Majorana 旋量场专题

第二节　二分量旋量与四分量旋量

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旋量表示生成元的约化

- **Dirac 旋量场**和 **Majorana 旋量场**都可以**分解**为**左手**和**右手**的 Weyl 旋量场
- 为了更深刻地认识旋量场，本节进一步研究 **Weyl 旋量**
- 用 $\sigma^\mu = (\mathbf{1}, \boldsymbol{\sigma})$ 和 $\bar{\sigma}^\mu = (\mathbf{1}, -\boldsymbol{\sigma})$ 定义 **2×2 矩阵**

$$\boxed{s^{\mu\nu} \equiv \frac{\mathrm{i}}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)}$$

- 由 $(\sigma^\mu)^\dagger = \sigma^\mu$ 和 $(\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu$ 推出

$$(s^{\mu\nu})^\dagger = -\frac{\mathrm{i}}{4}[(\bar{\sigma}^\nu)^\dagger(\sigma^\mu)^\dagger - (\bar{\sigma}^\mu)^\dagger(\sigma^\nu)^\dagger] = -\frac{\mathrm{i}}{4}(\bar{\sigma}^\nu \sigma^\mu - \bar{\sigma}^\mu \sigma^\nu) = \frac{\mathrm{i}}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)$$

- 从而将 **Weyl 表象**中的**旋量表示生成元**约化为

$$\mathcal{S}^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu] = \frac{\mathrm{i}}{4}\begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix} = \begin{pmatrix} s^{\mu\nu} & \\ & (s^{\mu\nu})^\dagger \end{pmatrix}$$

- 也就是说，**4×4 矩阵 $\mathcal{S}^{\mu\nu}$** 是 2×2 矩阵 $s^{\mu\nu}$ 和 $(s^{\mu\nu})^\dagger$ 的**直和**
- 因而 $s^{\mu\nu}$ 和 $(s^{\mu\nu})^\dagger$ 是两个 Lorentz 群 **2 维表示**的**生成元**

左手和右手 Weyl 旋量所处 2 维表示

- 对于 **Lorentz 变换** Λ 的一组**变换参数** $\omega_{\mu\nu}$，用 $s^{\mu\nu}$ 生成固有保时向有限变换

$$d(\Lambda) \equiv \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}s^{\mu\nu}\right)$$

- 它属于**左手 Weyl 旋量**所处的 **2 维表示**

- 相应的逆变换矩阵为 $d^{-1}(\Lambda) = \exp\left(\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}s^{\mu\nu}\right)$，取厄米共轭，得

$$d^{-1\dagger}(\Lambda) = \exp\left[-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}(s^{\mu\nu})^\dagger\right]$$

- 这是用 $(s^{\mu\nu})^\dagger$ 生成的固有保时向有限变换，属于**右手 Weyl 旋量**所处的 **2 维表示**

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- 于是，**旋量表示的 4×4 Lorentz 变换矩阵**分解为

$$D(\Lambda) = \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\mathcal{S}^{\mu\nu}\right) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\omega_{\mu\nu}s^{\mu\nu}/2} & \\ & \mathrm{e}^{-\mathrm{i}\omega_{\mu\nu}(s^{\mu\nu})^\dagger/2} \end{pmatrix} = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$$

- 因此，4 维旋量表示 $\{D(\Lambda)\}$ 是 2 维表示 $\{d(\Lambda)\}$ 和 $\{d^{-1\dagger}(\Lambda)\}$ 的**直和**

等价表示

🐕 利用 $\sigma^2\sigma^\mu\sigma^2 = (\bar{\sigma}^\mu)^{\mathrm{T}}$ 和 $\sigma^2\bar{\sigma}^\mu\sigma^2 = (\sigma^\mu)^{\mathrm{T}}$ 推出

$$\begin{aligned}
\sigma^2 s^{\mu\nu}\sigma^2 &= \frac{\mathrm{i}}{4}(\sigma^2\sigma^\mu\sigma^2\sigma^2\bar{\sigma}^\nu\sigma^2 - \sigma^2\sigma^\nu\sigma^2\sigma^2\bar{\sigma}^\mu\sigma^2) \\
&= \frac{\mathrm{i}}{4}[(\bar{\sigma}^\mu)^{\mathrm{T}}(\sigma^\nu)^{\mathrm{T}} - (\bar{\sigma}^\nu)^{\mathrm{T}}(\sigma^\mu)^{\mathrm{T}}] = -(s^{\mu\nu})^{\mathrm{T}} \\
\sigma^2 d(\Lambda)\sigma^2 &= \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\sigma^2 s^{\mu\nu}\sigma^2\right) \\
&= \exp\left[\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}(s^{\mu\nu})^{\mathrm{T}}\right] = \left[\exp\left(\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}s^{\mu\nu}\right)\right]^{\mathrm{T}} = d^{-1\mathrm{T}}(\Lambda)
\end{aligned}$$

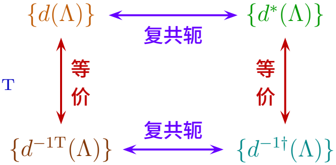
🥕 这里 $d^{-1\mathrm{T}}(\Lambda) = [d^{-1\dagger}(\Lambda)]^*$，线性表示 $\{d^{-1\mathrm{T}}(\Lambda)\}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 的**复共轭表示**

等价表示

- 利用 $\sigma^2\sigma^\mu\sigma^2 = (\bar{\sigma}^\mu)^{\mathrm{T}}$ 和 $\sigma^2\bar{\sigma}^\mu\sigma^2 = (\sigma^\mu)^{\mathrm{T}}$ 推出

$$\begin{aligned}\sigma^2 s^{\mu\nu}\sigma^2 &= \frac{\mathrm{i}}{4}(\sigma^2\sigma^\mu\sigma^2\sigma^2\bar{\sigma}^\nu\sigma^2 - \sigma^2\sigma^\nu\sigma^2\sigma^2\bar{\sigma}^\mu\sigma^2)\\ &= \frac{\mathrm{i}}{4}[(\bar{\sigma}^\mu)^{\mathrm{T}}(\sigma^\nu)^{\mathrm{T}} - (\bar{\sigma}^\nu)^{\mathrm{T}}(\sigma^\mu)^{\mathrm{T}}] = -(s^{\mu\nu})^{\mathrm{T}}\end{aligned}$$

$$\begin{aligned}\sigma^2 d(\Lambda)\sigma^2 &= \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\sigma^2 s^{\mu\nu}\sigma^2\right)\\ &= \exp\left[\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}(s^{\mu\nu})^{\mathrm{T}}\right] = \left[\exp\left(\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}s^{\mu\nu}\right)\right]^{\mathrm{T}} = d^{-1\mathrm{T}}(\Lambda)\end{aligned}$$



- 这里 $d^{-1\mathrm{T}}(\Lambda) = [d^{-1\dagger}(\Lambda)]^*$，线性表示 $\{d^{-1\mathrm{T}}(\Lambda)\}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 的**复共轭表示**
- 将 Pauli 矩阵 σ^2 看作一个**幺正变换矩阵**，满足 $(\sigma^2)^{-1} = (\sigma^2)^\dagger = \sigma^2$
- 则 $d(\Lambda)$ 与 $d^{-1\mathrm{T}}(\Lambda)$ 由一个**相似变换**联系起来，相似变换矩阵为 σ^2
- 因此，线性表示 $\{d(\Lambda)\}$ 和 $\{d^{-1\mathrm{T}}(\Lambda)\}$ 是**等价**的
- 由于 $(\sigma^2)^* = -\sigma^2$，$\sigma^2 d(\Lambda)\sigma^2 = d^{-1\mathrm{T}}(\Lambda)$ 的复共轭为 $\sigma^2 d^*(\Lambda)\sigma^2 = d^{-1\dagger}(\Lambda)$
- 可见，线性表示 $\{d(\Lambda)\}$ 的**复共轭表示** $\{d^*(\Lambda)\}$ 与 $\{d^{-1\dagger}(\Lambda)\}$ **等价**

左手 Weyl 旋量

于是，**左手 Weyl 旋量**

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

的固有保时向 Lorentz 变换为

$$\eta'_a = [d(\Lambda)]_a{}^b \eta_b, \quad a, b = 1, 2$$

η_a 是 $\{d(\Lambda)\}$ **表示空间**中的**列矢量**

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引入**反对称**的**二维 Levi-Civita 符号** ε^{ab}，定义为

$$\varepsilon^{12} = -\varepsilon^{21} = 1, \quad \varepsilon^{11} = \varepsilon^{22} = 0$$

