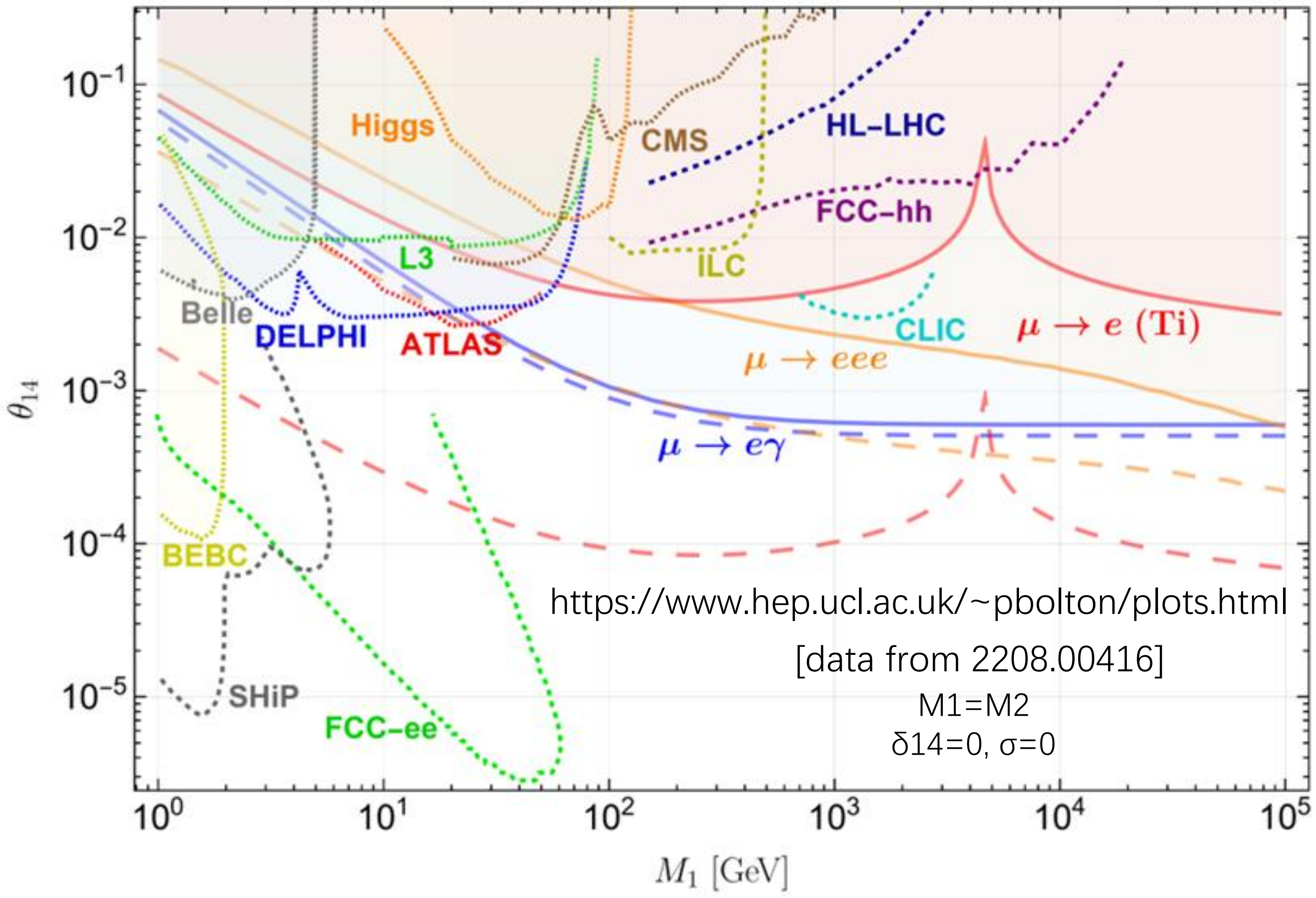
Conclusion: The MSM has 11 DoFs, through $\theta 14$ and $\delta 14$, we can explicitly depict the unitarity violation, and constrain the Light-Heavy mixing in turn and the following application.

