Frontiers in Nuclear Lattice EFT: From an initio Nuclear Structure to Reactions
01-03 March 2025, Beihang

Effective range expansion with the left-hand cut

Feng-Kun Guo

Institute of Theoretical Physics, Chinese Academy of Sciences

- Neutron-halo scattering — Xu Zhang, Hai-Long Fu, FKG, H.-W. Hammer, PRC 108 (2023) 044304
- $DD^*/D\bar{D}^*$ scattering — Meng-Lin Du et al., PRL 131 (2023) 131903; Teng Ji et al., arXiv:2502.04458
- Generalized ERE w/ lhc — Meng-Lin Du, FKG, Bing Wu, arXiv:2408.09375

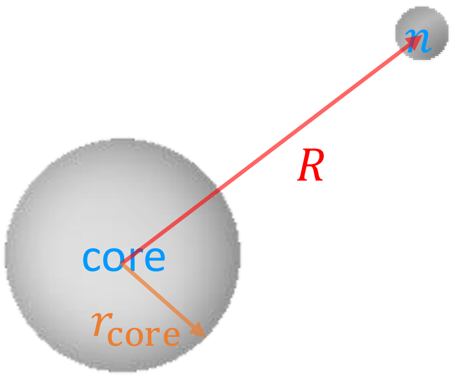
Neutron-halo scattering

- Halo EFT C. A. Bertulani, H.-W. Hammer, and U. van Kolck, NPA 712 (2002) 37; P. F. Bedaque, H. W. Hammer, and U. van Kolck, PLB 569 (2003) 159; …
  Reviews: H.-W. Hammer, C. Ji, D. R. Phillips, JPG 44 (2017) 103002; H.-W. Hammer, arXiv:2203.13074 [nucl-th]; …

- One-neutron halo nuclei

$$R \sim \frac{1}{Q} \sim \frac{1}{\sqrt{2\mu E_n}} \gg r_{\text{core}} \sim \frac{1}{\Lambda} \sim \frac{1}{\sqrt{2\mu E_c^*}}$$

S-wave 1-neutron halos	^{11}Be	^{15}C	^{19}C
Experiment			
J^P	$1/2^+$	$1/2^+$	$1/2^+$
S_{1n}[MeV]	0.50164(25)	1.2181(8)	0.58(9)
E_c^*[MeV]	3.36803(3)	6.0938(2)	1.62(2)
$\langle r^2 \rangle_{nc}^{1/2}$ [fm]	6.05(23)	4.15(50)	6.6(5)
Halo EFT			
Q/Λ	0.39	0.45	0.6
r_{nc}/a_{nc}	0.38	0.43	0.33
$\langle r^2 \rangle_{nc,\text{ theo}}^{1/2}$ [fm]	6.85	4.93	5.72

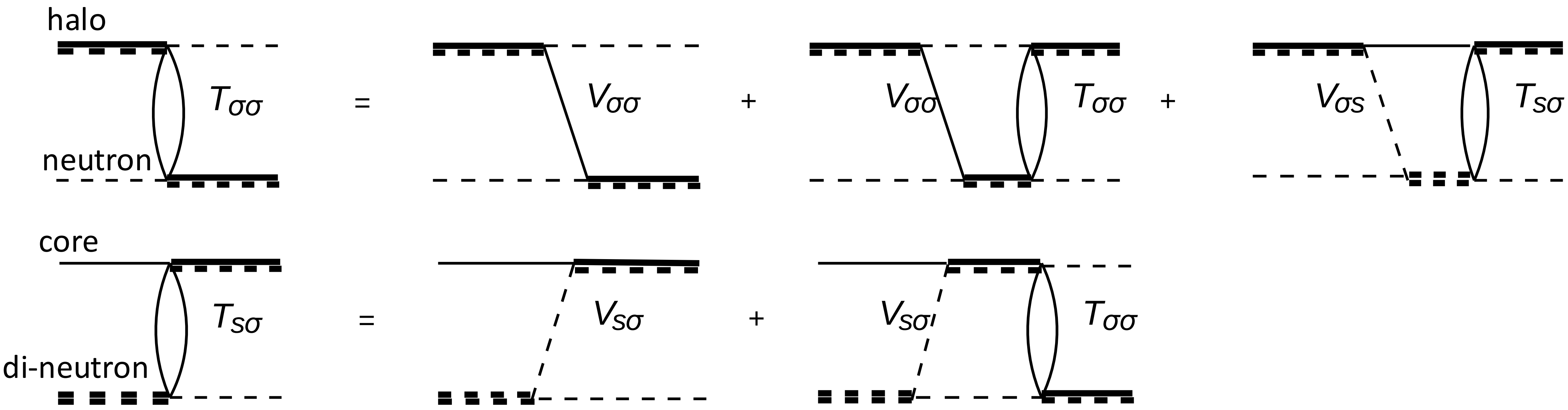


H.-W. Hammer, C. Ji, D. R. Phillips, JPG 44 (2017) 103002

Neutron-halo scattering

X. Zhang, H.-L. Fu, FKG, H.-W. Hammer, PRC 108 (2023) 044304

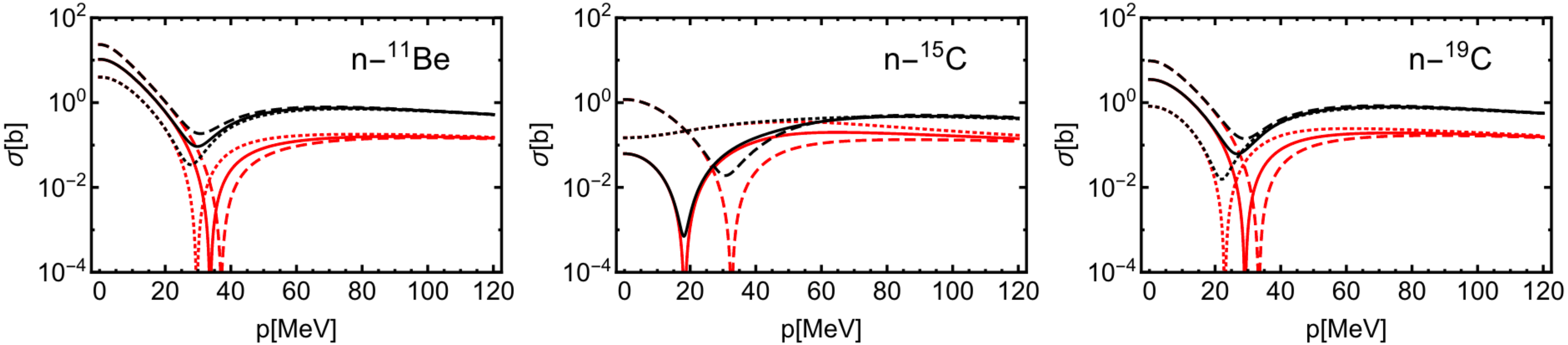
- Consider scattering between neutron and one-neutron-halo nucleus with total spin $J = 0$



- Inputs: a_s (nn sca. length), B_σ (for neutron halo), B_{2n} (2-neutron separation energy)

Neutron-halo scattering: amplitude zeros

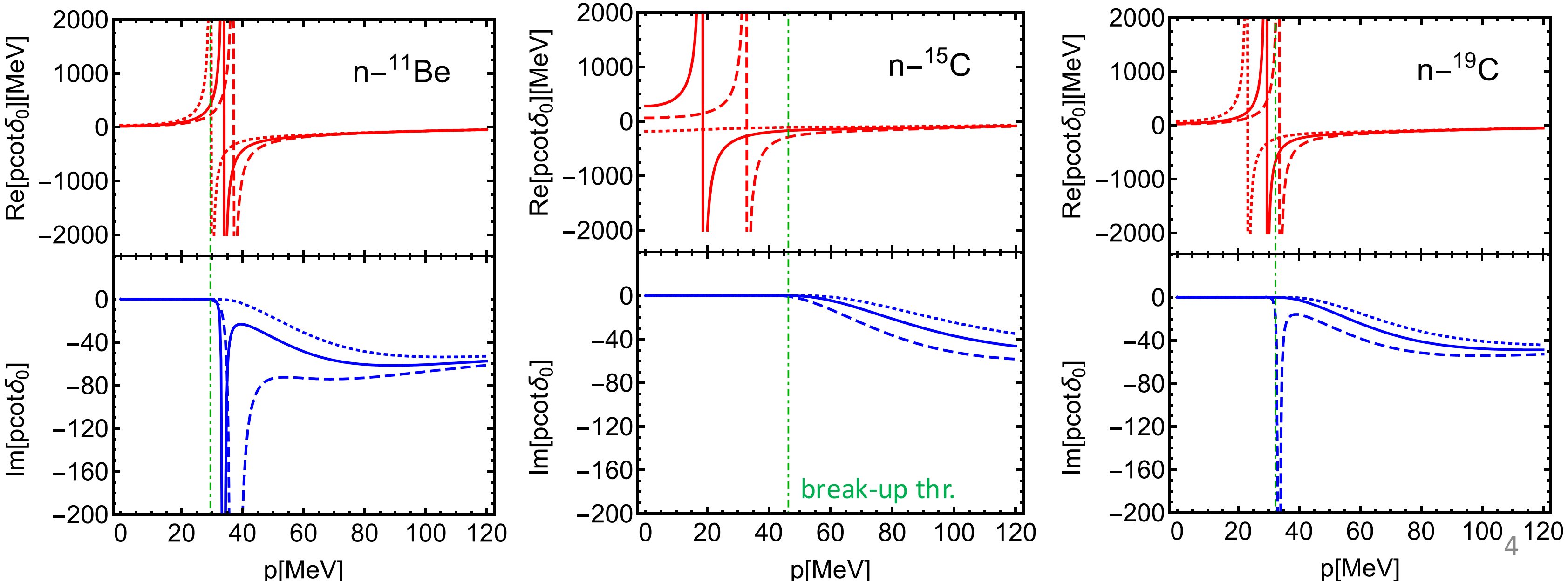
- Inputs: a_s (nn sca. length), B_σ (for neutron halo), B_{2n} (2-neutron separation energy)
- Total cross sections [red: s-wave, black: $l \leq 4$]