它与 Pauli 矩阵 σ^2 的关系是

$$\varepsilon^{ab} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \mathrm{i}\begin{pmatrix} & -\mathrm{i} \\ \mathrm{i} & \end{pmatrix} = (\mathrm{i}\sigma^2)^{ab}$$

等价的左手 Weyl 旋量

- 通过 ε^{ab} 定义

$$\eta^a \equiv \varepsilon^{ab}\eta_b = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ -\eta_1 \end{pmatrix}$$

- 则

$$\eta^1 = \eta_2, \quad \eta^2 = -\eta_1$$

- $\sigma^2 d(\Lambda)\sigma^2 = d^{-1\mathrm{T}}(\Lambda)$ 等价于 $\sigma^2 d(\Lambda) = d^{-1\mathrm{T}}(\Lambda)\sigma^2$，故 η^a 的 Lorentz 变换为

$$\begin{aligned} \eta'^a &= \varepsilon^{ab}\eta'_b = \varepsilon^{ab}[d(\Lambda)]_b{}^c\eta_c = \mathrm{i}[\sigma^2 d(\Lambda)]^{ac}\eta_c \\ &= \mathrm{i}[d^{-1\mathrm{T}}(\Lambda)\sigma^2]^{ac}\eta_c = [d^{-1\mathrm{T}}(\Lambda)]^a{}_b\varepsilon^{bc}\eta_c \end{aligned}$$

等价的左手 Weyl 旋量

通过 ε^{ab} 定义

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即

$$\boxed{\eta'^a = [d^{-1\mathrm{T}}(\Lambda)]^a{}_b\eta^b}$$

可见 η^a 是 $\{d^{-1\mathrm{T}}(\Lambda)\}$ **表示空间**中的**列矢量**

由于 $\{d^{-1\mathrm{T}}(\Lambda)\}$ **等价**于 $\{d(\Lambda)\}$，η^a 也是**左手 Weyl 旋量**

ε^{ab} 和 ε_{ab}

- 两种左手 Weyl 旋量 η_a 与 η^a 是**等价**的，它们之间的关系类似于 Lorentz 逆变矢量 A^μ 与协变矢量 $A_\mu = g_{\mu\nu}A^\nu$ 之间的关系
- ε^{ab} 的作用类似于度规 $g_{\mu\nu}$，相当于 **2 维旋量空间的"度规"**，用于**提升**旋量指标

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- ε^{ab} 的作用类似于度规 $g_{\mu\nu}$，相当于 **2 维旋量空间的"度规"**，用于**提升**旋量指标
- 用 $\varepsilon_{12} = -\varepsilon_{21} = -1$ 和 $\varepsilon_{11} = \varepsilon_{22} = 0$ 定义 ε_{ab}，则

$$\varepsilon_{ab} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -\mathrm{i}\begin{pmatrix} & -\mathrm{i} \\ \mathrm{i} & \end{pmatrix} = (-\mathrm{i}\sigma^2)_{ab}$$

- ε_{ab} 是 ε^{ab} 的**逆矩阵**，满足

$$\varepsilon_{ab}\varepsilon^{bc} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \delta_a{}^c$$

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于是，$\eta^1 = \eta_2$ 和 $\eta^2 = -\eta_1$ 表明

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta^2 \\ \eta^1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} = \varepsilon_{ab}\eta^b$$

也就是说，ε_{ab} 用于**下降**旋量指标

左手 Weyl 旋量的内积

🐖 任意两个**左手 Weyl 旋量** η_a 和 ζ_a 的**内积**

$$\eta^a\zeta_a = \varepsilon^{ab}\eta_b\zeta_a = \varepsilon_{ab}\eta^a\zeta^b$$

在固有保时向 Lorentz 变换下**不变**，满足

$$\eta'^a\zeta'_a = [d^{-1\mathrm{T}}(\Lambda)]^a{}_b\eta^b[d(\Lambda)]_a{}^c\zeta_c = \eta^b[d^{-1}(\Lambda)]_b{}^a[d(\Lambda)]_a{}^c\zeta_c = \eta^b\delta_b{}^c\zeta_c = \eta^a\zeta_a$$

👰 第二步用了**转置**性质 $[d^{-1\mathrm{T}}(\Lambda)]^a{}_b = [d^{-1}(\Lambda)]_b{}^a$，可见 $\eta^a\zeta_a$ 是 **Lorentz 标量**

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- 由 $\eta^1 = \eta_2$、$\eta^2 = -\eta_1$、$\zeta^1 = \zeta_2$ 和 $\zeta^2 = -\zeta_1$ 得

$$\eta^a\zeta_a = \eta^1\zeta_1 + \eta^2\zeta_2 = \eta_2\zeta_1 - \eta_1\zeta_2 = -\eta_2\zeta^2 - \eta_1\zeta^1 = -\eta_a\zeta^a$$

- 即参与缩并的**旋量指标一升一降**会多出一个**负号**
- 这种性质与 Lorentz 矢量内积 $A^\mu B_\mu = A_\mu B^\mu$ **截然不同**
- 原因在于旋量空间度规 ε^{ab} 是**反对称**的

Grassmann 数

- $\eta^a\zeta_a = -\eta_a\zeta^a$ 表明 $\boxed{\eta^a\eta_a = -\eta_a\eta^a}$，若 η_a 和 η^a 是普通的复数，则 $\eta^a\eta_a = 0$
- 为了使 $\eta^a\eta_a \neq 0$，必须要求左手 Weyl 旋量 η^a 与 η_a **反对易**
- 即它们是 **Grassmann 数**，**任意两个 Grassmann 数都是反对易的**
- 以复数作为组合系数，则若干个 Grassmann 数的**线性组合**也是 Grassmann 数
- 因此，η_a 是 Grassmann 数意味着 $\eta^a = \varepsilon^{ab}\eta_b$ 也是 Grassmann 数

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- 因此，η_a 是 Grassmann 数意味着 $\eta^a = \varepsilon^{ab} \eta_b$ 也是 Grassmann 数
- 虽然如此，Grassmann 数是**反对易的 c 数**，而不是 Hilbert 空间上的算符
- 对 Grassmann 数表达的旋量场进行**量子化**，才得到**旋量场算符**，而 Grassmann 数的反对易性质与旋量场算符的反对易关系**相匹配**
- ⚠ 旋量也**可以不是** Grassmann 数，旋量系数 $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 就是**普通的复数**

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- ⚠ 旋量也**可以不是** Grassmann 数，旋量系数 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 就是**普通的复数**
- 假设 η_a 和 ζ^a 都是 Grassmann 数，则 $\eta_a\zeta^a = -\zeta^a\eta_a$，相应地，将**省略旋量指标**的**内积**写成 $\eta\zeta \equiv \eta^a\zeta_a = -\eta_a\zeta^a = \zeta^a\eta_a = \zeta\eta$，即内积 $\eta\zeta$ 和 $\zeta\eta$ 是**相等**的
- 内积 $\eta^a\eta_a$ 有**等价**表达式

$$\eta\eta = \eta^a\eta_a = \varepsilon_{ab}\eta^a\eta^b = -\eta^1\eta^2 + \eta^2\eta^1 = -2\eta^1\eta^2$$
$$= 2\eta_2\eta_1 = \eta_2\eta_1 - \eta_1\eta_2 = -\varepsilon^{ab}\eta_a\eta_b = -\eta_a\eta^a$$

左手 Weyl 旋量的复共轭

- 将左手 Weyl 旋量 η_a 的复共轭记为 $\eta^\dagger_{\dot{a}} = \begin{pmatrix} \eta^\dagger_{\dot{1}} \\ \eta^\dagger_{\dot{2}} \end{pmatrix}$
- 量子化之后，算符 η_a 和 $\eta^\dagger_{\dot{a}}$ 互为厄米共轭
- 对 $\eta'_a = [d(\Lambda)]_a{}^b \eta_b$ 两边取复共轭，得到 $\eta^\dagger_{\dot{a}}$ 的 Lorentz 变换

$$\eta'^\dagger_{\dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} \eta^\dagger_{\dot{b}}$$

左手 Weyl 旋量的复共轭

🐏 将左手 Weyl 旋量 η_a 的复共轭记为 $\eta^\dagger_{\dot a} = \begin{pmatrix} \eta^\dagger_{\dot 1} \\ \eta^\dagger_{\dot 2} \end{pmatrix}$

