

粲强子衰变过程中轻标量介子的研究

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第三届BESIII-Belle II-LHCb粲强子物理联合研讨会

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轻强子理论研究

□轻介子

- 轻标量介子
- 胶球候选态: $X(2370)$, $f_0(1710)$
- Hybrid, $\pi(1400)$, $\pi(1600)$, $\eta_1(1855)$
- 重子-反重子阈值

□轻重子

- 丢失的重子激发态
- 低能重子激发态 $\Lambda(1405)$, $N(1535)$, $N(1440)$, $\Sigma(1/2^-)$, $\Xi(1/2^-)$

粲强子的多体衰变过程

□ 粲介子和粲重子 $A \rightarrow 1 + 2 + 3$

- 末态强子强子相互作用
- 粲介子和粲重子衰变为研究轻强子提供了较大的相空间
- **BESIII, Belle/Belle II, LHCb**提供了丰富的实验数据

□ 粲偶素三体衰变

- 强衰变过程
- 满足同位旋守恒
- 粲偶素是味道单态，可以利用 **SU(3)** 味对称性进行分析

粲介子多体衰变

- $D^0 \rightarrow \pi^0 \pi^0 \bar{K}^0$, *Phys.Rev.D* 110 (2024) 114050, BCKe, DMLi, **EW**
- $D_s^+ \rightarrow \pi^0 \pi^0 \pi^+$, *Chin.Phys.C* 47 (2023) 4, 043101, H Zhang, YH Lyu, LJ Liu, **EW**
- $D_s^+ \rightarrow \pi^0 K^+ \bar{K}_S^0$, *Phys.Rev.D* 107 (2023) 3, 034001, **EW**, DMLi, LSGeng, JJXie
- $D_s^+ \rightarrow \pi^+ \bar{K}_S^0 \bar{K}_S^0$, *Phys.Rev.D* 105 (2022) 11, 116010, **EW**, DMLi, LSGeng, JJXie
- $D_s^+ \rightarrow \pi^+ K^+ K^-$, *Phys.Lett.B* 821 (2021) 136617, JYWang, MYDuan, DMLi, **EW**
- $D^+ \rightarrow \pi^+ \pi^0 \eta$, *Eur.Phys.J.C* 80 (2020) 1041, MYDuan, Jywang, GYWang, DMLi, **EW**
- $D^+ \rightarrow VP \nu e^+$, *Eur.Phys.J.C* 80 (2020) 5, 388, GYWang, Roca, **EW**, WHLiang, Oset

粲重子多体衰变

- $\Xi_c \rightarrow \Lambda \bar{K} \pi$, PRD111 (2025)054011, GYWang, LKLi, WCYan, **EW**
- $\Lambda_c^+ \rightarrow \pi^+ \eta n$, *Phys.Rev.D* 111 (2025) 3, 034046, MYLi, WTLyu, **EW**
- $\Lambda_c^+ \rightarrow \pi^+ \eta \Lambda$, *Phys.Rev.D* 111 (2025) 1, 016004, *Phys.Rev.D* 110 (2024) 5, 054020, *Phys.Rev.D* 106 (2022) 5, 056001, **EW**, JJXie, JJWu, LSGeng, Oset, CWXiao
- $\Lambda_c^+ \rightarrow \pi^0 \eta p$, *Phys.Lett.B* 846 (2023) 138185, XCFeng, LLWei, MYDuan, DMLi, **EW**
- $\Lambda_c^+ \rightarrow p \bar{K}^0 \eta$, *Phys.Rev.D* 110 (2024) 7, 074010, LSGeng, JJXie, DMLi, **EW**
- $\Lambda_c^+ \rightarrow p K^- \pi^+$, MYDuan, *Eur.Phys.J.C* 84 (2024) 12, 1253, GYWang, WTLyu, **EW**
- $\Omega_c \rightarrow \pi^+ \Omega(2012) \rightarrow \pi^+ \bar{K} \Xi(\pi^+ \bar{K} \Xi \pi)$, *Phys.Rev.D* 102 (2020) 7, 076009, CHZeng, JXLu, **EW**, LSGeng, JJXie
- $\Lambda_c^+ \rightarrow p \pi^+ \pi^-$, $p K^+ K^-$, *Eur.Phys.J.C* 80 (2020) 9, 842, Zwang, **EW**, DMLi, JJXie

粲偶素多体衰变

- $J/\psi \rightarrow \bar{K}^0 K^+ \rho^-$, *Phys.Rev.D* 110 (2024) 1, 014032, **EW**, DMLi, LSGeng, JJXie
- $\eta_c \rightarrow \bar{K}^0 K^+ \pi^-$, *Phys.Rev.D* 108 (2023) 11, 114004, **EW**, DMLi, LSGeng, JJXie
- $\chi_{c0} \rightarrow \bar{p} K^+ \Lambda$, *Phys.Rev.D* 102 (2020) 3, 036003, GYWang, MYDuan, **EW**, DMLi
- $J/\psi \rightarrow K^+ K^- f_0(980)$, *Eur.Phys.J.C* 79 (2019) 5, 411, HXChen, WHLiang, **EW**, Oset
- $\chi_{c0} \rightarrow \bar{\Lambda} \Sigma \pi$, *Phys.Rev.D* 98 (2018) 11, 114017, LJLiu, **EW**, JJXie
- $\chi_{c0} \rightarrow \bar{\Sigma} \Sigma \pi$, *Phys.Lett.B* 753 (2016) 526, **EW**, JJXie, Oset

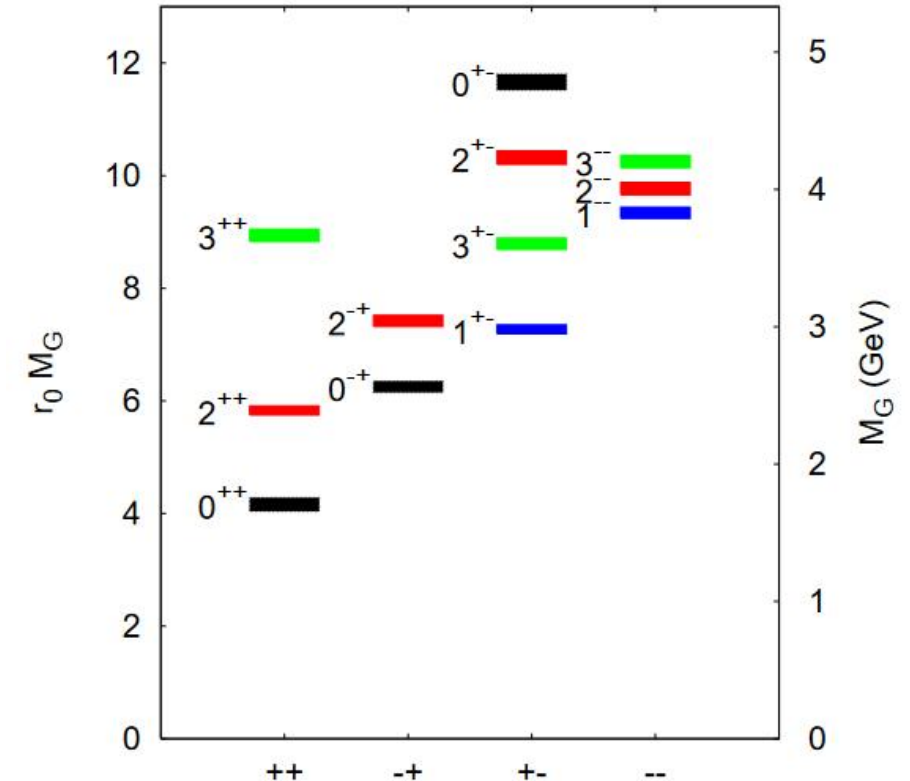
Scalar mesons

- **Masses puzzle:** $a_0(980)/f_0(980)$, $K(700)$, $f_0(500)$
- $a_0(980)/f_0(980)$: $q\bar{q}$, tetraquark, hadronic molecules
- $f_0(1370)$, $f_0(1500)$, $f_0(1710)$
- $f_0(1710)$: glueball, $s\bar{s}$, $K^*\bar{K}^*$

PDG: Tentative classification

	Γ [MeV]	isospin i	structure
$a_0(980)$	~ 50	1	$K\bar{K}, qq\bar{q}\bar{q}$
$f_0(980)$	~ 50	0	$K\bar{K}, qq\bar{q}\bar{q}$
$f_0(500)$	~ 800	0	$\pi\pi, qq\bar{q}\bar{q}$
$K_0^*(700)$	~ 600	$\frac{1}{2}$	$K\pi, qq\bar{q}\bar{q}$
$a_0(1450)$	265	1	$u\bar{d}, d\bar{u}, d\bar{d} - u\bar{u}$
$f_0(1370)$	~ 400	0	$d\bar{d} + u\bar{u}$
$f_0(1710)$	125	0	$s\bar{s}$
$K_0^*(1430)$	294	$\frac{1}{2}$	$u\bar{s}, d\bar{s}, s\bar{u}, s\bar{d}$

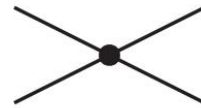
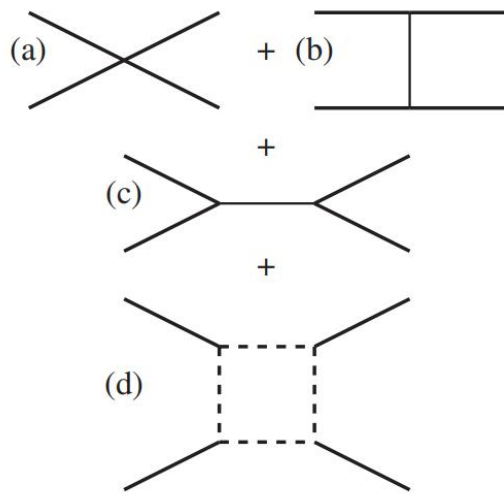
Y. Chen, PRD73(2006)014516



VV interaction from hidden-gauge lagrangians

□ The hidden-gauge Lagrangian Geng-Oset, PRD79(2009)074009

$$g = \frac{M_V}{2f},$$



$$\mathcal{L} = -\frac{1}{4}\langle \bar{V}_{\mu\nu} \bar{V}^{\mu\nu} \rangle + \frac{1}{2}M_V^2 \langle [V_\mu - (i/g)\Gamma_\mu]^2 \rangle,$$

$$\bar{V}_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - ig[V_\mu, V_\nu],$$

$$\Gamma_\mu = \frac{1}{2}\{u^\dagger[\partial_\mu - i(v_\mu + a_\mu)]u + u[\partial_\mu - i(v_\mu - a_\mu)]u^\dagger\},$$

$$\mathcal{L}_{VVVV} = \frac{1}{2}g^2 \langle [V_\mu, V_\nu] V^\mu V^\nu \rangle,$$

$$u^2 = U = \exp\left(\frac{i\sqrt{2}\Phi}{f}\right)$$

$$V_\mu = \begin{pmatrix} \frac{\omega + \rho^0}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & \frac{\omega - \rho^0}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix}_\mu, \quad \Phi = \begin{pmatrix} \frac{\eta}{\sqrt{6}} + \frac{\pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta}{\sqrt{6}} - \frac{\pi^0}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}.$$

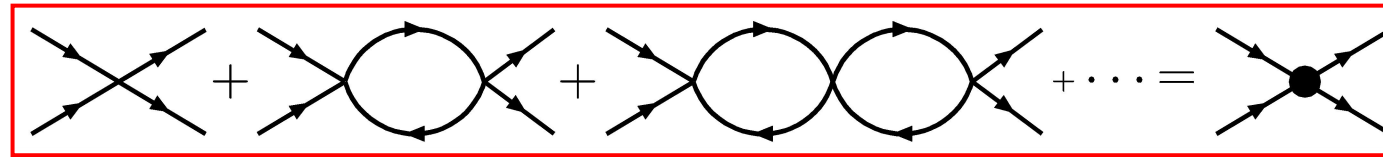
$$\begin{aligned} \mathcal{L}_{VVV} &= ig \langle (\partial_\mu V_\nu - \partial_\nu V_\mu) V^\mu V^\nu \rangle \\ &= ig \langle V^\mu \partial_\nu V_\mu V^\nu - \partial_\nu V_\mu V^\mu V^\nu \rangle \\ &= ig \langle (V^\mu \partial_\nu V_\mu - \partial_\nu V_\mu V^\mu) V^\nu \rangle. \end{aligned}$$

$$\mathcal{L}_{V\Phi\Phi} = -ig \langle V_\mu [\Phi, \partial^\mu \Phi] \rangle.$$

Bethe-Salpeter equation

□ Transition amplitude T

$$V + VGV + VGVGV + \dots = T$$



□ On-shell factorization

$$\begin{aligned} VGV &= \int d^4q V(q^2) \frac{i}{q^2 - m_1^2 + i\epsilon} \frac{1}{(P - q)^2 - m_2^2 + i\epsilon} V(q^2) \\ &= V(m^2)GV(m^2) \end{aligned}$$

$$V(q^2) = V(m^2) + \frac{\partial V}{\partial q^2} \Big|_{q^2=m^2} (q^2 - m^2) \quad G = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - M_{V1}^2} \frac{1}{q^2 - M_{V2}^2}$$

$$T = (1 - VG)^{-1}V$$

Free parameter

□ Loop function

$$G = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - M_{V_1}^2} \frac{1}{q^2 - M_{V_2}^2}$$

□ Subtraction constants in dimensional regularization (DR)

$$G = \frac{1}{16\pi^2} \left(\alpha + \text{Log} \frac{m_1^2}{\mu^2} + \frac{m_2^2 - m_1^2 + s}{2s} \text{Log} \frac{m_2^2}{m_1^2} \right. \\ \left. + \frac{p}{\sqrt{s}} \left(\text{Log} \frac{s - m_2^2 + m_1^2 + 2p\sqrt{s}}{-s + m_2^2 - m_1^2 + 2p\sqrt{s}} + \text{Log} \frac{s + m_2^2 - m_1^2 + 2p\sqrt{s}}{-s - m_2^2 + m_1^2 + 2p\sqrt{s}} \right) \right)$$

□ cutoff values in cutoff method

$$G = \int_0^{q_{max}} \frac{q^2 dq}{(2\pi)^2} \frac{\omega_1 + \omega_2}{\omega_1 \omega_2 [(P^0)^2 - (\omega_1 + \omega_2)^2 + i\epsilon]}$$

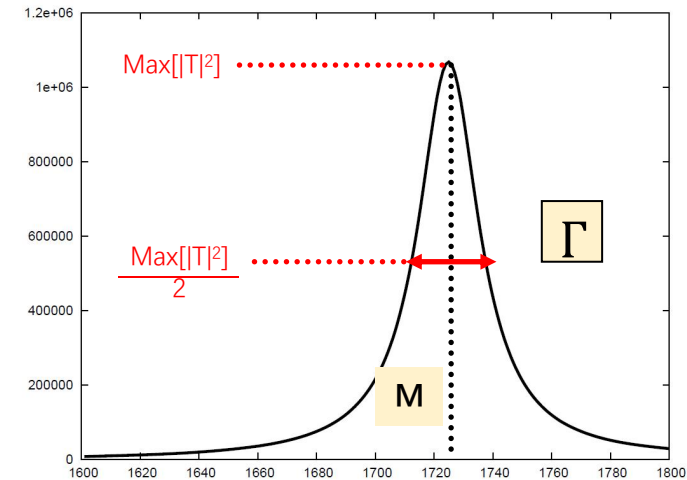
Since we work with hadronic phenomena, one often uses a cutoff value of about 1 GeV to guide calculations in DR method.

Dynamically generated state

- To identify resonances, one goes to the complex plane to look for poles.
- Around the pole position, the amplitude can be approximated by

$$T_{ij} = \frac{g_i g_j}{s - s_{\text{pole}}},$$

$$M = \text{Re}\sqrt{s}_{\text{pole}} \quad \text{and} \quad \Gamma = 2 \times \text{Im}\sqrt{s}_{\text{pole}}$$



Dynamically generated states

➤ **strangeness=0, isospin=0, and spin=2**

	pole position (no two p's)	real axis (with two p's)
$f_2(1270)$	(1275, 2)	(1276, 97)
$f'_2(1525)$	(1525, 6)	(1525, 45)

Two subtraction constants slightly tuned to reproduce exactly the masses!