- Zero in the s-wave cross section!

$$T^l_{\sigma\sigma}(p,p,E) = \frac{2\pi}{\mu_{n\sigma}} \frac{\eta e^{2i\delta^R_l(p)} - 1}{2ip}$$

- pole in $p\cot\delta^R_0(p)$



Neutron-halo scattering: amplitude zeros

- Found before for n-d, n-^{19}C scattering

  W.T.H. van Oers, J.D. Seagrave, PLB 24 (1967) 562; …
  M. T. Yamashita, T. Frederico, L. Tomio, PLB 670 (2008) 49; …

  - Leads to a modified effective range expansion (ERE)

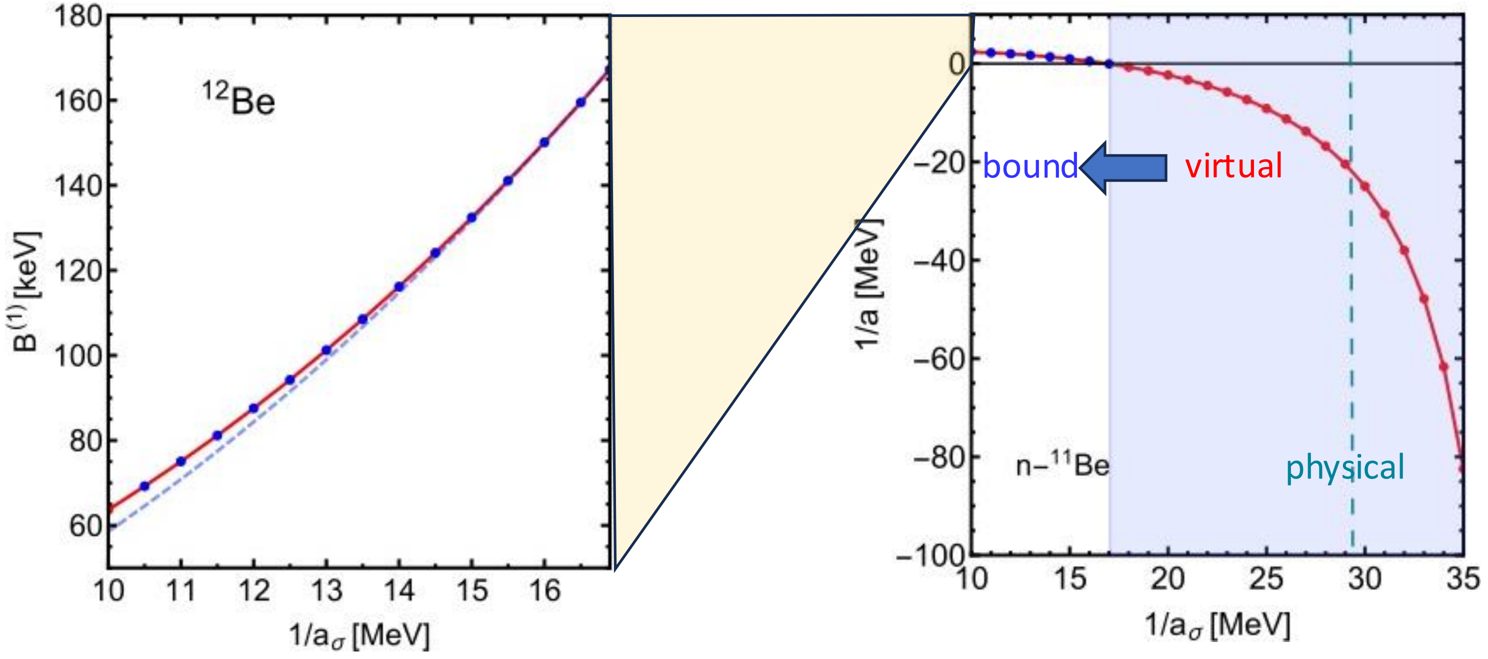
$$k\cot\delta_0 = -A + Bk^2 - \frac{C}{1+Dk^2}$$