🧢 量子化之后，算符 η_a 和 $\eta^\dagger_{\dot a}$ 互为厄米共轭

🕶️ 对 $\eta'_a = [d(\Lambda)]_a{}^b \eta_b$ 两边取复共轭，得到 $\eta^\dagger_{\dot a}$ 的 Lorentz 变换

$$\eta'^\dagger_{\dot a} = [d^*(\Lambda)]_{\dot a}{}^{\dot b} \eta^\dagger_{\dot b}$$

👕 引进指标上带着点号的二维 Levi-Civita 符号

$$\varepsilon^{\dot a \dot b} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = (\mathrm{i}\sigma^2)^{\dot a \dot b}, \quad \varepsilon_{\dot a \dot b} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = (-\mathrm{i}\sigma^2)_{\dot a \dot b}$$

🩳 其分量数值与 ε^{ab} 和 ε_{ab} 分别**相同**

🩴 定义 $\eta^{\dagger \dot a} \equiv \varepsilon^{\dot a \dot b} \eta^\dagger_{\dot b} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \eta^\dagger_{\dot 1} \\ \eta^\dagger_{\dot 2} \end{pmatrix} = \begin{pmatrix} \eta^\dagger_{\dot 2} \\ -\eta^\dagger_{\dot 1} \end{pmatrix}$，则有 $\eta^{\dagger \dot 1} = \eta^\dagger_{\dot 2}$ 和 $\eta^{\dagger \dot 2} = -\eta^\dagger_{\dot 1}$

右手 Weyl 旋量

$\sigma^2 d^*(\Lambda)\sigma^2 = d^{-1\dagger}(\Lambda)$ 等价于 $\sigma^2 d^*(\Lambda) = d^{-1\dagger}(\Lambda)\sigma^2$

故 $\eta^{\dagger\dot{a}}$ 的 Lorentz 变换为

$$\begin{aligned}\eta'^{\dagger\dot{a}} &= \varepsilon^{\dot{a}\dot{b}}\eta'^{\dagger}_{\dot{b}} = \varepsilon^{\dot{a}\dot{b}}[d^*(\Lambda)]_{\dot{b}}{}^{\dot{c}}\eta^{\dagger}_{\dot{c}} = \mathrm{i}[\sigma^2 d^*(\Lambda)]^{\dot{a}\dot{c}}\eta^{\dagger}_{\dot{c}} \\ &= \mathrm{i}[d^{-1\dagger}(\Lambda)\sigma^2]^{\dot{a}\dot{c}}\eta^{\dagger}_{\dot{c}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\varepsilon^{\dot{b}\dot{c}}\eta^{\dagger}_{\dot{c}}\end{aligned}$$

即

$$\eta'^{\dagger\dot{a}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\eta^{\dagger\dot{b}}$$

右手 Weyl 旋量

- $\sigma^2 d^*(\Lambda)\sigma^2 = d^{-1\dagger}(\Lambda)$ 等价于 $\sigma^2 d^*(\Lambda) = d^{-1\dagger}(\Lambda)\sigma^2$
- 故 $\eta^{\dagger\dot{a}}$ 的 Lorentz 变换为

$$\begin{aligned}\eta'^{\dagger\dot{a}} &= \varepsilon^{\dot{a}\dot{b}}\eta'^{\dagger}_{\dot{b}} = \varepsilon^{\dot{a}\dot{b}}[d^*(\Lambda)]_{\dot{b}}{}^{\dot{c}}\eta^{\dagger}_{\dot{c}} = \mathrm{i}[\sigma^2 d^*(\Lambda)]^{\dot{a}\dot{c}}\eta^{\dagger}_{\dot{c}} \\ &= \mathrm{i}[d^{-1\dagger}(\Lambda)\sigma^2]^{\dot{a}\dot{c}}\eta^{\dagger}_{\dot{c}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\varepsilon^{\dot{b}\dot{c}}\eta^{\dagger}_{\dot{c}}\end{aligned}$$

- 即

$$\boxed{\eta'^{\dagger\dot{a}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\eta^{\dagger\dot{b}}}$$

- 可见，$\eta^{\dagger\dot{a}}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ **表示空间**中的**列矢量**，因而是**右手 Weyl 旋量**
- 由于表示 $\{d^*(\Lambda)\}$ **等价**于 $\{d^{-1\dagger}(\Lambda)\}$，$\eta^{\dagger}_{\dot{a}}$ 也是**右手 Weyl 旋量**
- 因此，在这套符号约定中，**不带点**的旋量指标对应于**左手 Weyl 旋量**及其**表示**
- 而**带点**的旋量指标对应于**右手 Weyl 旋量**及其**表示**

右手 Weyl 旋量的内积

任意两个**右手 Weyl 旋量** $\eta^{\dagger\dot{a}}$ 和 $\zeta^{\dagger\dot{a}}$ 的**内积**

$$\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = \varepsilon_{\dot{a}\dot{b}}\eta^{\dagger\dot{b}}\zeta^{\dagger\dot{a}} = \varepsilon^{\dot{a}\dot{b}}\eta^\dagger_{\dot{a}}\zeta^\dagger_{\dot{b}}$$

在固有保时向 Lorentz 变换下**不变**，满足

$$\eta'^\dagger_{\dot{a}}\zeta'^{\dagger\dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}}\eta^\dagger_{\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{b}}[d^\dagger(\Lambda)]^{\dot{b}}{}_{\dot{a}}[d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{b}}\delta^{\dot{b}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}}$$

第二步用了**转置**性质 $[d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} = [d^\dagger(\Lambda)]^{\dot{b}}{}_{\dot{a}}$，可见 $\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}}$ 是 **Lorentz 标量**

右手 Weyl 旋量的内积

任意两个**右手 Weyl 旋量** $\eta^{\dagger\dot{a}}$ 和 $\zeta^{\dagger\dot{a}}$ 的**内积**

$$\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = \varepsilon_{\dot{a}\dot{b}}\eta^{\dagger\dot{b}}\zeta^{\dagger\dot{a}} = \varepsilon^{\dot{a}\dot{b}}\eta^\dagger_{\dot{a}}\zeta^\dagger_{\dot{b}}$$

在固有保时向 Lorentz 变换下**不变**，满足

$$\eta'^\dagger_{\dot{a}}\zeta'^{\dagger\dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}}\eta^\dagger_{\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{b}}[d^\dagger(\Lambda)]^{\dot{b}}{}_{\dot{a}}[d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{b}}\delta^{\dot{b}}{}_{\dot{c}}\zeta^{\dagger\dot{c}} = \eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}}$$

第二步用了**转置**性质 $[d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} = [d^\dagger(\Lambda)]^{\dot{b}}{}_{\dot{a}}$，可见 $\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}}$ 是 **Lorentz 标量**

由 $\eta^{\dagger\dot{1}} = \eta^\dagger_{\dot{2}}$、$\eta^{\dagger\dot{2}} = -\eta^\dagger_{\dot{1}}$、$\zeta^{\dagger\dot{1}} = \zeta^\dagger_{\dot{2}}$ 和 $\zeta^{\dagger\dot{2}} = -\zeta^\dagger_{\dot{1}}$ 得

$$\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = \eta^\dagger_{\dot{1}}\zeta^{\dagger\dot{1}} + \eta^\dagger_{\dot{2}}\zeta^{\dagger\dot{2}} = -\eta^{\dagger\dot{2}}\zeta^{\dagger\dot{1}} + \eta^{\dagger\dot{1}}\zeta^{\dagger\dot{2}} = -\eta^{\dagger\dot{2}}\zeta^\dagger_{\dot{2}} - \eta^{\dagger\dot{1}}\zeta^\dagger_{\dot{1}} = -\eta^{\dagger\dot{a}}\zeta^\dagger_{\dot{a}}$$

即参与缩并的**带点旋量指标一升一降**会多出一个**负号**

右手 Weyl 旋量的内积

任意两个**右手 Weyl 旋量** $\eta^{\dagger\dot a}$ 和 $\zeta^{\dagger\dot a}$ 的**内积**

$$\eta^\dagger_{\dot a}\zeta^{\dagger\dot a} = \varepsilon_{\dot a\dot b}\eta^{\dagger\dot b}\zeta^{\dagger\dot a} = \varepsilon^{\dot a\dot b}\eta^\dagger_{\dot a}\zeta^\dagger_{\dot b}$$