Application:

Constrains as HNL:

Basical nature of N_R:

- **To suppress m_ν: Heavy**
- **Gauge singlet: Neutral**
- **Lepton** HNL
- **ν-N in CC: $U = AU_0$**



$$-\mathcal{L}_{cc} = \frac{g}{\sqrt{2}}\, \overline{(e \;\; \mu \;\; \tau)_L}\, \gamma^\mu \left[U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L + R \begin{pmatrix} \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix}_L \right] W_\mu^- + \text{h.c.}$$

=> Non-unitarity

cLFV:

Processes	Current bounds	Future sensitivities
$Br(\mu \to e\gamma)$	3.1×10^{-13} [18]	6.0×10^{-14} [21]
$Br(\mu \to eee)$	1.0×10^{-12} [19]	1.0×10^{-16} [22]
$R_{\mu\to e}$ (Ti)	1.0×10^{-18} [20]	1.0×10^{-18} [23, 24]

Constrains on Non-Unitarity: $(AU_0)(AU_0)^\dagger = AA^\dagger = I - RR^\dagger$

Deg. $(M_1 \simeq M_2)$

Heir. $(M_1 \ll M_2)$

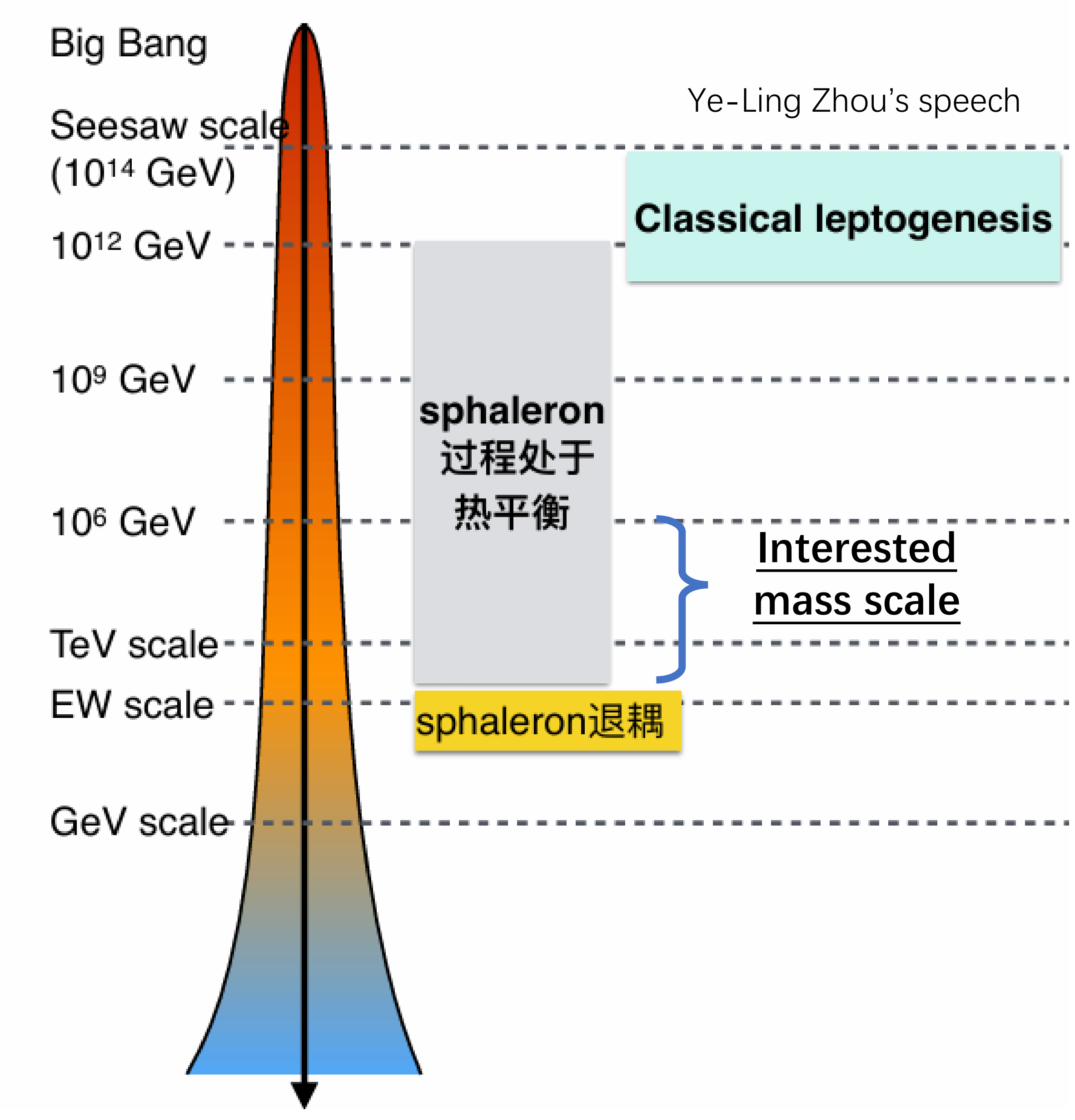
Non-unitary para.	ν oscillation	EWPT
$1-\alpha_{ee}$	$3.9\cdot 10^{-2}$	$9.4\cdot 10^{-6}$
$1-\alpha_{\mu\mu}$	$6.2\cdot 10^{-3}$	$1.3\cdot 10^{-4}$
$1-\alpha_{\tau\tau}$	$1.4\cdot 10^{-1}$	$2.1\cdot 10^{-4}$
$\lvert\alpha_{\mu e}\rvert$	$1.8\cdot 10^{-2}$	$2.4\cdot 10^{-5}$
$\lvert\alpha_{\tau e}\rvert$	$4.2\cdot 10^{-2}$	$4.4\cdot 10^{-5}$
$\lvert\alpha_{\tau\mu}\rvert$	$1.1\cdot 10^{-2}$	$2.6\cdot 10^{-4}$