  - Manifestation of a 3-body virtual state

    B. A. Girard, M. G. Fuda, PRC 19 (1979) 579; …

    - Increasing the nh scattering length a_σ => excited Efimov state

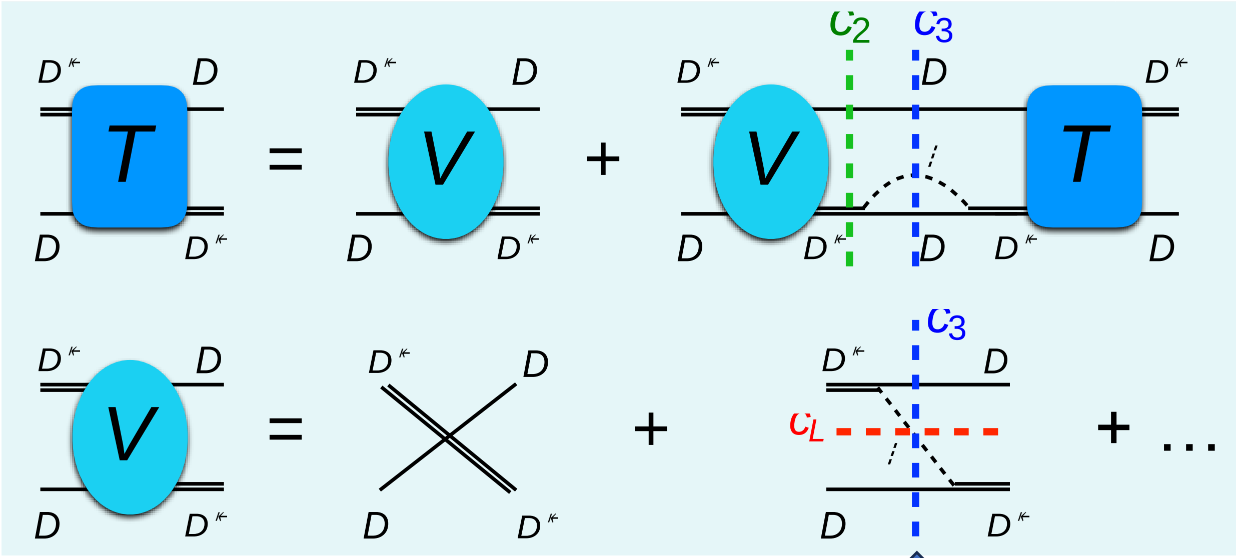
nh scattering length changes sign



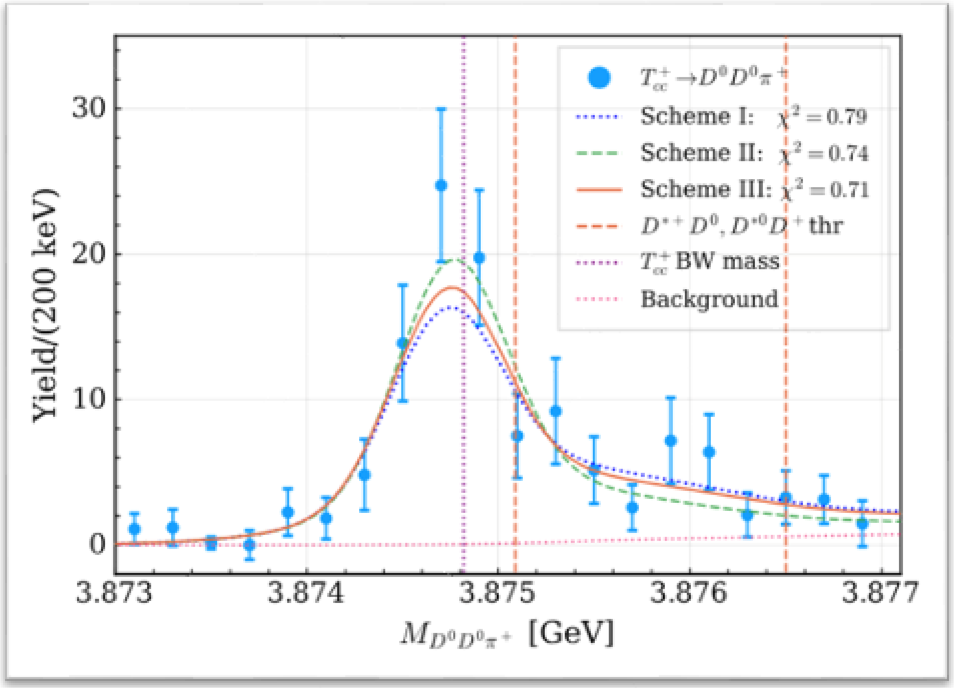
DD^* scattering

M.-L. Du et al., PRD 105 (2022) 014024

- Similar finding in hadronic physics: DD^* scattering with unphysical pion mass
  - $D^* \to D\pi$ in P-wave: D^* as a $D\pi$ dimer

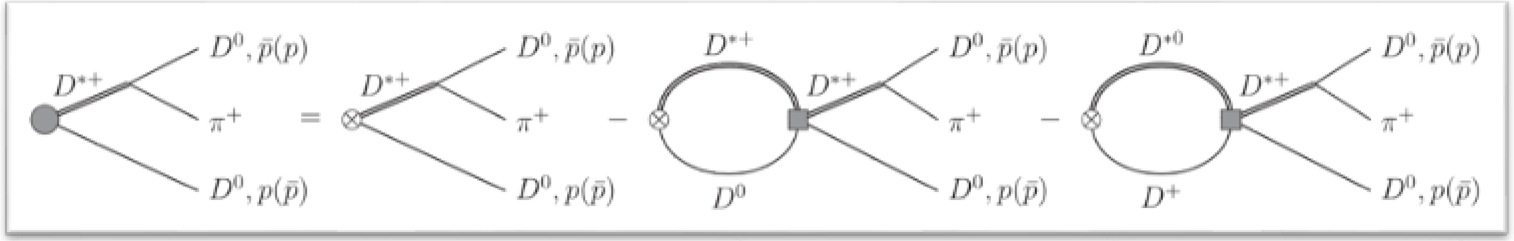


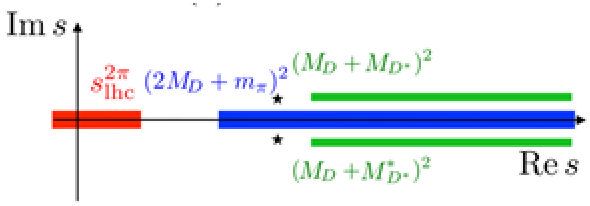
c_3: 3-body cut for $m_{D^*} > m_D + M_\pi$



Data: LHCb, Nature Commun. 13 (2022) 3351

  - $T_{cc}(3875)$: double-charm tetraquark, DD^* molecular state

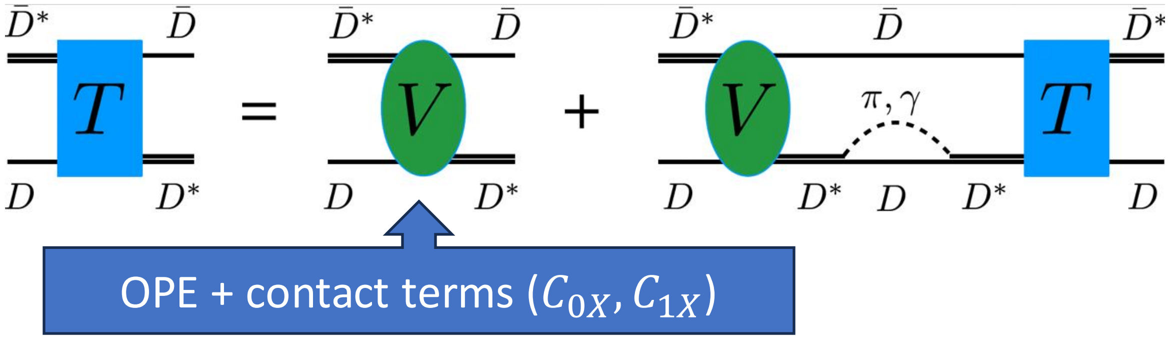


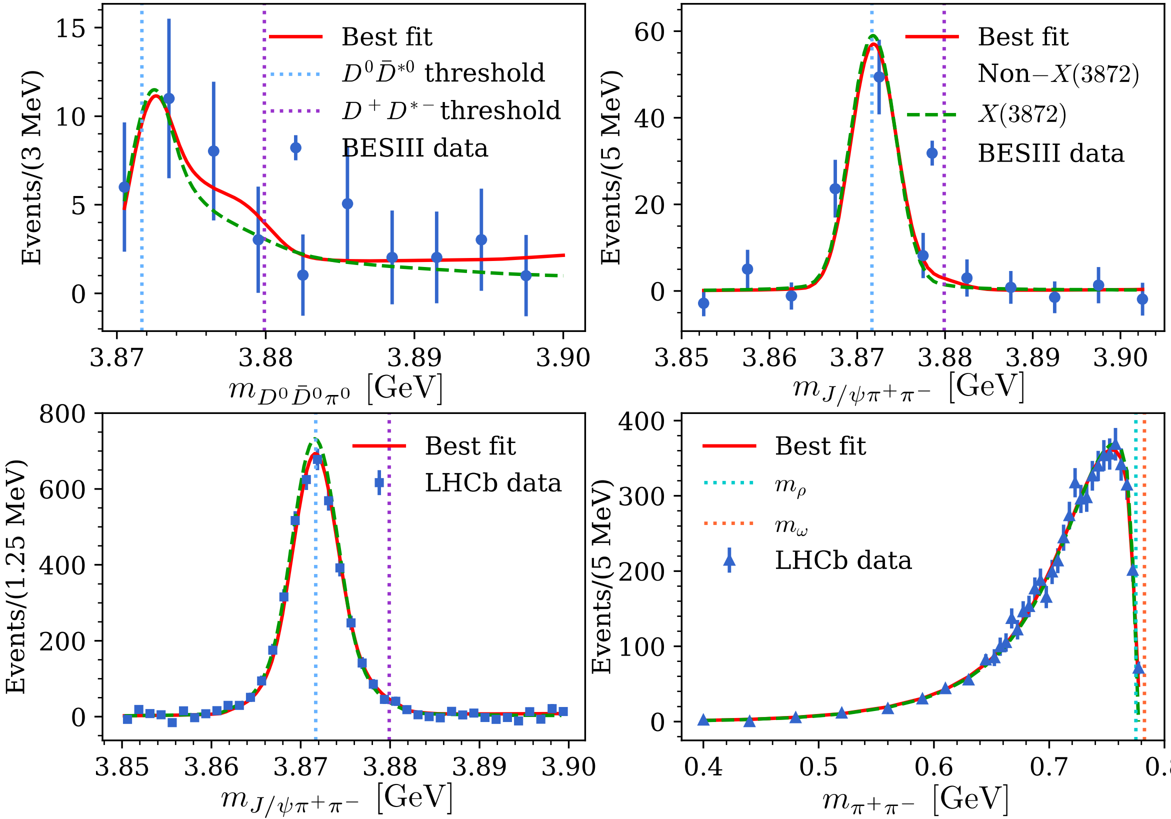


$D\bar{D}^*$ scattering

Zhen-Hua Zhang et al., JHEP 08 (2024) 130; Teng Ji et al., arXiv:2502.04458

- $D\bar{D}^*$ scattering is similar





- Main findings:
  - $X(3872)$ bound state at a 5σ level

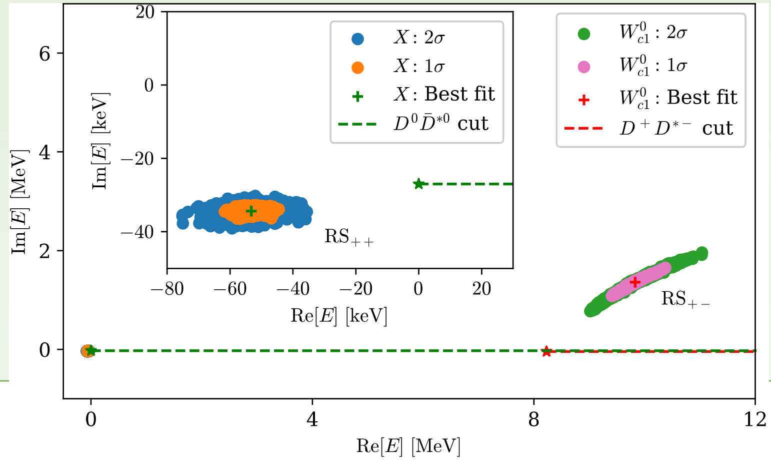
$$E_X = \left(-53^{+10}_{-25} - i34^{+2}_{-12}\right)\ \text{keV}$$