在固有保时向 Lorentz 变换下**不变**，满足

$$\eta'^\dagger_{\dot a}\zeta'^{\dagger\dot a} = [d^*(\Lambda)]_{\dot a}{}^{\dot b}\eta^\dagger_{\dot b}[d^{-1\dagger}(\Lambda)]^{\dot a}{}_{\dot c}\zeta^{\dagger\dot c} = \eta^\dagger_{\dot b}[d^\dagger(\Lambda)]^{\dot b}{}_{\dot a}[d^{-1\dagger}(\Lambda)]^{\dot a}{}_{\dot c}\zeta^{\dagger\dot c} = \eta^\dagger_{\dot b}\delta^{\dot b}{}_{\dot c}\zeta^{\dagger\dot c} = \eta^\dagger_{\dot a}\zeta^{\dagger\dot a}$$

第二步用了**转置**性质 $[d^*(\Lambda)]_{\dot a}{}^{\dot b} = [d^\dagger(\Lambda)]^{\dot b}{}_{\dot a}$，可见 $\eta^\dagger_{\dot a}\zeta^{\dagger\dot a}$ 是 **Lorentz 标量**

由 $\eta^{\dagger\dot 1} = \eta^\dagger_{\dot 2}$、$\eta^{\dagger\dot 2} = -\eta^\dagger_{\dot 1}$、$\zeta^{\dagger\dot 1} = \zeta^\dagger_{\dot 2}$ 和 $\zeta^{\dagger\dot 2} = -\zeta^\dagger_{\dot 1}$ 得

$$\eta^\dagger_{\dot a}\zeta^{\dagger\dot a} = \eta^\dagger_{\dot 1}\zeta^{\dagger\dot 1} + \eta^\dagger_{\dot 2}\zeta^{\dagger\dot 2} = -\eta^{\dagger\dot 2}\zeta^{\dagger\dot 1} + \eta^{\dagger\dot 1}\zeta^{\dagger\dot 2} = -\eta^{\dagger\dot 2}\zeta^\dagger_{\dot 2} - \eta^{\dagger\dot 1}\zeta^\dagger_{\dot 1} = -\eta^{\dagger\dot a}\zeta^\dagger_{\dot a}$$

即参与缩并的**带点旋量指标一升一降**会多出一个**负号**

假设右手 Weyl 旋量 $\eta^{\dagger\dot a}$ 和 $\zeta^\dagger_{\dot a}$ 都是 **Grassmann 数**，则 $\eta^{\dagger\dot a}\zeta^\dagger_{\dot a} = -\zeta^\dagger_{\dot a}\eta^{\dagger\dot a}$

将**省略**带点旋量指标的**内积**写成 $\eta^\dagger\zeta^\dagger \equiv \eta^\dagger_{\dot a}\zeta^{\dagger\dot a} = -\eta^{\dagger\dot a}\zeta^\dagger_{\dot a} = \zeta^\dagger_{\dot a}\eta^{\dagger\dot a} = \zeta^\dagger\eta^\dagger$

则内积 $\eta^\dagger\zeta^\dagger$ 和 $\zeta^\dagger\eta^\dagger$ **相等**

Lorentz 不变量和 Weyl 旋量算符

- 可以看到，只要将不带点和带点的旋量指标分别**缩并完毕**，就得到 **Lorentz 标量**
- 另一方面，**缩并**一个不带点的指标和一个带点的指标并**不能**得到 Lorentz 不变量
- 比如，$\eta^a \zeta^\dagger_{\dot{a}}$ 和 $\eta^{\dagger\dot{a}} \zeta_a$ 都**不是 Lorentz 标量**

Lorentz 不变量和 Weyl 旋量算符

- 可以看到，只要将**不带点**和**带点**的旋量指标分别**缩并完毕**，就得到 **Lorentz 标量**
- 另一方面，**缩并**一个**不带点**的指标和一个**带点**的指标并**不能**得到 Lorentz 不变量
- 比如，$\eta^a\zeta^\dagger_{\dot a}$ 和 $\eta^{\dagger\dot a}\zeta_a$ 都**不是 Lorentz 标量**
- 对于 **Weyl 旋量算符** η_a 和 ζ_a，有

$$(\eta\zeta)^\dagger = (\eta^a\zeta_a)^\dagger = (\zeta_a)^\dagger(\eta^a)^\dagger = \zeta^\dagger_{\dot a}\eta^{\dagger\dot a} = \zeta^\dagger\eta^\dagger$$

- 即 $\zeta^\dagger\eta^\dagger$ 是 $\eta\zeta$ 的**厄米共轭**算符
- 厄米共轭操作将**左手**和**右手** Weyl 旋量算符**相互转换**

Dirac 旋量场的分解

依照上述关于旋量指标的约定，将 **Dirac 旋量场** $\psi(x)$ 分解成**左手 Weyl 旋量场** $\eta_a(x)$ 和**右手 Weyl 旋量场** $\zeta^{\dagger\dot{a}}(x)$ ，形式为

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \zeta^{\dagger\dot{a}}(x) \end{pmatrix}$$

在量子化之前，$\eta_a(x)$ 和 $\zeta^{\dagger\dot{a}}(x)$ 是 Grassmann 数，因而 $\psi(x)$ 也是 **Grassmann 数**

这是前面**转置两个旋量场**必须添加一个额外**负号**的原因

根据 $D(\Lambda) = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$，$\psi(x)$ 的**固有保时向 Lorentz 变换**表达成

$$\begin{pmatrix} \eta'_a(x') \\ \zeta'^{\dagger\dot{a}}(x') \end{pmatrix} = \psi'(x') = D(\Lambda)\psi(x) = \begin{pmatrix} [d(\Lambda)]_a{}^b \eta_b(x) \\ [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\zeta^{\dagger\dot{b}}(x) \end{pmatrix}$$

$\psi(x)$ 的 **Dirac 共轭**是 $\bar{\psi} = \psi^\dagger\gamma^0 = \begin{pmatrix} \eta^\dagger_{\dot{b}} & \zeta^b \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}{}_{\dot{a}} \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta^\dagger_{\dot{a}} \end{pmatrix}$

Dirac 矩阵的指标形式

保持**旋量指标平衡**，则 **Dirac 方程** $(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0$ 化为

$$\begin{pmatrix} -m\delta_a{}^b & \mathrm{i}(\sigma^\mu)_{a\dot{b}}\partial_\mu \\ \mathrm{i}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu & -m\delta^{\dot{a}}{}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dagger\dot{b}} \end{pmatrix} = 0$$

因而 **Dirac 矩阵**的指标形式是

$$\gamma^\mu = \begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}$$

注意，γ^μ 中的 γ^0 与 Dirac 共轭 $\bar{\psi} = \psi^\dagger\gamma^0 = \begin{pmatrix} \eta^\dagger_{\dot{a}} & \zeta^a \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}{}_{\dot{a}} \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta^\dagger_{\dot{a}} \end{pmatrix}$

中的 γ^0 具有**不同的指标结构**

两者本质不同，有些书将后者记为 β 以示区别

σ^μ 和 $\bar{\sigma}^\mu$ 的 Lorentz 变换规则

- 于是，γ^μ 的 **Lorentz 变换规则** $D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$ 左边变成

$$
\begin{aligned}
&D^{-1}(\Lambda)\gamma^\mu D(\Lambda) \\
&= \begin{pmatrix} [d^{-1}(\Lambda)]_a{}^c & \\ & [d^\dagger(\Lambda)]^{\dot a}{}_{\dot c} \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{c\dot d} \\ (\bar{\sigma}^\mu)^{\dot c d} & \end{pmatrix} \begin{pmatrix} [d(\Lambda)]_d{}^b & \\ & [d^{-1\dagger}(\Lambda)]^{\dot d}{}_{\dot b} \end{pmatrix} \\
&= \begin{pmatrix} & [d^{-1}(\Lambda)]_a{}^c(\sigma^\mu)_{c\dot d}[d^{-1\dagger}(\Lambda)]^{\dot d}{}_{\dot b} \\ [d^\dagger(\Lambda)]^{\dot a}{}_{\dot c}(\bar{\sigma}^\mu)^{\dot c d}[d(\Lambda)]_d{}^b & \end{pmatrix}
\end{aligned}
$$

- 右边化为

$$
\Lambda^\mu{}_\nu\gamma^\nu = \begin{pmatrix} & \Lambda^\mu{}_\nu(\sigma^\nu)_{a\dot b} \\ \Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot a b} & \end{pmatrix}
$$

