Boson-fermion universality of mesoscopic entanglement fluctuations in free systems

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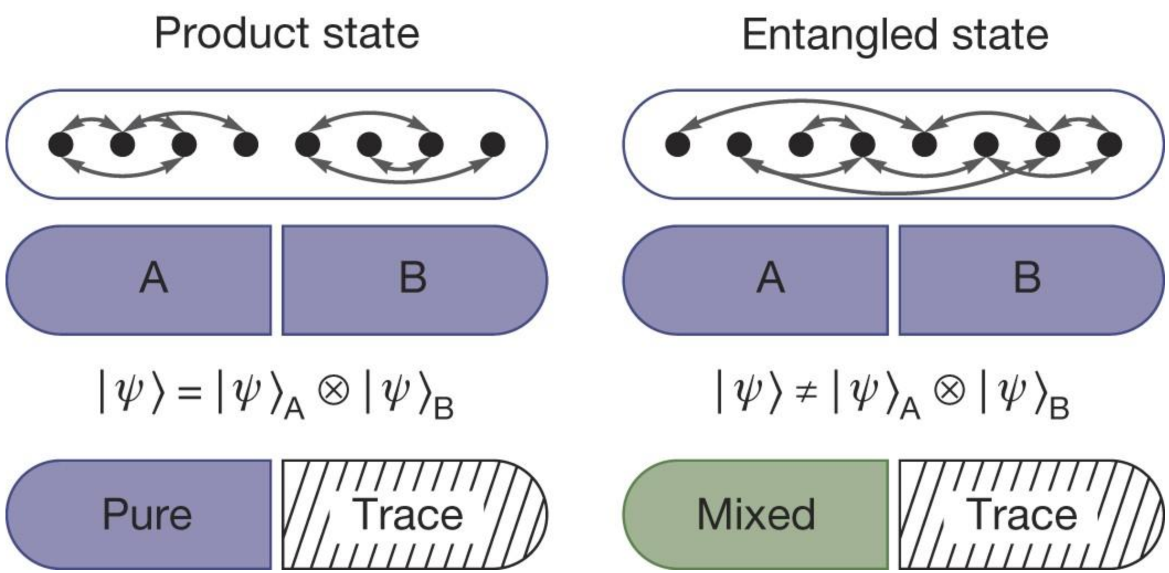
Outline

- Motivations
- Model
- Setup and method
- Emergent random structure
- Fluctuations of entanglement
- Numerical verification

Motivations

Why we study entanglement?

- Quantum phase transitions
- Topological order
- Entropy of the black hole
- Practical applications: quantum computing



For the pure state $|\psi\rangle$

Density operator:

$$\rho \equiv |\psi\rangle\langle\psi|$$

Reduced density matrix:

$$\rho_A \equiv \mathrm{Tr}_B(\rho)$$

Entanglement entropy:

$$S(\rho_A) \equiv -\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

Entangled state: ρ_A describes a mixed state, $S(\rho_A) > 0$.

Product state: ρ_A describes a pure state, $S(\rho_A) = 0$.

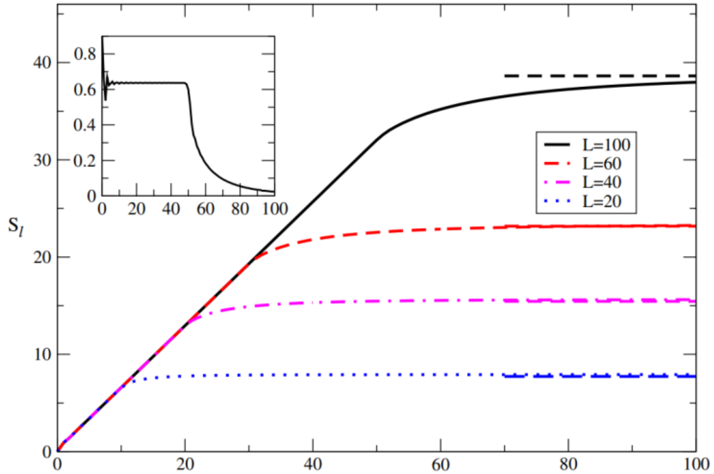
Motivations

Dynamics of the entanglement after a quantum quenching

Quench protocols: at time $t = 0$, the Hamiltonian of an isolated system is changed instantaneously, the initial state $|\psi_0\rangle$ will do an unitary evolution under the new Hamiltonian H:

$$|\psi(t)\rangle = e^{-i\frac{H}{\hbar}t}|\psi_0\rangle. \Rightarrow \rho(t), \rho_A(t), S(t).$$

