# Derivation and determination of nuclear matrix element for neutrinoless double beta decay

Dong-Liang Fang

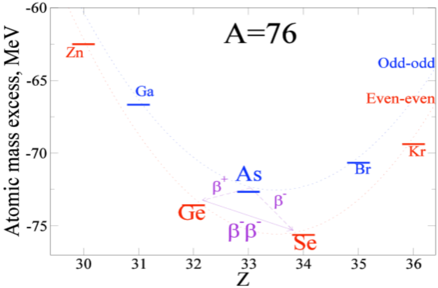
Institute of Modern Physics CAS, Lanzhou

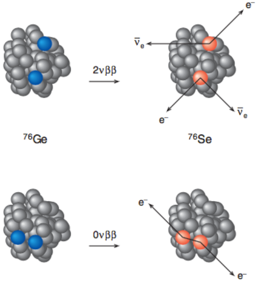
August 21, 2025

# Content

# Why $\beta\beta$-decay

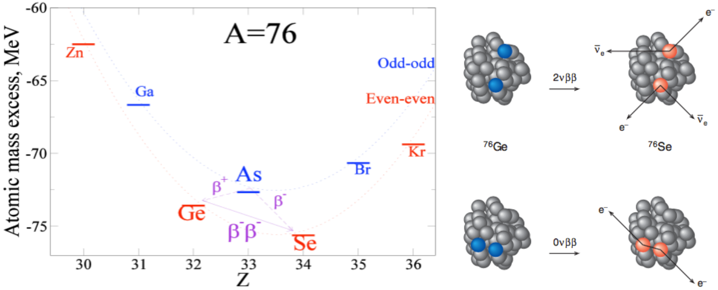
- Strong nuclear pairing in nuclei for neutron-neutron and proton-proton



# Why $\beta\beta$-decay

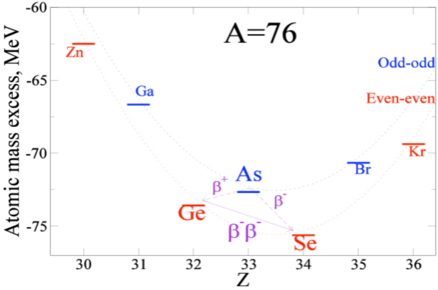
- Strong nuclear pairing in nuclei for neutron-neutron and proton-proton

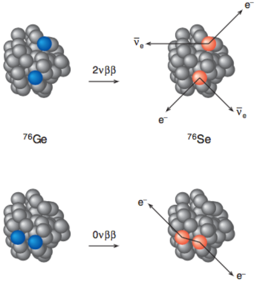



- Double beta ($\beta\beta$) decay is originating from the mass staggering

# Why $\beta\beta$-decay

- Strong nuclear pairing in nuclei for neutron-neutron and proton-proton




- Double beta ($\beta\beta$) decay is originating from the mass staggering
- Neutrinoless $\beta\beta$-decay is possible if $\nu = \bar{\nu}$ and $m_\nu \neq 0$

# Second order process in nucleus

- The decay width of a free particle are obtained with a plane wave:

$$d\Gamma = \frac{1}{2m_i}\prod_f\left(\frac{d^3p_f}{(2\pi)^3 2E_f}\right)|\mathcal{M}(m_i \to \sum_f p_f)|^2(2\pi)^4\delta^4(p_i - \sum_f p_f) \quad (1)$$

- For bound system such as nucleus, we can separate the wave functions into the co-moving and intrinsic coordinates:

$$|I(F)\rangle = \sqrt{2E_{I(F)}}e^{i\vec{q}_{I(F)}\cdot\vec{R}}|i(f)\rangle \quad (2)$$

- Here $|i(f)\rangle$ are nuclear states with finite spins.
- A more general expression for decay width:

$$d\Gamma = \frac{1}{2M_I}\prod_f\frac{d\vec{\mathbf{k}}_f}{(2\pi)^3 2E_f}\frac{d\vec{\mathbf{P}}_F}{(2\pi)^3 2E_F}$$
$$\times\ |\langle\prod_f k_f, F|\frac{1}{2!}T\int d^4x d^4y \mathcal{H}_{\text{int}}(x)\mathcal{H}_{\text{int}}(y)|I\rangle|^2 \quad (3)$$

# Second order process in nucleus

- To separate the intrinsic and co-moving coordinate, on insert $\int d^3R|R\rangle\langle R|$ and redefine $\vec{\mathbf{x}}' = \vec{\mathbf{x}} - \vec{\mathbf{R}}$ for $x$ and $y$. Noticing $\langle F|R\rangle = e^{-i\vec{\mathbf{P}}_F\cdot\vec{\mathbf{R}}}$ etc.
- Therefore after integrating over R and $x^0$, $y^0$, we have:

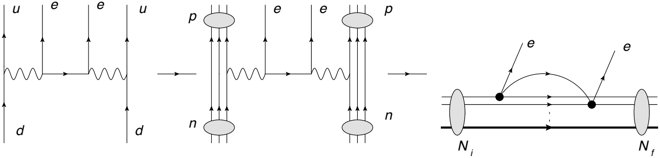
$$
\begin{aligned}
d\Gamma &= \prod_f \frac{d\vec{\mathbf{k}}_f}{(2\pi)^3 2E_f}\frac{d\vec{\mathbf{P}}_F}{(2\pi)^3}(2\pi)^4\delta^4(\sum_f k_f - P_F) \\
&\times |\langle\prod_f k_f, f|\int d^3x' d^3y' \mathcal{H}_{\rm int}(x)\mathcal{H}_{\rm int}(y)\sum_{contr}\frac{1}{\sum E_f(x) - M_I}|i\rangle|^2
\end{aligned}
$$

  the denominator sums over all the possible contractions and all the energies of the particle at the x vertex.
- For nuclear community, one usually redefine the decay width after integrating out the momentum of nucleus as:

$$
d\Gamma = \prod_f \frac{d\vec{\mathbf{k}}_f}{(2\pi)^3}2\pi\delta(\sum_f E_f + E_F - M_I)|R|^2 \qquad (4)
$$

# Background

- The underlying mechanism with L-R symmetry
  - left-handed and right-handed neutrino mixing
  - $SU(2)_L$ and $SU(2)_R$ gauge boson mixing




As an example, we show the derivation of $0\nu\beta\beta$ decay width with the L-R symmetry model (LRSM)

# Background

The guage symmetry of LRSM:

$$SU_L(2) \otimes SU_R(2) \otimes U_{B-L}(1) \tag{5}$$

Fermions are assigned as fundamental representation of $SU(2)$:

$$SU_L(2) : \begin{pmatrix} u_L \\ d_L \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \cdots \quad SU_R(2) : \begin{pmatrix} u_R \\ d_R \end{pmatrix} \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} \cdots \tag{6}$$