➤ **strangeness=0, isospin=0, and spin=0**

	pole position (no two p's)	real axis (two p's)
$f_0(1370)$	(1512, 51)	(1523, 257)
$f_0(1710)$	(1726, 28)	(1721, 133)

Table: Branching ratios of the $f_0(1710)$ in comparison with data.

	Our model	PDG
$\Gamma(\pi\pi)/\Gamma(K\bar{K})$	< 1%	< 11% at 95% C.L.
$\Gamma(\eta\eta)/\Gamma(K\bar{K})$	~ 50%	(48 ± 15)%

No tuning of the parameters, their positions are fixed by those of the f_2 states!

Dynamically generated states

Notation: (mass, width) in MeV

$I^G(J^{PC})$	Theory			PDG data		
	pole position	real axis		name	mass	width
		$\Lambda_b = 1.4 \text{ GeV}$	$\Lambda_b = 1.5 \text{ GeV}$			
$0^+(0^{++})$	(1512,51)	(1523,257)	(1517,396)	$f_0(1370)$	1200~1500	200~500 ●
$0^+(0^{++})$	(1726,28)	(1721,133)	(1717,151)	$f_0(1710)$	1724 ± 7	137 ± 8 ●
$0^+(1^{++})$	(1802,78)	(1802,49)		f_1		●
$0^+(2^{++})$	(1275,2)	(1276,97)	(1275,111)	$f_2(1270)$	1275.1 ± 1.2	$185.0^{+2.9}_{-2.4}$ ●
$0^+(2^{++})$	(1525,6)	(1525,45)	(1525,51)	$f'_2(1525)$	1525 ± 5	73^{+6}_{-5} ●
$1^-(0^{++})$	(1780,133)	(1777,148)	(1777,172)	a_0		●
$1^+(1^{+-})$	(1679,235)	(1703,188)		b_1		●
$1^-(2^{++})$	(1569,32)	(1567,47)	(1566,51)	$a_2(1700)??$		●
$1/2(0^+)$	(1643,47)	(1639,139)	(1637,162)	K		●
$1/2(1^+)$	(1737,165)	(1743,126)		$K_1(1650)?$		●
$1/2(2^+)$	(1431,1)	(1431,56)	(1431,63)	$K_2^*(1430)$	1429 ± 1.4	104 ± 4 ●

Effects of PP coupled channels

PHYSICAL REVIEW D **104**, 114001 (2021)

Further study of $f_0(1710)$ with the coupled-channel approach and the hadron molecular picture

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The $f_0(1710)$ was previously proposed to be a dynamically generated state with interactions between vector mesons. We extend the study of $f_0(1710)$ by including its coupling to channels of pseudoscalar mesons within the coupled-channel approach. The channels involved are $K^*\bar{K}^*$, $\rho\rho$, $\omega\omega$, $\phi\phi$, $\omega\phi$, $\pi\pi$, $K\bar{K}$, $\eta\eta$. We show that the pole assigned to $f_0(1710)$ does not change much. Then we calculate the partial decay widths of $f_0(1710) \rightarrow K^*\bar{K}^* \rightarrow \pi\pi, K\bar{K}, \eta\eta$ as the coupled channel dynamically generated state as well as assuming it to be a pure $K^*\bar{K}^*$ molecule. In both cases the ratios of partial decay widths agree fairly with that in PDG.

Other predictions for $a_0(1710)$

PHYSICAL REVIEW D **83**, 016007 (2011)

Low-lying even-parity meson resonances and spin-flavor symmetry

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(Received 6 May 2010; published 20 January 2011)*

$\sqrt{s_R}$	$\eta\pi$	$\bar{K}K$	$\omega\rho$	$\phi\rho$	$\bar{K}^*K^*$
(991, -46)	2906	3831	775	4185	5541
(1442, -5)	907	285	10898	677	3117
(1760, -12)	790	1241	667	5962	5753

PHYSICAL REVIEW D **97**, 034030 (2018)

Strong decays of the higher isovector scalar mesons

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Department of Physics, Zhengzhou University, Zhengzhou, Henan 450001, China

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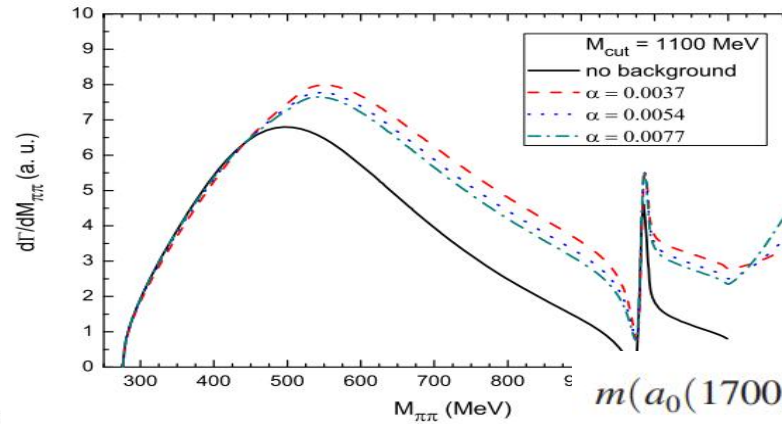
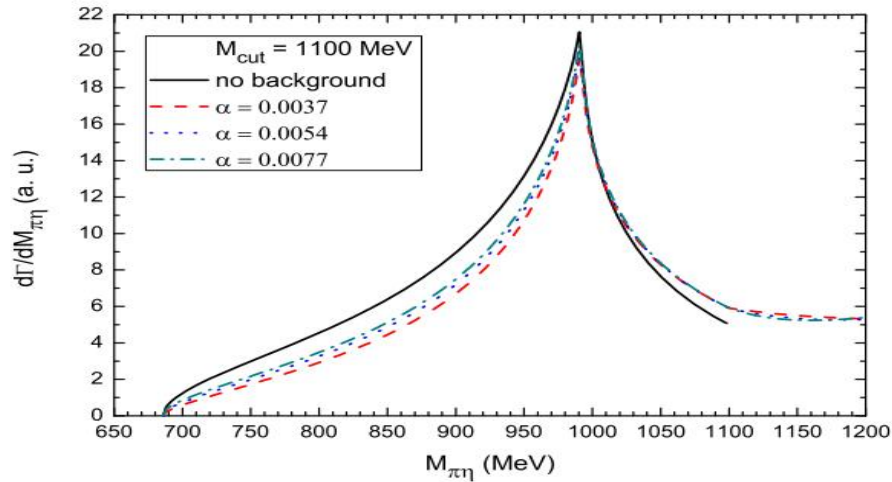
TABLE IV. Decay widths of $a_0(2^3P_0)$ (in MeV). The initial state mass is set to be 1744 MeV.

Channel	Mode	$\Gamma_i(2^3P_0)$
$0^+ \rightarrow 0^-0^-$	$\pi\eta$	7.91
	$\pi\eta'$	2.46
	$\pi\eta(1475)$	19.25
	$\pi\eta(1295)$	20.86
	$K\bar{K}$	1.07
$0^+ \rightarrow 0^-1^+$	$\pi b_1(1235)$	213.08
	$\pi f_1(1285)$	38.54
	$\pi f_1(1420)$	1.02
$0^+ \rightarrow 1^-1^-$	$\rho\omega$	59.96
Total width		364.12

$a_0(1710)$

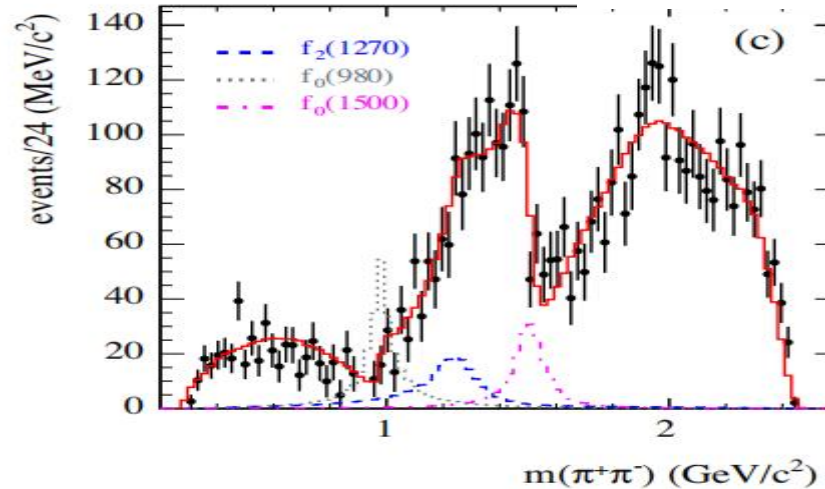
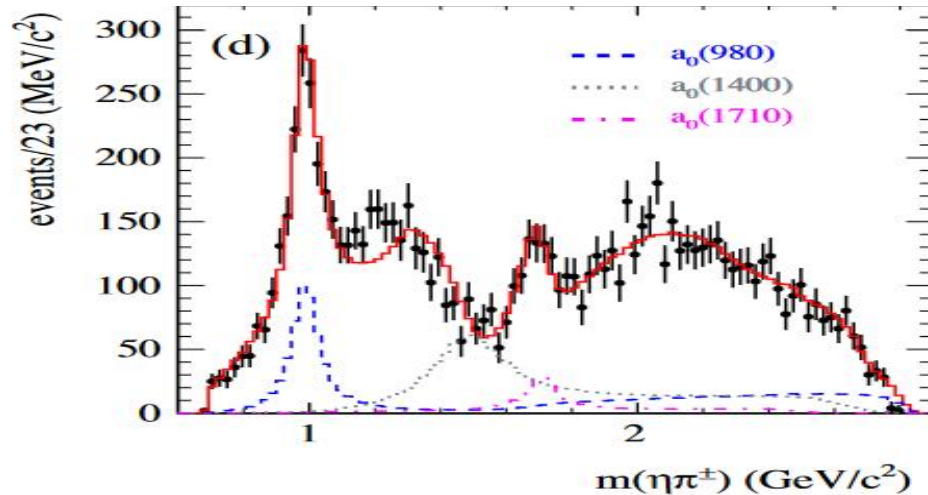
□ BaBar: $\eta_c \rightarrow \eta \pi^+ \pi^-$

Liang-Xie-Oset PLB766 (2017) 59



$$m(a_0(1700)) = 1704 \pm 5_{\text{stat}} \pm 2_{\text{sys}} \text{ MeV}/c^2,$$

$$\Gamma(a_0(1700)) = 110 \pm 15_{\text{stat}} \pm 11_{\text{sys}} \text{ MeV}/c^2.$$

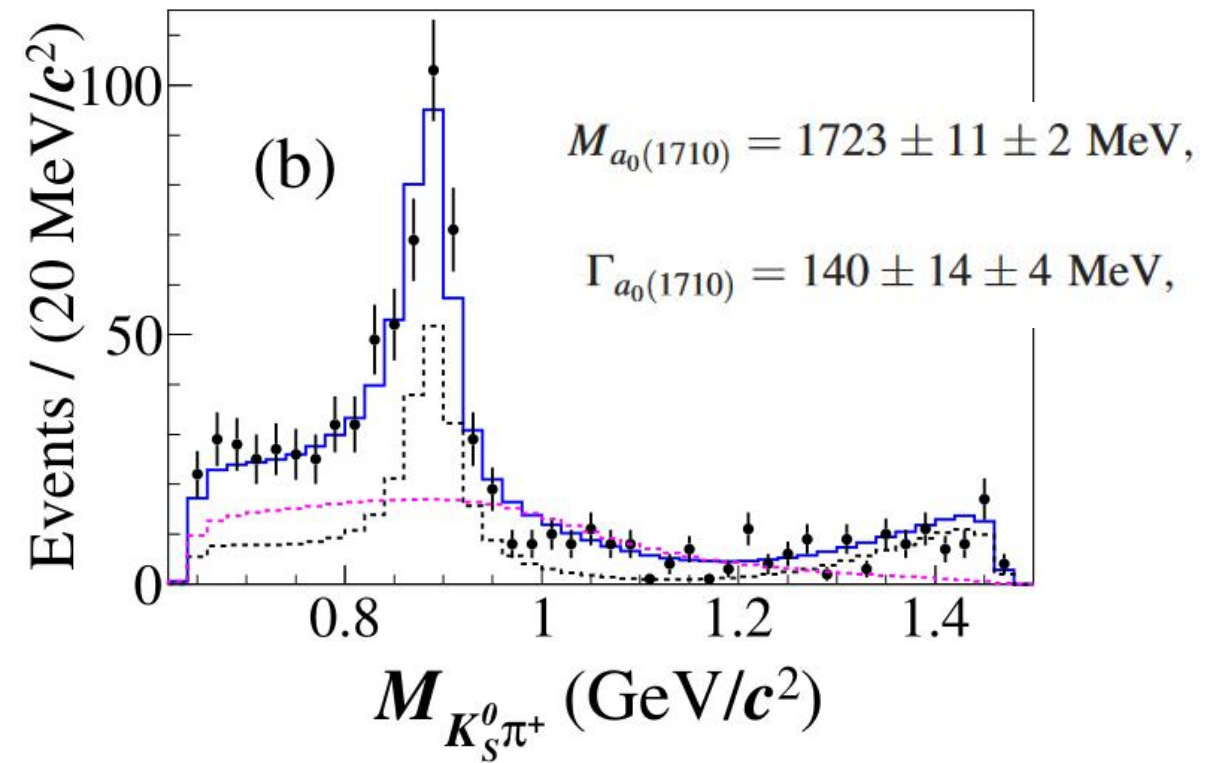
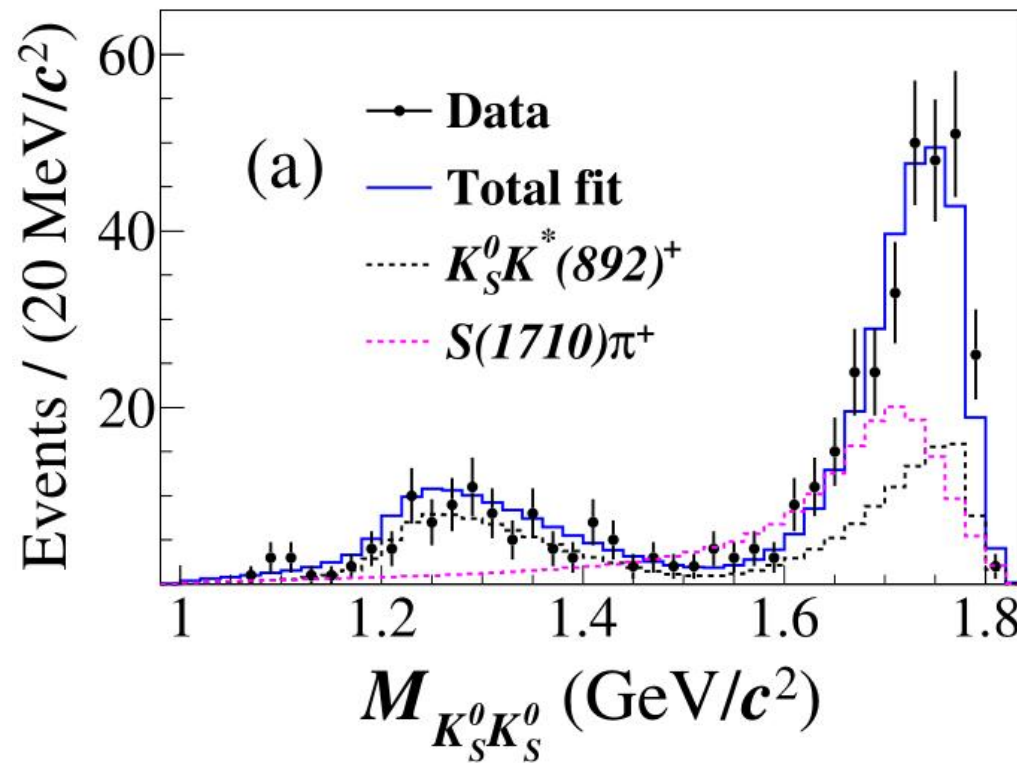


BaBar:
PRD104(2021)072002

BESIII measurements

$$\square D_s^+ \rightarrow K_s^0 K_s^0 \pi^+$$

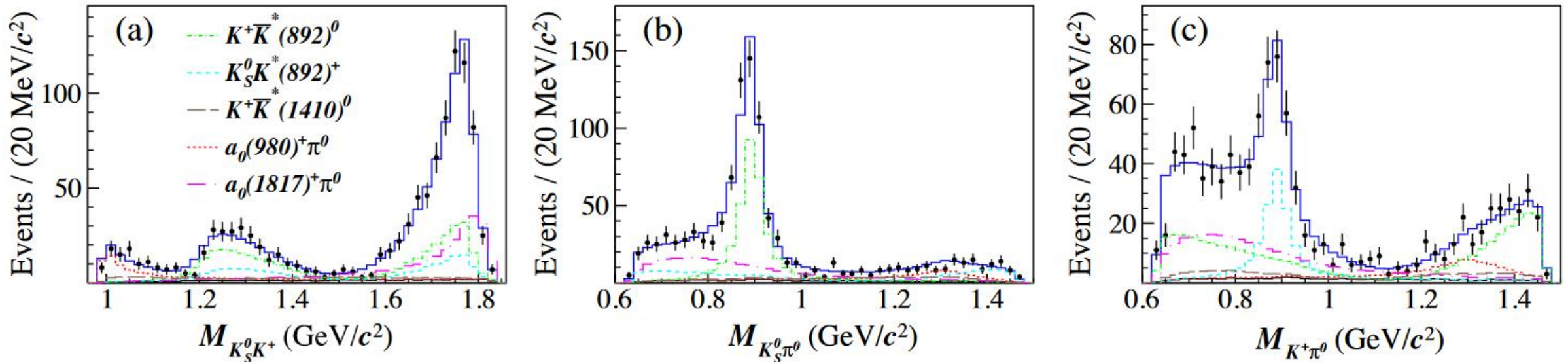
BESIII: PRD105 (2022) 5, L051103



BESIII measurements

$$\square D_s^+ \rightarrow K_S^0 K^+ \pi^0$$

BESIII: PRL129, 182001



$$M_{a_0(1710)} = 1817 \pm 8 \pm 20 \text{ MeV},$$

$$\Gamma_{a_0(1710)} = 97 \pm 22 \pm 15 \text{ MeV}.$$

Too close to the boundary region!!!