TFIC with infinite system size

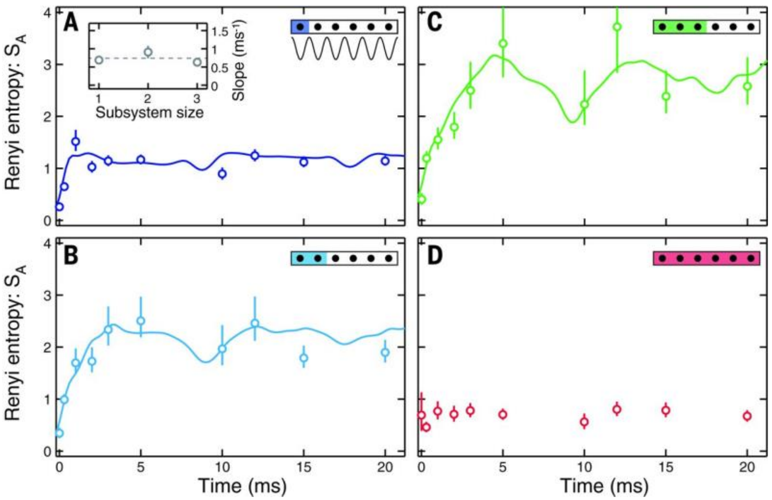


Universal linear growth in time

No temporal fluctuations at late time

Calabrese, Cardy. (2005)

Bose–Hubbard model



Growth at beginning and fluctuations at late time

Experiment is performed at finite system!

M. Greiner's group. (2016)

Model

1D harmonic oscillator chains:

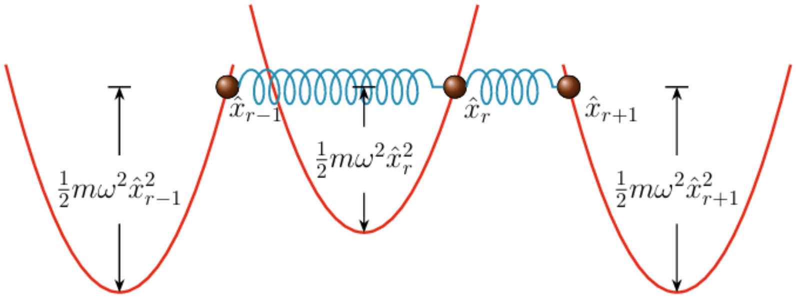
$$\hat{H} = \sum_{j=0}^{L-1}\left(\frac{1}{2m}p_j^2 + \frac{m\omega^2}{2}x_j^2 + \frac{K}{2}\left(x_j - x_{j+1}\right)^2\right)$$

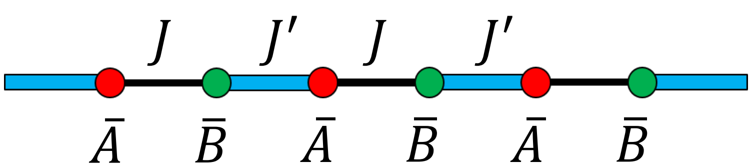
m: the mass of a oscillator
ω: the frequency of a oscillator
K: the coupling strength

The Rice-Mele model:

$$H_{RM} = -\sum_{i=1}^{L}(Jc_{i\bar{A}}^{\dagger}c_{i\bar{B}} + J'c_{i\bar{B}}^{\dagger}c_{(i+1)\bar{A}} + \text{h.c.})$$
$$+M\sum_{i=1}^{L}(c_{i\bar{A}}^{\dagger}c_{i\bar{A}} - c_{i\bar{B}}^{\dagger}c_{i\bar{B}}).$$

J and J': the hopping amplitudes
M: the staggered onsite mass

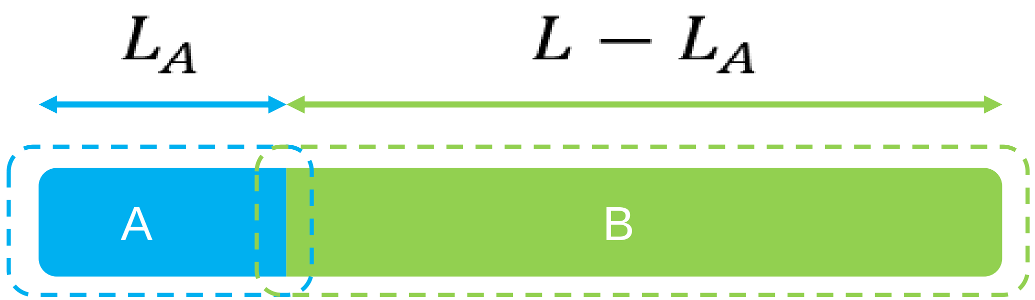




Quench dynamics: $|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$,

where $|\psi_0\rangle$ is the ground state of the Hamiltonian before the quantum quench.