After successive symmetry broken:

$$SU_L(2) \otimes SU_R(2) \otimes U_{B-L}(1) \to SU_L(2) \otimes U_Y(1) \to U_{EM}(1) \tag{7}$$

Which lead to the neutrino mass with $N_e = C\bar{\nu}_R^T$:

$$\mathcal{L}^{\nu}_{mass} = \begin{pmatrix} \nu^T & N^T \end{pmatrix} C \begin{pmatrix} M_L & M_D \\ M_D & M_R \end{pmatrix} \begin{pmatrix} \nu \\ N \end{pmatrix} \tag{8}$$

Here neutrino has three generations $\nu^T = \begin{pmatrix} \nu_{eL} & \nu_{\mu L} & \nu_{\tau L} \end{pmatrix}$

# Background

After diagonalization, we could have:

$$\begin{pmatrix} \nu_W \\ N_W \end{pmatrix} = \begin{pmatrix} U & S \\ T & V \end{pmatrix} \begin{pmatrix} \nu \\ N \end{pmatrix} \quad (9)$$

*i.e.* $\nu_{eL} = \sum_j U_{ej}\nu_j + \sum_J S_{eJ}N_J$ is the weak eigenstates
Symmetry Broken also leads to L-R gauge boson mixing:

$$\begin{aligned} W_L &= \cos\xi W_1 - \sin\xi W_2 \\ W_R &= \cos\xi W_2 + \sin\xi W_1 \end{aligned} \quad (10)$$

Where $\xi$ is the mixing angle.

# Background

After diagonalization, we could have:

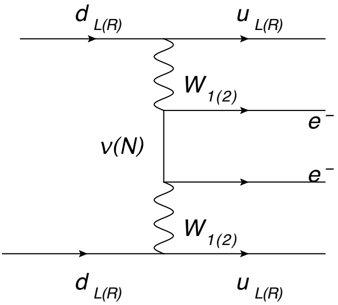
$$\begin{pmatrix} \nu_W \\ N_W \end{pmatrix} = \begin{pmatrix} U & S \\ T & V \end{pmatrix} \begin{pmatrix} \nu \\ N \end{pmatrix} \tag{9}$$

*i.e.* $\nu_{eL} = \sum_j U_{ej}\nu_j + \sum_J S_{eJ}N_J$ is the weak eigenstates

Symmetry Broken also leads to L-R gauge boson mixing:

$$\begin{aligned} W_L &= \cos\xi W_1 - \sin\xi W_2 \\ W_R &= \cos\xi W_2 + \sin\xi W_1 \end{aligned} \tag{10}$$

Where $\xi$ is the mixing angle.
This would lead to more $0\nu\beta\beta$ diagrams.



# Background

M. Doi *et. al.* Prog. Theo. Phys. Suppl. 83,1(1985)

- Hamiltonian for interactions from LRSM relevant to $0\nu\beta\beta$:

$$H_{\rm int} = \frac{G_F \cos\theta_C}{\sqrt{2}}(J_L^\mu j_{L\mu} + \kappa J_L^\mu j_{R\mu} + \eta J_R^\mu j_{L\mu} + \lambda J_R^\mu j_{R\mu}) \tag{11}$$

  Here $\kappa = \eta \approx \tan\zeta$ and $\lambda \approx (\frac{M_{W1}}{M_{W2}})^2$, suggesting that the latter three terms are suppressed.

- Besides this, we have also six fermion interactions coming from Yukawa couplings with Triplet Higgs bosons:

$$H_{\rm int} = \sum_{l_1,l_2 l_3} C_{l_1,l_2,l_3} j_{l_1} J_{l_2}^\mu J_{l_3\mu} \tag{12}$$

  Here $l_1$, $l_2$ and $l_3$ could be either L or R, and the coefficients C's are usually suppressed by triplet higgs mass, we usually neglect their contributions

# Nuclear currents

- The nuclear current has the form by inserting the intermediate states

$$J_{IJ}^{\rho\sigma} = \langle f|J_{WI}^{\rho}|m\rangle\langle m|J_{WJ}^{\sigma}|i\rangle$$

- Here I and J could either be L or R, and the left-handed and right-handed weak current of quark has the form:

$$\begin{aligned} J_L^{\mu} &= \bar{u}(1-\gamma_5)\gamma^{\mu}d \\ J_R^{\mu} &= \bar{u}(1+\gamma_5)\gamma^{\mu}d \end{aligned} \qquad (13)$$

- At the nucleon level, the current may be more complicated with induced components:

$$J_{L(R)}^{\mu} = g_V(q^2)\gamma^{\mu} - ig_M(q^2)\frac{\sigma^{\mu\nu}}{2m_p}q_{\nu} \mp g_A(q^2)\gamma^{\mu}\gamma_5 \pm g_P(q^2)q^{\nu}\gamma_5 \qquad (14)$$

- Here $g_V(0) = 1$ and $g_A(0) = 1.27$ and the form factors are generally assumed to be dipole form: $g(q^2) = g/(1+q^2/\Lambda^2)^2$

- The weak current of lepton can be written as:

$$j^{\mu}_{L(R)} = \bar{e}\gamma^{\mu}(1 \mp \gamma_5)\nu \tag{15}$$

Therefore the double electron emission + neutrino propagator has the form:

$$\begin{aligned} j^{\mu}_{L(R)}(\vec{\mathbf{x}})j^{\nu}_{L(R)}(\vec{\mathbf{y}}) &= \bar{e}\gamma^{\mu}(1 \mp \gamma_5)N_i\bar{e}\gamma^{\nu}(1 \mp \gamma_5)N_j \\ &= -\bar{e}\gamma^{\mu}(1 \mp \gamma_5)N_iN_j^T(1 \mp \gamma_5^T)\gamma^{\nu\,T}\bar{e}^T \\ &= -i\delta_{ij}\int \frac{d^4q}{2\pi^4}\frac{e^{iq(x-y)}}{q^2 - m_i^2} \\ &\times \bar{e}\gamma^{\mu}(1 \mp \gamma_5)(\gamma_\rho q_\rho + m_i)(1 \mp \gamma_5)\gamma^{\nu}\bar{e}^C \end{aligned} \tag{16}$$

- Be aware of the properties of $\gamma$-matrices, $(1-\gamma_5)(1+\gamma_5) = 0$ and $(1-\gamma_5)\gamma_\rho(1-\gamma_5) = 0$,
- We have two types of terms, namely the mass terms(same chirality on both sides of the propagator) and $V + A$ terms (different chirality on the two sides of the propagator)