Collaboration	$M_{a_0(1710)}$	$\Gamma_{a_0(1710)}$	Ref.
BABAR	$1704 \pm 5 \pm 2$	$110 \pm 15 \pm 11$	[8]
BESIII	$1723 \pm 11 \pm 2$	$140 \pm 14 \pm 4$	[9]
BESIII	$1817 \pm 8 \pm 20$	$97 \pm 22 \pm 15$	[10]

VV interactions



Eur. Phys. J. C (2022) 82:509
<https://doi.org/10.1140/epjc/s10052-022-10460-4>

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Regular Article - Theoretical Physics

Two dynamical generated a_0 resonances by interactions between vector mesons

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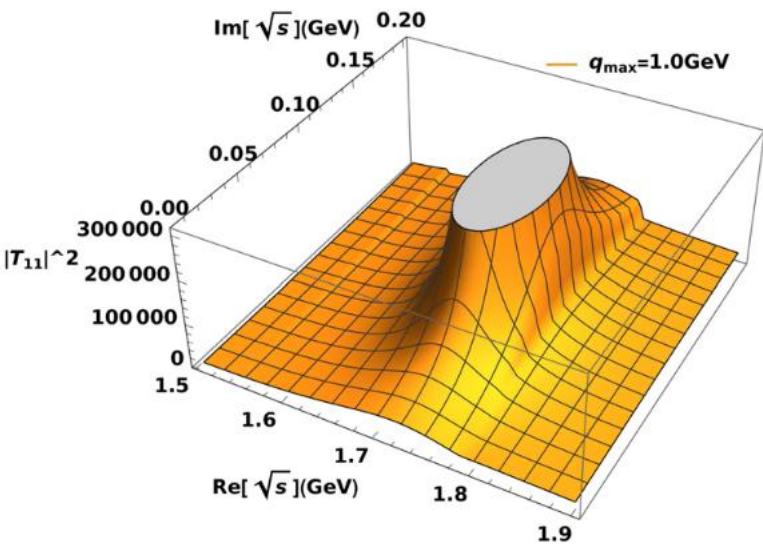
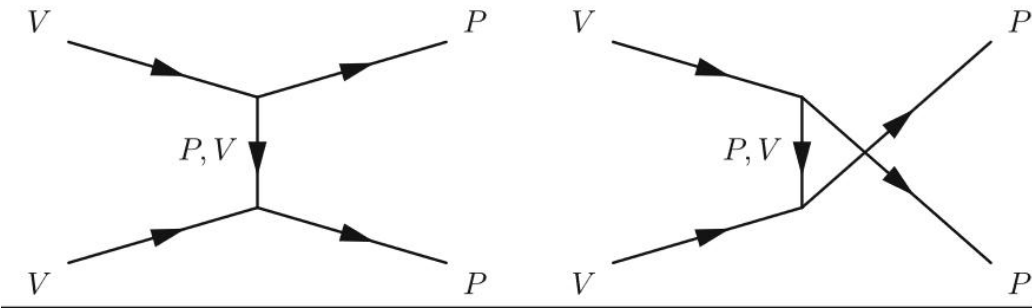


Table 3 The resonance pole for different cutoffs

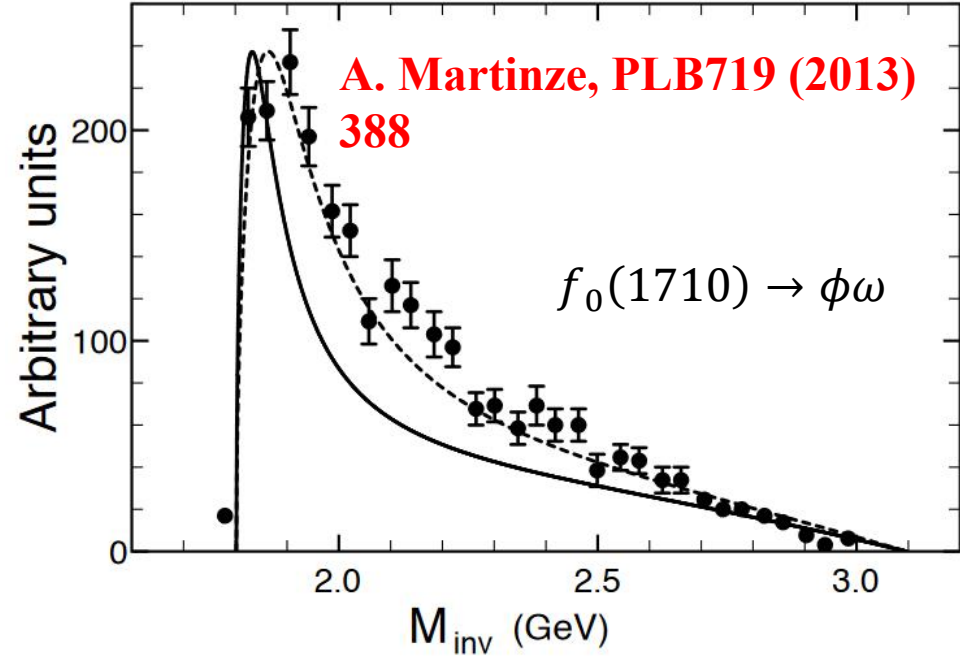
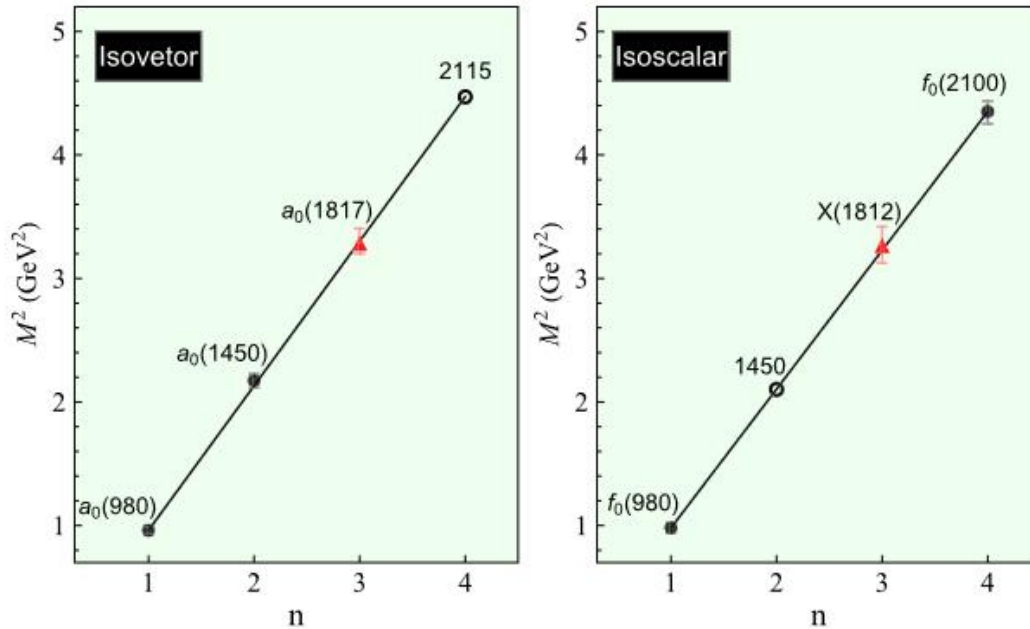
q_{max} (GeV)	0.9	1.0	1.1
Pole (GeV)	$1.76 - 0.09i$	$1.72 - 0.10i$	$1.69 - 0.11i$

$a_0(1817)$

□ $a_0(1817)$ & $X(1812)$ ($J/\psi \rightarrow \gamma \phi \omega$)

D.Guo, PRD105(2022)114014

BESIII: Phys.Rev.Lett. 96 (2006)
162002, PRD87(2013)032008



Other studies

□ Molecule

- Z.Y. Wang, Y.W. Peng, J.Y Yi, W.C. Luo, C. W. Xiao, PRD107 (2023) 1116018
- Oset-Dai-Geng, EPJC82 (2022) 3, 225, Sci.Bull. 68 (2023) 243

□ Four quark in MIT model

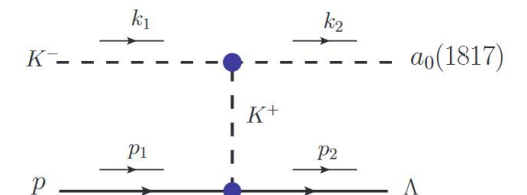
- N.N. Achasov, Phys.Rev.D 108 (2023) 3, 036018

□ Search for $a_0(1710/1817)$

- Abreu-Wang-Oset, Eur.Phys.J.C 83 (2023) 3, 243
- EW-JJXie-LSGeng-DMLi, PRD108(2023)114004
- Xiao-Yun Wang, Hui-Fang Zhou, Xiang Liu, Phys.Rev.D 108 (2023) 3, 034015
- EW-JJXie-LSGeng-DMLi, Phys.Rev.D 110 (2024) 1, 014032, $J/\psi \rightarrow \bar{K}^0 K^+ \rho^-$,

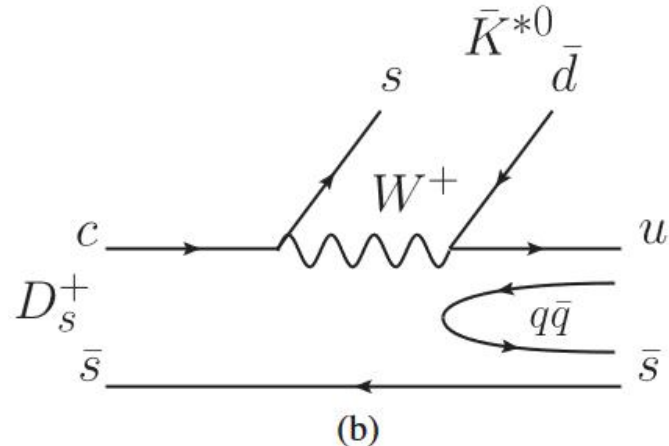
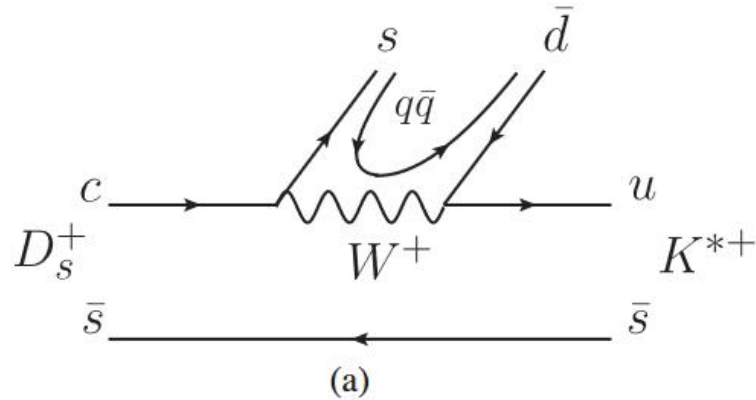
$$J/\psi \rightarrow \phi K^+ \bar{K}^- (K^0 \bar{K}^0)$$

$$\eta_c \rightarrow \bar{K}^0 K^+ \pi^-$$



Mechanism of the $D_s^+ \rightarrow K_S^0 K_S^0 \pi^+$

□ Weakly decay



$$D_s^+ \rightarrow V_1 [s\bar{d} \rightarrow s(\bar{u}u + \bar{d}d + \bar{s}s)\bar{d}] (u\bar{s} \rightarrow K^{*+}),$$

$$D_s^+ \rightarrow V_2 [u\bar{s} \rightarrow u(\bar{u}u + \bar{d}d + \bar{s}s)\bar{s}] (\bar{d}s \rightarrow \bar{K}^{*0}),$$

$$\sum_{i=u,d,s} s\bar{q}_i q_i \bar{d} = M_{3i} M_{i2} = (M^2)_{32},$$

$$\sum_{i=u,d,s} u\bar{q}_i q_i \bar{s} = M_{1i} M_{i3} = (M^2)_{13},$$

$$M = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}$$

Mechanism of the $D_s^+ \rightarrow K_S^0 K_S^0 \pi^+$

□Weakly decay+Hadronization

$$P = \begin{pmatrix} \frac{\eta}{\sqrt{3}} + \frac{\pi^0}{\sqrt{2}} + \frac{\eta'}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta}{\sqrt{3}} - \frac{\pi^0}{\sqrt{2}} + \frac{\eta'}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{\eta}{\sqrt{3}} + \frac{\sqrt{6}\eta'}{3} \end{pmatrix},$$

$$V = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix}.$$

$$(M^2)_{32} \rightarrow (V \cdot P)_{32} = \pi^+ K^{*-} - \frac{1}{\sqrt{2}} \pi^0 \bar{K}^{*0},$$

$$(M^2)_{13} \rightarrow (P \cdot V)_{13} = \pi^+ K^{*0} + \frac{1}{\sqrt{2}} \pi^0 K^{*+},$$

$$D_s^+ \rightarrow V_1 \left(\pi^+ K^{*-} K^{*+} - \frac{1}{\sqrt{2}} \pi^0 \bar{K}^{*0} K^{*+} \right),$$

$$D_s^+ \rightarrow V_2 \left(\pi^+ K^{*0} \bar{K}^{*0} + \frac{1}{\sqrt{2}} \pi^0 K^{*+} \bar{K}^{*0} \right).$$

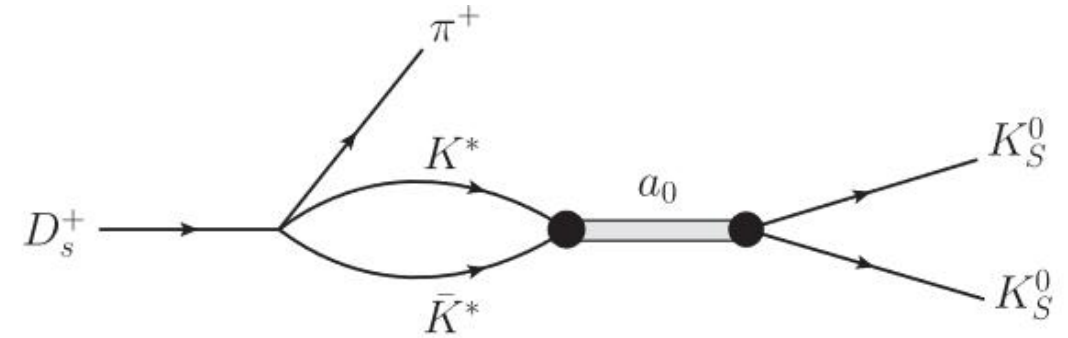
Mechanism of the $D_s^+ \rightarrow K_S^0 K_S^0 \pi^+$