- 两相比较，推出

$$
[d^{-1}(\Lambda)]_a{}^c(\sigma^\mu)_{c\dot d}[d^{-1\dagger}(\Lambda)]^{\dot d}{}_{\dot b} = \Lambda^\mu{}_\nu(\sigma^\nu)_{a\dot b}, \quad [d^\dagger(\Lambda)]^{\dot a}{}_{\dot c}(\bar{\sigma}^\mu)^{\dot c d}[d(\Lambda)]_d{}^b = \Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot a b}
$$

- 这分别是 σ^μ 和 $\bar{\sigma}^\mu$ 的 **Lorentz 变换规则**

Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar{\sigma}^\mu\zeta$

- 对**任意 Weyl 旋量** η 和 ζ，定义 $\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{a\dot{b}}\zeta^{\dagger\dot{b}}, \quad \eta^\dagger\bar{\sigma}^\mu\zeta \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\zeta_b$
- 它们都是 **Lorentz 矢量**，相应的固有保时向 Lorentz 变换为

$$\eta'\sigma^\mu\zeta'^\dagger = [d^{-1\mathrm{T}}(\Lambda)]^a{}_c\eta^c(\sigma^\mu)_{a\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_{\dot{d}}\zeta^{\dagger\dot{d}} = \eta^c[d^{-1}(\Lambda)]_c{}^a(\sigma^\mu)_{a\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_{\dot{d}}\zeta^{\dagger\dot{d}}$$

$$= \eta^c\Lambda^\mu{}_\nu(\sigma^\nu)_{c\dot{d}}\zeta^{\dagger\dot{d}} = \Lambda^\mu{}_\nu\eta\sigma^\nu\zeta^\dagger$$

$$\eta'^\dagger\bar{\sigma}^\mu\zeta' = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{c}}\eta^\dagger_{\dot{c}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d = \eta^\dagger_{\dot{c}}[d^\dagger(\Lambda)]^{\dot{c}}{}_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d$$

$$= \eta^\dagger_{\dot{c}}\Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot{c}d}\zeta_d = \Lambda^\mu{}_\nu\eta^\dagger\bar{\sigma}^\mu\zeta$$

Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar{\sigma}^\mu\zeta$

对**任意 Weyl 旋量** η 和 ζ，定义 $\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{a\dot{b}}\zeta^{\dagger\dot{b}}$, $\quad \eta^\dagger\bar{\sigma}^\mu\zeta \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\zeta_b$

它们都是 **Lorentz 矢量**，相应的固有保时向 Lorentz 变换为

$$\begin{aligned}\eta'\sigma^\mu\zeta'^\dagger &= [d^{-1\mathrm{T}}(\Lambda)]^a{}_c\eta^c(\sigma^\mu)_{a\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_{\dot{d}}\zeta^{\dagger\dot{d}} = \eta^c[d^{-1}(\Lambda)]_c{}^a(\sigma^\mu)_{a\dot{b}}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_{\dot{d}}\zeta^{\dagger\dot{d}}\\ &= \eta^c\Lambda^\mu{}_\nu(\sigma^\nu)_{c\dot{d}}\zeta^{\dagger\dot{d}} = \Lambda^\mu{}_\nu\eta\sigma^\nu\zeta^\dagger\\ \eta'^\dagger\bar{\sigma}^\mu\zeta' &= [d^*(\Lambda)]_{\dot{a}}{}^{\dot{c}}\eta^\dagger_{\dot{c}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d = \eta^\dagger_{\dot{c}}[d^\dagger(\Lambda)]^{\dot{c}}{}_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d\\ &= \eta^\dagger_{\dot{c}}\Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot{c}d}\zeta_d = \Lambda^\mu{}_\nu\eta^\dagger\bar{\sigma}^\mu\zeta\end{aligned}$$

由 $\sigma^2\sigma^\mu\sigma^2 = (\bar{\sigma}^\mu)^\mathrm{T}$ 得 $(\mathrm{i}\sigma^2)\sigma^\mu(\mathrm{i}\sigma^2) = -(\bar{\sigma}^\mu)^\mathrm{T}$，相应的指标形式为

$$\varepsilon^{ac}(\sigma^\mu)_{c\dot{d}}\varepsilon^{\dot{d}\dot{b}} = -[(\bar{\sigma}^\mu)^\mathrm{T}]^{a\dot{b}} = -(\bar{\sigma}^\mu)^{\dot{b}a}$$

对于 **Weyl 旋量场** $\eta_a(x)$ 和 $\zeta^{\dagger\dot{a}}(x)$，有 **Grassmann 数性质**

$$\begin{aligned}[\eta^a(\sigma^\mu)_{a\dot{b}}\zeta^{\dagger\dot{b}}]^\dagger &= \zeta^b(\sigma^\mu)_{b\dot{a}}\eta^{\dagger\dot{a}} = -\eta^{\dagger\dot{a}}(\sigma^\mu)_{b\dot{a}}\zeta^b = -\varepsilon^{\dot{a}\dot{c}}\eta^\dagger_{\dot{c}}(\sigma^\mu)_{b\dot{a}}\varepsilon^{bd}\zeta_d\\ &= \eta^\dagger_{\dot{c}}\varepsilon^{db}(\sigma^\mu)_{b\dot{a}}\varepsilon^{\dot{a}\dot{c}}\zeta_d = -\eta^\dagger_{\dot{c}}(\bar{\sigma}^\mu)^{\dot{c}d}\zeta_d = -[\zeta^\dagger_{\dot{d}}(\bar{\sigma}^\mu)^{\dot{d}c}\eta_c]^\dagger\end{aligned}$$

即 $(\eta\sigma^\mu\zeta^\dagger)^\dagger = \zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\zeta = -(\zeta^\dagger\bar{\sigma}^\mu\eta)^\dagger$

Lorentz 张量 $\eta\sigma^\mu\bar{\sigma}^\nu\zeta$ 和 $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger$

🐓 类似地，$\eta\sigma^\mu\bar{\sigma}^\nu\zeta \equiv \eta^a(\sigma^\mu)_{a\dot{b}}(\bar{\sigma}^\nu)^{\dot{b}c}\zeta_c$ 和 $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}(\sigma^\nu)_{b\dot{c}}\zeta^{\dagger\dot{c}}$ 都是二阶 Lorentz 张量

🎁 由 $\sigma^2\bar{\sigma}^\mu\sigma^2 = (\sigma^\mu)^{\mathrm{T}}$ 得 $(-\mathrm{i}\sigma^2)\bar{\sigma}^\mu(-\mathrm{i}\sigma^2) = -(\sigma^\mu)^{\mathrm{T}}$，相应的指标形式为

$$\varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\mu)^{\dot{c}d}\varepsilon_{db} = -[(\sigma^\mu)^{\mathrm{T}}]_{\dot{a}b} = -(\sigma^\mu)_{b\dot{a}}$$

💎 再利用 $\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c$ 和 $\varepsilon^{ac}(\sigma^\mu)_{c\dot{d}}\varepsilon^{\dot{d}\dot{b}} = -[(\bar{\sigma}^\mu)^{\mathrm{T}}]^{a\dot{b}} = -(\bar{\sigma}^\mu)^{\dot{b}a}$ 推出

$$\begin{aligned}\varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}(\sigma^\mu)_{d\dot{e}}\varepsilon^{\dot{e}\dot{b}} &= \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}\delta_d{}^f(\sigma^\mu)_{f\dot{e}}\varepsilon^{\dot{e}\dot{b}} = \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}\varepsilon_{dg}\varepsilon^{gf}(\sigma^\mu)_{f\dot{e}}\varepsilon^{\dot{e}\dot{b}}\\ &= (-\sigma^\nu)_{g\dot{a}}(-\bar{\sigma}^\mu)^{\dot{b}g} = (\bar{\sigma}^\mu)^{\dot{b}g}(\sigma^\nu)_{g\dot{a}}\end{aligned}$$

💠 故 $$\begin{aligned}[\eta^a(\sigma^\mu)_{a\dot{b}}(\bar{\sigma}^\nu)^{\dot{b}c}\zeta_c]^\dagger &= \zeta^\dagger_{\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\eta^{\dagger\dot{a}} = -\eta^{\dagger\dot{a}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\zeta^\dagger_{\dot{c}}\\ &= -\varepsilon^{\dot{a}\dot{d}}\eta^\dagger_{\dot{d}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\varepsilon_{\dot{c}\dot{e}}\zeta^{\dagger\dot{e}} = \eta^\dagger_{\dot{d}}\varepsilon_{\dot{e}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\varepsilon^{\dot{a}\dot{d}}\zeta^{\dagger\dot{e}}\\ &= \eta^\dagger_{\dot{d}}(\bar{\sigma}^\mu)^{\dot{d}g}(\sigma^\nu)_{g\dot{e}}\zeta^{\dagger\dot{e}} = [\zeta^e(\sigma^\nu)_{e\dot{g}}(\bar{\sigma}^\mu)^{\dot{g}d}\eta_d]^\dagger\end{aligned}$$