Setup and method



(Bosonic)

Covariance matrix $\gamma(t)$:

$$\gamma_{ij}(t) \equiv \frac{1}{2}\langle\psi(t)|\{\hat{\mathbf{r}}_i, \hat{\mathbf{r}}_j\}|\psi(t)\rangle$$

$$\hat{\boldsymbol{r}} = (x_1, x_2, \cdots x_L, p_1, p_2 \cdots p_L)^T$$

(Fermionic)

Correlation function matrix $C(t)$:

$$C_{ij}(t) \equiv \langle\psi(t)|\boldsymbol{c}_i^\dagger \boldsymbol{c}_j|\psi(t)\rangle$$

$$\boldsymbol{c} = (c_{1\bar{A}}, c_{2\bar{A}}, \cdots c_{L\bar{A}}, c_{1\bar{B}}, c_{2\bar{B}} \cdots c_{L\bar{B}})^T$$

When all indexes are limited to subsystem A, we get submatrices $\gamma_A(t)$ and $C_A(t)$.

$\{\lambda_b(t)\}$: Symplectic eigenvalues of $\gamma_A(t)$.

$\{\lambda_f(t)\}$: eigenvalues of $C_A(t)$.

$b, f = 1, 2, \cdots 2L_A$.

$\{\lambda_b(t)\}, \{\lambda_f(t)\} \xrightarrow[\text{calculate}]{}$ Entanglement entropy $S(t)$

Emergent random structure

The example of elements of matrix $\gamma_A(t)$:

$$\langle\psi(t)|x_r x_s|\psi(t)\rangle = \frac{1}{mL}\sum_k e^{-ik(s-r)}\frac{1}{\omega_k}\left(E_+(k)+E_-(k)\cos\left(2\omega_k t\right)\right)$$

(Bosonic)

The elements of matrix $C_A(t)$:

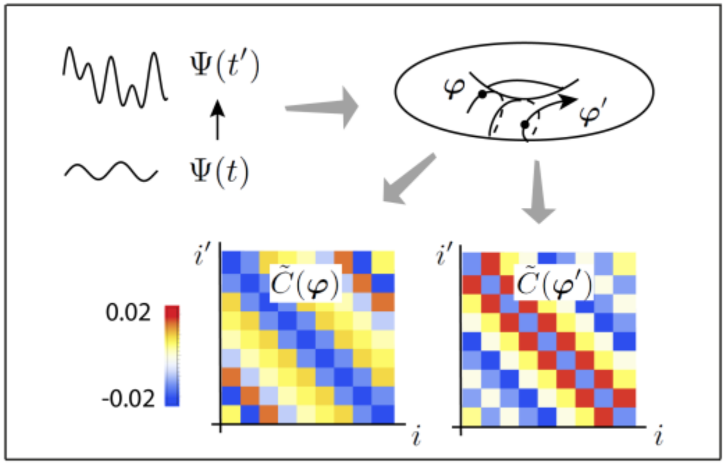
$$C_{i\sigma,i'\sigma'}(t) = \frac{1}{L}\sum_k \mathrm{e}^{ik\,(i-i')}\Big(\gamma_{\sigma\sigma'}(k)+\alpha_{\sigma\sigma'}(k)\cos(2E_{fk}t)+\beta_{\sigma\sigma'}(k)\sin(2E_{fk}t)\Big)$$

(Fermionic)

The time parameter enters through the dynamical phase factor $\boldsymbol{\varphi}$.

$$\boldsymbol{\varphi} \equiv 2\boldsymbol{\omega_b}t,\ \boldsymbol{\omega_b} \equiv \{\omega_{k_0},\omega_{k_1},\cdots,\omega_{k_{N-1}}\}.\ \boldsymbol{\varphi} \equiv 2\boldsymbol{\omega_f}t,\ \boldsymbol{\omega_f} \equiv \{E_{fk_0},E_{fk_1},\cdots,E_{fk_{N-1}}\}.$$

Since $\boldsymbol{\omega_b}$ and $\boldsymbol{\omega_f}$ are incommensurate, the evolution of the matrix $\gamma_A(t)$ and $C_A(t)$ can be viewed as a ergodic rotation on a torus $\mathbb{T}^N$, described by $\boldsymbol{\varphi}$.



By the ergodic theorem, in long time scale, the statistics of time series are equivalent to the statistics of random uniform sampling on the torus $\mathbb{T}^N$.

Fluctuations of entanglement

Because of the equivalence of statistics: $\gamma_A(\boldsymbol{\varphi}), C_A(\boldsymbol{\varphi}) \xrightarrow[\text{replace}]{} \gamma_A(t), C_A(t)$

Entanglement entropy:

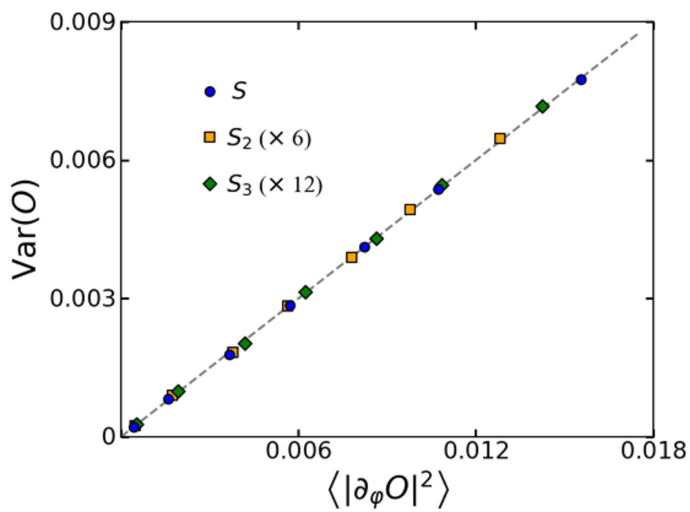
$$\mathrm{S}(\boldsymbol{\varphi}) = \frac{1}{2}\int_{\frac{1}{4}}^{\infty} d\lambda\, e(\lambda)\mathrm{Tr}_A\delta\left(\lambda\mathbb{I} - \tilde{C}_A^2(\boldsymbol{\varphi})\right), \text{ where } \tilde{C}_A(\boldsymbol{\varphi}) \equiv i\mathbb{J}\gamma_A(\boldsymbol{\varphi}). \quad \text{(Bosonic)}$$

$$\mathrm{S}(\boldsymbol{\varphi}) = \int_0^1 d\lambda\, e'(\lambda)\mathrm{Tr}_A\delta\big(\lambda\mathbb{I} - C_A(\boldsymbol{\varphi})\big). \quad \text{(Fermionic)} \qquad \mathbb{J} \equiv \begin{pmatrix} 0 & \mathbb{I}_{L_A} \\ -\mathbb{I}_{L_A} & 0 \end{pmatrix}$$

From the concentration-of-measure theory, we have $Var(S) \propto \langle |\partial_\varphi S|^2 \rangle$.

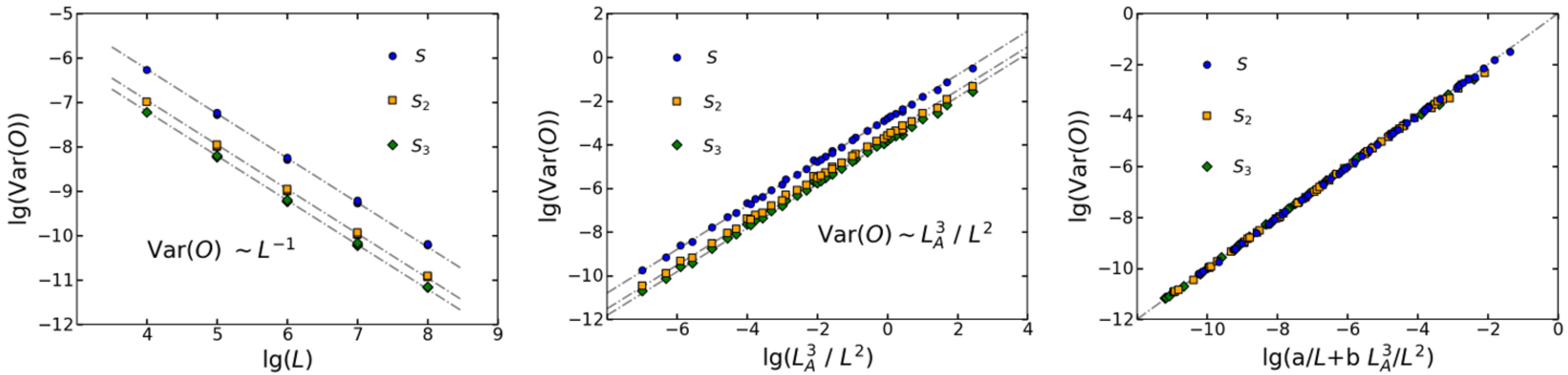
Analytically, we obtain $Var(S) = \dfrac{a}{L} + b\dfrac{L_A^3}{L^2}$.

This is irrespective of particle statistics!
(for free fermions and Bosons)



Numerical verification: 1D harmonic oscillator chains

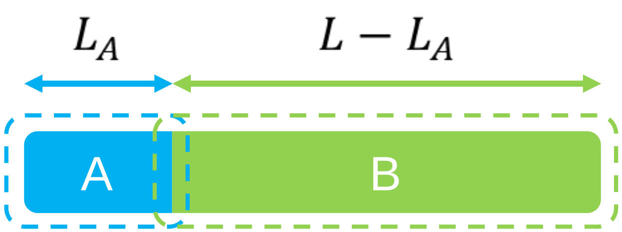
Free Fermionic systems give the same results. See Ref: *Nat Commun* **15**, 1775 (2024)



Region: $L_A \ll L$

Region: $L_A > L^{1/3}$

$$\mathrm{Var(O)} = \frac{a}{L} + b\,\frac{L_A^3}{L^2}$$



Total system size L: $10^2 \sim 10^8$

Subsystem size L_A: $2 \sim 3000$

1. The scaling law is universal with respect to entanglement entropy and Rényi entropy.

2. The scaling law is insensitive to particle statistics. The insensitivity of entanglement fluctuations to the particle statistics is dubbed boson-fermion universality.