□ Final state interaction

$$|K^{*0} \bar{K}^{*0}\rangle = \frac{1}{\sqrt{2}} (|K^* \bar{K}^*, I=1\rangle - |K^* \bar{K}^*, I=0\rangle),$$

$$|K^{*+} K^{*-}\rangle = -\frac{1}{\sqrt{2}} (|K^* \bar{K}^*, I=1\rangle + |K^* \bar{K}^*, I=0\rangle).$$

$$\begin{aligned} & V_1 K^{*+} K^{*-} + V_2 K^{*0} \bar{K}^{*0} \\ &= -\frac{V_1}{\sqrt{2}} (|K^* \bar{K}^*, I=1\rangle + |K^* \bar{K}^*, I=0\rangle) \\ & \quad + \frac{V_2}{\sqrt{2}} (|K^* \bar{K}^*, I=1\rangle - |K^* \bar{K}^*, I=0\rangle) \\ &= \frac{V_2 - V_1}{\sqrt{2}} |K^* \bar{K}^*, I=1\rangle - \frac{V_2 + V_1}{\sqrt{2}} |K^* \bar{K}^*, I=0\rangle. \end{aligned}$$



$$\mathcal{M}_a = \frac{V_2 - V_1}{4} \tilde{G}_{K^* \bar{K}^*}(M_{K_S^0 K_S^0}) \times \frac{g_{K^* \bar{K}^*} g_{K \bar{K}}}{M_{K_S^0 K_S^0}^2 - M_{a_0(1710)}^2 + i M_{a_0(1710)} \Gamma_{a_0(1710)}},$$

Set	$M_{a_0(1710)}$	$\Gamma_{a_0(1710)}$	$g_{K^* \bar{K}^*}$	$\Gamma_{K \bar{K}}$
I (Refs. [13,15])	1777	148	(7525, -i1529)	36
II (Ref. [28])	1720	200	(8731, -i2200)	74

Mechanism of the $D_s^+ \rightarrow K_S^0 K_S^0 \pi^+$

□G function

$$G_{K^* \bar{K}^*}(M_{K_S^0 K_S^0}) = \int_{m_-^2}^{m_+^2} \int_{m_-^2}^{m_+^2} d\tilde{m}_1^2 d\tilde{m}_2^2 \\ \times \omega(\tilde{m}_1^2) \omega(\tilde{m}_2^2) \tilde{G}(M_{K_S^0 K_S^0}, \tilde{m}_1^2, \tilde{m}_2^2),$$

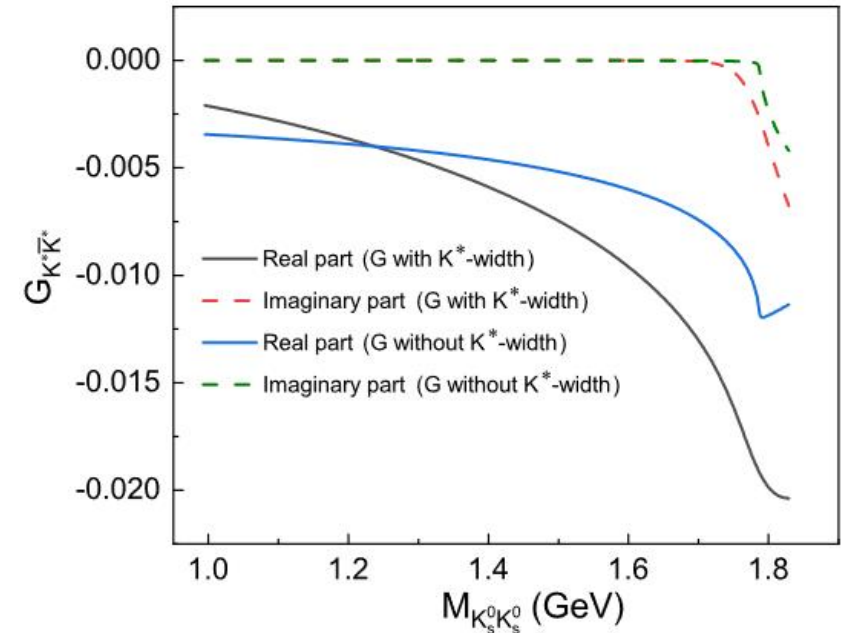
$$\omega(\tilde{m}_1^2) = \frac{1}{N} \text{Im} \left(\frac{1}{\tilde{m}_1^2 - m_{K^*}^2 + i\Gamma(\tilde{m}_1^2)\tilde{m}_1} \right)$$

$$N = \int_{m_-^2}^{m_+^2} d\tilde{m}_1^2 \text{Im} \left(\frac{1}{\tilde{m}_1^2 - m_{K^*}^2 + i\Gamma(\tilde{m}_1^2)\tilde{m}_1} \right),$$

$$\tilde{G} = \frac{1}{16\pi^2} \left\{ a_\mu + \ln \frac{m_1^2}{\mu^2} + \frac{m_2^2 - m_1^2 + s}{2s} \ln \frac{m_2^2}{m_1^2} \right. \\ \times \frac{p}{\sqrt{s}} [\ln(s - (m_2^2 - m_1^2) + 2p\sqrt{s}) \\ + \ln(s + (m_2^2 - m_1^2) + 2p\sqrt{s}) \\ - \ln(-s + (m_2^2 - m_1^2) + 2p\sqrt{s}) \\ \left. - \ln(-s - (m_2^2 - m_1^2) + 2p\sqrt{s})] \right\}$$

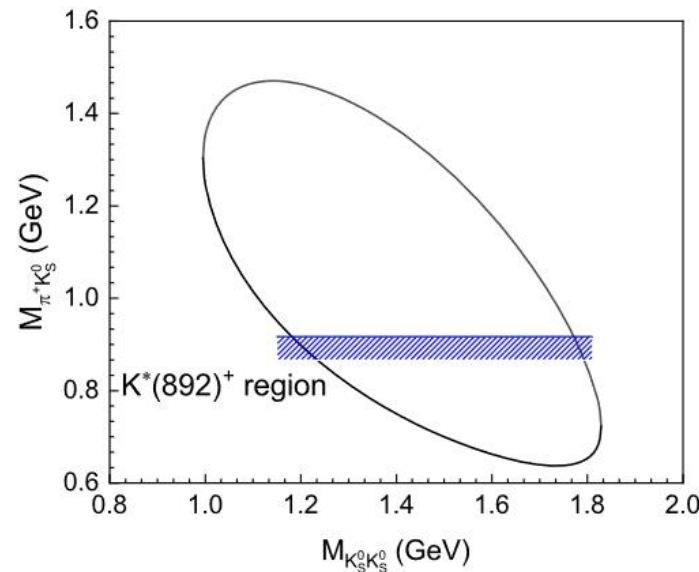
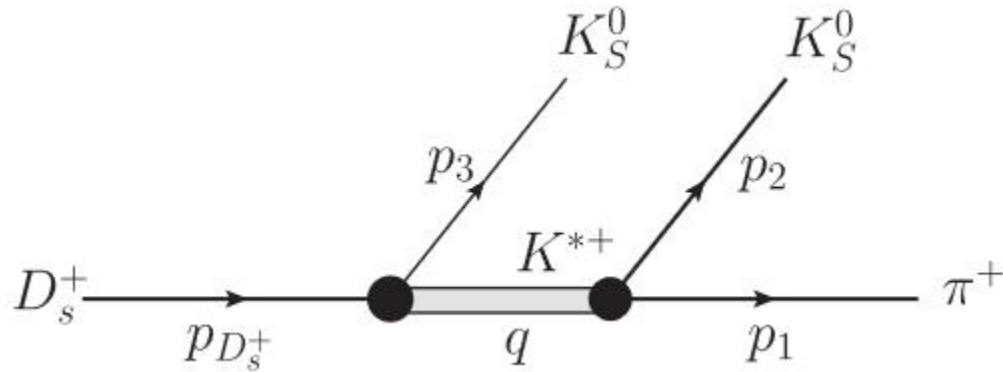
$$\Gamma(\tilde{m}_1^2) = \Gamma_{K^*} \frac{\tilde{k}^3}{k^3},$$

$$\tilde{k} = \frac{\lambda(\tilde{m}_1^2, m_\pi^2, m_K^2)}{2\tilde{m}_1},$$



Mechanism of the $D_s^+ \rightarrow K_S^0 K_S^0 \pi^+$

□ The contribution of K^*



$$\begin{aligned} \mathcal{M}_b = & \frac{g_{D_s \bar{K} K^*} g_{K^* K \pi}}{2} \frac{1}{q^2 - m_{K^{*+}}^2 + i m_{K^{*+}} \Gamma_{K^{*+}}} \\ & \times \left[(m_{K_S^0}^2 - m_{\pi^+}^2) \left(1 - \frac{q^2}{m_{K^{*+}}^2} \right) \right. \\ & + 2 p_1 \cdot p_3 \frac{m_{\pi^+}^2 - m_{K_S^0}^2 - m_{K^{*+}}^2}{m_{K^{*+}}^2} \\ & \left. + 2 p_2 \cdot p_3 \frac{m_{\pi^+}^2 - m_{K_S^0}^2 + m_{K^{*+}}^2}{m_{K^{*+}}^2} \right] \\ & + (\text{exchange term with } p_2 \leftrightarrow p_3), \end{aligned}$$

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b,$$

$$\frac{d^2 \Gamma}{dM_{K_S^0 K_S^0} dM_{\pi K_S^0}} = \frac{M_{K_S^0 K_S^0} M_{\pi K_S^0}}{128 \pi^3 m_{D_s^+}^3} (|\mathcal{M}_a|^2 + |\mathcal{M}_b|^2),$$

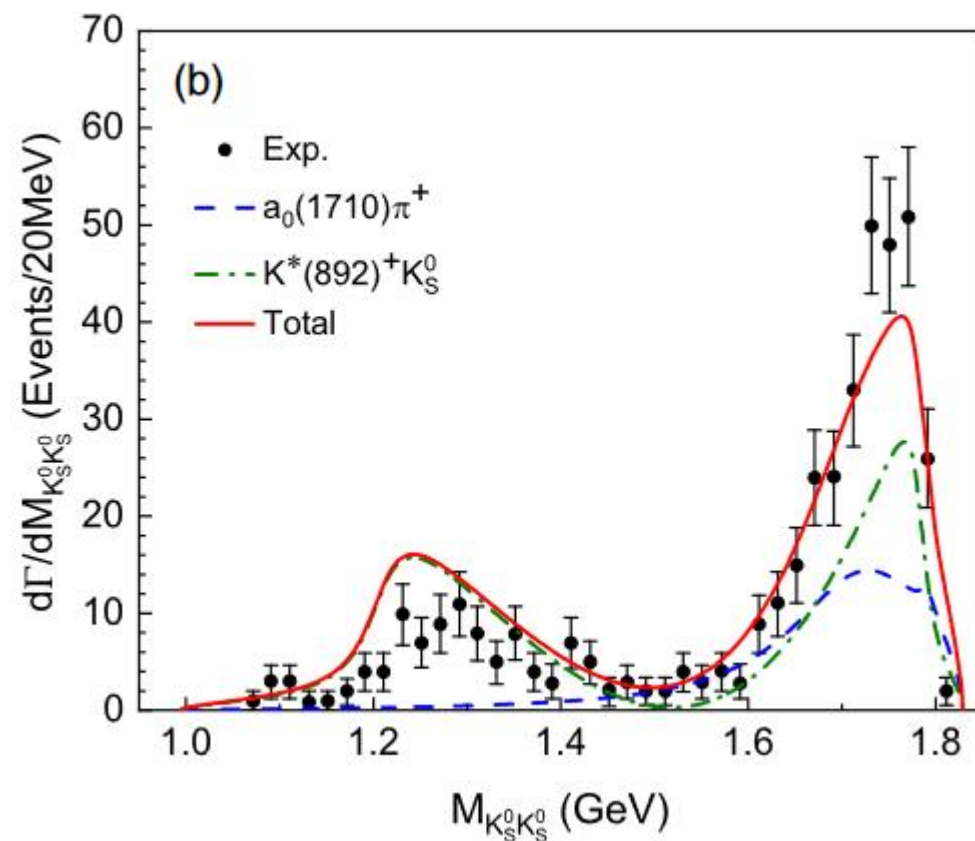
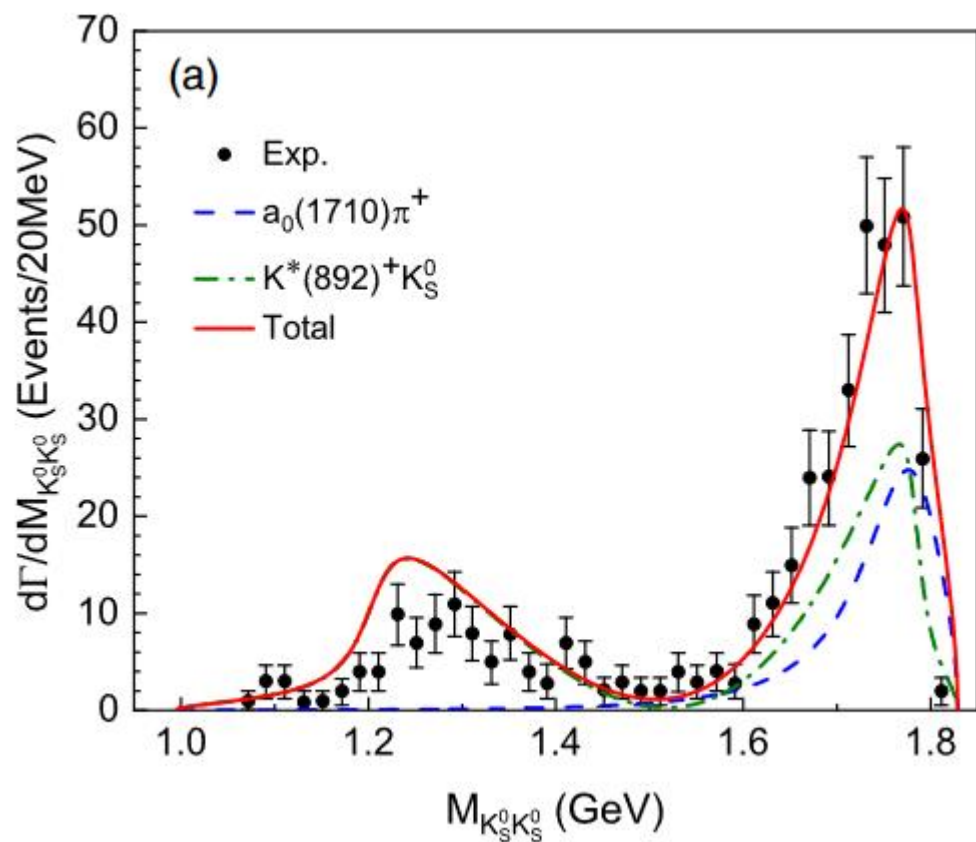
Results

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$$\square D_s^+ \rightarrow K_S^0 K_S^0 \pi^+$$

Set	$M_{a_0(1710)}$	$\Gamma_{a_0(1710)}$	$g_{K^* \bar{K}^*}$	$\Gamma_{K \bar{K}}$
I (Refs. [13,15])	1777	148	(7525, $-i1529$)	36
II (Ref. [28])	1720	200	(8731, $-i2200$)	74