  - It has an isospin-1 partner, $W_{c1}^{0,\pm}$
    - Neutral one rel. to D^+D^{*-} threshold:

$$\left(1.6^{+0.7}_{-0.9} + i1.4^{+0.3}_{-0.6}\right)\ \text{MeV}$$

    - Virtual state ⇒ threshold cusp
    - Should be more pronounced in B^0 decays
    - Supported by lattice QCD calculation

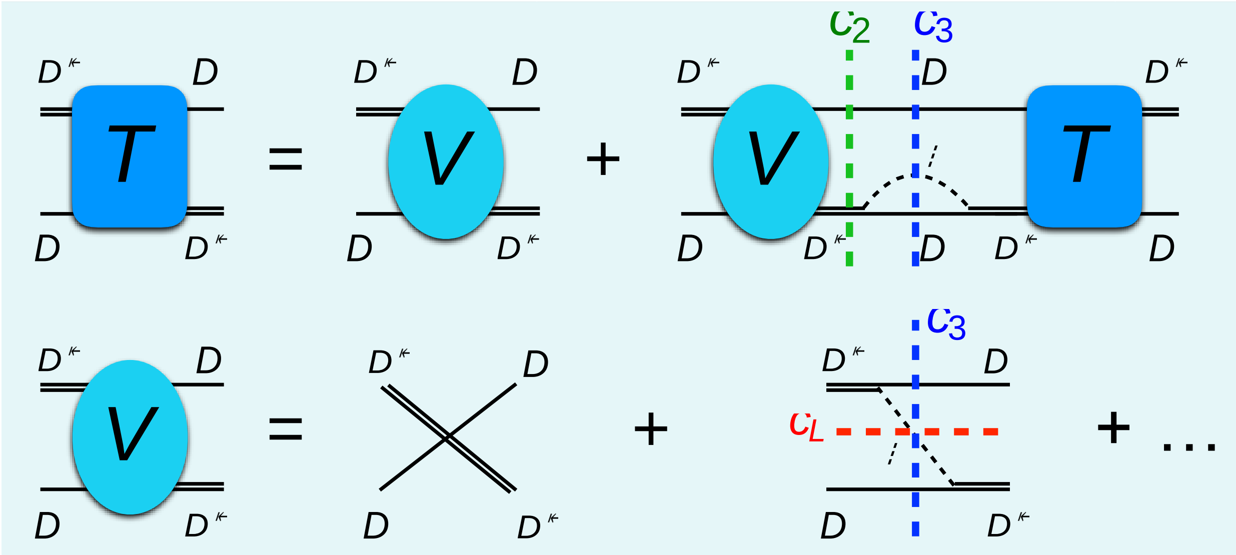
M. Sadl et al., arXiv:2406.09842

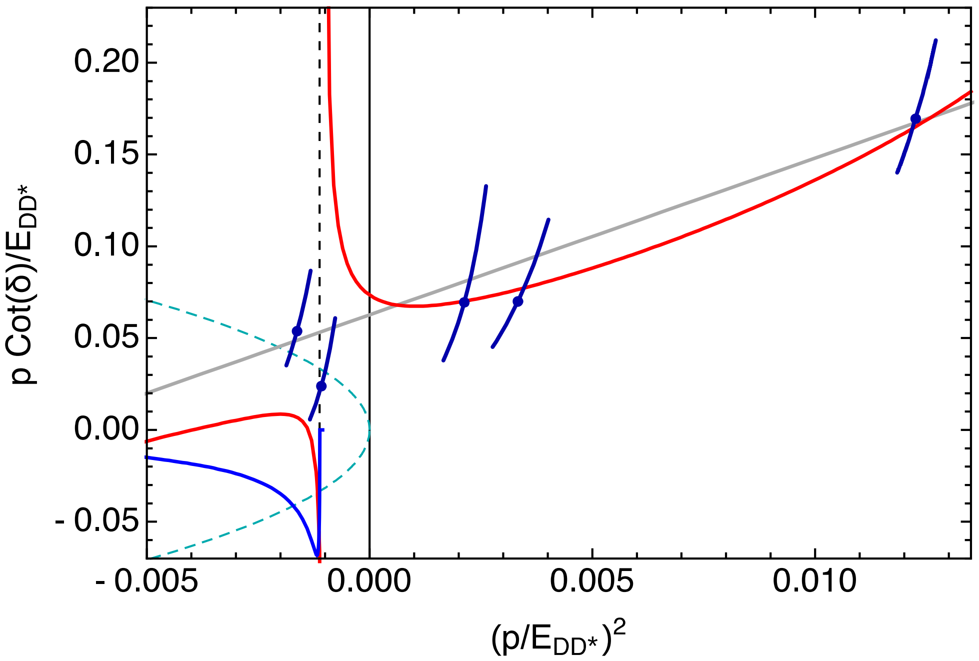


DD^* scattering

M.-L. Du et al., PRL 131 (2023) 131903

- Similar finding in hadronic physics: DD^* scattering with unphysical pion mass
  - ☐ $D^* \to D\pi$ in p-wave: D^* as a $D\pi$ dimer



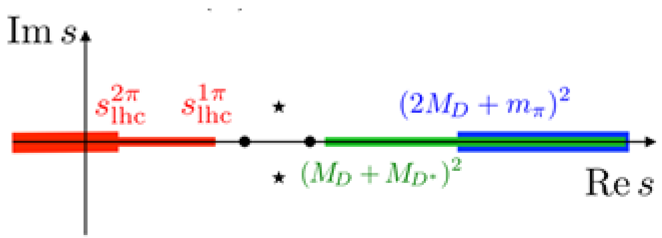


Lattice QCD data ($M_\pi = 280$ MeV):
M. Padmanath, S. Prelovsek, PRL 129 (2022) 032002

- ☐ Usual effective range expansion

$$k \cot \delta_0 = \frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \cdots \qquad \text{for short} - \text{range int.}$$

  - ➢ Generally used to extract pole position from lattice QCD calculations
- ☐ For unphysically large M_π, $m_{D^*} < m_D + M_\pi$, left-hand-cut

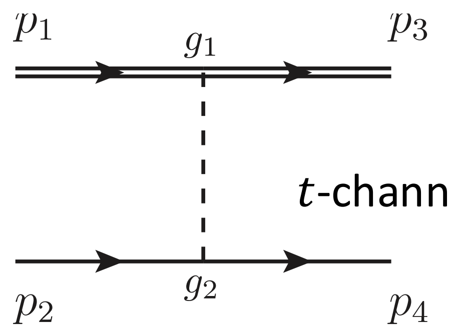


ERE convergence radius

- Convergence radius for ERE: the nearest singularity
  - Branch point of the left-hand cut: Endpoint singularity of projection of t/u-ch. amplitude
    - t-channel

$$L_t\left(k^2\right) \equiv \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{t-m_{\mathrm{ex}}^2} = -\frac{1}{4k^2}\log\frac{m_{\mathrm{ex}}^2/4+k^2}{m_{\mathrm{ex}}^2/4}$$

ERE would fail at $k^2 = -\frac{m_{\mathrm{ex}}^2}{4}$



    - u-channel

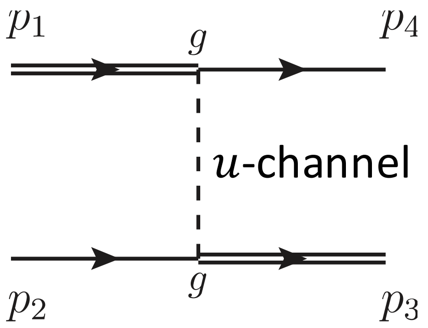
$$L_u\left(k^2\right) \equiv \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{u-m_{\mathrm{ex}}^2} \approx -\frac{1}{4k^2}\log\frac{\mu_+^2/4+k^2}{\mu_+^2/4+\eta^2k^2}$$