# Lepton currents

- After contracting with external lepton legs and integrating over $q^0$, we can obtain the two lepton terms with the form

$$
\begin{aligned}
S_{L\rho\sigma}(\vec{\mathbf{x}}, \vec{\mathbf{y}}; a) &= \frac{\bar{e}(\epsilon_1, \vec{\mathbf{x}})\gamma_\rho(1-\gamma_5)\gamma_\sigma e^C(\epsilon_2, \vec{\mathbf{y}})}{\omega + E_m + (\epsilon_2 - \epsilon_1)/2} \\
&- \frac{\bar{e}(\epsilon_2, \vec{\mathbf{x}})\gamma_\rho(1-\gamma_5)\gamma_\sigma e^C(\epsilon_1, \vec{\mathbf{y}})}{\omega + E_m + (\epsilon_1 - \epsilon_2)/2} \\
V_{L\rho\sigma}(\vec{\mathbf{x}}, \vec{\mathbf{y}}; a) &= \frac{q^\mu \bar{e}(\epsilon_1, \vec{\mathbf{x}})\gamma_\rho(1-\gamma_5)\gamma_\mu\gamma_\sigma e^C(\epsilon_2, \vec{\mathbf{y}})}{\omega + E_m + (\epsilon_2 - \epsilon_1)/2} \\
&- \frac{q^\mu \bar{e}(\epsilon_2, \vec{\mathbf{x}})\gamma_\rho(1-\gamma_5)\gamma_\mu\gamma_\sigma e^C(\epsilon_1, \vec{\mathbf{y}})}{\omega + E_m + (\epsilon_1 - \epsilon_2)/2}
\end{aligned}
$$

- Here $\omega = \sqrt{\vec{\mathbf{q}}^2 + m_j^2}$ is the neutrino energy and for light neutrino $\omega \approx |\vec{\mathbf{q}}|$
- $e(\epsilon, \vec{\mathbf{x}})$ is Coulomb distorted electron wave function and can be obtained by the solution of Dirac equations

- The general decay width of $0\nu\beta\beta$ decay can be written as below following the S-matrix theory as we have shown:

$$d\Gamma = 2\pi \sum_{spin} |R|^2 \delta(\epsilon_1 + \epsilon_2 + E_f - M_i) \frac{d\vec{\mathbf{p}}_1}{(2\pi)^3} \frac{d\vec{\mathbf{p}}_2}{(2\pi)^3} \quad (17)$$

- Here the R-matrix can be written as follows for general LRSM (we focus on the light neutrino mediated mechanism)

$$\begin{aligned} R &= (\frac{G_F \cos\theta_C}{\sqrt{2}})^2 \sum_j \int d\vec{\mathbf{x}} \int d\vec{\mathbf{y}} \int \frac{d\vec{\mathbf{q}}}{2\omega(2\pi)^3} e^{i\vec{\mathbf{q}}\cdot(\vec{\mathbf{x}}-\vec{\mathbf{y}})} \\ &\times \sum_a [(J^{\rho\sigma}_{LL} S_{L\rho\sigma} + J^{\rho\sigma}_{RR} S_{R\rho\sigma}) + (J^{\rho\sigma}_{LR} V_{L\rho\sigma} + J^{\rho\sigma}_{RL} V_{R\rho\sigma})] \quad (18) \end{aligned}$$

# Electron wave functions

- The electron wave function can be obtained from solutions of Dirac equations:

$$H\Psi = (\vec{\alpha} \cdot \vec{\mathbf{p}} - \beta - V)\Psi = W\Psi \tag{19}$$

For central field, we have a polar form:

$$H\Psi = [i\gamma_5\sigma_r(\frac{\partial}{\partial r} - \frac{1}{r} - \beta K) + V(r) + \beta]\Psi \tag{20}$$

$K = \vec{\sigma} \cdot \vec{\mathbf{l}} + 1$ commute with H and its eigenvalues are $\kappa = -|j| - 1/2$ for $j = l + 1/2$ and $\kappa = |j| + 1/2$ for $j = l - 1/2$.

- The solution are with the general form $\Psi^T = a_{\kappa\mu}(g_\kappa \chi^T_{\kappa\mu}, if_\kappa \chi^T_{-\kappa\mu})$

$$\begin{aligned} \frac{df}{dr} &= \frac{\kappa - 1}{r} f - (W - 1 - V)g \\ \frac{dg}{dr} &= (W - V + 1)f - \frac{\kappa + 1}{r} g \end{aligned} \tag{21}$$

$a_{k\mu}$ is determined by matching the plane wave solution at infinity

# Background

So the electron wave function can be expressed as:

$$\Psi(\epsilon, \vec{\mathbf{x}}) = \Psi^{(S)}(\epsilon, \vec{\mathbf{x}}) + \Psi^{(P)}(\epsilon, \vec{\mathbf{x}}) + \cdots \tag{22}$$

The s-wave has the form:

$$\Psi^{(S)}(\epsilon, \vec{\mathbf{x}}) = \begin{pmatrix} \tilde{g}_{-1}\chi_s \\ (\sigma \cdot \hat{p})\tilde{f}_1\chi_s \end{pmatrix} \tag{23}$$

For $S - S$ electrons, we could have for example:

$$\begin{aligned} & \bar{e}(\epsilon_1, \vec{\mathbf{x}})\gamma_\mu(1-\gamma_5)\gamma_\nu e^C(\epsilon_1, \vec{\mathbf{x}}) \\ = & \begin{pmatrix} \tilde{g}^*_{-1}\chi^\dagger_s & (\sigma \cdot \hat{p})^\dagger \tilde{f}^*_1 \chi^\dagger_s \end{pmatrix} \gamma_0\gamma_\mu(1-\gamma_5)\gamma_\nu i\gamma_2 \begin{pmatrix} \tilde{g}^*_{-1}\chi^*_s \\ (\sigma \cdot \hat{p})^* \tilde{f}^*_1 \chi^*_s \end{pmatrix} \end{aligned} \tag{24}$$

For decay to ground states, only $\gamma_0\gamma_0$ term or $[\gamma_i \otimes \gamma_j]^0$.
Besides, we have also contributions from $S - P$ electrons

# Background

- For derivation of the final decay width, several assumptions are used:
- **long wavelength approximation**: $e^S(\vec{\mathbf{x}}) = e^S(R)$, $e^L(\mathbf{x}) \sim (kx)^L, \cdots$
- **equal energy approximation**: $\epsilon_1 \approx \epsilon_2$
- Under such assumption, for example, for the mass mechanism:

$$
\begin{aligned}
|R|^2 &\approx |\bar{e}^S(\epsilon_1, R)\gamma_\mu(1-\gamma_5)\gamma_\nu e^{SC}(\epsilon_2, R)|^2 \\
&\times |\sum_j U_{ej}^2 m_j \sum_m \int d\vec{\mathbf{x}} d\vec{\mathbf{y}} \int d\vec{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{\omega_j(\omega_j + A_m)} \langle f|J_L^\mu|m\rangle\langle m|J_L^\nu|i\rangle|^2 \\
&= |\bar{e}^S(\epsilon_1, R)(1+\gamma_5)e^{SC}(\epsilon_2, R)|^2 \\
&\times |\sum_j U_{ej}^2 m_j \sum_m \int d\vec{\mathbf{x}} d\vec{\mathbf{y}} \int d\vec{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{\omega_j(\omega_j + A_m)} \langle f|J_L^\mu|m\rangle\langle m|J_{L\mu}|i\rangle|^2
\end{aligned}
$$