Thank you for your attention!

Model

1D harmonic oscillator chains:

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The Rice-Mele model:

$$H_{RM} = -\sum_{i=1}^{L}(Jc_{i\bar{A}}^{\dagger}\, c_{i\bar{B}} + J'c_{i\bar{B}}^{\dagger}\, c_{(i+1)\bar{A}} + \text{h.c.})$$
$$+M\sum_{i=1}^{L}(c_{i\bar{A}}^{\dagger}\, c_{i\bar{A}} - c_{i\bar{B}}^{\dagger}\, c_{i\bar{B}}).$$

Second quantization:

$$\hat{H} = \sum_k \omega_k\left(\alpha_k^{\dagger}\alpha_k + \frac{1}{2}\right)$$

$$\omega_k = \sqrt{\omega^2 + 2K(1-\cos k)/m}$$

Diagonalization:

$$H_{RM} = \sum_k\left(-E_{fk}c_{k,\bar{A}}^{\dagger}c_{k,\bar{A}} + E_{fk}c_{k,\bar{B}}^{\dagger}c_{k,\bar{B}}\right)$$

$$E_{fk} = \sqrt{M^2 + J^2 + J'^2 + 2JJ'\cos k}.$$

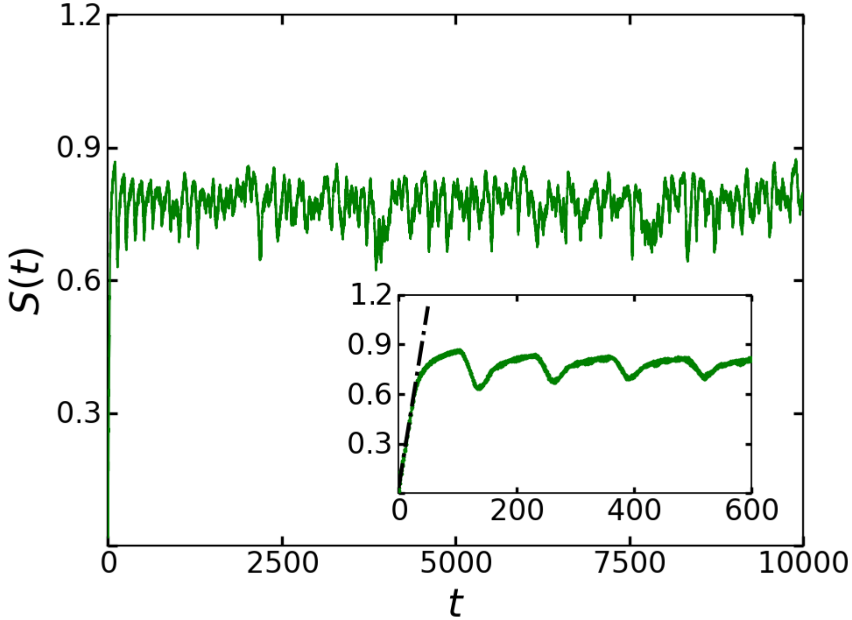
Ground state: $|\psi_0\rangle = \prod_k \otimes \left|0\right\rangle_k.$ $\quad |\psi_0\rangle = \prod_k \otimes\, c_{k,\bar{A}}^{\dagger}|0\rangle.$

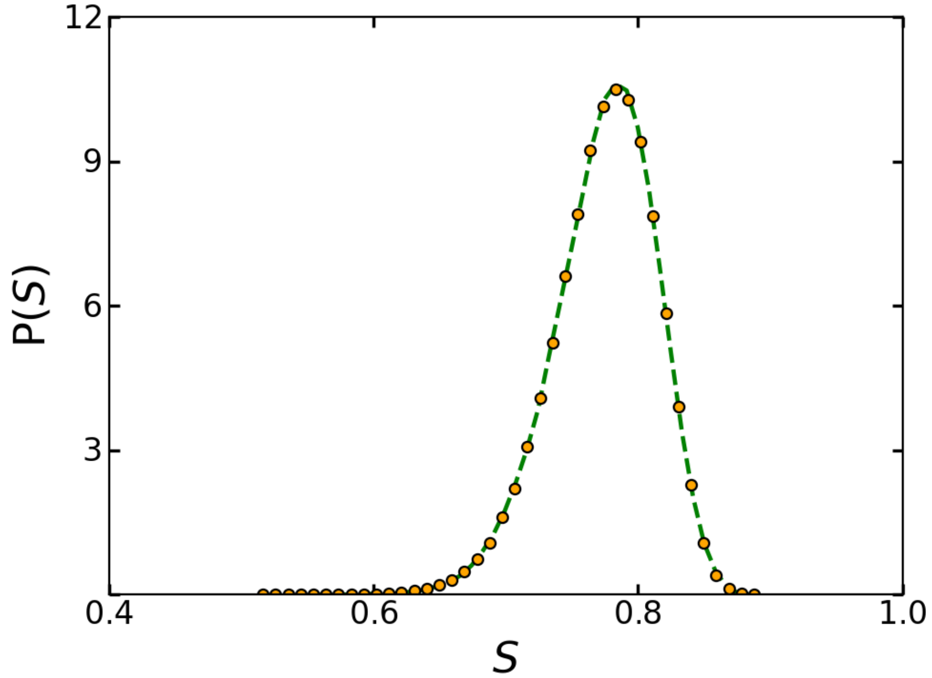
Quench dynamics: $\hat{H}_0(m,\omega_0,K_0) \longrightarrow \hat{H}(m,\omega,K)$ $\quad \hat{H}_0(J_0,J_0',M_0) \longrightarrow \hat{H}(J,J',M)$

