Intersection theory in Feynman parametrization

Reduction in Feynman parametrization via intersection theory

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January 10, 2025

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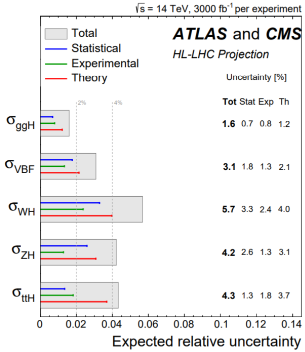
Part I: Reduction of Feynman Integrals

High-precision computation

The rapid development of HEP experiments has brought us high-precision data in decades, which in turn demands high-precision calculations in theory (Let $\mu = \dfrac{\sigma_{\text{Exp}}}{\sigma_{\text{SM}}}$):

- At 2012: $\mu = 1.4 \pm 0.3(\text{Exp}) \pm (\text{negligible Th})$.
- At 2022: $\mu = 1.05 \pm 0.04(\text{Exp}) \pm 0.04(\text{Th})$.



The future experiments (Run III of LHC, HL-LHC) require reducing theoretical uncertainties by at least a factor of 5-10 (1-2 higher orders in α_s).

Reduction of Feynman Integrals

The computation of Feynman integrals involves two parts:

- Reduce loop integrals to certain basis (which is called master integrals) by using Integration-By-Parts identities (IBPs).

$$I(\nu_1, \nu_2, \cdots, \nu_n) = \sum_i c_i \mathcal{I}_i \,, \quad \text{where } \mathcal{I}_i\text{'s are master integrals}\,.$$

- Calculate MIs. For analytic calculation, there are Mellin-Barnes method, canonical form method. And for numeric calculation, we have tools like AMFlow, sector decomposition, dimensional difference equations and numerical differential equations. All these method depend on IBP.

Problems	Massless double box	Massless double-pentagon	4-loop $g + g \to H$
Reduction time	18h	12d	860d

Part II: Introduction of Intersection Theory

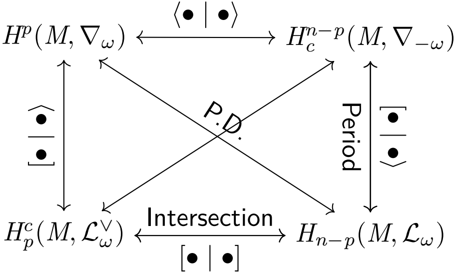
Introduction of Intersection Theory

The intersection theory reveals the vector space structure of a class of integrals known as *period integrals*, by treating these integrals as a non-degenerate bilinear form between the integration domain and a meromorphic differential form. [arXiv: 1711.00469][arXiv: 1810.03818]

$$I = \int_{\mathcal{C}_R} u\, \varphi_L \equiv \langle \varphi_L | \mathcal{C}_R] \ .$$

- u: A given multivalued function $u = \prod_{i=1}^{m} P_i(x)^{\alpha_i}$ called *twist*, which defines an n-dimensional affine variety M as $M := \mathbb{CP}^n \setminus \bigcup_{i=1}^{m} \{P_i(x) = 0\}$ and a single-valued holomorphic 1-form $\omega = \mathrm{d}\log u$.
- φ: A smooth p-form on M, governed by cohomology group $H^p(M, \nabla_\omega)$ with covariant derivative $\nabla_\omega = \mathrm{d} + \omega\wedge$.
- $\mathcal{C}$: A p-simplex of M, governed by homology group $H_p(M, \mathcal{L}_\omega^\vee)$ with the local system $\mathcal{L}_\omega^\vee$ generating by all branches of u.

One may find following isomorphism according to Poincaré duality and de Rham theorem: (See [K. Aomoto & M. Kita, *Theory of hypergeometric functions.*] for more details.)



And hence the dual integrals are introduced as

$$I^\vee = \int_{\mathcal{C}_L} u^{-1}\, \varphi_R \equiv [\mathcal{C}_L | \varphi_R\rangle \ .$$

Due to the isomorphism between $H^p(M, \nabla_\omega)$ and $H_c^{n-p}(M, \nabla_{-\omega})$, one can introduce a non-degenerate bilinear pairing between (bra) vectors and dual (ket) vectors, known as the *intersection number* $\langle \varphi_L | \varphi_R \rangle_\omega$.

$$\langle \varphi_L | \varphi_R \rangle_\omega \equiv \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathcal{C}} \iota(\varphi_L) \wedge \varphi_R = \frac{(-1)^n}{(2\pi \mathrm{i})^n} \int_{\mathcal{C}} \varphi_L \wedge \iota(\varphi_R) \,,$$

With

$$\iota(\varphi_L) = \varphi_L - \nabla_\omega(h\psi_L) \,, \quad \iota(\varphi_R) = \varphi_L - \nabla_{-\omega}(h\psi_R) \,,$$

$$h = \sum_{p \in \mathcal{P}_\omega} (1 - \theta_{x,p}) \,, \quad \theta_{x,p} = \theta(|x - p| - \epsilon) \,, \quad \mathcal{P}_\omega = \{\text{poles of } \omega\} \,, \quad \nabla_\omega \psi_L = \varphi_L \,, \quad \nabla_{-\omega} \psi_R = \varphi_R \,.$$