Nonrelativistic approx.

$$\mu_+^2 \equiv \frac{4m_1m_2}{(m_1+m_2)^2}\mu_{\mathrm{ex}}^2$$

$$\mu_{\mathrm{ex}}^2 \equiv m_{\mathrm{ex}}^2-(m_1-m_2)^2$$

$$\eta^2 \equiv \frac{(m_1-m_2)^2}{(m1+m_2)^2}$$

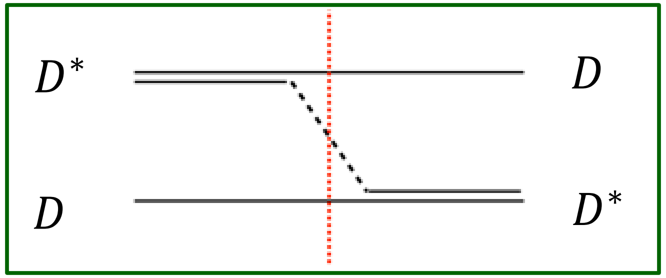


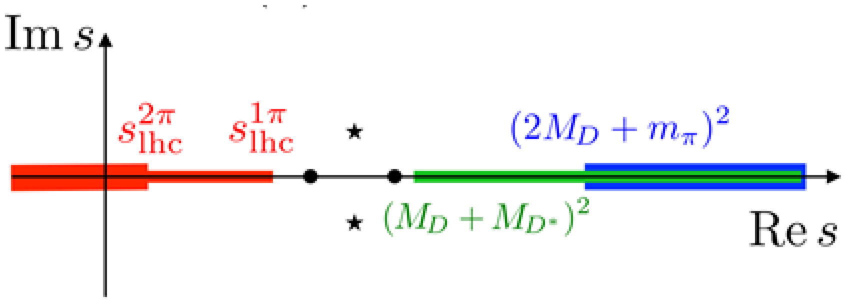
      - ✧ Nearby branch point: $k^2 = -\frac{\mu_+^2}{4}$
      - ✧ Farther branch point: $k^2 = -\frac{\mu_+^2}{4\eta^2}$
  - Amplitude zero: pole of $k\cot\delta$ may be present closer to threshold

Left-hand cut for DD^* scattering

M.-L. Du et al., PRL 131 (2023) 131903

- u-channel one-pion exchange





☞ two-body branch point:

$$E = m_D + m_{D^*}$$

$$\Longrightarrow p_{\rm rhc_2}^2 = 0$$

☞ three-body branch point:

$$E = m_D + m_{D^*} + M_\pi$$

$$\Longrightarrow \left(\frac{p_{\rm rhc_3}}{E_{DD^*}}\right)^2 = +0.019$$

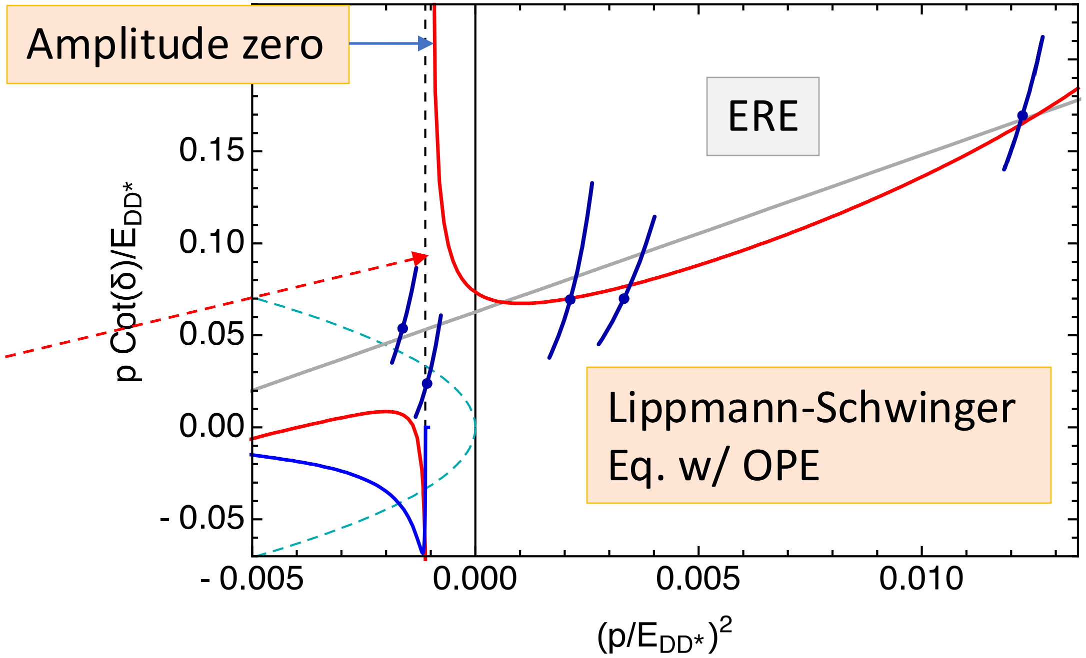
☞ left-hand cut branch point:

$$\Longrightarrow \left(\frac{p_{\rm lhc}^{1\pi}}{E_{DD^*}}\right)^2 = -0.001 \qquad -\frac{\mu_+^2}{4}$$

$$\left(\frac{\tilde{p}_{\rm lhc}^{1\pi}}{E_{DD^*}}\right)^2 = -0.190 \qquad -\frac{\mu_+^2}{4\eta^2}$$

Lattice QCD data ($M_\pi = 280$ MeV, $m_D = 1927$ MeV, $m_{D^*} = 2049$ MeV):

M. Padmanath, S. Prelovsek, PRL 129 (2022) 032002

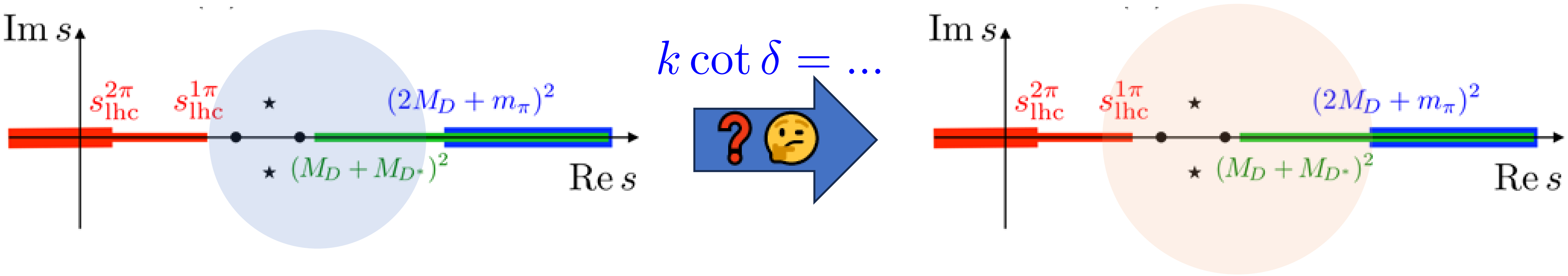


Generalization of ERE

- Recent works dealing with the left-hand cut in finite volume

  Meng, Epelbaum, JHEP (2021); Meng et al., PRD (2024); Hansen, Raposo, JHEP (2024); Dawid et al., PRD (2023); Hansen et al., PRD (2024); Bubna et al., JHEP (2024)

- Can we analytically generalize ERE to achieve a larger convergence radius?