# Background

- For derivation of the final decay width, several assumptions are used:
- **long wavelength approximation**: $e^S(\vec{\mathbf{x}}) = e^S(R)$, $e^L(\mathbf{x}) \sim (kx)^L, \cdots$
- **equal energy approximation**: $\epsilon_1 \approx \epsilon_2$
- Under such assumption, for example, for the mass mechanism:

$$
\begin{aligned}
|R|^2 \approx & \; |\bar{e}^S(\epsilon_1, R)\gamma_\mu(1-\gamma_5)\gamma_\nu e^{SC}(\epsilon_2, R)|^2 \\
& \times \; |\sum_j U_{ej}^2 m_j \sum_m \int d\vec{\mathbf{x}} d\vec{\mathbf{y}} \int d\vec{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{\omega_j(\omega_j + A_m)} \langle f|J_L^\mu|m\rangle\langle m|J_L^\nu|i\rangle|^2 \\
& = \; |\bar{e}^S(\epsilon_1, R)(1+\gamma_5)e^{SC}(\epsilon_2, R)|^2 \\
& \times \; |\sum_j U_{ej}^2 m_j \sum_m \int d\vec{\mathbf{x}} d\vec{\mathbf{y}} \int d\vec{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{\omega_j(\omega_j + A_m)} \langle f|J_L^\mu|m\rangle\langle m|J_{L\mu}|i\rangle|^2
\end{aligned}
$$

- For decay to ground states, this then can be written as:

$$
\sum_{spin} |R|^2 = (f_{11}^0 + f_{11}^1 \cos\theta)|m_{\beta\beta}^\nu|^2|M|^2 \quad (25)
$$

- $\cos\theta$ term will not contribute to the total decay rate but is important

# phase space factor

- Here $f$'s are functionals of electron wave functions:

$$\begin{aligned} f_{11}^{0} &= |g_{-1}(k_1R)g_{-1}(k_2R)|^2 + |f_1(k_1R)f_1(k_2R)|^2 \\ &+ |g_{-1}(k_1R)f_1(k_2R)|^2 + |f_1(k_1R)g_{-1}(k_2R)|^2 \\ f_{11}^{1} &= -2Re[g_{-1}(k_1R)g_{-1}(k_2R)(f_1(k_1R)f_1(k_2R))^* \\ &\quad +g_{-1}(k_1R)f_1(k_2R)(f_1(k_1R)g_{-1}(k_2R))]^* \end{aligned} \tag{26}$$

- After integration over electron momenta,

$$G^{01} = C\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(f_{11}^{0} + f_{11}^{1}\cos\theta) \tag{27}$$

- the decay width are well separated into three parts:

$$\Gamma = G^{01}(m_{\beta\beta}^{\nu})^2|M|^2 \tag{28}$$

- Such formalism works for different mechanism with different decay opeartors

$$\Gamma_{\rm tot} = \sum_{ij} Re(C_iC_j)G_{ij}M_iM_j \tag{29}$$

# Background

- A complete expression for LRSM:

$$\begin{aligned}\Gamma^{0\nu}(0^+ \to 0^+) &= G^{01}(\langle m_\nu \rangle M_l + \langle \eta_N \rangle M_h)^2 + \langle \lambda \rangle^2 (G^{02} M^2_{\omega^-} \\ &+ G^{011} M^2_{q^+} - 2G^{010} M_{\omega^-} M_{q^+}) \\ &+ \langle \eta \rangle^2 (G^{02} M^2_{\omega^+} + G^{011} M^2_{q^-} - 2G^{010} M_{\omega^+} M_{q^-} \\ &+ G^{08} M^2_P + G^{09} M^2_R - G^{07} M_P M_R) \\ &+ \cdots \end{aligned} \tag{30}$$

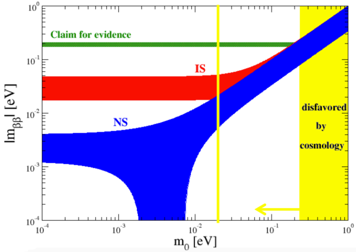
$\langle m_\nu \rangle = |\sum_j (U_{ej})^2 m_j|$,
$\langle \eta_N \rangle = |\sum_J \frac{(S_{eJ})^2 m_p}{M_J}|$,
$\langle \lambda \rangle = |\tan\xi \sum_j U_{ej} T^*_{ej} (g'_V / g_V)|$,
$\langle \eta \rangle = |(M_{W1}/M_{W2})^2 \sum_j U_{ej} T^*_{ej}|$
are new physics parameters

## Background

- A complete expression for LRSM:

$$\begin{aligned}\Gamma^{0\nu}(0^+ \to 0^+) &= G^{01}(\langle m_\nu\rangle M_l + \langle\eta_N\rangle M_h)^2 + \langle\lambda\rangle^2(G^{02}M^2_{\omega^-} \\ &+ G^{011}M^2_{q^+} - 2G^{010}M_{\omega^-}M_{q^+}) \\ &+ \langle\eta\rangle^2(G^{02}M^2_{\omega^+} + G^{011}M^2_{q^-} - 2G^{010}M_{\omega^+}M_{q^-} \\ &+ G^{08}M_P^2 + G^{09}M_R^2 - G^{07}M_P M_R) \\ &+ \cdots \end{aligned} \quad (30)$$

$\langle m_\nu\rangle = |\sum_j (U_{ej})^2 m_j|$,
$\langle\eta_N\rangle = |\sum_J \frac{(S_{eJ})^2 m_p}{M_J}|$,
$\langle\lambda\rangle = |\tan\xi \sum_j U_{ej}T^*_{ej}(g'_V/g_V)|$,
$\langle\eta\rangle = |(M_{W1}/M_{W2})^2 \sum_j U_{ej}T^*_{ej}|$
are new physics parameters



# Background

- In above expression, we first consider the neutrino mass mechanism

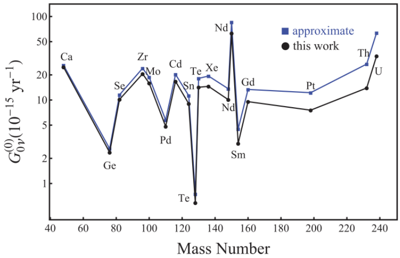
$$\Gamma^{0\nu} = |\langle m_\nu \rangle|^2 G^{01} M_m^2 \tag{31}$$