💍 即 $$(\eta\sigma^\mu\bar{\sigma}^\nu\zeta)^\dagger = \zeta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger = \eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger = (\zeta\sigma^\nu\bar{\sigma}^\mu\eta)^\dagger$$

旋量双线性型的分解

将 **Dirac 旋量双线性型**分解成由 **Weyl 旋量**表达的 **Lorentz 张量**，有

$$\bar{\psi}\psi = \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}\eta_a \\ \zeta^{\dagger\dot{a}}\end{pmatrix} = \zeta^a\eta_a + \eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = \zeta\eta + \eta^\dagger\zeta^\dagger$$

$$\bar{\psi}\gamma^5\psi = \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}-\delta_a{}^b & \\ & \delta^{\dot{a}}{}_{\dot{b}}\end{pmatrix}\begin{pmatrix}\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix} = -\zeta^a\eta_a + \eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = -\zeta\eta + \eta^\dagger\zeta^\dagger$$

$$\begin{aligned}\bar{\psi}\gamma^\mu\psi &= \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}\begin{pmatrix}\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix} = \zeta^a(\sigma^\mu)_{a\dot{b}}\zeta^{\dagger\dot{b}} + \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\eta_b \\ &= \zeta\sigma^\mu\zeta^\dagger + \eta^\dagger\bar{\sigma}^\mu\eta\end{aligned}$$

$$\begin{aligned}\bar{\psi}\gamma^\mu\gamma^5\psi &= \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}\begin{pmatrix}-\delta_b{}^c & \\ & \delta^{\dot{b}}{}_{\dot{c}}\end{pmatrix}\begin{pmatrix}\eta_c \\ \zeta^{\dagger\dot{c}}\end{pmatrix} \\ &= \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}\begin{pmatrix}-\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix} = \zeta^a(\sigma^\mu)_{a\dot{b}}\zeta^{\dagger\dot{b}} - \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\eta_b \\ &= \zeta\sigma^\mu\zeta^\dagger - \eta^\dagger\bar{\sigma}^\mu\eta\end{aligned}$$

旋量双线性型的分解

还有

$$\bar{\psi}\sigma^{\mu\nu}\psi = \frac{\mathrm{i}}{2}\begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}(\sigma^\mu\bar{\sigma}^\nu-\sigma^\nu\bar{\sigma}^\mu)_a{}^b & \\ & (\bar{\sigma}^\mu\sigma^\nu-\bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{b}}\end{pmatrix}\begin{pmatrix}\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix}$$

$$= \frac{\mathrm{i}}{2}\zeta^a(\sigma^\mu\bar{\sigma}^\nu-\sigma^\nu\bar{\sigma}^\mu)_a{}^b\eta_b + \frac{\mathrm{i}}{2}\eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu\sigma^\nu-\bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{b}}\zeta^{\dagger\dot{b}}$$

$$= \frac{\mathrm{i}}{2}\zeta(\sigma^\mu\bar{\sigma}^\nu-\sigma^\nu\bar{\sigma}^\mu)\eta + \frac{\mathrm{i}}{2}\eta^\dagger(\bar{\sigma}^\mu\sigma^\nu-\bar{\sigma}^\nu\sigma^\mu)\zeta^\dagger$$

进一步推出

$$\bar{\psi}_\mathrm{R}\psi_\mathrm{L} = \frac{1}{2}\bar{\psi}(1-\gamma^5)\psi = \zeta\eta$$

$$\bar{\psi}_\mathrm{L}\psi_\mathrm{R} = \frac{1}{2}\bar{\psi}(1+\gamma^5)\psi = \eta^\dagger\zeta^\dagger$$

$$\bar{\psi}_\mathrm{L}\gamma^\mu\psi_\mathrm{L} = \frac{1}{2}\bar{\psi}(\gamma^\mu-\gamma^\mu\gamma^5)\psi = \eta^\dagger\bar{\sigma}^\mu\eta$$

$$\bar{\psi}_\mathrm{R}\gamma^\mu\psi_\mathrm{R} = \frac{1}{2}\bar{\psi}(\gamma^\mu+\gamma^\mu\gamma^5)\psi = \zeta\sigma^\mu\zeta^\dagger$$

拉氏量的分解

- 另一方面，**自由 Dirac 旋量场**的**拉氏量**分解为

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix}\zeta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}-m\delta_a{}^b & \mathrm{i}(\sigma^\mu)_{a\dot{b}}\partial_\mu \\ \mathrm{i}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu & -m\delta^{\dot{a}}{}_{\dot{b}}\end{pmatrix}\begin{pmatrix}\eta_b \\ \zeta^{\dagger\dot{b}}\end{pmatrix} \\ &= -m\zeta^a\eta_a + \mathrm{i}\zeta^a(\sigma^\mu)_{a\dot{b}}\partial_\mu\zeta^{\dagger\dot{b}} + \mathrm{i}\eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu\eta_b - m\eta^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} \\ &= \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta + \mathrm{i}\zeta\sigma^\mu\partial_\mu\zeta^\dagger - m(\zeta\eta + \eta^\dagger\zeta^\dagger)\end{aligned}$$

- 这里的质量项涉及两个**不同的 Weyl 旋量场** $\eta_a(x)$ 和 $\zeta_a(x)$，称为 **Dirac 质量项**
- 如果**质量** $m = 0$，则

$$\mathcal{L}_\mathrm{L} = \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta \quad 和 \quad \mathcal{L}_\mathrm{R} = \mathrm{i}\zeta\sigma^\mu\partial_\mu\zeta^\dagger$$

分别描述自由的**左手 Weyl 旋量场** $\eta_a(x)$ 和**右手 Weyl 旋量场** $\zeta^{\dagger\dot{a}}(x)$

- 相应的运动方程是两个 **Weyl 方程**：

$$\mathrm{i}\bar{\sigma}^\mu\partial_\mu\eta = 0, \quad \mathrm{i}\sigma^\mu\partial_\mu\zeta^\dagger = 0$$

Weyl 旋量场的 C 变换

- 接下来讨论 **Weyl 旋量场**的 **C 变换**
- 首先，**电荷共轭矩阵**的指标形式为 $\mathcal{C} = \begin{pmatrix} -i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}$
- 将 $\psi(x)$ 的**电荷共轭场** $\psi^{\mathrm{C}}(x)$ 分解成 Weyl 旋量场，得到

$$\psi^{\mathrm{C}}(x) = \mathcal{C}\bar{\psi}^{\mathrm{T}}(x) = \mathcal{C}\begin{pmatrix} \zeta^b(x) & \eta^\dagger_{\dot{b}}(x) \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}\begin{pmatrix} \zeta^b(x) \\ \eta^\dagger_{\dot{b}}(x) \end{pmatrix} = \begin{pmatrix} \zeta_a(x) \\ \eta^{\dagger\dot{a}}(x) \end{pmatrix}$$

- 从而，**Dirac 旋量场** $\psi(x)$ 的 **C 变换**化为

$$\begin{pmatrix} C^{-1}\eta_a(x)C \\ C^{-1}\zeta^{\dagger\dot{a}}(x)C \end{pmatrix} = C^{-1}\psi(x)C = \zeta_C^*\psi^{\mathrm{C}}(x) = \begin{pmatrix} \zeta_C^*\zeta_a(x) \\ \zeta_C^*\eta^{\dagger\dot{a}}(x) \end{pmatrix}$$

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$$\psi^{\mathrm{C}}(x) = \mathcal{C}\bar{\psi}^{\mathrm{T}}(x) = \mathcal{C}\begin{pmatrix} \zeta^b(x) & \eta^\dagger_{\dot{b}}(x) \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \zeta^b(x) \\ \eta^\dagger_{\dot{b}}(x) \end{pmatrix} = \begin{pmatrix} \zeta_a(x) \\ \eta^{\dagger\dot{a}}(x) \end{pmatrix}$$