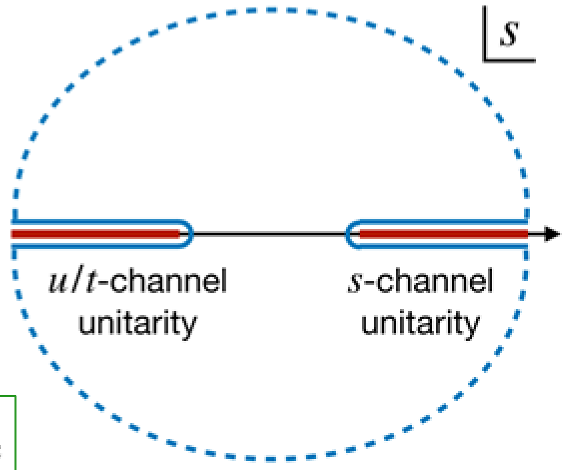


- ☐ Let's solve the problem in continuum

The N/D method

Meng-Lin Du, FKG, Bing Wu, arXiv:2408.09375

- N/D method of dispersion relation for 2-body scattering:
  - N: left-hand cut (lhc), no right-hand cut
  - D: right-hand cut (rhc), no left-hand cut



$$T(s) = \frac{N(s)}{D(s)}$$ Chew, Mandelstam (1960)

$$\mathrm{Im} D = \mathrm{Im}\frac{N}{T} = N\mathrm{Im}\frac{1}{T} = \begin{cases} -N\rho, & s > s_{\rm thr} \\ 0, & s < s_{\rm thr} \end{cases}$$

$$\mathrm{Im} N = \begin{cases} \mathrm{Im} T D, & s < s_{\rm lhc} \\ 0, & s > s_{\rm lhc} \end{cases}$$

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + \sum_{m=0}^{n-1} a_m s^m - \frac{(s-s_0)^n}{\pi}\int_{s_{\rm thr}}^{\infty} ds' \frac{\rho(s')N(s')}{(s'-s)(s'-s_0)^n},$$

$$N(s) = \sum_{m=0}^{n-\ell-1} b_m s^m + \frac{(s-s_0)^{n-\ell}}{\pi}\int_{-\infty}^{s_{\rm left}} ds' \frac{\mathrm{Im}T(s')D(s')}{(s'-s_0)^{n-\ell}(s'-s)}.$$

If $N = 1$ ⇒

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + P(s) + G(s)$$

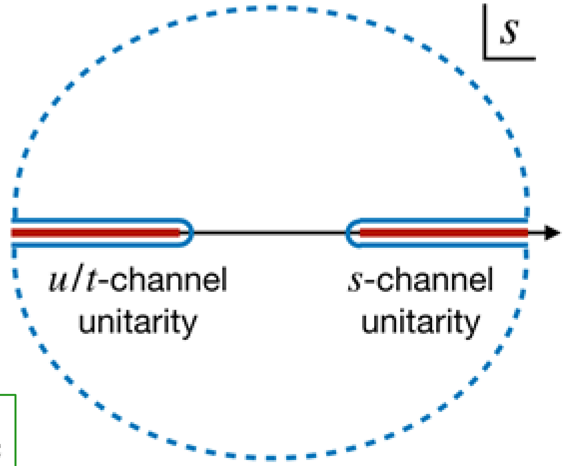
$$T(s) = \frac{1}{D(s)}$$

Neglect CDD pole,
low-momentum expansion

$$d(k^2) = P(k^2) - ik = \frac{1}{a} + \frac{1}{2}rk^2 + \mathcal{O}(k^4) - ik$$

The N/D method

- N/D method of dispersion relation for 2-body scattering:
  - N: left-hand cut (lhc), no right-hand cut
  - D: right-hand cut (rhc), no left-hand cut



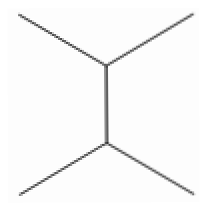
$$T(s) = \frac{N(s)}{D(s)}$$ Chew, Mandelstam (1960)

$$\mathrm{Im} D = \mathrm{Im}\frac{N}{T} = N \mathrm{Im}\frac{1}{T} = \begin{cases} -N\rho, & s > s_{\mathrm{thr}} \\ 0, & s < s_{\mathrm{thr}} \end{cases}$$

$$\mathrm{Im} N = \begin{cases} \mathrm{Im} T D, & s < s_{\mathrm{lhc}} \\ 0, & s > s_{\mathrm{lhc}} \end{cases}$$

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + \sum_{m=0}^{n-1} a_m s^m - \frac{(s-s_0)^n}{\pi} \int_{s_{\mathrm{thr}}}^{\infty} ds' \frac{\rho(s')N(s')}{(s'-s)(s'-s_0)^n},$$

$$N(s) = \sum_{m=0}^{n-\ell-1} b_m s^m + \frac{(s-s_0)^{n-\ell}}{\pi} \int_{-\infty}^{s_{\mathrm{left}}} ds' \frac{\mathrm{Im}T(s')D(s')}{(s'-s_0)^{n-\ell}(s'-s)}.$$

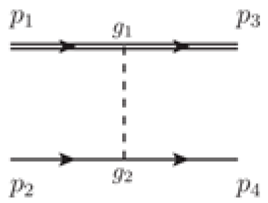


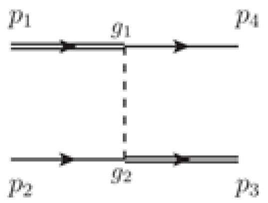
$$L_t\left(k^2\right) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\mathrm{ex}}^2} = -\frac{1}{4k^2}\log\frac{m_{\mathrm{ex}}^2/4 + k^2}{m_{\mathrm{ex}}^2/4}$$

e.g., along the lhc, $k^2 < -\frac{m_{\mathrm{ex}}^2}{4} \Rightarrow \mathrm{Im}\, T \neq 0$

Left-hand cut from one-particle exchange (OPE)

- Starting point: Restrict to OPE, then the lhc of T equals to the lhc of OPE potential
  - Singularity of integral must come from that of integrand
  - If the contour does not touch singularity, then integral is regular
    - Singularities of integrand becomes sing. of integral on other Riemann sheets
    - lhc of V: $k^2 < -\mu_+^2/4$ (u channel) or $k^2 < -m_{\rm ex}^2/4$ (t channel)

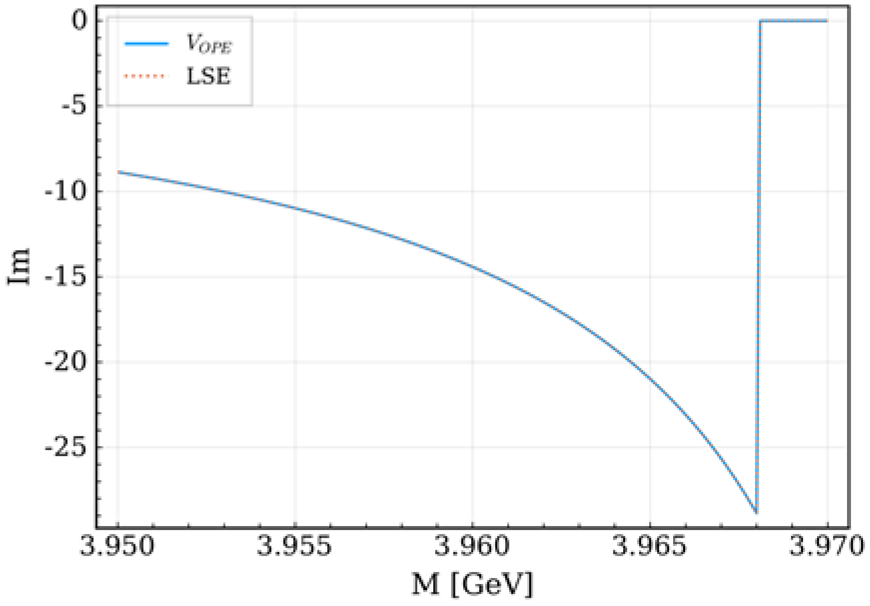




$$T\left(E;k,k\right)=V\left(E;k,k\right)+\int_0^\infty \frac{l^2dl}{2\pi^2}V\left(E;k,l\right)G\left(E;l\right)T\left(E;l,k\right)$$

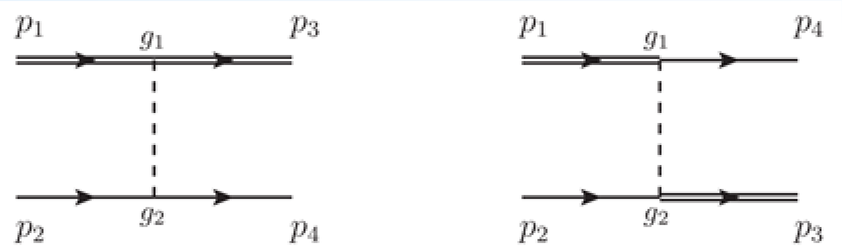
$$\operatorname{Im} f\left(k^2\right)=c\operatorname{Im} L\left(k^2\right)=-\frac{c}{4k^2}\pi, \quad \text{for } k^2<k_{\rm lhc}^2$$

c: strength of the lhc, containing product of coupling constants



N/D

- Couplings in different waves do not change the structure



S-wave couplings

$$L_t(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\text{ex}}^2} = -\frac{1}{4k^2}\log\frac{m_{\text{ex}}^2/4 + k^2}{m_{\text{ex}}^2/4},$$

$$L_u(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{u - m_{\text{ex}}^2} \approx -\frac{1}{4k^2}\log\frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2},$$