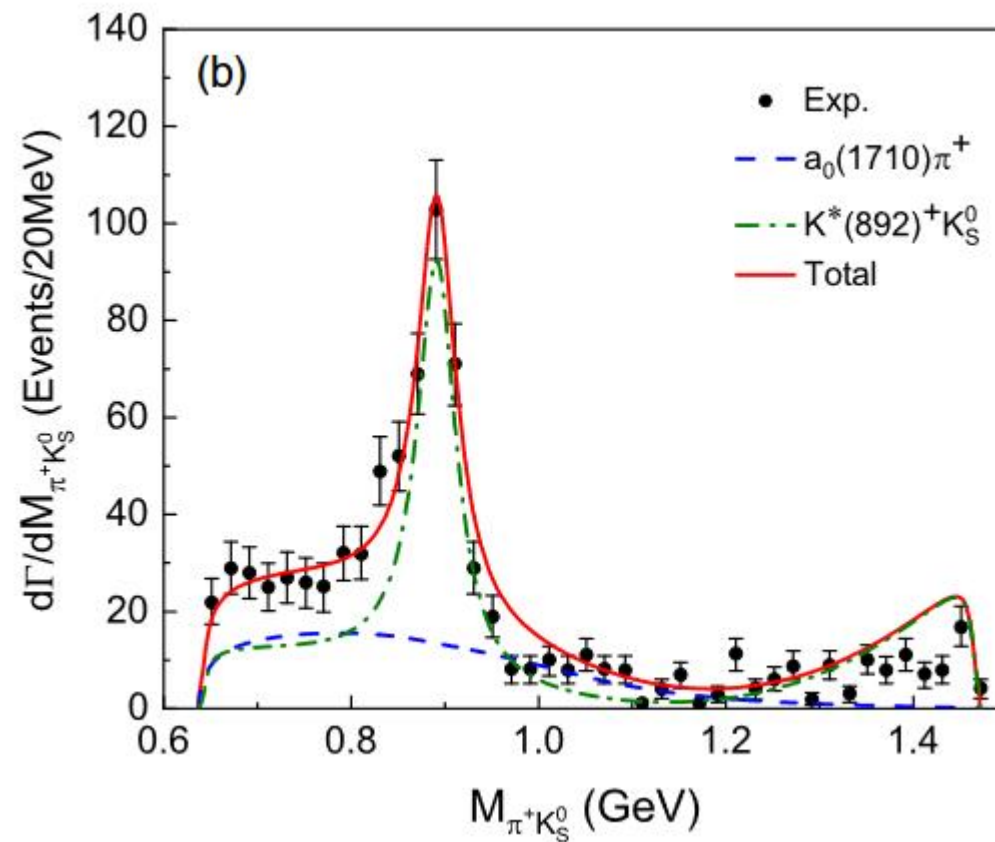
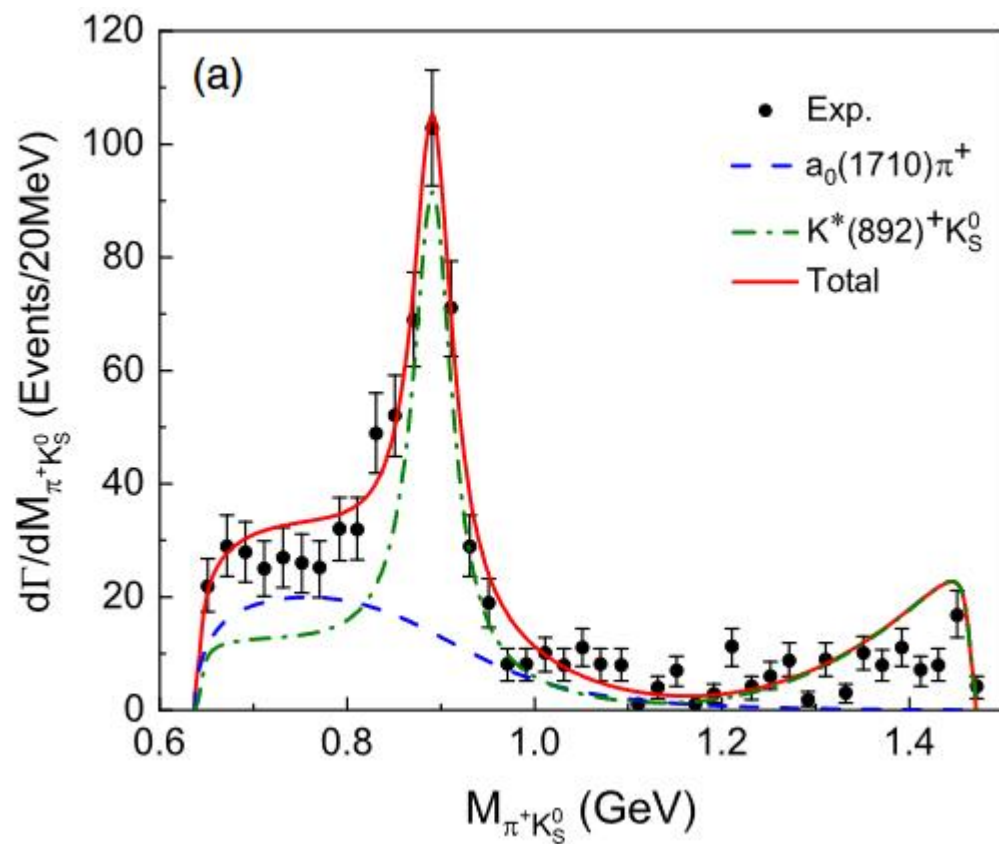


Results

Phys.Rev.D 105 (2022) 11, 116010



$$\square D_s^+ \rightarrow K_S^0 K_S^0 \pi^+$$



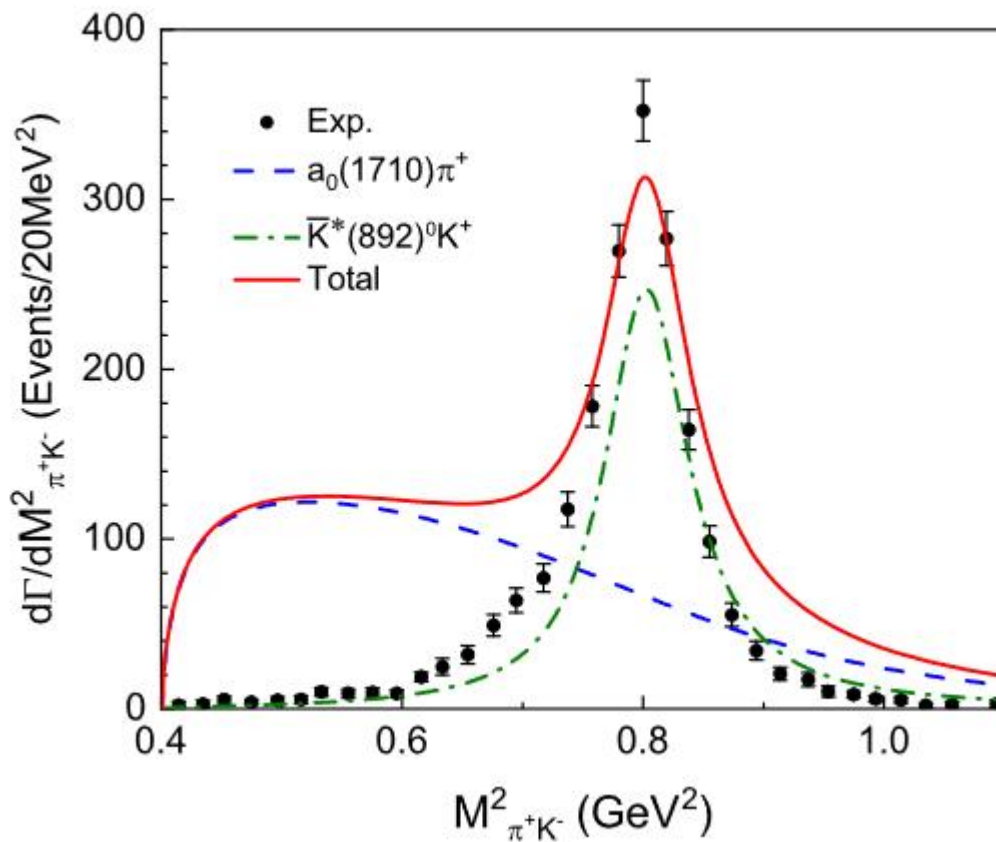
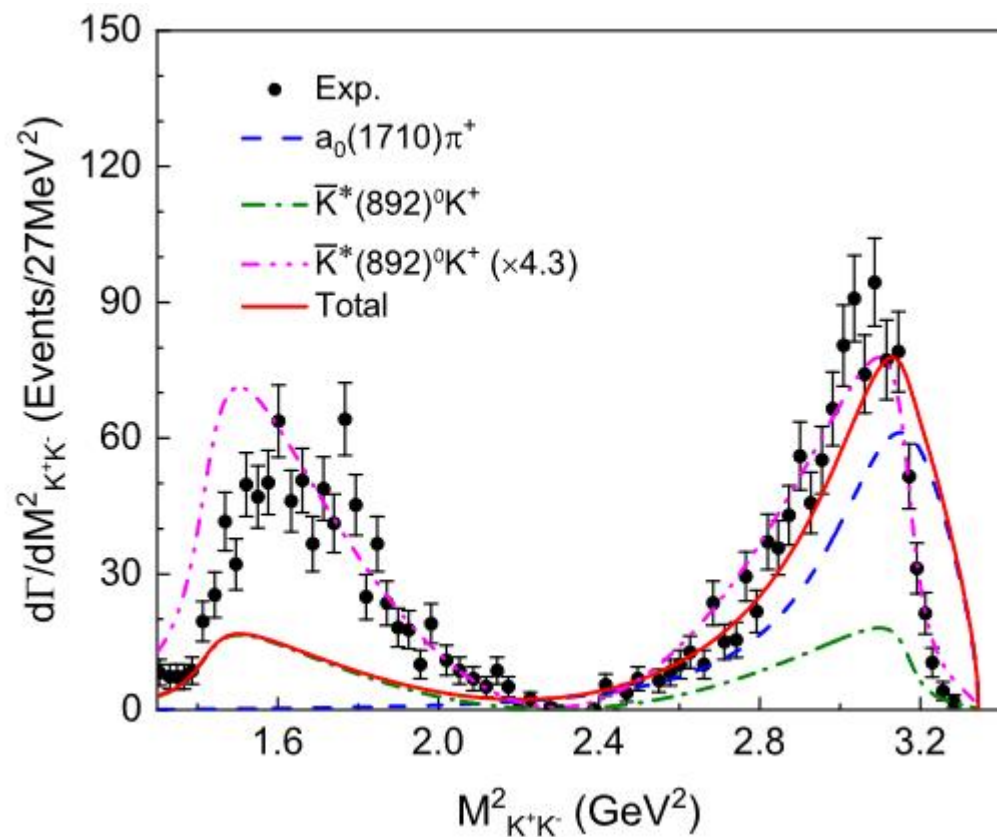
Results

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$$\square D_s^+ \rightarrow K^+ K^- \pi^+$$

BESIII: PRD104(2021)012016

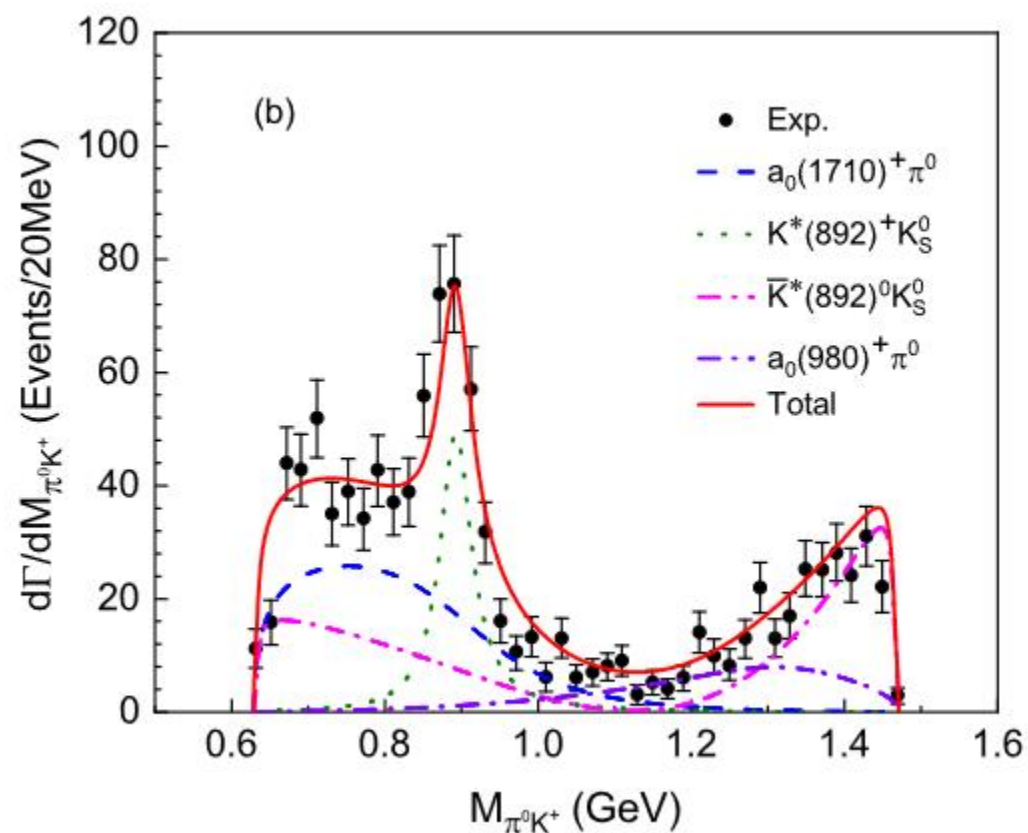
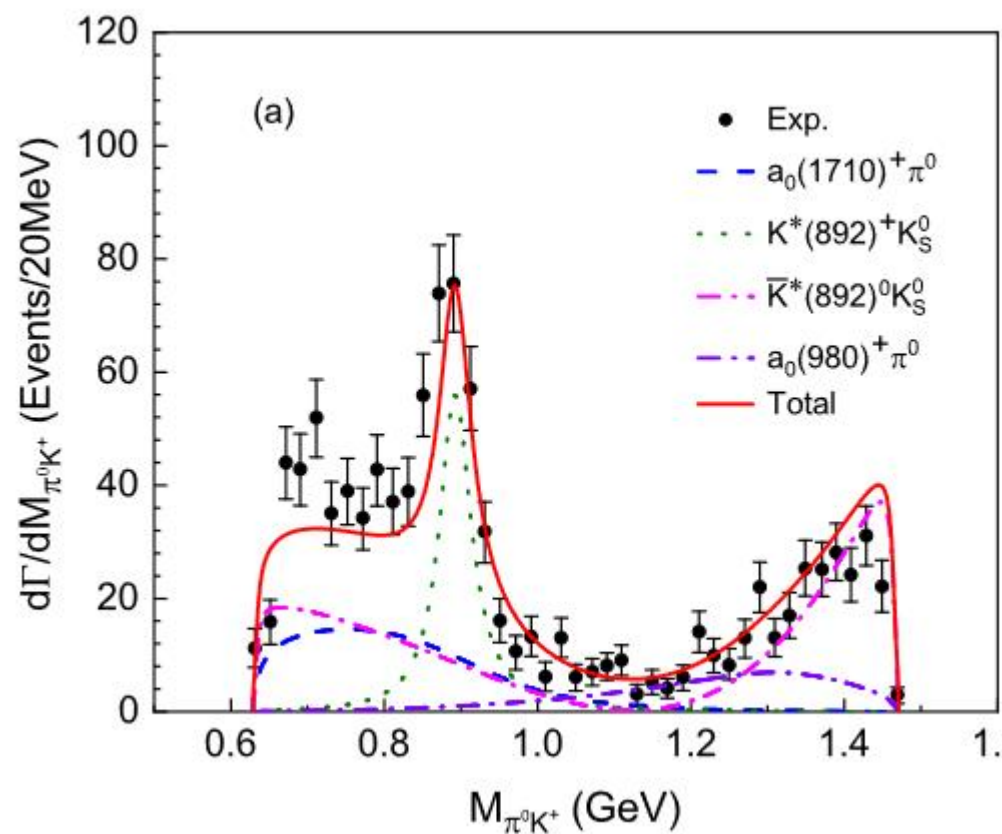