[2306.01040]
[2103.01998]

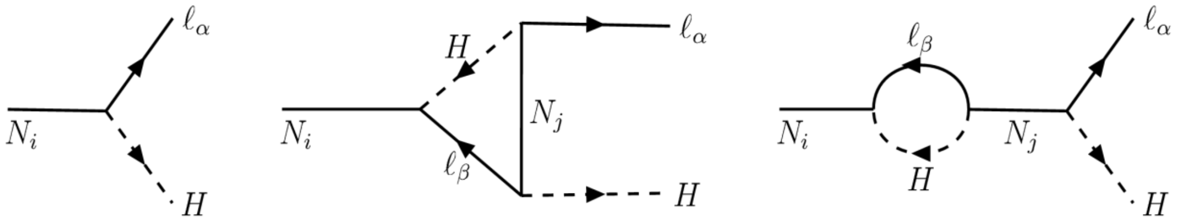
95%C.L., where $\alpha \equiv A$

Mixing angles	ν oscillation		EWPT	
	Degeneracy	Hierarchy	Degeneracy	Hierarchy
θ_{14}	$4.4\times 10^{-2}\,(2.5^\circ)$	$9.0\times 10^{-2}\,(5.2^\circ)$	$3.0\times 10^{-3}\,(0.17^\circ)$	$4.3\times 10^{-3}\,(0.25^\circ)$
θ_{15}		—		—
θ_{24}	$7.9\times 10^{-2}\,(4.5^\circ)$	$1.1\times 10^{-1}\,(6.4^\circ)$	$1.1\times 10^{-2}\,(0.64^\circ)$	$1.6\times 10^{-2}\,(0.92^\circ)$
θ_{25}		—		—
θ_{34}	$1.0\times 10^{-1}\,(5.9^\circ)$	$1.8\times 10^{-1}\,(10.4^\circ)$	$7.2\times 10^{-3}\,(0.41^\circ)$	$8.9\times 10^{-3}\,(0.51^\circ)$
θ_{35}		—		—