# Background

- In above expression, we first consider the neutrino mass mechanism

$$\Gamma^{0\nu} = |\langle m_\nu \rangle|^2 G^{01} M_m^2 \qquad (31)$$

Calculations of PSFs using numerical electron wave functions (Kotila *et al.* PRC85,034316(2012))




| Nucleus | $G_{0\nu}^{(0)}$ ($10^{-15}$ yr$^{-1}$) | $G_{0\nu}^{(1)}$ ($10^{-15}$ yr$^{-1}$) | $Q_{\beta\beta}$ (MeV) |
|---|---|---|---|
| $^{48}$Ca | 24.81 | −23.09 | 4.27226(404) |
| $^{76}$Ge | 2.363 | −1.954 | 2.03904(16) |
| $^{82}$Se | 10.16 | −9.074 | 2.99512(201) |
| $^{96}$Zr | 20.58 | −18.67 | 3.35037(289) |
| $^{100}$Mo | 15.92 | −14.25 | 3.03440(17) |
| $^{110}$Pd | 4.815 | −4.017 | 2.01785(64) |
| $^{116}$Cd | 16.70 | −14.83 | 2.81350(13) |
| $^{124}$Sn | 9.040 | −7.765 | 2.28697(153) |
| $^{128}$Te | 0.5878 | −0.3910 | 0.86587(131) |
| $^{130}$Te | 14.22 | −12.45 | 2.52697(23) |
| $^{136}$Xe | 14.58 | −12.73 | 2.45783(37) |
| $^{148}$Nd | 10.10 | −8.506 | 1.92875(192) |
| $^{150}$Nd | 63.03 | −57.76 | 3.37138(20) |
| $^{154}$Sm | 3.015 | −2.295 | 1.21503(125) |
| $^{160}$Gd | 9.559 | −7.932 | 1.72969(126) |
| $^{198}$Pt | 7.556 | −5.868 | 1.04717(311) |
| $^{232}$Th | 13.93 | −10.95 | 0.84215(246) |
| $^{238}$U | 33.61 | −28.13 | 1.14498(125) |

# Formalism

Calculation of the nuclear part (NME) depends on the nuclear structure theory. Modern nuclear structure calculations face two obstacles:

- **many-body methods**
  - exact Configuration Interaction approaches
  - approximate approaches with Configuration truncations: QRPA, DFT, IBM, $\cdots$

# Formalism

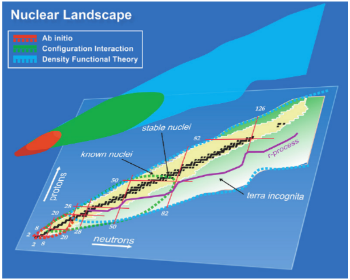
Calculation of the nuclear part (NME) depends on the nuclear structure theory. Modern nuclear structure calculations face two obstacles:

- **many-body methods**
  - exact Configuration Interaction approaches
  - approximate approaches with Configuration truncations: QRPA, DFT, IBM, $\cdots$
- **nuclear force**
  - *ab initio*:
    - phenomenological realistic forces
    - Chiral forces
  - Effective interactions:
    - Skyrme, Gogny, Relativistic mean fields,$\cdots$

# Formalism

Calculation of the nuclear part (NME) depends on the nuclear structure theory. Modern nuclear structure calculations face two obstacles:

- **many-body methods**
  - exact Configuration Interaction approaches
  - approximate approaches with Configuration truncations: QRPA, DFT, IBM, $\cdots$
- **nuclear force**
  - *ab initio*:
    - phenomenological realistic forces
    - Chiral forces
  - Effective interactions:
    - Skyrme, Gogny, Relativistic mean fields,$\cdots$



# Formalism

QRPA methods based on G-matrix with CD-Bonn potential

- WS meanfield + pairing and residual interactions from G-matrix
- deformation of nuclei is taken into consideration

Pros:

- QRPA is capable of dealing with intermediates states
- Closure approximation is used for other approaches

Cons:

- only one phonon excitations are considered
- meanfield interactions and residual interactions are of different types

NSM method starts from G-matrix but fitted by nuclear properties

Pros:

- exact solutions for many-body problem
- all possible excitations included

Cons:

- large computation burden and only applicable for limited nuclei
- usually severe model space truncation leads to uncontrolled errors

# Many-body calculations

- The nuclear many body wave functions can be written as a Slater determinant which fulfills the permutation symmetry of Fermions:

$$\phi(x_1, ..., x_A) = \frac{1}{A!} \begin{vmatrix} \phi_1(x_1) & ... & \phi_A(x_A) \\ ... & & ... \\ \phi_1(x_A) & ... & \phi_A(x_1) \end{vmatrix} \tag{32}$$

- or equivalently in second-quantized form:

$$|\phi\rangle = \prod_i c_i^\dagger |0\rangle \tag{33}$$

- Usually the operator can be written as:

$$\begin{aligned} O_{1b} &= \sum_{\tau_1\tau_2} \langle\tau_1|\mathcal{O}_i|\tau_2\rangle c_{\tau_1}^\dagger c_{\tau_2} \\ O_{2b} &= \sum_{\tau_1\tau_2\tau_3\tau_4} \langle\tau_1\tau_2|\mathcal{O}_i|\tau_3\tau_4\rangle c_{\tau_1}^\dagger c_{\tau_2}^\dagger c_{\tau_4} c_{\tau_3} \\ & ... \end{aligned} \tag{34}$$

# Many-body calculations

- Usually the single particle wave functions are expanded on certain basis, *e.g.* Harmonic oscillator basis

$$\phi_i(\vec{x}) = \sum_k C_{ik}\phi_k(x) \tag{35}$$

- in actual calculations, one first calculate the so-called reduced density from the wave functions

$$\langle J_f||[c_k^\dagger \tilde{c}_{k'}]_J||J_i\rangle = \sum_{\tau_1\tau_2} C_{\tau_1 k} C_{\tau_2 k'} \langle J_f||[c_{\tau_1}^\dagger \tilde{c}_{\tau_2}]_J||J_i\rangle \tag{36}$$

- Here the Wigner-Eckart theorem is used

$$\langle J_f m_f|[c^\dagger c]_{Jm}|J_i m_i\rangle = \frac{(-1)^{j_i - m_i} C^{Jm}_{J_f m_f J_i - m_i} \langle J_f||[c^\dagger c]_J||J_i\rangle}{\sqrt{2J+1}} \tag{37}$$

- Besides the reduced one body density, we have also two body, three body, ..., density defined in a similar way

# Many-body calculations

- Therefore, using the reduced densities, we could calculate the nuclear transition amplitude:

$$\mathcal{A}_{1b} = \sum_{m_i m_f m} \langle J_f m_f | [c^\dagger_{\tau_1} \tilde{c}_{\tau_2}]_{Jm} | J_i m_i \rangle \langle \tau_1 | \mathcal{O}_{Jm} | \tau_2 \rangle$$

$$= \frac{\langle J_f || [c^\dagger_k \tilde{c}_{k'}]_J || J_i \rangle}{\sqrt{2J+1}} \langle k || \mathcal{O}_J || k' \rangle \qquad (38)$$

- This formalism can actually be used in various occasions, if $\tau_1, \tau_2$ are with the same species, this is charge conserving transition (e.g. electromagnetic transition), otherwise charge exchange transition (e.g. $\beta$-decay)
- Similar expressions can be obtained for two-, three-, ..., body transitions.