The evolution of states: $|\psi(t)\rangle = e^{-i\hat{H}t}|\psi_0\rangle.$

Here and after the Planck constant $\hbar$ is set to unity.

Numerical simulation: 1D harmonic oscillator chains





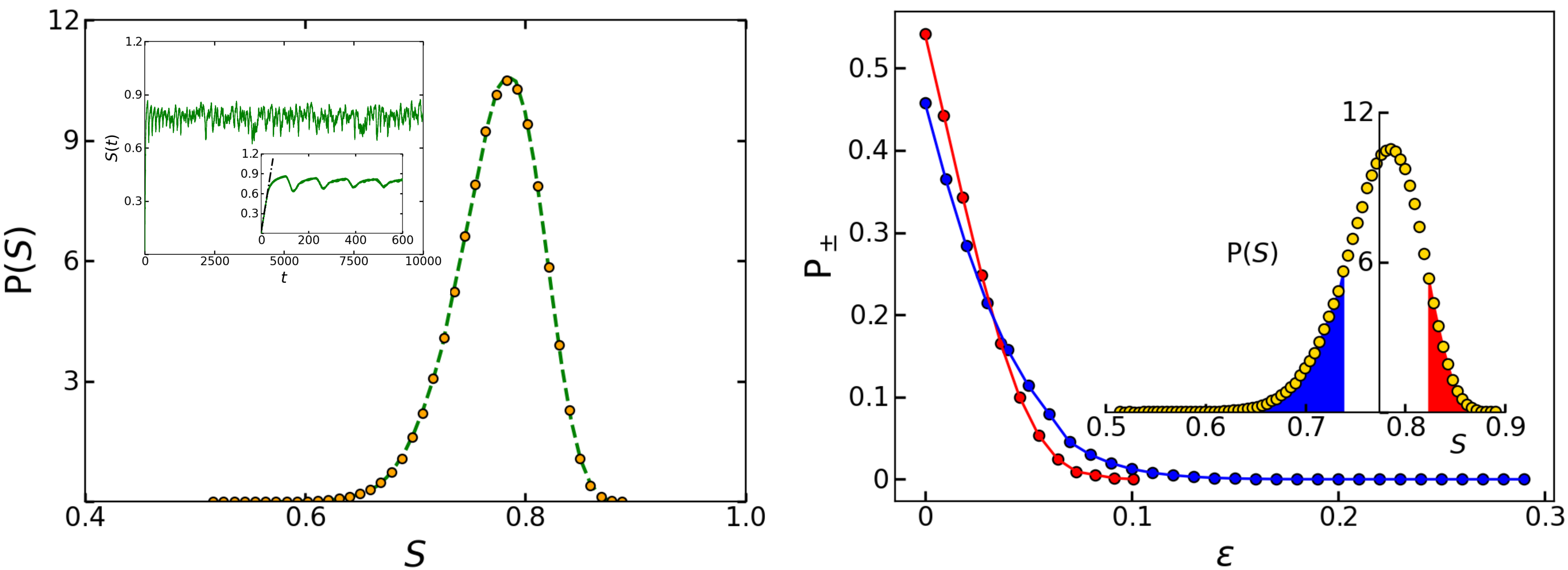
$$L = 124, L_A = 25$$

Green doted line: time series

Orange dots: uniformly drawn from the torus $\mathbb{T}^N$

Quench parameter: $\widehat{H}_0(\omega_0^2 = 1.5,\ K_0 = 1) \longrightarrow \widehat{H}(\omega^2 = 2.5, K = 1)$.