Leptogenesis (Based on **Ulysses** Package V2.0.1 and “2BE3F”, “2RESsp” mode)

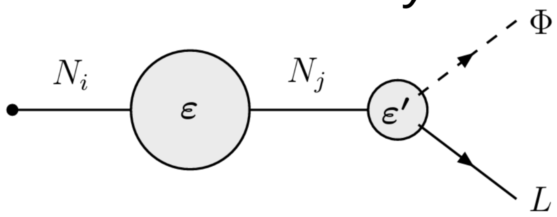


- Classical thermal CP asymmetry:



[1909.09610]

- Resonant CP asymmetry:



Anisimov, Buchmuller, Drewes, Mendizabal, 1012.5821

{ Mixing
Oscillation

- Boltzmann Equation:

Re: Spector Washout

$$\boxed{-p_{1\alpha} W_1 \sum_\beta \left(C^l_{\alpha\beta} + C^\Phi_\beta\right) N_{\beta\beta}}$$

$$\frac{dN_{N_1}}{dz} = -D_1\left(N_{N_1} - N^{\rm eq}_{N_1}\right)$$

$$\frac{dN_{\alpha\alpha}}{dz} = \sum_{\alpha=e,\mu,\tau}\left(\epsilon^{(1)}_{\alpha\alpha} D_1\left(N_{N_1} - N^{\rm eq}_{N_1}\right) - p_{1\alpha} W_1 N_{\alpha\alpha}\right)$$

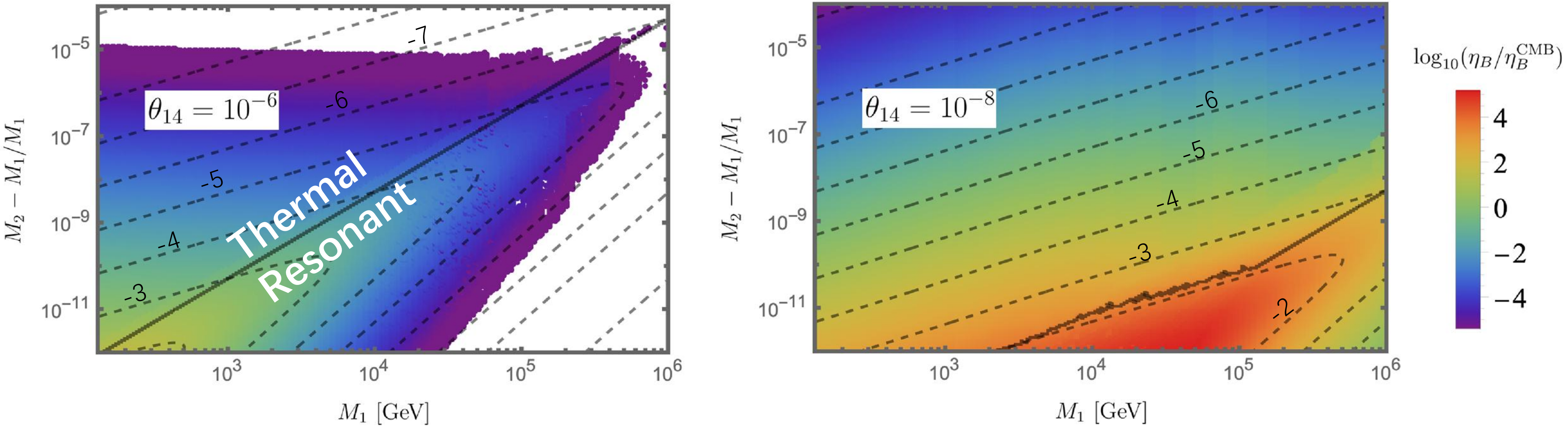
Leptogenesis (Based on **Ulysses** V2.0.1 and "2BE3F", "2RESsp" mode) [2007.09150] [2301.05722]