# Nuclear current operator

The induced weak current under the impulse approximation:

$$J^{\mu}_{L(R)} = \sum_{1}^{A} \tau^{+}[g^{\mu 0} g_V(q^2) \pm g^{\mu j}(g_A(q^2)\sigma_j \pm i g_M(q^2)\frac{(\sigma \times \vec{\mathbf{q}})_j}{2m_p} - g_P(q^2)\frac{q_j \vec{\sigma}\cdot\vec{\mathbf{q}}}{2m_p})]\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_n) \tag{39}$$

The non-relativistic reduction is needed for a non-relativistic system, time component is a scalar and spatial component is a 3-vector

$$J_{L(R),0}(\mathbf{r}) = \sum_{n} g_V(q^2)\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_n)$$

$$J_{L(R),i}(\mathbf{r}) = \mp[g_A(q^2)\sigma_i - \frac{q_j \vec{\sigma}\cdot\vec{\mathbf{q}}}{2m_p} + i g_M(q^2)\frac{(\sigma \times \vec{\mathbf{q}})_i}{2m_p}]\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_n) \tag{40}$$

# NME

Leads to:

$$\begin{aligned} M &= \sum_m \int d\vec{\mathbf{x}} d\vec{\mathbf{y}} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{\omega_j(\omega_j + A)} \langle f|J_L^\mu|m\rangle\langle m|J_{L\mu}|i\rangle \\ &= \sum_m \int d\vec{\mathbf{x}} d\vec{\mathbf{y}} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{\omega_j(\omega_j + A)} (\langle f|J_{L0}|m\rangle\langle m|J_{L0}|i\rangle - \sum_i \langle f|J_{Li}|m\rangle\langle m|J_{Li}|i\rangle) \end{aligned} \tag{41}$$

Substitute the detailed forms of $J_\mu$ into above formula, we obtain:

$$M = \langle H_F(r) + H_{GT}(r)\sigma_1 \cdot \sigma_2 + H_T(r)(\sigma_1 \otimes \sigma_2)^2 : (\vec{\mathbf{q}} \otimes \vec{\mathbf{q}})^2 \rangle$$

here the most important part is the "neutrino potential":

$$H_{iGT}(r) = \frac{2R}{\pi} \int_0^\infty j_0(qr) h_i(q^2) dq \tag{42}$$

$i$ could be AA, AP, PP and MM

Short-range correlation functions are usually multiplied

$$f(r) = c(1 - be^{-ar^2}) \tag{43}$$

# NME calculation details

particle-particle vs. particle-hole channel

pp (most approaches):

- calculations in particle-particle channel adopts the closure approximation

$$\sum_m \frac{|m\rangle\langle m|}{\omega + A_m} \approx \frac{1}{\omega + \tilde{A}} \tag{44}$$

- The NME can be expressed by the sum of two-body density

$$M = \sum_{pp'nn',J} \langle 0_f^+||[pp']_J[nn']_J||0_i^+\rangle\langle pp'J||\mathcal{O}(\tilde{A})||nn'J\rangle \tag{45}$$

ph (mostly QRPA):

- The intermediate states are accounted explicitly

$$M = \sum_{pp'nn'} \sum_{Jm} \frac{\langle 0_f^+|||[c_p^\dagger \tilde{c}_n]_J||Jm\rangle\langle Jm||[c_p^\dagger \tilde{c}_n]_J||0_i^+\rangle}{2J+1} \langle p||\mathcal{O}_f||n\rangle\langle p'||\mathcal{O}_i||n'\rangle$$

The two body operator and the two one body operators are connected by a specific transformation

# Results

**DLF** *et al.* Phys. Rev. C97,045503(2018)

| | | AV18 | | | | | | | | CD Bonn | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $g_A = g_{A0}$ | | | | $g_A = 0.75g_{A0}$ | | | | $g_A = g_{A0}$ | | | | $g_A = 0.75g_{A0}$ | | | |
| | | $M_F^{0\nu}$ | $M_{GT}^{0\nu}$ | $M_T^{0\nu}$ | $M_l^{\prime 0\nu}$ | $M_{F,l}^{0\nu}$ | $M_{GT,l}^{0\nu}$ | $M_{T,l}^{0\nu}$ | $M_l^{\prime 0\nu}$ | $M_F^{0\nu}$ | $M_{GT}^{0\nu}$ | $M_T^{0\nu}$ | $M_l^{\prime 0\nu}$ | $M_{F,l}^{0\nu}$ | $M_{GT,l}^{0\nu}$ | $M_{T,l}^{0\nu}$ | $M_l^{\prime 0\nu}$ |
| $^{76}Ge \rightarrow ^{76}Se$ | a | -1.09 | 3.11 | -0.44 | 3.34 | -1.09 | 3.94 | -0.46 | 2.63 | -1.10 | 2.99 | -0.40 | 3.27 | -1.09 | 3.90 | -0.42 | 2.64 |
| | b | -1.06 | 2.92 | -0.45 | 3.12 | -1.06 | 3.70 | -0.47 | 2.48 | -1.15 | 3.09 | -0.41 | 3.40 | -1.15 | 4.00 | -0.43 | 2.72 |
| $^{82}Se \rightarrow ^{82}Kr$ | a | -1.00 | 2.86 | -0.41 | 3.07 | -1.00 | 3.61 | -0.43 | 2.41 | -1.00 | 2.76 | -0.37 | 3.01 | -1.00 | 3.58 | -0.42 | 2.41 |
| | b | -0.98 | 2.68 | -0.42 | 2.86 | -0.97 | 3.39 | -0.38 | 2.26 | -1.05 | 2.85 | -0.38 | 3.13 | -1.05 | 3.67 | -0.39 | 2.49 |
| $^{130}Te \rightarrow ^{130}Xe$ | a | -1.17 | 2.95 | -0.52 | 3.16 | -1.16 | 3.37 | -0.55 | 2.31 | -1.15 | 2.85 | -0.46 | 3.10 | -1.15 | 3.29 | -0.49 | 2.29 |
| | b | -1.13 | 2.73 | -0.53 | 2.90 | -1.13 | 3.11 | -0.56 | 2.13 | -1.21 | 2.95 | -0.47 | 3.22 | -1.21 | 3.38 | -0.50 | 2.37 |
| $^{136}Xe \rightarrow ^{136}Ba$ | a | -0.37 | 1.12 | -0.17 | 1.18 | -0.37 | 1.39 | -0.17 | 0.91 | -0.33 | 1.05 | -0.13 | 1.12 | -0.33 | 1.29 | -0.14 | 0.85 |
| | b | -0.36 | 1.06 | -0.17 | 1.11 | -0.36 | 1.31 | -0.17 | 0.86 | -0.35 | 1.10 | -0.14 | 1.18 | -0.35 | 1.33 | -0.14 | 0.89 |
| $^{150}Nd \rightarrow ^{150}Sm$ | a | -1.35 | 2.98 | -0.53 | 3.28 | -1.35 | 3.54 | -0.56 | 2.52 | -1.36 | 2.89 | -0.45 | 3.28 | -1.37 | 3.45 | -0.52 | 2.50 |
| | b | -1.32 | 2.74 | -0.55 | 3.01 | -1.31 | 3.26 | -0.57 | 2.33 | -1.43 | 3.00 | -0.46 | 3.43 | -1.43 | 3.55 | -0.53 | 2.59 |