Results



$$\square D_s^+ \rightarrow K_S^0 K^+ \pi^0$$

Phys.Rev.D 107 (2023) 3, 034001

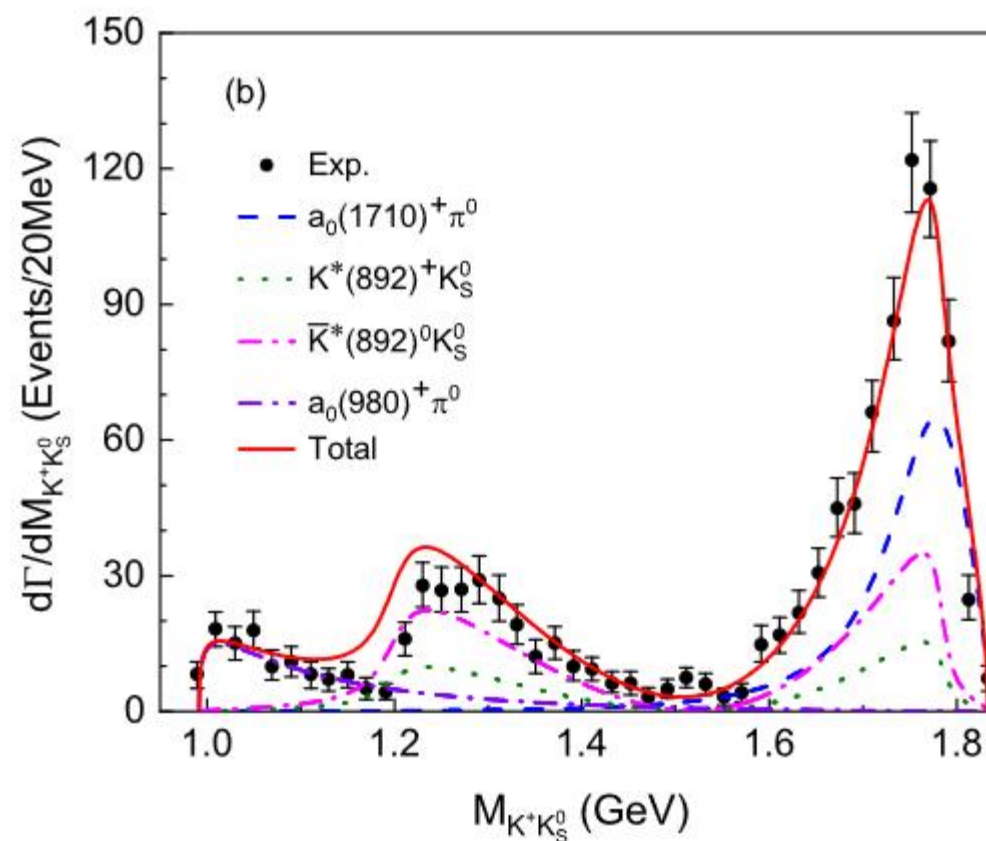
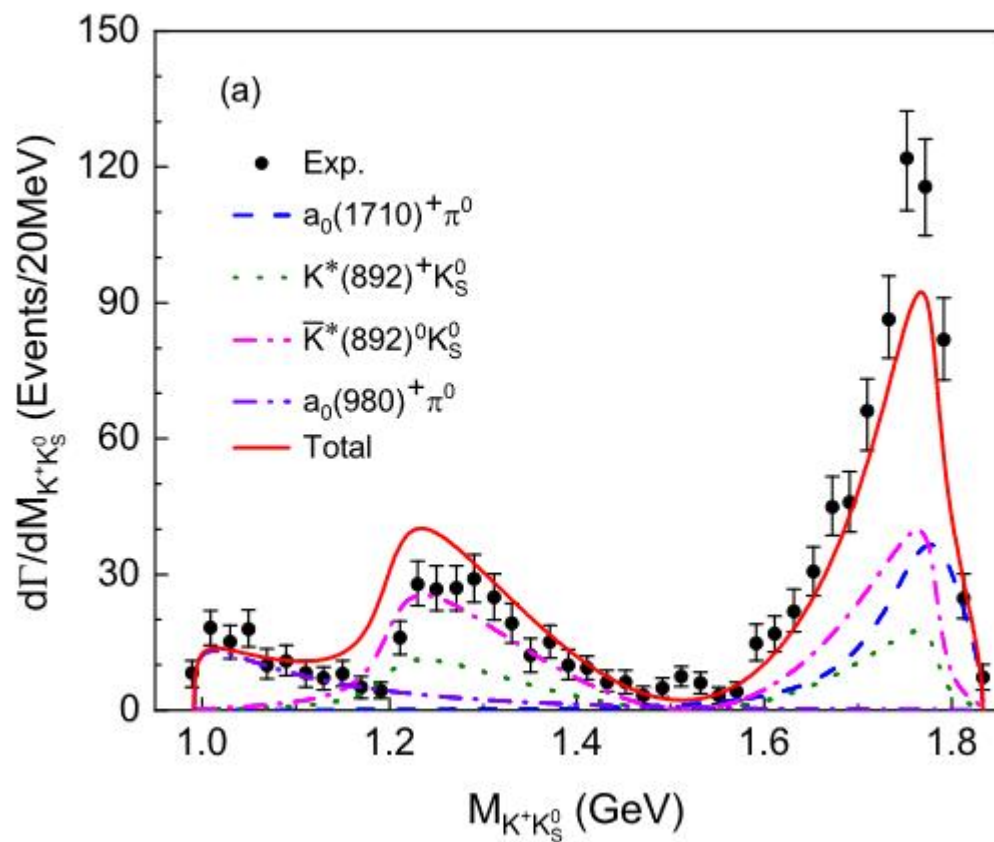


Results



$$\square D_s^+ \rightarrow K_S^0 K^+ \pi^0$$

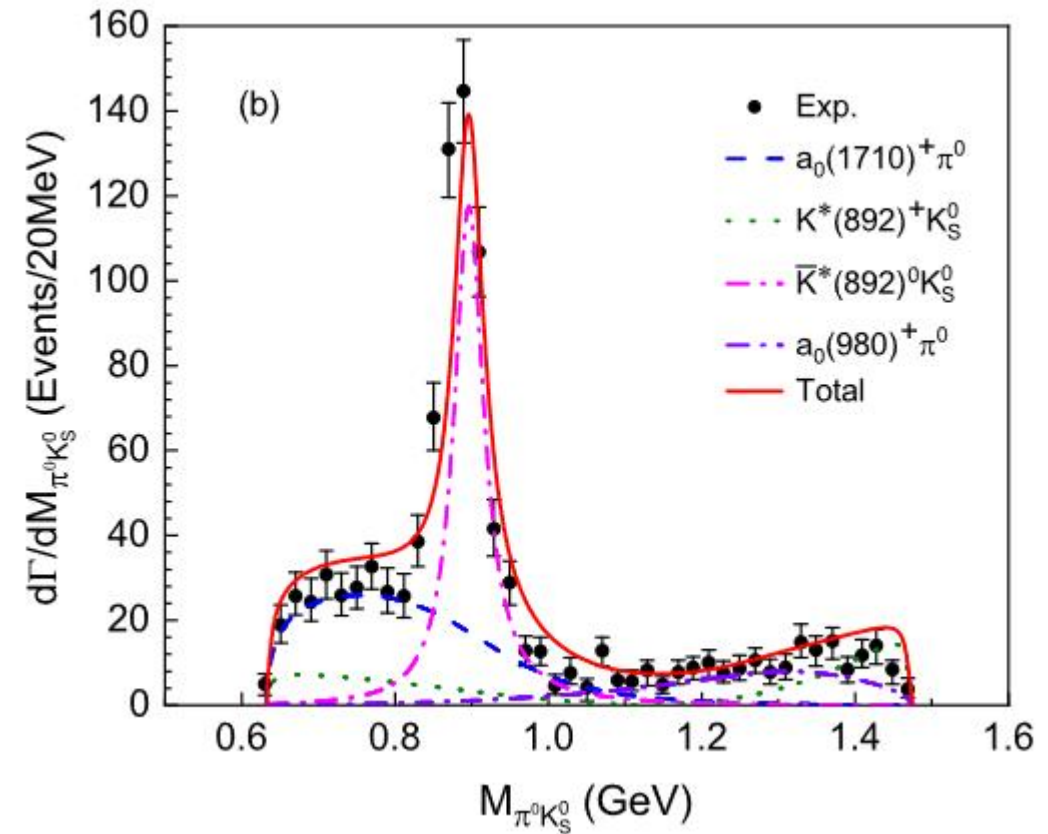
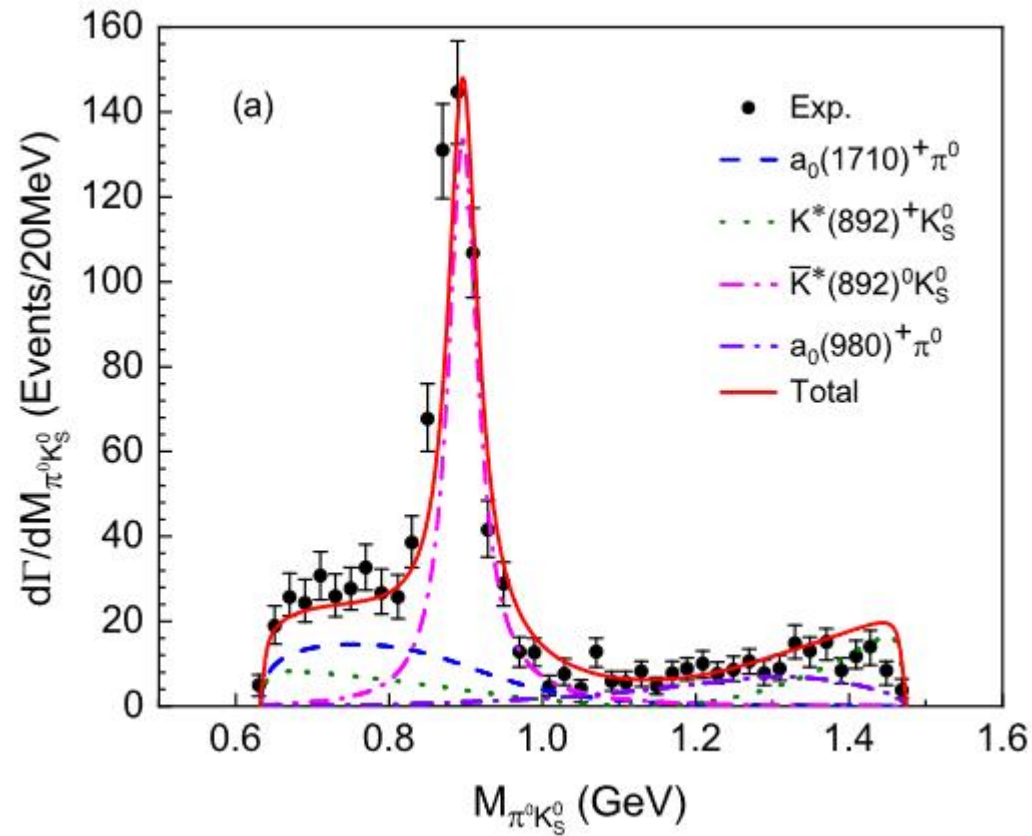
Phys.Rev.D 107 (2023) 3, 034001



Results

$$\square D_s^+ \rightarrow K_S^0 K^+ \pi^0$$

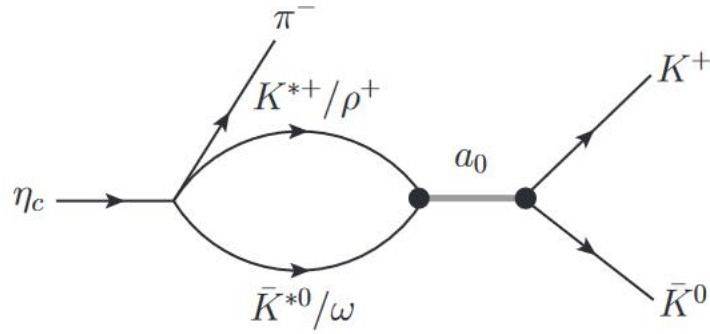
Phys.Rev.D 107 (2023) 3, 034001



$a_0(1710)$ in $\eta_c \rightarrow KK\pi$

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$\eta_c \rightarrow KK\pi$



$$V = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix},$$

$$P = \begin{pmatrix} \frac{\eta}{\sqrt{3}} + \frac{\pi^0}{\sqrt{2}} + \frac{\eta'}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta}{\sqrt{3}} - \frac{\pi^0}{\sqrt{2}} + \frac{\eta'}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{\eta}{\sqrt{3}} + \frac{\sqrt{6}\eta'}{3} \end{pmatrix}$$

$$\begin{aligned} & \langle VVP \rangle \\ &= (VV)_{12} P_{21} \\ &= \pi^- \sum_i V_{1i} V_{i2} \\ &= \pi^- \left[\rho^+ \left(\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} \right) + \left(-\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} \right) \rho^+ + \bar{K}^{*0} K^{*+} \right] \\ &= \pi^- \left[\sqrt{2} \rho^+ \omega + \bar{K}^{*0} K^{*+} \right]. \end{aligned} \quad (3)$$

$$\mathcal{M}_a = V_p \times (G_{\bar{K}^{*0} K^{*+}} t_{\bar{K}^{*0} K^{*+} \rightarrow \bar{K}^0 K^+} + \sqrt{2} G_{\omega \rho^+} t_{\omega \rho^+ \rightarrow \bar{K}^0 K^+}),$$

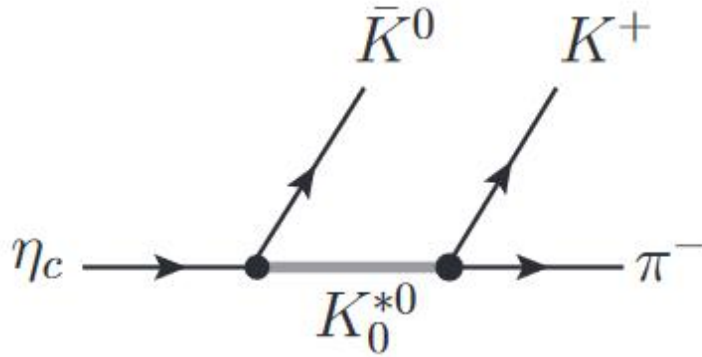
$$G_i(M_{\bar{K}^0 K^+}) = \int_{m_{1-}^2}^{m_{1+}^2} \int_{m_{2-}^2}^{m_{2+}^2} d\tilde{m}_1^2 d\tilde{m}_2^2 \times$$

$$\omega(\tilde{m}_1^2) \omega(\tilde{m}_2^2) \tilde{G}(M_{\bar{K}^0 K^+}, \tilde{m}_1^2, \tilde{m}_2^2),$$

$$\omega(\tilde{m}_i^2) = \frac{1}{N} \text{Im} \left[\frac{1}{\tilde{m}_i^2 - m_{V_i}^2 + i\Gamma(\tilde{m}_i^2) \tilde{m}_i} \right],$$

$$N = \int_{\tilde{m}_{i-}^2}^{\tilde{m}_{i+}^2} d\tilde{m}_i^2 \text{Im} \left[\frac{1}{\tilde{m}_i^2 - m_{V_i}^2 + i\Gamma(\tilde{m}_i^2) \tilde{m}_i} \right],$$

$\square K_0(1430)$



$$\mathcal{L} = g_{\eta_c K K_0^*} \eta_c K K_0^*$$

$$\mathcal{L} = g_{K_0^* K \pi} K_0^* K \pi.$$

$$\Gamma_{\eta_c \rightarrow K_0^* K} = \frac{g_{\eta_c K_0^* K}^2}{8\pi} \frac{|\mathbf{P}|}{m_{\eta_c}^2},$$

$$\Gamma_{K_0^* \rightarrow K \pi} = \frac{g_{K_0^* K \pi}^2}{8\pi} \frac{|\mathbf{P}|}{m_{K_0^*}^2},$$

$$\mathcal{M}_b = \frac{g_{\eta_c K + K_0^{*-}} g_{K_0^{*-} \bar{K}^0 \pi^-}}{M_{\bar{K}^0 \pi^-}^2 - M_{K_0^*}^2 + i M_{K_0^*} \Gamma_{K_0^*}},$$

$$\mathcal{M}_c = \frac{g_{\eta_c \bar{K}^0 K_0^{*0}} g_{K_0^{*0} K^+ \pi^-}}{M_{K^+ \pi^-}^2 - M_{K_0^*}^2 + i M_{K_0^*} \Gamma_{K_0^*}},$$