$$\varepsilon^{\rm th}_{i\alpha} = \frac{1}{8\pi (Y_\nu^\dagger Y_\nu)_{ii}} \sum_{j\neq i} \left\{ \mathrm{Im}\left[(Y_\nu^*)_{\alpha i} (Y_\nu)_{\alpha j} (Y_\nu^\dagger Y_\nu)_{ij} \right] \mathcal{F}(x_{ji}) + \mathrm{Im}\left[(Y_\nu^*)_{\alpha i} (Y_\nu)_{\alpha j} (Y_\nu^\dagger Y_\nu)^*_{ij} \right] \mathcal{G}(x_{ji}) \right\}$$

$$\varepsilon^{\rm re}_{i\alpha} = \sum_{j\neq i} \frac{\mathrm{Im}\left[Y^\dagger_{i\alpha} Y_{\alpha j} \left(Y^\dagger Y\right)_{ij} \right] + \frac{M_i}{M_j} \mathrm{Im}\left[Y^\dagger_{i\alpha} Y_{\alpha j} \left(Y^\dagger Y\right)_{ji} \right]}{\left(Y^\dagger Y\right)_{ii} \left(Y^\dagger Y\right)_{jj}} \left(f^{\rm mix}_{ij} + f^{\rm osc}_{ij} \right)$$

$$\varepsilon_\alpha = \sum_{i=1}^{2} \varepsilon_{i\alpha} = \begin{cases} \varepsilon^{\rm th}_{i\alpha} & M_2 - M_1 \gg \Gamma_i \\ \varepsilon^{\rm re}_{i\alpha} & M_2 - M_1 \lesssim \Gamma_i\,. \end{cases}$$

$$\eta_B \in (6.12 \pm 0.03) \times 10^{-10}$$
[PLANCK, 2018]



Conclusion:

- **EMSM** solve the exact Seesaw relation in MSM,
- Return to approximated scenario ✓
- Correspondence with C-I and NR’s ✓
- Make a general table list for 11 params.

<u>Application</u>

- Constrains to ν-N mixing angles and HNL masses scale
- Probing Leptogenesis

<u>May for further Research</u>

- 1-loop applicability
- Light-sterile neutrino models

Thanks for your listening!

Questions are Welcomed

- 3-Step **Parameterizing** MSM ($m_{\text{lightest}}=0$):

$$\begin{pmatrix} 0 & M_{\rm D} \\ M_{\rm D}^T & M_{\rm R} \end{pmatrix} = \{u\} \begin{pmatrix} D_\nu & 0 \\ 0 & D_N \end{pmatrix} u^T$$

Step1 DoFs counting:

$$\left\{ \begin{pmatrix} I & 0 \\ 0 & U_0' \end{pmatrix} \underline{\begin{pmatrix} A & R \\ S & B \end{pmatrix} \begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix}} \right\}^{\rm T} \begin{bmatrix} (\nu_{\rm L})^c \\ N_{\rm R} \end{bmatrix} \qquad O_{12} \equiv \begin{pmatrix} c_{12} & s_{12}e^{i\delta_{12}} & \cdots \\ -s_{12}e^{-i\delta_{12}} & c_{12} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Buy ten-5×5 rotation matrices in the complex plane：

$$\begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix} = O_{23}O_{13}O_{12}, \quad \begin{pmatrix} I & 0 \\ 0 & U_0' \end{pmatrix} = O_{45}, \quad \begin{pmatrix} A & R \\ S & B \end{pmatrix} = O_{35}O_{25}O_{15}O_{34}O_{24}O_{14}$$