Numerical simulation: 1D harmonic oscillator chains

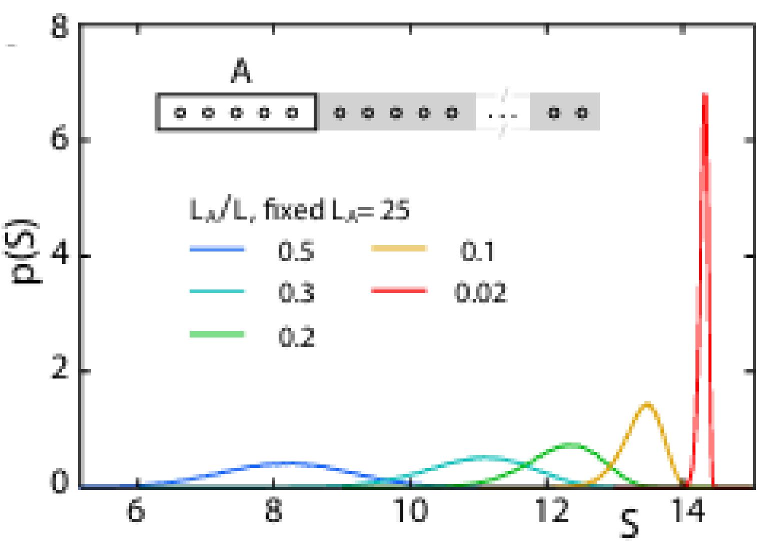


$$L = 124, L_A = 25$$

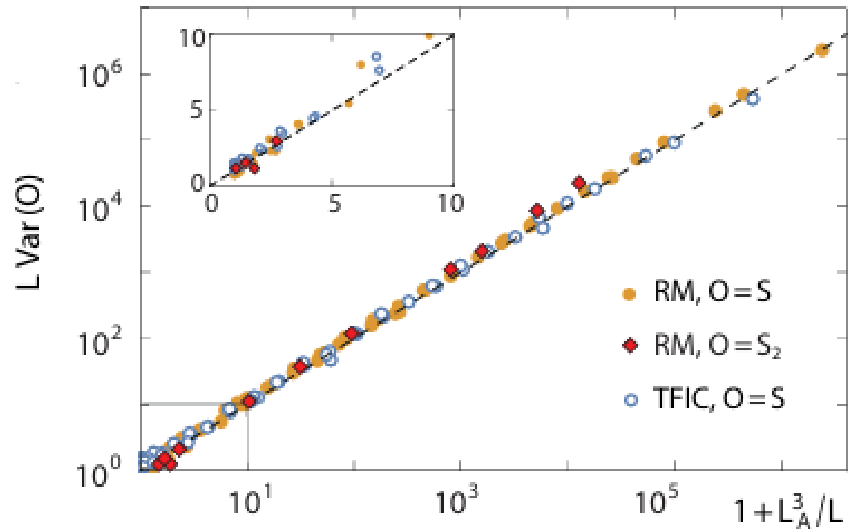
Quench parameter: $\widehat{H}_0(\omega_0^2 = 1.5,\ K_0 = 1) \longrightarrow \widehat{H}(\omega^2 = 2.5, K = 1)$.

$$\mathrm{P}(|S - \langle S\rangle| \geq \epsilon) = \begin{cases} e^{-\frac{\epsilon^2}{b_+}}, & \text{for } S - \langle S\rangle > 0, \quad \text{Sub-Gaussian upper tail} \\ e^{-\frac{\varepsilon^2}{(b_- + c\epsilon)}}, & \text{for } S - \langle S\rangle < 0, \quad \text{Sub-Gamma lower tail} \end{cases}$$

Numerical simulation: 1D free Fermionic systems



Asymmetric distribution



Scaling law

Free Fermionic systems give the same results.

The universality arises from common probabilistic structures, namely a product probability measure due to incommensurate $\boldsymbol{\omega_b}$ and $\boldsymbol{\omega_f}$. $\mathbb{T}^N = \mathbb{T}_0 \times \cdots \times \mathbb{T}_{N-1}$

1D XXZ spin chains

The Hamiltonian of the 1D XXZ spin chain is

$$\hat{H} = \sum_{\mathrm{i}=0}^{L-1} \frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + \Delta \sum_{i=0}^{L-1} (S_i^z S_{i+1}^z - \frac{1}{4})$$