P-wave couplings

$$\frac{1}{2}\int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_3)^2}{t - m_{\text{ex}}^2}\mathrm{d}\cos\theta = -\frac{m_{\text{ex}}^2}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{t - m_{\text{ex}}^2} - 1$$

$$\frac{1}{2}\int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_4)^2}{u - m_{\text{ex}}^2}\mathrm{d}\cos\theta \approx -\frac{\mu_{\text{ex}}^2}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{u - m_{\text{ex}}^2} - 1$$

$\mathcal{F}_\ell/2$

- N/D construction of the amplitude:

$$f(k^2) = \frac{n(k^2)}{d(k^2)} \qquad \begin{aligned} \operatorname{Im} d(k^2) &= -k\, n(k^2), && \text{for } k^2 > 0, \\ \operatorname{Im} n(k^2) &= d(k^2)\, \operatorname{Im} f(k^2), && \text{for } k^2 < k_{\text{lhc}}^2. \end{aligned}$$

N/D

No singularity along lhc, polynomial in k'^2

$= c\,\mathrm{Im}\,L$

$$n\left(k^2\right) = n_m\left(k^2\right) + \frac{\left(k^2\right)^m}{\pi}\int_{-\infty}^{k^2_{\rm lhc}} \frac{d\left(k'^2\right)\mathrm{Im}\,f\left(k'^2\right)}{\left(k'^2-k^2\right)\left(k'^2\right)^m}\,\mathrm{d}k'^2$$

$$n\left(k^2\right) = n'_m\left(k^2\right) + \frac{P\left(k^2\right)}{\pi}\int_{-\infty}^{k^2_{\rm lhc}} \frac{\mathrm{Im}\,f\left(k'^2\right)}{k'^2-k^2}\,\mathrm{d}k'^2 = n'_m\left(k^2\right) + P\left(k^2\right)cL\left(k^2\right)$$

$$\begin{aligned} n\left(k^2\right) &= n_0 + n_1k^2 + \frac{k^2}{\pi}\int_{-\infty}^{k^2_{\rm lhc}} \frac{\left(d_0+d_1k'^2\right)\mathrm{Im}\,f\left(k'^2\right)}{\left(k'^2-k^2\right)k'^2}\,\mathrm{d}k'^2 \\ &= n_0 + n_1k^2 - cL_0 + \left(d_0+d_1k^2\right)\frac{c}{\pi}\int_{-\infty}^{k^2_{\rm lhc}} \frac{\mathrm{Im}\,L\left(k'^2\right)}{k'^2-k^2}\,\mathrm{d}k'^2 \\ &= n'_0 + n_1k^2 + \left(d_0+d_1k^2\right)cL\left(k^2\right) \end{aligned}$$

$$\propto \tilde{n}\left(k^2\right) + \tilde{g}\left(L\left(k^2\right) - L_0\right)$$

$$L_0 = L(k^2=0) = -1/\mu_{\rm ex}^2$$

divide by a polynomial, so that the normalization is $\tilde{n}(0) = 1$

N/D

$$d(k^2) = d_n(k^2) - \frac{(k^2 - k_0^2)^n}{\pi} \int_0^\infty \frac{k' n(k'^2)\, \mathrm{d}k'^2}{(k'^2 - k^2)(k'^2 - k_0^2)^n}$$

$$d\left(k^2\right) = \tilde{d}\left(k^2\right) - ik\left(\tilde{n}\left(k^2\right) - \tilde{g}L_0\right) - \frac{\tilde{g}}{\pi} \int_0^\infty \frac{k' L\left(k'^2\right)}{k'^2 - k^2}\, \mathrm{d}k'^2$$

$$= \tilde{d}\left(k^2\right) - ikn\left(k^2\right) - \tilde{g} d^{\mathrm{R}}\left(k^2\right)$$

- cancelation along lhc, so that $d(k^2)$ is real
- along rhc, $\mathrm{Im}\, d(k^2) = -k\, n(k^2)$

$$d_t^{\mathrm{R}}\left(k^2\right) = \frac{i}{4k} \log \frac{m_{\mathrm{ex}}/2 + ik}{m_{\mathrm{ex}}/2 - ik},$$

$$d_u^{\mathrm{R}}\left(k^2\right) = \frac{i}{4k} \left(\log \frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log \frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k} \right)$$

Generalized ERE with lhc

Meng-Lin Du, FKG, Bing Wu, arXiv:2408.09375

- Then we obtain the generalized ERE with OPE lhc in an analytic form:

$$\frac{1}{f(k^2)} = \frac{\tilde{d}(k^2) - \tilde{g} d^R(k^2)}{\tilde{n}(k^2) + \tilde{g}\left(L(k^2) - L_0\right)} - ik$$

$$L_t\left(k^2\right) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\rm ex}^2} = -\frac{1}{4k^2}\log\frac{m_{\rm ex}^2/4 + k^2}{m_{\rm ex}^2/4}$$

$$L_u\left(k^2\right) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{u - m_{\rm ex}^2} \approx -\frac{1}{4k^2}\log\frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2}$$

$$d_t^{\rm R}\left(k^2\right) = \frac{i}{4k}\log\frac{m_{\rm ex}/2 + ik}{m_{\rm ex}/2 - ik},$$

$$d_u^{\rm R}\left(k^2\right) = \frac{i}{4k}\left(\log\frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log\frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k}\right)$$

$\tilde{d}, \tilde{n}$: rational functions

$\tilde{d}, \tilde{n}$: take as polynomials (neglect CDD poles)

$$\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i k^{2i} - \tilde{g} d^{\rm R}(k^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j k^{2j} + \tilde{g}\left(L(k^2) - L_0\right)} - ik$$

Pade-like expansion

[0,1] approximant

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\rm R}(k^2)}{1 + \tilde{g}\left(L(k^2) - L_0\right)} - ik\right]^{-1}$$

$\tilde{g} \to 0$

no exchange

$$\frac{1}{f(k^2)} = \frac{1}{a} + \frac{1}{2} r k^2 - ik$$

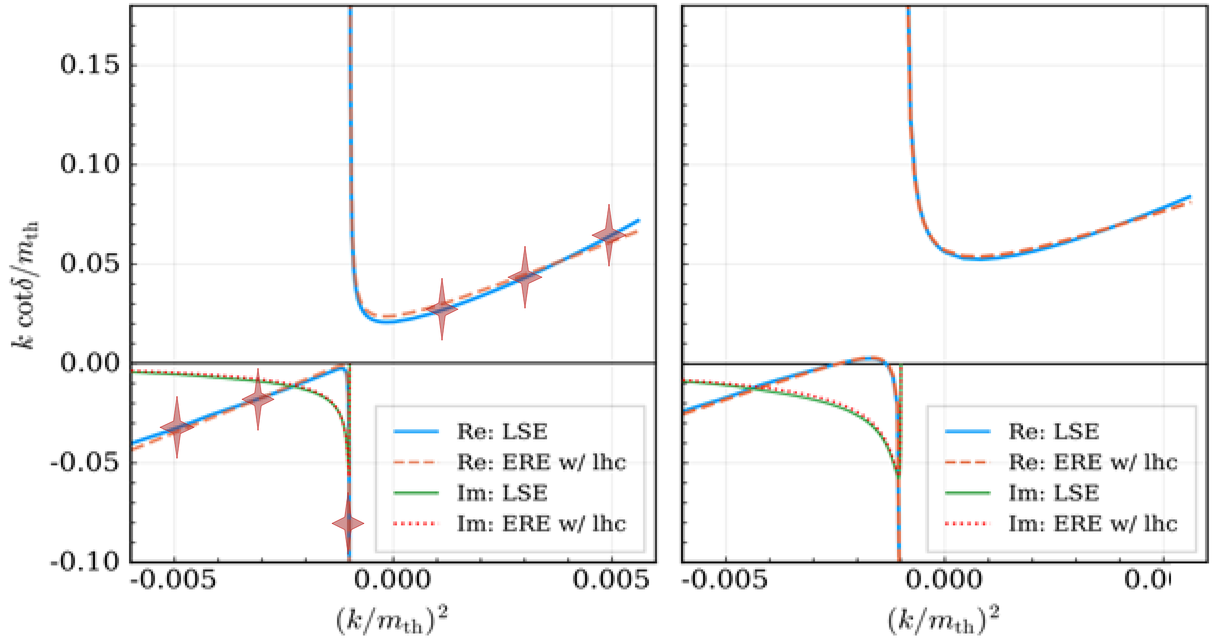
- Scattering length $a = f(k^2 = 0) = \left[\tilde{d}_0 + \frac{\tilde{g}}{\mu_+}(1 - \eta)\right]^{-1}$
- Effective range $r = \frac{\mathrm{d}^2(1/f + ik)}{\mathrm{d}k^2}\Big|_{k=0} = 2\tilde{d}_1 - \frac{8\tilde{g}}{3\mu_+^3}\left(1 - \eta^3\right) - \frac{4\tilde{g}}{\mu_+^4 a_u}\left(1 - \eta^4\right)$