DoFs	Light-light	Heavy-heavy	Interplay (v-N)
Complex phases	3	1	6
Mixing angles	3	1	6
Summed: 7	5: -1 Up. phase	0: -2 basis select.	2: -9 Cond.-1 Up

11 DoFs For MSM

7-from light v Exp.

$$0 = UD_\nu U^{\rm T} + RD_N R^{\rm T}$$

$$\text{NH}: \text{diag}\{0, m_2, m_3\}$$

$$\tilde{R} = A^{-1}R$$

M_D √ $= (AU_0)D_\nu(U_0S)^{\rm T} + RD_NB^{\rm T}$

$$-M_\nu \equiv -U_0 D_\nu U_0^T = \tilde{R} D_N \tilde{R}^T$$

$\boxed{\theta_{14}, \delta_{14}}$ 4-from heavy

$\boxed{\text{diag}\{M_1, M_2\}}$

Appendix. A Solution:

$$O_{12} \equiv \begin{pmatrix} c_{12} & s_{12}e^{i\delta_{12}} & \dots \\ -s_{12}e^{-i\delta_{12}} & c_{12} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$\tilde{R}_{12} = \eta_0\sqrt{-\frac{m_{11}}{M_2} - \frac{M_1}{M_2}\tilde{R}_{11}^2},$$

$$\tilde{R}_{21} = \frac{-(m_{11}w_1^2 + m_{22}w_2^2 - m_{33}w_3^2)\,\tilde{R}_{11} + 2\,\eta_1\sqrt{m_2m_3}\,W_1\,\tilde{R}_{12}}{2m_{11}w_1w_2},$$

$$\tilde{R}_{31} = \frac{-(m_{11}w_1^2 - m_{22}w_2^2 + m_{33}w_3^2)\,\tilde{R}_{11} - 2\,\eta_1\sqrt{m_2m_3}\,W_1\,\tilde{R}_{12}}{2m_{11}w_1w_3},$$

$$\tilde{R}_{22} = \frac{-(m_{11}w_1^2 + m_{22}w_2^2 - m_{33}w_3^2)\,\tilde{R}_{12} - 2\,\eta_1\sqrt{m_2m_3}\,W_2\,\tilde{R}_{11}}{2m_{11}w_1w_2},$$

$$\tilde{R}_{32} = \frac{-(m_{11}w_1^2 - m_{22}w_2^2 + m_{33}w_3^2)\,\tilde{R}_{12} + 2\,\eta_1\sqrt{m_2m_3}\,W_2\,\tilde{R}_{11}}{2m_{11}w_1w_3}$$

$$\tilde{R} = \begin{pmatrix} \frac{\hat{s}_{14}^*}{c_{14}} & \frac{\hat{s}_{15}^*}{c_{14}c_{15}} \\ \frac{\hat{s}_{24}^*}{c_{14}c_{24}} & \frac{\hat{s}_{14}\hat{s}_{24}^*\hat{s}_{15}^*}{c_{14}c_{15}c_{24}} + \frac{\hat{s}_{25}^*}{c_{24}c_{15}c_{25}} \\ \frac{\hat{s}_{34}^*}{c_{14}c_{24}c_{34}} & \frac{\hat{s}_{14}\hat{s}_{34}^*\hat{s}_{15}^*}{c_{14}c_{24}c_{15}c_{34}} + \frac{\hat{s}_{24}\hat{s}_{34}^*\hat{s}_{25}^*}{c_{24}c_{34}c_{15}c_{25}} + \frac{\hat{s}_{35}^*}{c_{34}c_{15}c_{25}c_{35}} \end{pmatrix}$$

$$\sin\theta_{ik} = \frac{|p_{ik}|}{\sqrt{1+|p_{ik}|^2}}\,, \quad \delta_{ik} = -\arg p_{ik}$$

$$p_{24} = c_{14}\tilde{R}_{21}\,,$$

$$p_{34} = c_{14}c_{24}\tilde{R}_{31}\,,$$

$$p_{15} = c_{14}\tilde{R}_{12}\,,$$

$$p_{25} = c_{15}c_{24}\big(\tilde{R}_{22} - c_{14}^2\tilde{R}_{11}^*\tilde{R}_{12}\tilde{R}_{21}\big)\,,$$

$$p_{35} = c_{15}c_{25}c_{34}\big[\tilde{R}_{32} - c_{14}^2c_{24}^2(\tilde{R}_{11}^*\tilde{R}_{12} + \tilde{R}_{21}^*\tilde{R}_{22})\tilde{R}_{31}\big]$$