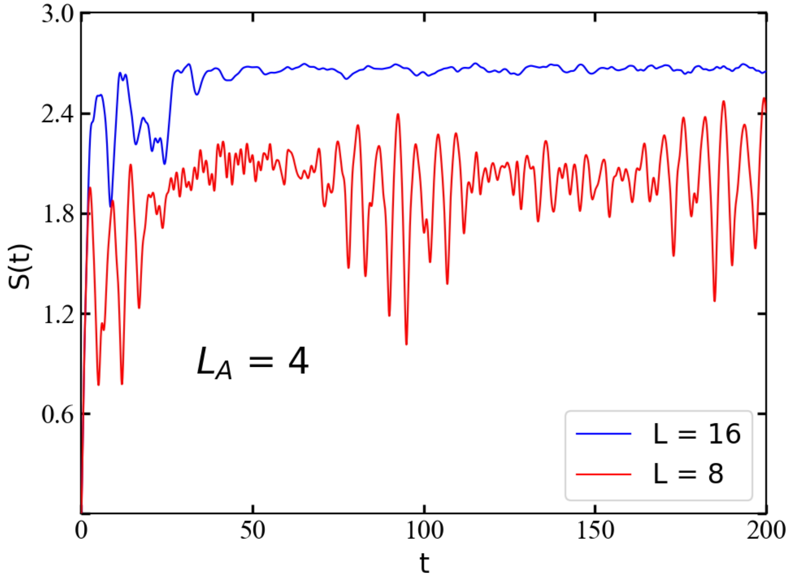
We study the entanglement dynamics of two different initial states, namely, Neel state and random phase state.

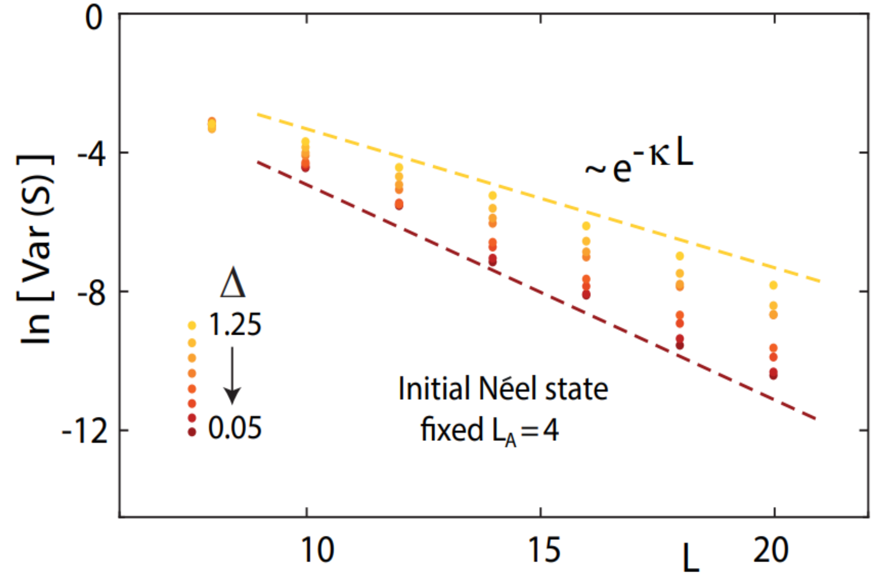
Neel state: $|\uparrow\downarrow\uparrow\downarrow \cdots \uparrow\downarrow\rangle$

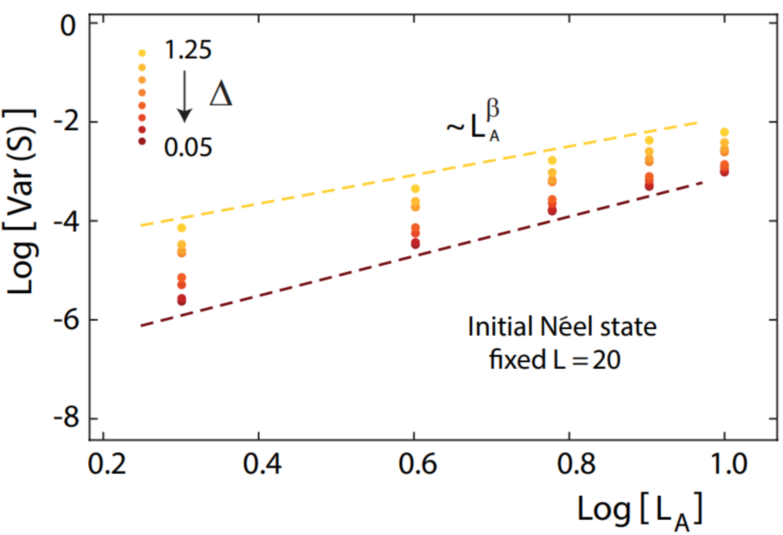
Random phase state: $|\psi\rangle = \sum_{n=0} \frac{e^{i\varphi_n}}{\sqrt{D}} |n\rangle$

The simulation is processed by exact diagonalization. And the conserved quantity total magnetization S_z and translational invariant symmetry have been taken into account.

1D XXZ spin chains







$$Var(S) \sim e^{-\kappa L} L_A^{\beta}$$

$$S(\rho_A) = -\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

$$\rho_A(t) = \sum_{m,n} \mathrm{e}^{-i(\omega_m - \omega_n)t} \chi_m \chi_n^* \mathrm{Tr}_B \left|\Psi_m\right\rangle \left\langle\Psi_n\right|$$