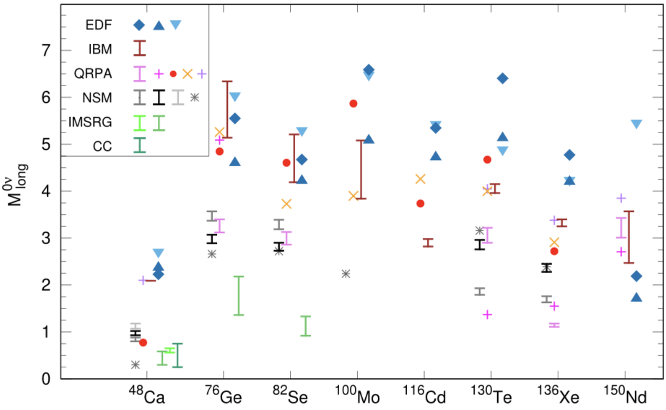
$M_F = -\frac{1}{3} M_{GT}$ approximately hold, Tensor component is about 1/10 and its contribution is at sub leading order

Uncertainties of the calculations

- nuclear force, quenching of $g_A$ in nuclei, SRC

# Results

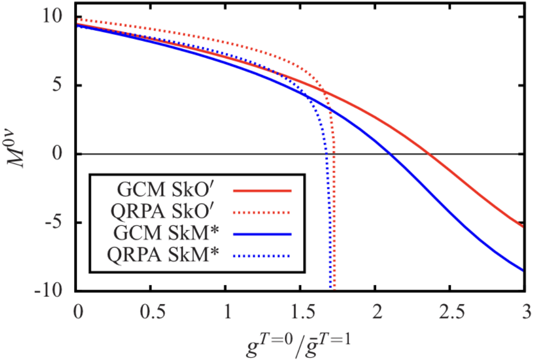
Agostini *et al.* Rev. Mod. Phys. 95, 025002(2023)




Deviations between different calculations are still large

# Origins of deviations

The deviations of different many calculations may come from different factors, either from the nuclear force or many-body correlations. Some can be qualitatively discussed.



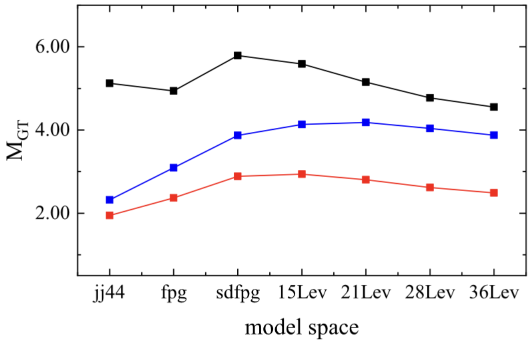

For example, the lack of isoscalar pairing in EDF calculations leads to overestimation of the NME

Hinohara *et al.*, Phys.Rev.C90, 031301(R)(2014)

# Comparative study

Brown *et al.* Phys. Rev. C91, 041301(R)(2015)

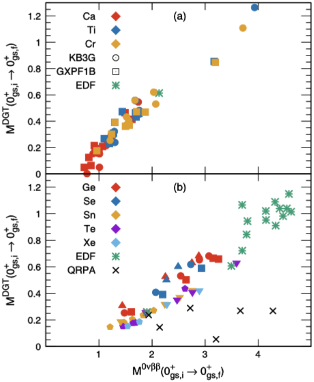
- The most important error for NSM comes from model space truncation




- Internal errors may come from the choice of Hamiltonian, closure approximation
- External errors from src, $g_A$ quenching, etc.
- $M^{0\nu}(^{76}Ge) = [2.6(3)][0.89(4)][1.01(3)][1.28(3)] = 3.0(4)$

# double GT

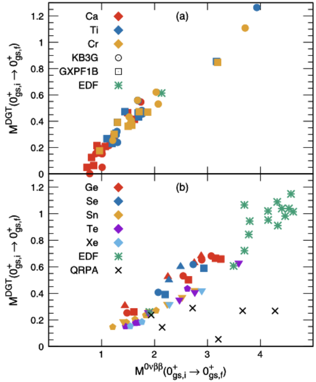
- For shell model calculations, one finds the correlation between double GT transition strength and $0\nu\beta\beta$ NME
- They suggest that it comes from a similar radial dependence of the two transition operator   Shumizu et al. Phys. Rev. Lett. 120. 142502(2018)

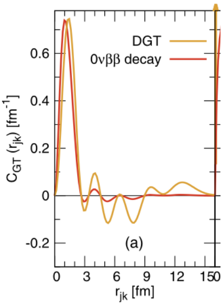


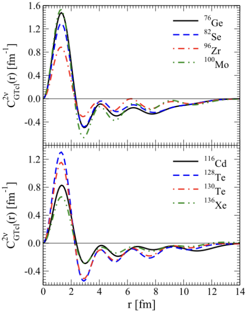
# double GT

- For shell model calculations, one finds the correlation between double GT transition strength and $0\nu\beta\beta$ NME
- They suggest that it comes from a similar radial dependence of the two transition operator

Shumizu et al. Phys. Rev. Lett. 120. 142502(2018)