- 从而，**Dirac 旋量场** $\psi(x)$ 的 **C 变换**化为

$$\begin{pmatrix} C^{-1}\eta_a(x)C \\ C^{-1}\zeta^{\dagger\dot{a}}(x)C \end{pmatrix} = C^{-1}\psi(x)C = \zeta_C^*\psi^{\mathrm{C}}(x) = \begin{pmatrix} \zeta_C^*\zeta_a(x) \\ \zeta_C^*\eta^{\dagger\dot{a}}(x) \end{pmatrix}$$

- 即**左右**手 Weyl 旋量场的 **C 变换**是

$$C^{-1}\eta_a(x)C = \zeta_C^*\zeta_a(x), \quad C^{-1}\zeta^{\dagger\dot{a}}(x)C = \zeta_C^*\eta^{\dagger\dot{a}}(x)$$

- 可见，电荷共轭变换将 **η 和 ζ 相互转换**。取厄米共轭，得 $C^{-1}\eta^\dagger_{\dot{b}}(x)C = \zeta_C\zeta^\dagger_{\dot{b}}(x)$ 及 $C^{-1}\zeta^b(x)C = \zeta_C\eta^b(x)$，分别与 $\varepsilon^{\dot{a}\dot{b}}$ 和 ε_{ab} 缩并，推出

$$C^{-1}\eta^{\dagger\dot{a}}(x)C = \zeta_C\zeta^{\dagger\dot{a}}(x), \quad C^{-1}\zeta_a(x)C = \zeta_C\eta_a(x)$$

Majorana 旋量场的分解

🦙 下面讨论 **Majorana 旋量场**，**Majorana 条件**意味着 $\begin{pmatrix} \eta_a \\ \zeta^{\dagger\dot{a}} \end{pmatrix} = \psi = \mathcal{C}\bar{\psi}^{\mathrm{T}} = \begin{pmatrix} \zeta_a \\ \eta^{\dagger\dot{a}} \end{pmatrix}$

🎰 即 $\eta = \zeta$，这表明 Majorana 旋量场中的**左手**和**右手** Weyl 旋量场是**相关**的

🎮 因此，可以将 Majorana 旋量场 $\psi(x)$ 分解成

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \eta^{\dagger\dot{a}}(x) \end{pmatrix}$$

Majorana 旋量场的分解

下面讨论 **Majorana 旋量场**，**Majorana 条件**意味着 $\begin{pmatrix}\eta_a\\ \zeta^{\dagger\dot{a}}\end{pmatrix} = \psi = \mathcal{C}\bar{\psi}^{\mathrm{T}} = \begin{pmatrix}\zeta_a\\ \eta^{\dagger\dot{a}}\end{pmatrix}$

即 $\eta = \zeta$，这表明 Majorana 旋量场中的**左手**和**右手** Weyl 旋量场是**相关**的

因此，可以将 Majorana 旋量场 $\psi(x)$ 分解成 $\psi(x) = \begin{pmatrix}\eta_a(x)\\ \eta^{\dagger\dot{a}}(x)\end{pmatrix}$

而**自由 Majorana 旋量场**的**拉氏量**分解为

$$\mathcal{L} = \frac{1}{2}\bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = \frac{1}{2}\begin{pmatrix}\eta^a & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}-m\delta_a{}^b & \mathrm{i}(\sigma^\mu)_{a\dot{b}}\partial_\mu\\ \mathrm{i}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu & -m\delta^{\dot{a}}{}_{\dot{b}}\end{pmatrix}\begin{pmatrix}\eta_b\\ \eta^{\dagger\dot{b}}\end{pmatrix}$$
$$= \frac{1}{2}[\mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta + \mathrm{i}\eta\sigma^\mu\partial_\mu\eta^\dagger - m(\eta\eta + \eta^\dagger\eta^\dagger)]$$

利用 $\zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\zeta$ 将方括号中**第二项**化为

$$\mathrm{i}\eta\sigma^\mu\partial_\mu\eta^\dagger = \mathrm{i}\partial_\mu(\eta\sigma^\mu\eta^\dagger) - \mathrm{i}(\partial_\mu\eta)\sigma^\mu\eta^\dagger = \mathrm{i}\partial_\mu(\eta\sigma^\mu\eta^\dagger) + \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta$$

扔掉全散度项 $\mathrm{i}\partial_\mu(\eta\sigma^\mu\eta^\dagger)$，拉氏量变成 $\mathcal{L} = \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta - \frac{1}{2}m(\eta\eta + \eta^\dagger\eta^\dagger)$

这里的质量项只涉及**一个 Weyl 旋量场** $\eta_a(x)$，称为 **Majorana 质量项**

Majorana 旋量场的 $\bar{\psi}\gamma^\mu\psi$ 和 $\bar{\psi}\sigma^{\mu\nu}\psi$

- $\zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\zeta$、$\eta\sigma^\mu\bar{\sigma}^\nu\zeta = \zeta\sigma^\nu\bar{\sigma}^\mu\eta$ 和 $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger = \zeta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger$ 意味着

$$\eta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\eta, \quad \eta\sigma^\mu\bar{\sigma}^\nu\eta = \eta\sigma^\nu\bar{\sigma}^\mu\eta, \quad \eta^\dagger\bar{\sigma}^\mu\sigma^\nu\eta^\dagger = \eta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger$$

- 对于 **Majorana 旋量场**，$\eta = \zeta$，$\bar{\psi}\gamma^\mu\psi = \zeta\sigma^\mu\zeta^\dagger + \eta^\dagger\bar{\sigma}^\mu\eta$ 化为

$$\bar{\psi}\gamma^\mu\psi = \eta\sigma^\mu\eta^\dagger + \eta^\dagger\bar{\sigma}^\mu\eta = -\eta^\dagger\bar{\sigma}^\mu\eta + \eta^\dagger\bar{\sigma}^\mu\eta = 0$$

- $\bar{\psi}\sigma^{\mu\nu}\psi = \frac{\mathrm{i}}{2}\zeta(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)\eta + \frac{\mathrm{i}}{2}\eta^\dagger(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)\zeta^\dagger$ 化为

$$\bar{\psi}\sigma^{\mu\nu}\psi = \frac{\mathrm{i}}{2}(\eta\sigma^\mu\bar{\sigma}^\nu\eta - \eta\sigma^\nu\bar{\sigma}^\mu\eta) + \frac{\mathrm{i}}{2}(\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\eta^\dagger - \eta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger) = 0$$

- 这样就验证了上一节的**结论**

手征旋量场

- 本小节从**手征旋量场**的角度分析 **Dirac 质量项**和 **Majorana 质量项**的构造过程
- 用 **Weyl 旋量场** $\eta_a(x)$ 和 $\zeta^{\dagger\dot{a}}(x)$ 将四分量**左手旋量场** $\psi_{\mathrm{L}}(x)$ 和**右手旋量场** $\psi_{\mathrm{R}}(x)$ 表达为

$$\psi_{\mathrm{L}} = \begin{pmatrix} \eta_a \\ 0 \end{pmatrix}, \quad \psi_{\mathrm{R}} = \begin{pmatrix} 0 \\ \zeta^{\dagger\dot{a}} \end{pmatrix}$$

$$\bar{\psi}_{\mathrm{L}} = (\psi_{\mathrm{L}})^\dagger\gamma^0 = \begin{pmatrix} \eta^\dagger_{\dot{b}} & 0 \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}{}_{\dot{a}} \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} 0 & \eta^\dagger_{\dot{a}} \end{pmatrix}$$

$$\bar{\psi}_{\mathrm{R}} = (\psi_{\mathrm{R}})^\dagger\gamma^0 = \begin{pmatrix} 0 & \zeta^b \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}{}_{\dot{a}} \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & 0 \end{pmatrix}$$

- 从而，拉氏量

$$\mathcal{L}_{\mathrm{L}} = \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta = \mathrm{i}\begin{pmatrix} 0 & \eta^\dagger_{\dot{a}} \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix} \partial_\mu \begin{pmatrix} \eta_b \\ 0 \end{pmatrix} = \mathrm{i}\bar{\psi}_{\mathrm{L}}\gamma^\mu\partial_\mu\psi_{\mathrm{L}}$$

$$\mathcal{L}_{\mathrm{R}} = \mathrm{i}\zeta\sigma^\mu\partial_\mu\zeta^\dagger = \mathrm{i}\begin{pmatrix} \zeta^a & 0 \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix} \partial_\mu \begin{pmatrix} 0 \\ \zeta^{\dagger\dot{b}} \end{pmatrix} = \mathrm{i}\bar{\psi}_{\mathrm{R}}\gamma^\mu\partial_\mu\psi_{\mathrm{R}}$$