Example

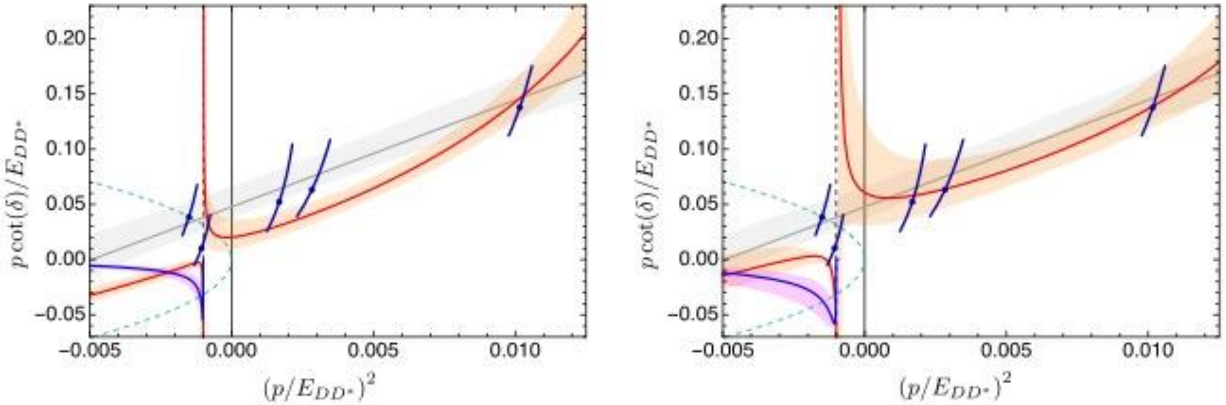
Meng-Lin Du, FKG, Bing Wu, arXiv:2408.09375

- DD^* scattering with lattice masses ($M_\pi = 280$ MeV, $m_D = 1927$ MeV, $m_{D^*} = 2049$ MeV)
  - Take the $[0,1]$ approximant, 3-parameter fit ($\tilde{d}_0, \tilde{d}_1, \tilde{g}$) to 6 chosen points from solving LSE
  - The amplitude zero is also well captured

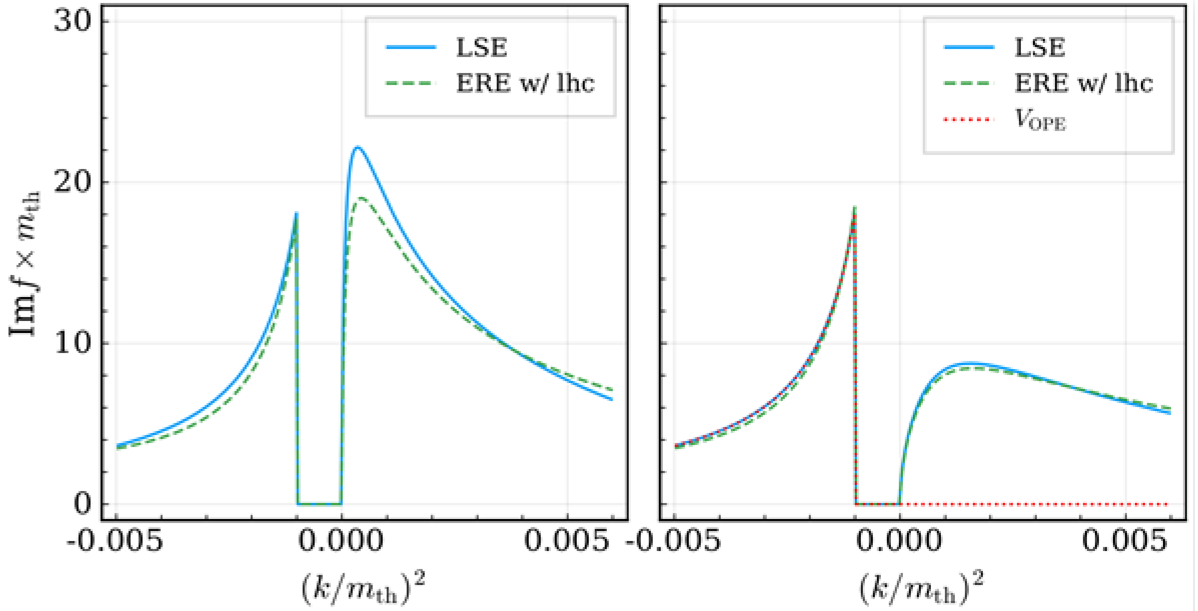
$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \tilde{g}\left(L(k^2) - L_0\right)} - ik\right]^{-1}$$



Left, right: corresponding to two fits to lattice QCD data performed in M.-L. Du et al., PRL 131 (2023) 131903



Example

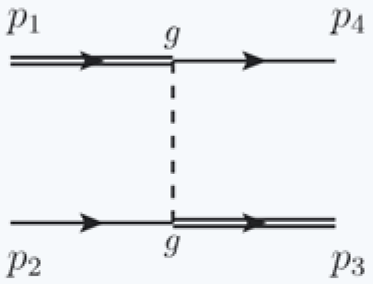


$9.0\ \mathrm{GeV}^{-2}$

$9.4\ \mathrm{GeV}^{-2}$ fit result

$$g_P = -\frac{2\pi\tilde{g}}{\mu d^{0,\mathrm{lhc}}\mathcal{F}_\ell}$$

$$g^2_{D^*D\pi}/(4F^2) = 9.2\ \mathrm{GeV}^{-2}$$



$$-\frac{2\pi}{\mu}\mathrm{Im} f = \mathrm{Im} T = \mathrm{Im} V_{\rm OPE}(k^2) = g_P \frac{-\pi}{4k^2}\mathcal{F}_\ell, \quad \text{for } k^2 < k^2_{\rm lhc}$$

$$\mathrm{Im}\, n(k^2) = -\tilde{g}\frac{\pi}{4k^2}, \quad \text{for } k^2 < k^2_{\rm lhc}$$

$$d_u^{0,\mathrm{lhc}} = \tilde{d}_0 - \frac{\tilde{d}_1\mu_+^2}{4} + \frac{\mu_+}{2}\left(1 + \frac{\tilde{g}}{\mu_{\rm ex}^2}\right) + \frac{\tilde{g}\log[2/(1+\eta)]}{\mu_+}$$

Amplitude zeros

- The amplitude zeros are also captured in this parametrization
  - Take the leading order, $\tilde{n}(k^2) = 1$, e.g., the [0,1] approximant

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \tilde{g}\left(L(k^2) - L_0\right)} - ik\right]^{-1}$$

  - For a general u-channel exchange, the amplitude has a zero at

$$1 + \tilde{g}\left[L_u\left(k^2_{u,\mathrm{zero}}\right) + \frac{1}{\mu^2_{\mathrm{ex}}}\right] = 0$$

  - For $|m_1 - m_2| \ll m_1 + m_2$, we have $\eta \ll 1$. Neglecting the η terms ⇒ analytic solution also for t-channel exchange

$$\eta^2 \equiv \frac{(m_1 - m_2)^2}{(m1 + m_2)^2}$$

$$k^2_{t,\ \mathrm{zero}} = -\frac{m^2_{\mathrm{ex}}}{4}\left[1 + \frac{1}{y} W\left(-e^{-y} y\right)\right]$$

Here, W: the Lambert W function, $y \equiv 1 + m^2_{\mathrm{ex}}/\tilde{g}$

The zero is from interplay between long-range (OPE) and short-range ($\tilde{d}_0, \tilde{d}_1$ in relation between $\tilde{g}$ and couplings g_P; or a_t and r_t) interactions

$$y = 1 + \frac{1 + \frac{4}{3} a_t m_{\mathrm{ex}}(1 - \log 4) - \frac{4\pi a_t m^2_{\mathrm{ex}}}{\mu g_P \mathcal{F}_\ell}}{2 + a_t m_{\mathrm{ex}}\left(1 - m_{\mathrm{ex}} r_t/4\right)}$$

Summary

- We have derived a generalized ERE, taking into account explicitly the left-hand-cut due to one-particle exchange (either t- or u-channel)

$$\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i k^{2i} - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j k^{2j} + \tilde{g}\left(L(k^2) - L_0\right)} - ik$$

- Should be applicable in a wide range of systems
  - ☐ Hadron physics
  - ☐ Nuclear physics
  - ☐ Cold atom physics?

Thank you for your attention!