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b + \mathcal{M}_c,$$

$$\frac{d^2\Gamma}{dM_{\bar{K}^0 K^+} dM_{K^+ \pi^-}} = \frac{M_{\bar{K}^0 K^+} M_{K^+ \pi^-}}{128\pi^3 m_{\eta_c}^3} |\mathcal{M}|^2,$$

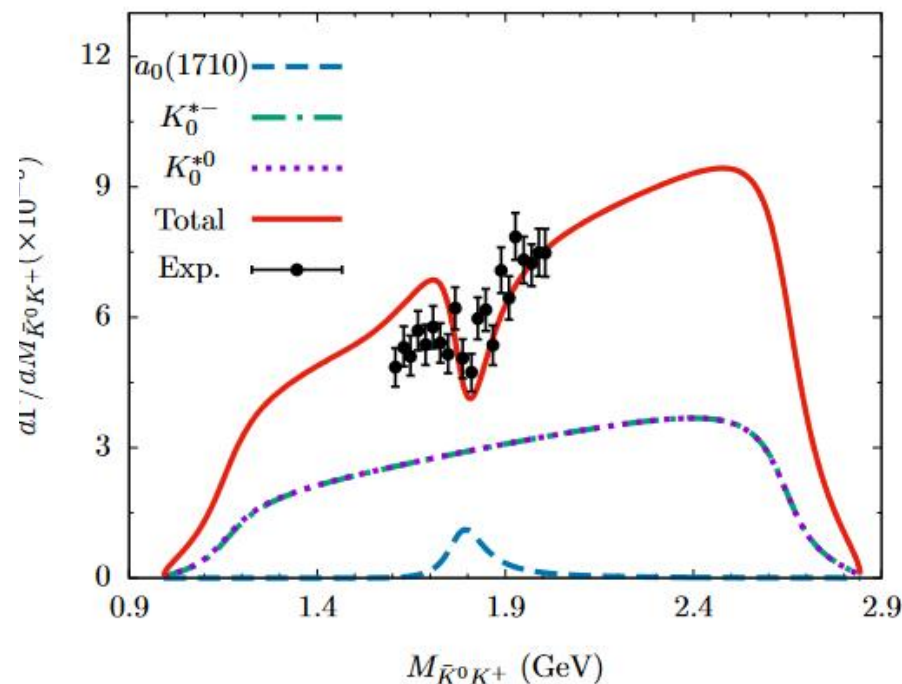
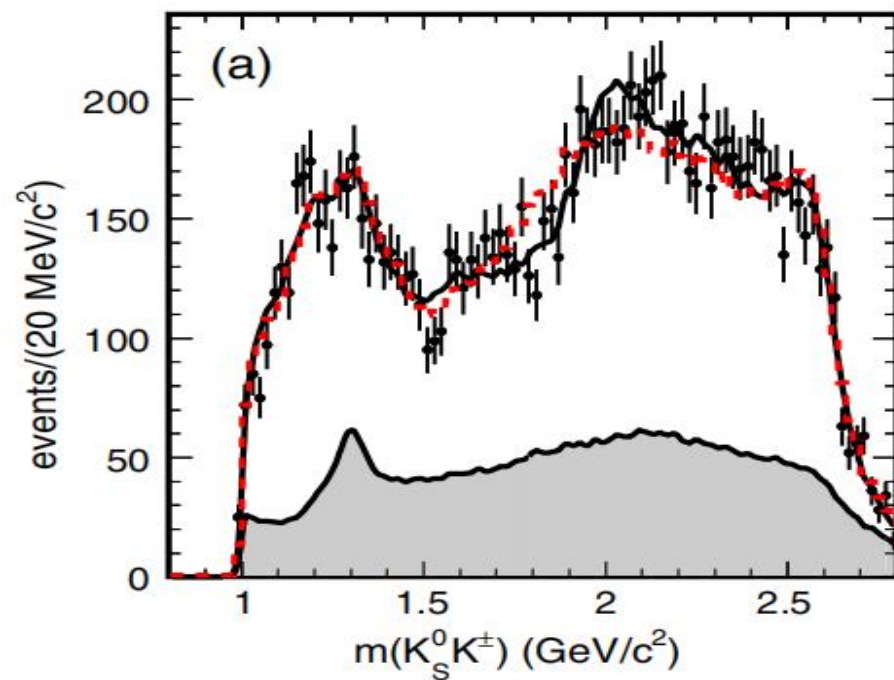
$$\frac{d^2\Gamma}{dM_{\bar{K}^0 K^+} dM_{\bar{K}^0 \pi^-}} = \frac{M_{\bar{K}^0 K^+} M_{\bar{K}^0 \pi^-}}{128\pi^3 m_{\eta_c}^3} |\mathcal{M}|^2.$$

Results

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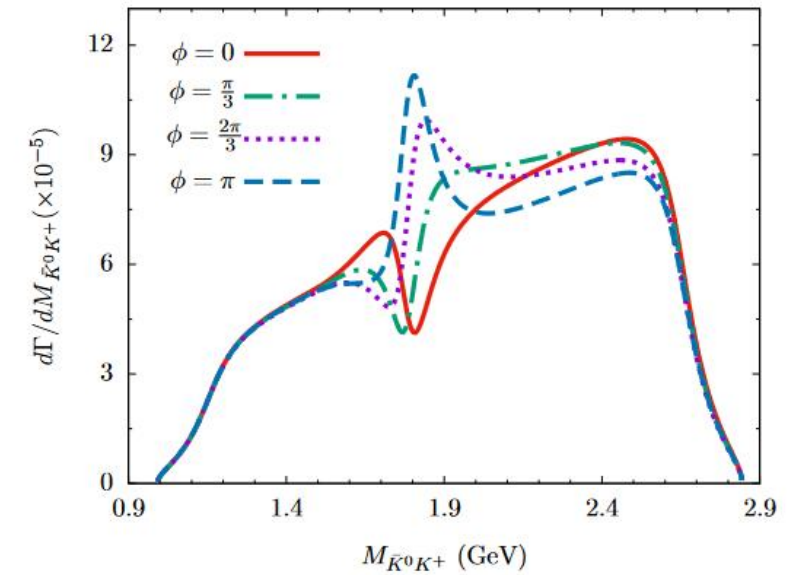
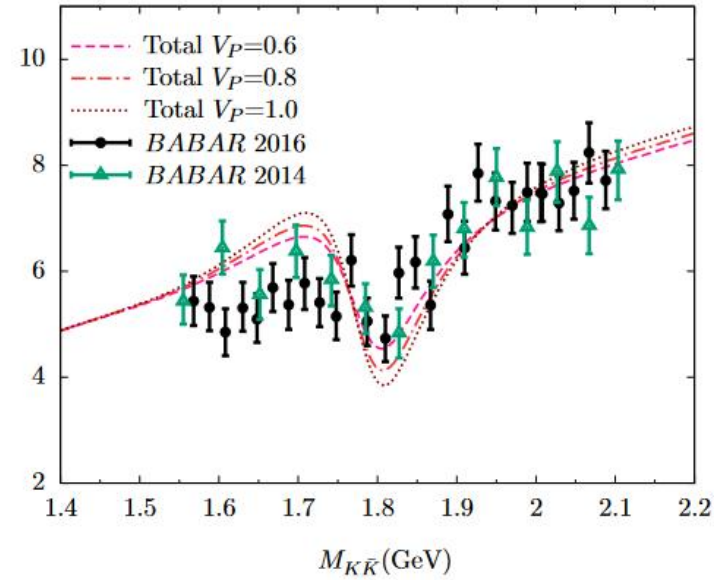
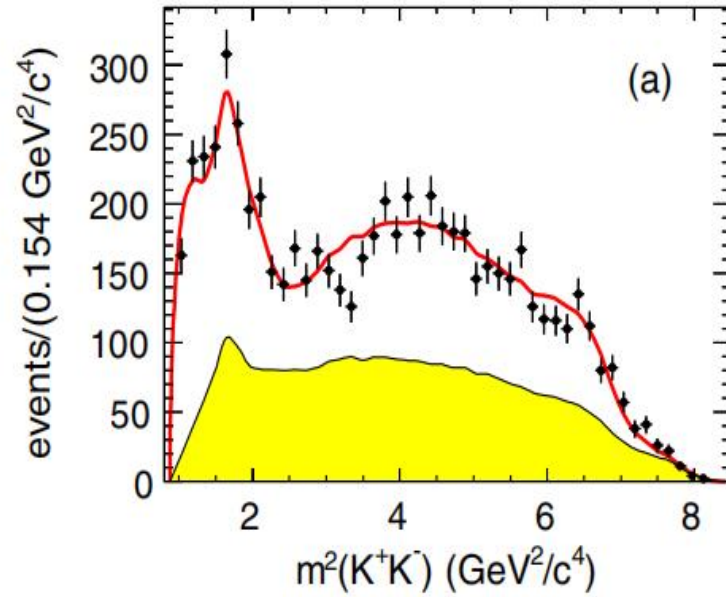


- $\bar{K}_S^0 K^\pm$ invariant mass distribution



BABAR: PRD 93(2016)012005

• $\eta_c \rightarrow K^+ K^- \pi^0$



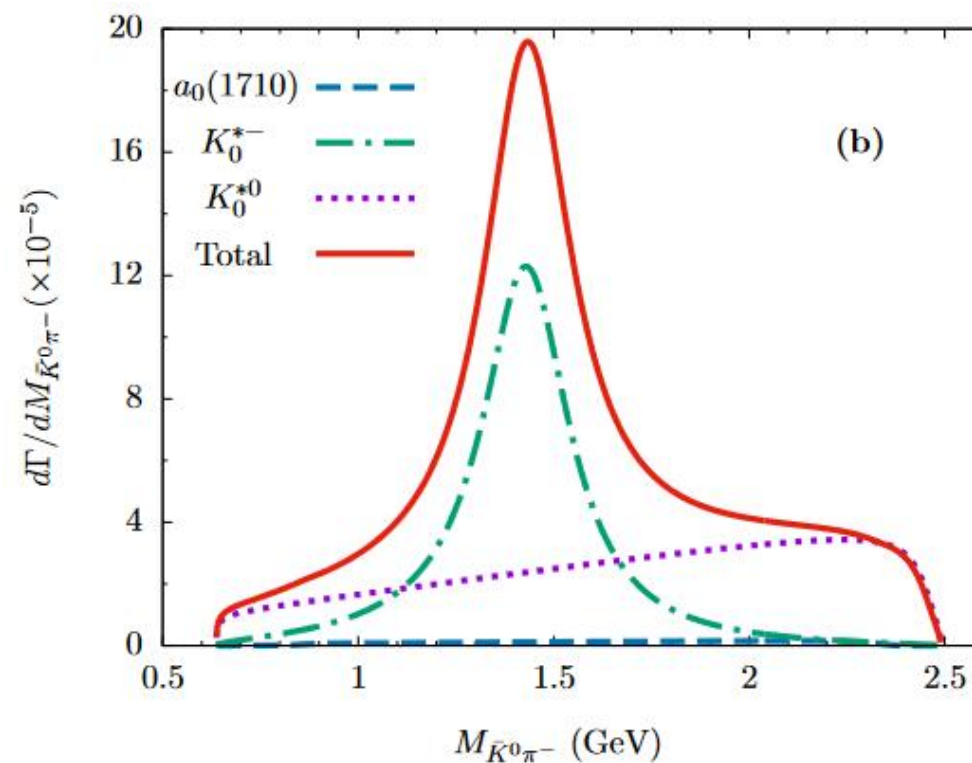
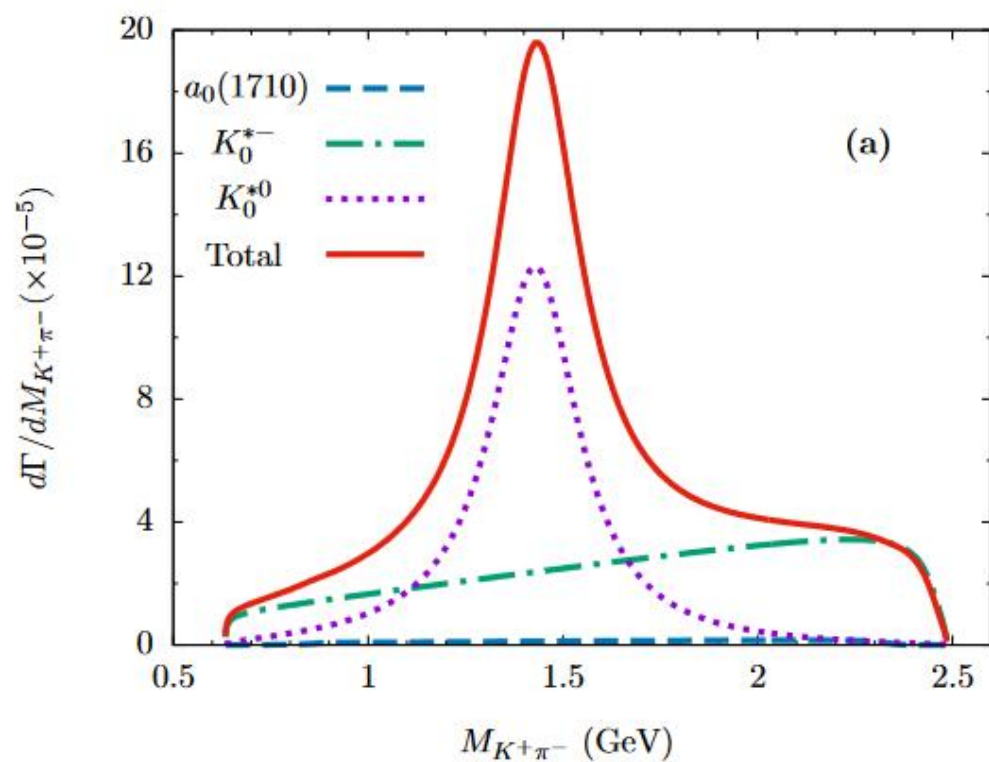
Belle: PRD89(2014)112004

Results

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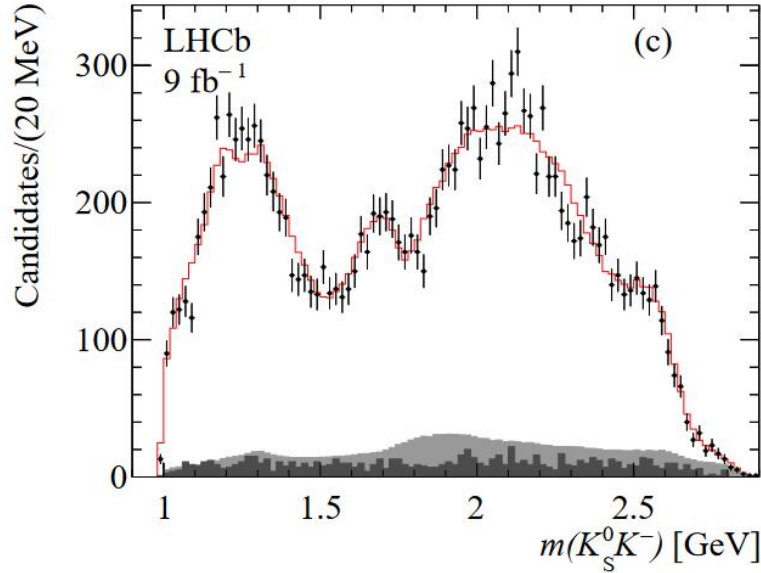