- 分别描述**自由**的**无质量左手旋量场** ψ_{L} 和**右手旋量场** ψ_{R}

构造 Dirac 质量项

如果要构造 **Dirac 质量项**，需采用前面给出的 $\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} = \zeta\eta$ 和 $\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} = \eta^\dagger\zeta^\dagger$，得到**自由 Dirac 旋量场** $\psi(x) = \psi_{\mathrm{L}}(x) + \psi_{\mathrm{R}}(x)$ 的拉氏量

$$\begin{aligned}\mathcal{L}_{\mathrm{D}} &= \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta + \mathrm{i}\zeta\sigma^\mu\partial_\mu\zeta^\dagger - m(\zeta\eta + \eta^\dagger\zeta^\dagger) \\ &= \mathrm{i}\bar{\psi}_{\mathrm{L}}\gamma^\mu\partial_\mu\psi_{\mathrm{L}} + \mathrm{i}\bar{\psi}_{\mathrm{R}}\gamma^\mu\partial_\mu\psi_{\mathrm{R}} - m(\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}})\end{aligned}$$

如果要构造 **Majorana 质量项**，需用到 ψ_{L} 和 ψ_{R} 的**电荷共轭场**：

$$\begin{aligned}
(\psi_{\mathrm{L}})^{\mathrm{C}} &= \mathcal{C}(\bar{\psi}_{\mathrm{L}})^{\mathrm{T}} = \mathcal{C}\begin{pmatrix}0 & \eta^\dagger_{\dot{b}}\end{pmatrix}^{\mathrm{T}} = \begin{pmatrix}\varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}}\end{pmatrix}\begin{pmatrix}0 \\ \eta^\dagger_{\dot{b}}\end{pmatrix} = \begin{pmatrix}0 \\ \eta^{\dagger\dot{a}}\end{pmatrix} \\
(\psi_{\mathrm{R}})^{\mathrm{C}} &= \mathcal{C}(\bar{\psi}_{\mathrm{R}})^{\mathrm{T}} = \mathcal{C}\begin{pmatrix}\zeta^b & 0\end{pmatrix}^{\mathrm{T}} = \begin{pmatrix}\varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}}\end{pmatrix}\begin{pmatrix}\zeta^b \\ 0\end{pmatrix} = \begin{pmatrix}\zeta_a \\ 0\end{pmatrix} \\
\overline{(\psi_{\mathrm{L}})^{\mathrm{C}}} &= [(\psi_{\mathrm{L}})^{\mathrm{C}}]^\dagger\gamma^0 = \begin{pmatrix}0 & \eta^b\end{pmatrix}\begin{pmatrix} & \delta^{\dot{b}}{}_{\dot{a}} \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix}\eta^a & 0\end{pmatrix} \\
\overline{(\psi_{\mathrm{R}})^{\mathrm{C}}} &= [(\psi_{\mathrm{R}})^{\mathrm{C}}]^\dagger\gamma^0 = \begin{pmatrix}\zeta^\dagger_{\dot{b}} & 0\end{pmatrix}\begin{pmatrix} & \delta^{\dot{b}}{}_{\dot{a}} \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix}0 & \zeta^\dagger_{\dot{a}}\end{pmatrix}
\end{aligned}$$

构造 Majorana 质量项

由此推出

$$\overline{(\psi_\mathrm{L})^\mathrm{C}}(\psi_\mathrm{L})^\mathrm{C} = \begin{pmatrix}\eta^a & 0\end{pmatrix}\begin{pmatrix}0\\ \eta^{\dagger\dot{a}}\end{pmatrix} = 0, \qquad \overline{(\psi_\mathrm{R})^\mathrm{C}}(\psi_\mathrm{R})^\mathrm{C} = \begin{pmatrix}0 & \zeta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}\zeta_a\\ 0\end{pmatrix} = 0$$

$$\overline{(\psi_\mathrm{R})^\mathrm{C}}\psi_\mathrm{L} = \begin{pmatrix}0 & \zeta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}\eta_a\\ 0\end{pmatrix} = 0, \qquad \overline{(\psi_\mathrm{L})^\mathrm{C}}\psi_\mathrm{R} = \begin{pmatrix}\eta^a & 0\end{pmatrix}\begin{pmatrix}0\\ \zeta^{\dagger\dot{a}}\end{pmatrix} = 0$$

$$\bar{\psi}_\mathrm{R}(\psi_\mathrm{L})^\mathrm{C} = \begin{pmatrix}\zeta^a & 0\end{pmatrix}\begin{pmatrix}0\\ \eta^{\dagger\dot{a}}\end{pmatrix} = 0, \qquad \bar{\psi}_\mathrm{L}(\psi_\mathrm{R})^\mathrm{C} = \begin{pmatrix}0 & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}\zeta_a\\ 0\end{pmatrix} = 0$$

以上 6 个算符**不能**用于构造质量项。可用的 **Majorana 质量项算符**是

$$\overline{(\psi_\mathrm{L})^\mathrm{C}}\psi_\mathrm{L} = \begin{pmatrix}\eta^a & 0\end{pmatrix}\begin{pmatrix}\eta_a\\ 0\end{pmatrix} = \eta\eta, \qquad \bar{\psi}_\mathrm{L}(\psi_\mathrm{L})^\mathrm{C} = \begin{pmatrix}0 & \eta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}0\\ \eta^{\dagger\dot{a}}\end{pmatrix} = \eta^\dagger\eta^\dagger$$

$$\overline{(\psi_\mathrm{R})^\mathrm{C}}\psi_\mathrm{R} = \begin{pmatrix}0 & \zeta^\dagger_{\dot{a}}\end{pmatrix}\begin{pmatrix}0\\ \zeta^{\dagger\dot{a}}\end{pmatrix} = \zeta^\dagger\zeta^\dagger, \qquad \bar{\psi}_\mathrm{R}(\psi_\mathrm{R})^\mathrm{C} = \begin{pmatrix}\zeta^a & 0\end{pmatrix}\begin{pmatrix}\zeta_a\\ 0\end{pmatrix} = \zeta\zeta$$

$\overline{(\psi_\mathrm{L})^\mathrm{C}}\psi_\mathrm{L}$ 与 $\bar{\psi}_\mathrm{L}(\psi_\mathrm{L})^\mathrm{C}$ 互为厄米共轭，而 $\overline{(\psi_\mathrm{R})^\mathrm{C}}\psi_\mathrm{R}$ 与 $\bar{\psi}_\mathrm{R}(\psi_\mathrm{R})^\mathrm{C}$ 互为厄米共轭

组合成 Majorana 旋量场

将**自由 Majorana 旋量场**的拉氏量改写为

$$\mathcal{L}_{\mathrm{M1}} = \mathrm{i}\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta - \frac{m}{2}(\eta\eta+\eta^\dagger\eta^\dagger) = \mathrm{i}\bar{\psi}_{\mathrm{L}}\gamma^\mu\partial_\mu\psi_{\mathrm{L}} - \frac{m}{2}\left[\overline{(\psi_{\mathrm{L}})^{\mathrm{C}}}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{L}}(\psi_{\mathrm{L}})^{\mathrm{C}}\right]$$

它描述 **Majorana 旋量场** $\psi_1 \equiv \psi_{\mathrm{L}} + (\psi_{\mathrm{L}})^{\mathrm{C}} = \begin{pmatrix} \eta_a \\ \eta^{\dagger\dot{a}} \end{pmatrix}$ 的自由运动

另一方面，描述 **Majorana 旋量场** $\psi_2 \equiv \psi_{\mathrm{R}} + (\psi_{\mathrm{R}})^{\mathrm{C}} = \begin{pmatrix} \zeta_a \\ \zeta^{\dagger\dot{a}} \end{pmatrix}$ 自由运动的拉氏量是

$$\mathcal{L}_{\mathrm{M2}} = \mathrm{i}\zeta\sigma^\mu\partial_\mu\zeta^\dagger - \frac{m}{2}(\zeta\zeta+\zeta^\dagger\zeta^\dagger) = \mathrm{i}\bar{\psi}_{\mathrm{R}}\gamma^\mu\partial_\mu\psi_{\mathrm{R}} - \frac{m}{2}\left[\bar{\psi}_{\mathrm{R}}(\psi_{\mathrm{R}})^{\mathrm{C}} + \overline{(\psi_{\mathrm{R}})^{\mathrm{C}}}\psi_{\mathrm{R}}\right]$$