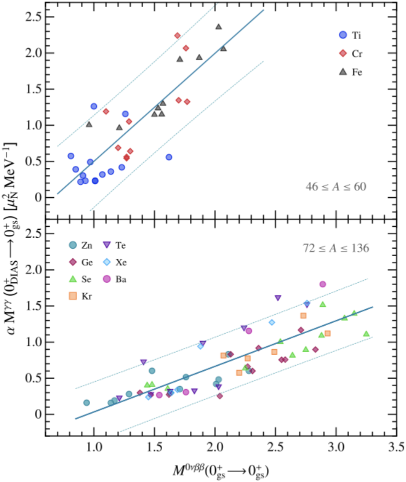


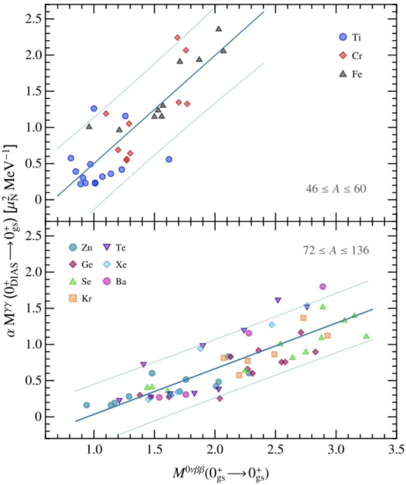
# $\gamma\gamma$-decay

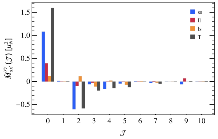
- Similar conclusion has been drawn for the correlation between $\gamma\gamma$-decay NME and $0\nu\beta\beta$ NME Romeo et al. Phys. Lett. B827, 136985(2022)

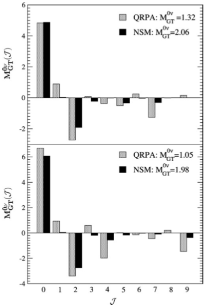


# $\gamma\gamma$-decay

- Similar conclusion has been drawn for the correlation between $\gamma\gamma$-decay NME and $0\nu\beta\beta$ NME Romeo et al. Phys. Lett. B827, 136985(2022)

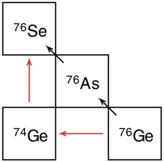


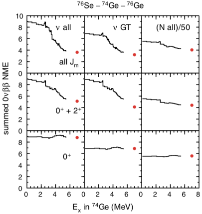



# two nucleon removal reaction

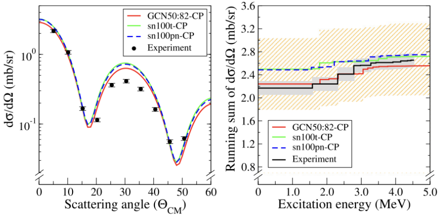
- Making use of the $0^+$ pair dominance for $0\nu\beta\beta$-NME
- two nucleon transfer reaction to the ground states could constrain the NME




Brown et al. Phys.Rev.Lett.113, 262501(2014)




Rebeiro et al. Phys.Lett.B809, 135702(2020)




- Pioneer work has been done for $^{138}$Ba, which has a similar sturcture as $^{136}$Ba
- Decent agreement between experiments and calculations is achieved

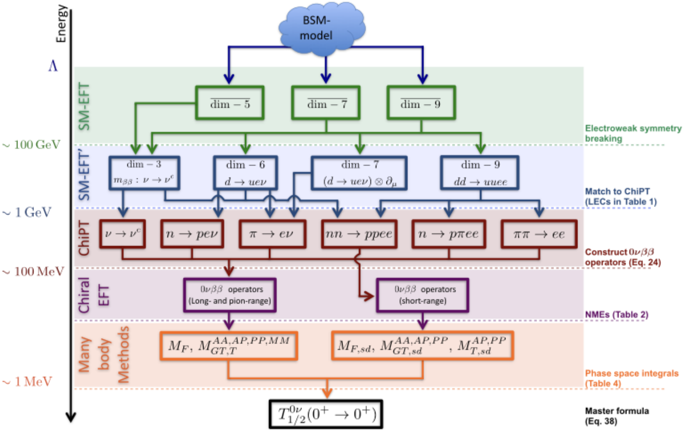
# Other NMEs

| | $^{76}$Ge | | | $^{82}$Se | | |
|---|---|---|---|---|---|---|
| | AV18 | cd-Bonn | w/o | AV18 | cd-Bonn | w/o |
| $M_F$ | -1.482 | -1.600 | -1.522 | -1.360 | -1.463 | -1.390 |
| $M_{GT}$ | 4.667 | 5.169 | 5.024 | 4.051 | 4.491 | 4.353 |
| $M_T$ | -0.775 | -0.774 | -0.752 | -0.730 | -0.728 | -0.709 |
| $M_{\omega F}$ | -1.458 | -1.571 | -1.499 | -1.333 | -1.432 | -1.365 |
| $M_{\omega GT+}$ | 4.604 | 5.087 | 4.961 | 4.041 | 4.462 | 4.342 |
| $M_{\omega GT-}$ | 3.607 | 3.868 | 3.627 | 3.156 | 3.383 | 3.164 |
| $M_{\omega T+}$ | -0.752 | -0.750 | -0.729 | -0.708 | -0.706 | -0.688 |
| $M_{\omega T-}$ | -0.464 | -0.463 | -0.455 | -0.440 | -0.440 | -0.432 |
| $M_{qF}$ | -0.944 | -0.971 | -0.857 | -0.886 | -0.910 | -0.806 |
| $M_{qGT+}$ | 4.364 | 4.611 | 4.237 | 3.826 | 4.042 | 3.705 |
| $M_{qGT-}$ | 1.671 | 1.682 | 1.440 | 1.431 | 1.440 | 1.223 |
| $M_{qT+}$ | 2.065 | 2.062 | 2.022 | 1.956 | 1.951 | 1.919 |
| $M_{qT-}$ | 2.331 | 2.328 | 2.271 | 2.196 | 2.194 | 2.140 |
| $RGT$ | 8.873 | 11.240 | 12.756 | 8.045 | 10.165 | 11.510 |
| $RT$ | -2.783 | -2.780 | -2.646 | -2.641 | -2.638 | -2.514 |
| $P$ | -0.672 | -0.682 | -0.630 | -0.635 | -0.643 | -0.598 |

Other NMEs needed for LRSM with QRPA calculations

# perspective

A model independent route for neutrinoless double beta decay




Cirigliano *et al.* JHEP12, 097(2018)

# Conclusion

- Neutrinoless double beta decay is very good probe for new physics beyond Standard Model
- Calculations of NME is important for the determinations of new physics parameters
- NMEs from various nuclear many-body approaches don't converge at present
- We need to understand the underlying mechanisms of this rare process

Thank You