Appendix. B Solution:

$$A = \begin{pmatrix} c_{14}c_{15} & 0 & 0 \\ -c_{14}\hat{s}_{15}\hat{s}^*_{25} - \hat{s}_{14}\hat{s}^*_{24}c_{25} & c_{24}c_{25} & 0 \\ \begin{array}{l} -c_{14}\hat{s}_{15}c_{25}\hat{s}^*_{35} - \hat{s}_{14}c_{24}s^*_{34}c_{35} \\ +\hat{s}_{14}\hat{s}^*_{24}\hat{s}_{25}\hat{s}^*_{35} \end{array} & -c_{24}\hat{s}_{25}\hat{s}^*_{35} - \hat{s}_{24}\hat{s}^*_{34}c_{35} & c_{34}c_{35} \end{pmatrix}$$

$$R = \begin{pmatrix} \hat{s}^*_{14}c_{15} & \hat{s}^*_{15} \\ -\hat{s}^*_{14}\hat{s}_{15}\hat{s}^*_{25} + c_{14}\hat{s}^*_{24}c_{25} & c_{15}\hat{s}^*_{25} \\ -\hat{s}^*_{14}\hat{s}_{15}c_{25}\hat{s}^*_{35} - c_{14}\hat{s}^*_{24}\hat{s}_{25}\hat{s}^*_{35} + c_{14}c_{24}\hat{s}^*_{34}c_{35} & c_{15}c_{25}\hat{s}^*_{35} \end{pmatrix},$$

$$S = \begin{pmatrix} -\hat{s}_{14}c_{24}c_{34} & -\hat{s}_{24}c_{34} & -\hat{s}_{34} \\ \begin{array}{l} \hat{s}_{14}c_{24}\hat{s}^*_{34}\hat{s}_{35} + \hat{s}_{14}\hat{s}^*_{24}\hat{s}_{25}c_{35} \\ -c_{14}\hat{s}_{15}c_{25}c_{35} \end{array} & \hat{s}_{24}\hat{s}^*_{34}\hat{s}_{35} - c_{24}\hat{s}_{25}c_{35} & -c_{34}\hat{s}_{35} \end{pmatrix}$$

$$B = \begin{pmatrix} c_{14}c_{24}c_{34} & 0 \\ -c_{14}c_{24}\hat{s}^*_{34}\hat{s}_{35} - c_{14}\hat{s}^*_{24}\hat{s}_{25}c_{35} - \hat{s}^*_{14}\hat{s}_{15}c_{25}c_{35} & c_{15}c_{25}c_{25} \end{pmatrix},$$

Appendix. C loop function

$$f_{ij}^{\mathrm{mix}} = \frac{\left(M_i^2 - M_j^2\right) M_i \Gamma_j}{\left(M_i^2 - M_j^2\right)^2 + M_i^2 \Gamma_j^2}, \quad f_{ij}^{\mathrm{osc}} = \frac{\left(M_i^2 - M_j^2\right) M_i \Gamma_j}{\left(M_i^2 - M_j^2\right)^2 + \left(M_i \Gamma_i + M_j \Gamma_j\right)^2 \frac{\det\left[\mathrm{Re}\left(Y^\dagger Y\right)\right]}{\left(Y^\dagger Y\right)_{ii}\left(Y^\dagger Y\right)_{jj}}}$$

Appendix. D 1-loop level (Feynman Gauge)

Shun Zhou's