- $K^{\pm}\pi^{-}$ invariant mass distribution

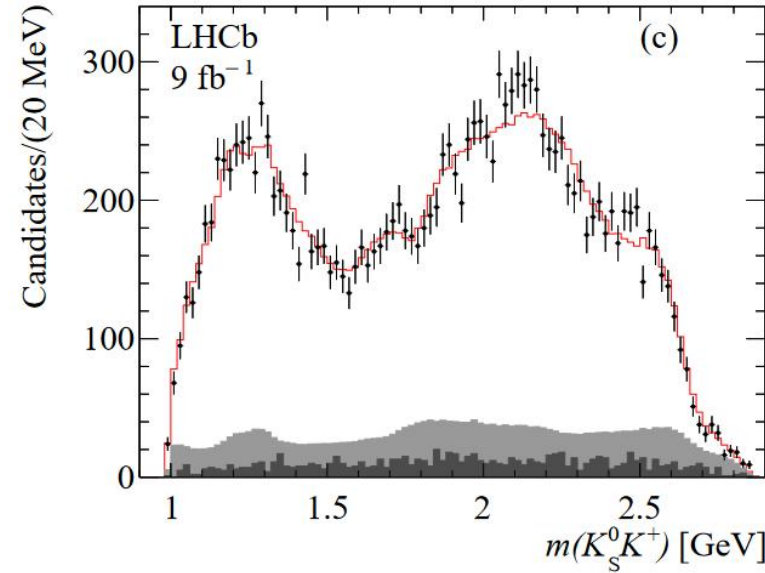


LHCb measurements 2304.14891

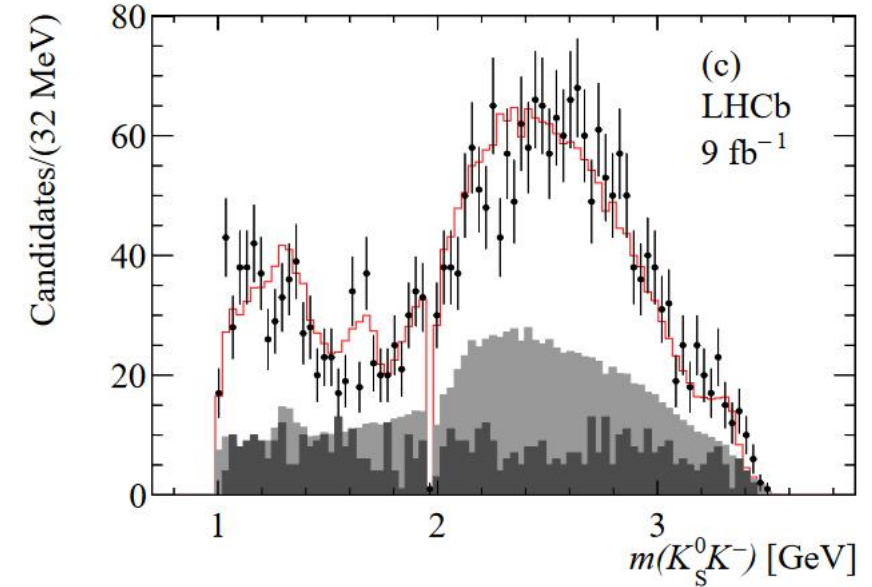
$$\eta_c \rightarrow K_S^0 K^- \pi^+,$$



$$\eta_c \rightarrow K_S^0 K^+ \pi^-,$$



$$\eta_c(2S) \rightarrow K_S^0 K \pi$$

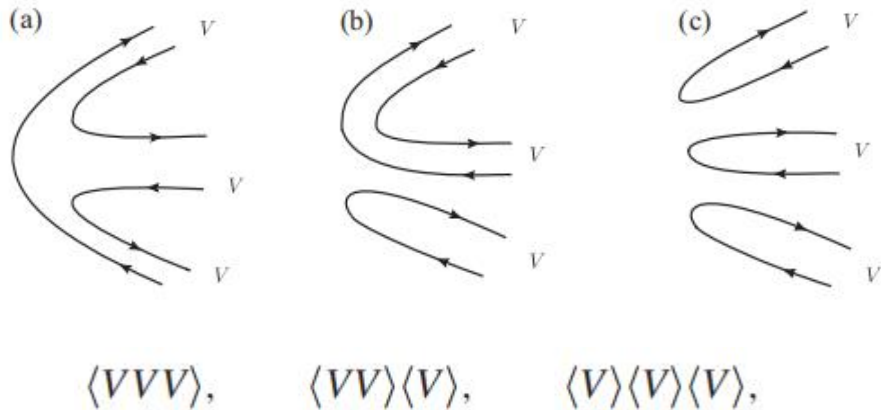
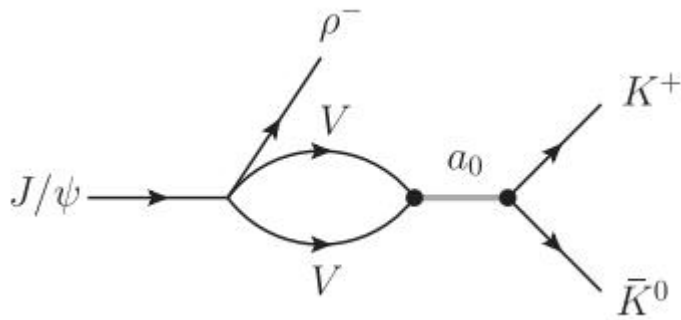


Resonance	Mass [MeV]			Γ [MeV]	$\Delta(2 \log \mathcal{L})$	Significance
$K_0^*(1430)$	$1493 \pm 4 \pm 7$			$215 \pm 7 \pm 4$	-	-
$K_0^*(1950)$	$1980 \pm 14 \pm 19$			$229 \pm 26 \pm 16$	316	17.8σ
$a_0(1700)$	$1736 \pm 10 \pm 12$			$134 \pm 17 \pm 61$	161	12.7σ
$\kappa(2600)$	$2662 \pm 59 \pm 201$			$480 \pm 47 \pm 72$	1338	36.6σ

$a_0(1710)$ in $J/\psi \rightarrow \bar{K}^0 K^+ \rho^-$

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• Reaction mechanism



$$V = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix},$$

$$\langle VVV \rangle: \alpha \times \left[\frac{\rho^+ \rho^0}{\sqrt{2}} + 3\sqrt{2}\omega\rho^+ + 3\bar{K}^{*0}K^{*+} \right] \rho^-, \quad \beta/\alpha = 0.32$$

PRD79(2009)07
4009

$$\langle VV \rangle \langle V \rangle: \beta \times [2\sqrt{2}\omega\rho^+ + 2\phi\rho^+] \rho^-.$$

$$\mathcal{M}_a = V_p \times [3\alpha G_{\bar{K}^{*0}K^{*+}} t_{\bar{K}^{*0}K^{*+} \rightarrow \bar{K}^0 K^+} + (2\sqrt{2}\beta + 3\sqrt{2}\alpha) G_{\omega\rho^+} t_{\omega\rho^+ \rightarrow \bar{K}^0 K^+} + 2\beta G_{\phi\rho^+} t_{\phi\rho^+ \rightarrow \bar{K}^0 K^+}],$$

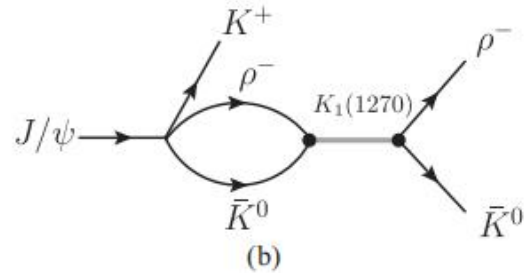
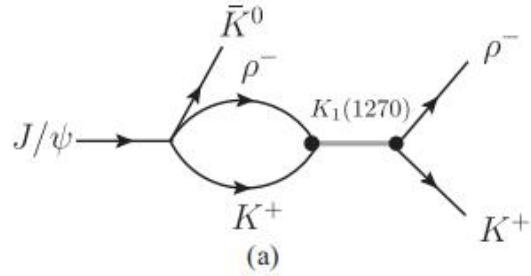
$$G_i(M_{\bar{K}^0 K^+}) = \int_{m_{1-}^2}^{m_{1+}^2} \int_{m_{2-}^2}^{m_{2+}^2} d\tilde{m}_1^2 d\tilde{m}_2^2 \times \omega(\tilde{m}_1^2) \omega(\tilde{m}_2^2) \tilde{G}(M_{\bar{K}^0 K^+}, \tilde{m}_1^2, \tilde{m}_2^2),$$

$$t_{i \rightarrow \bar{K}^0 K^+} = \frac{g_i \times g_{K\bar{K}}}{M_{\bar{K}^0 K^+}^2 - M_{a_0(1710)}^2 + iM_{a_0(1710)}\Gamma_{a_0(1710)}},$$

$a_0(1710)$ in $J/\psi \rightarrow \bar{K}^0 K^+ \rho^-$

Phys.Rev.D 110 (2024) 1, 014032

• $K_1(1270)$ & $a_0(980)$



$$\mathcal{M}_b = V'_p \times G_{K^+\rho^-} t_{K^+\rho^- \rightarrow K^+\rho^-},$$

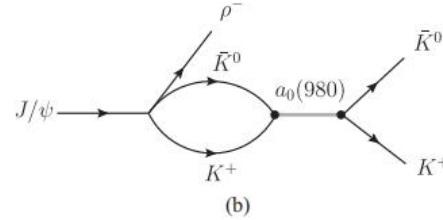
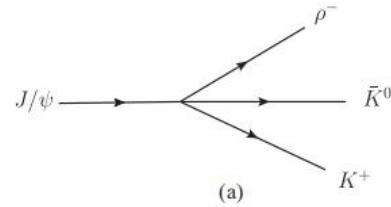


TABLE IV. Pole positions and coupling constants of the two poles of the $K_1(1270)$ [46]. All values are in units of MeV.

	First pole	Second pole
Pole position $\sqrt{s_0}$	$1195 - i123$	$1284 - i73$
$g_{K\rho}$	$-1671 + i1599$	$4804 + i395$

$$\mathcal{M}_d = V'_p [1 + G_{K\bar{K}} t_{\bar{K}^0 K^+ \rightarrow \bar{K}^0 K^+}],$$

$$T = [1 - VG]^{-1} V,$$

PLB821(2021)136
617

$$V_{K\bar{K} \rightarrow K\bar{K}} = -\frac{1}{4f^2} s,$$

$$V_{K\bar{K} \rightarrow \pi\eta} = \frac{\sqrt{6}}{12f^2} \left(3s - \frac{8}{3} m_K^2 - \frac{1}{3} m_\pi^2 - m_\eta^2 \right),$$

$$V_{\pi\eta \rightarrow K\bar{K}} = V_{K\bar{K} \rightarrow \pi\eta},$$

$$V_{\pi\eta \rightarrow \pi\eta} = -\frac{1}{3f^2} m_\pi^2,$$

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b + \mathcal{M}_c + \mathcal{M}_d,$$

$$\frac{d^2\Gamma}{dM_{\bar{K}^0 K^+} dM_{K^+ \rho^-}} = \frac{M_{\bar{K}^0 K^+} M_{K^+ \rho^-}}{128\pi^3 m_{J/\psi}^3} |\mathcal{M}|^2,$$

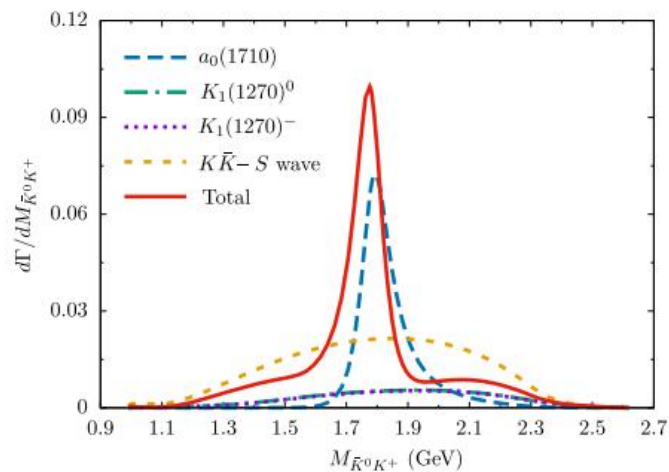
$$\frac{d^2\Gamma}{dM_{\bar{K}^0 K^+} dM_{\bar{K}^0 \rho^-}} = \frac{M_{\bar{K}^0 K^+} M_{\bar{K}^0 \rho^-}}{128\pi^3 m_{J/\psi}^3} |\mathcal{M}|^2.$$

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Results

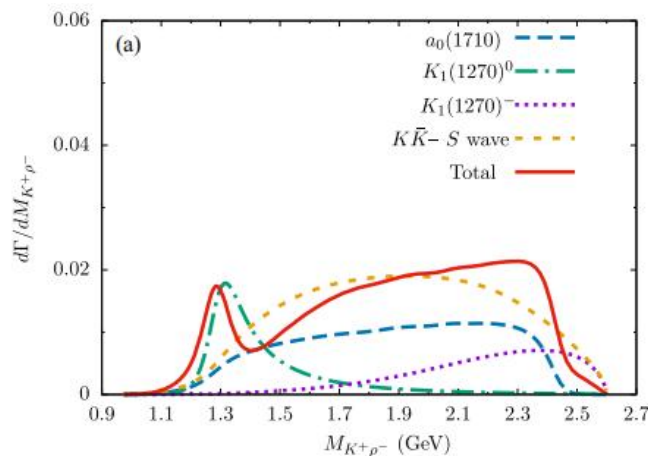


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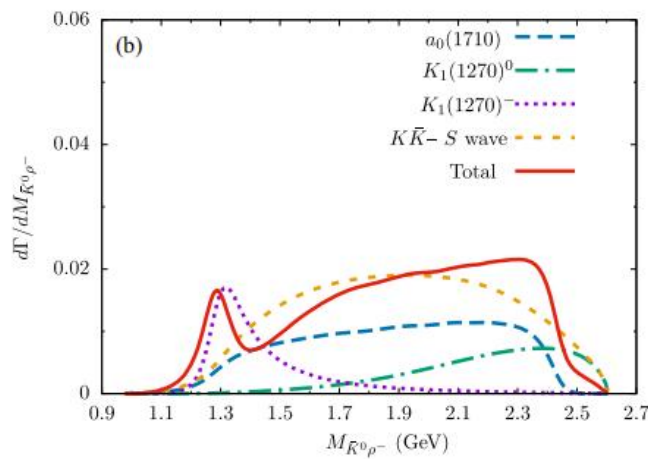
BABAR:

$$\mathcal{B}(J/\psi \rightarrow K_S^0 K^\pm \rho^\mp) = (1.87 \pm 0.18 \pm 0.34) \times 10^{-3}$$



➤ BESIII

$$(10.09 \pm 0.04) \times 10^9 J/\psi$$



➤ STCF per year

$$3.4 \times 10^{12} J/\psi$$

Front. Phys. 19, 14701 (2024).

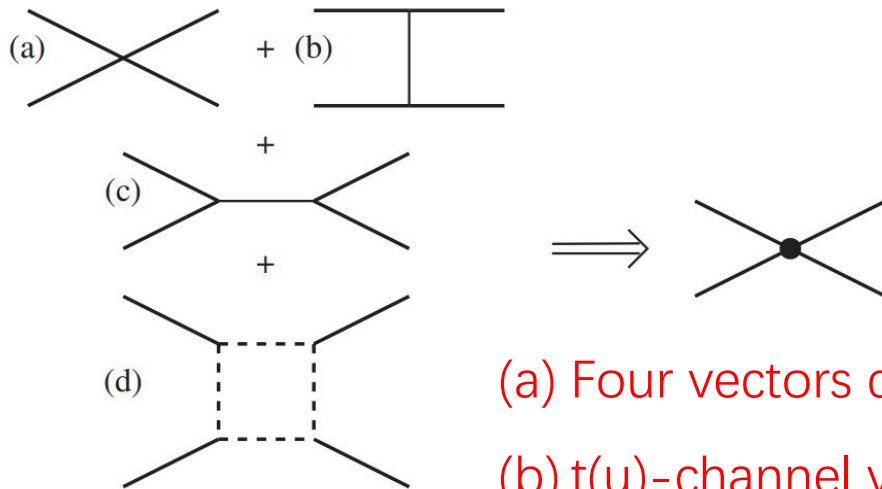
Summary

- Our results are in good agreement with the BESIII measurements, which supports the $K^*\bar{K}^*$ molecule of $a_0(1710/1817)$.
- The $a_0(1710/1817)$ mass and width are crucial to understand its internal structure.
- Precise experimental measurements are necessary.

Thank you very much!

VV interaction from hidden-gauge lagrangians

□ Tree level transition amplitudes



(a) Four vectors contact

(b) t(u)-channel vector exchange

(c) S-channel vector exchange

(d) Box diagram

Most important! Provide attraction!

Basically p-wave, neglected!

➤ Provides decays to two pseudoscalars;

➤ Contributes only to spin 0 and 2;

➤ Real part small, only imaginary part considered!