After choosing the basis of cohomology group, denoted as $\{\langle e_i|\}$ and $\{|h_i\rangle\}$, one have

$$\langle \varphi_L | = \sum_{i,j} \langle \varphi_L | h_i \rangle_\omega \left(\boldsymbol{C}^{-1} \right)_{ij} \langle e_j | \,, \quad \boldsymbol{C}_{ij} \equiv \langle e_i | h_j \rangle_\omega \,.$$

Relative cohomology

One can introduce relative cohomology group by the isomorphism [arXiv: 1804.00366]

$$H^n(\mathbb{CP}^n \setminus \mathcal{P}_\omega, \{D=0\}, \nabla_{-\omega}) \simeq H^n_c(\mathbb{CP}^n \setminus \mathcal{P}_\omega \cup \{D=0\}, \nabla_{-\omega}) \, .$$

Here boundaries are divided into two parts [arXiv: 2104.06898][arXiv: 2112.00055]

- Twisted boundaries: Given by $\mathcal{P}_\omega$. In their neighborhoods $\nabla_{\pm\omega}$ is locally invertible. They are boundaries regulated by u.
- Relative boundaries: Given by $\{D=0\}$. In their neighborhoods $\nabla_{\pm\omega}$ is only invertible up to an integration constant. They are singularities introduced by φ and not regulated by u.

Leray coboundaries are introduced instead of regulators in twist.

$$\delta_{x_{m+1},\cdots,x_n}\phi_R = \frac{u(x_1,\cdots,x_m,x_{m+1},\cdots,x_n)}{u(x_1,\cdots,x_m,0,\cdots,0)}\,\phi_R \wedge \mathrm{d}\theta_{x_{m+1},0} \wedge \cdots \wedge \mathrm{d}\theta_{x_n,0} \, .$$

And intersection numbers are

$$\left\langle \varphi_L \middle| \delta_{x_{m+1},\cdots,x_n}\phi_R \right\rangle_\omega = \left\langle \mathrm{Res}_{x_{m+1}=\cdots=x_n=0} \frac{u}{u_0}\varphi_L \middle| \phi_R \right\rangle_{\omega_0} \, .$$

Part III: Intersection Theory in Feynman Parametrization

Feynman & LP parametrization

Consider the LP parametrization

$$I(\nu_1, \cdots, \nu_n) = e^{\epsilon\gamma_E L} \frac{(-1)^\nu \Gamma(d/2)}{\Gamma\left((L+1)d/2 - \nu\right)} \int_0^\infty \left(\prod_i \frac{x_i^{\nu_i - 1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) (\mathcal{U} + \mathcal{W})^{-d/2} .$$

When some of the ν_i's are zero, meaning they are kind of integrals belong to sub-sectors within the integral family, we can insert a $\delta(x_i)$ into the integration measure, and get rid of the $\Gamma(\nu_i)$ in the denominator by

$$\frac{x_i^{\rho-1}}{\Gamma(\rho)} = \delta(x_i) + \mathcal{O}(\rho^1) .$$

And we can express integrals in all sectors of the family using a unified LP polynomial $G \equiv \mathcal{U} + \mathcal{W}$. Stripping the pre-factors we consider

$$J(\nu_1, \cdots, \nu_n) = \int_0^\infty \left(\prod_{i, \nu_i > 0} x_i^{\nu_i - 1} \, \mathrm{d}x_i \right) \left(\prod_{i, \nu_i = 0} \delta(x_i) \, \mathrm{d}x_i \right) (\mathcal{U} + \mathcal{W})^{-d/2} .$$

For those $\nu_i < 0$, i.e., integrals with numerators, we can integrate by parts and obtain

$$\int_0^\infty \mathrm{d}x_i \, \frac{x_i^{\rho+\nu_i-1}}{\Gamma(\rho+\nu_i)} \, G^{-d/2} = \int_0^\infty \mathrm{d}x_i \, \frac{x_i^{\rho-1}}{\Gamma(\rho)} \left(-\frac{\partial}{\partial x_i}\right)^{-\nu_i} G^{-d/2}$$
$$= \int_0^\infty \mathrm{d}x_i \, \delta(x_i) \left(-\frac{\partial}{\partial x_i}\right)^{-\nu_i} G^{-d/2} + \mathcal{O}(\rho^1) \, .$$

Therefore, we can again express the integrals using the same G polynomial.

Advantages in intersection theory

- We do **NOT** need to consider the spacetime dimension shift cause by partial derivatives in usual IBP reduction methods, since integrands with extra factors of G in the denominator are treated automatically in a unified way.
- The Symanzik polynomials are usually simpler than the Baikov polynomial since they are homogeneous polynomials of the Feynman parameters, and can be naturally interpreted in a projective space.

Intersection theory in Feynman parametrization

In our approach, we'd like to consider

$$\begin{aligned}
\text{Feynman integrands} \quad & \in H^n(\mathbb{CP}^n \setminus \mathcal{P}_\omega, \{D=0\}, \nabla_\omega) && \text{(relative forms)} \\
& \simeq H_c^n(\mathbb{CP}^n \setminus \mathcal{P}_\omega \cup \{D=0\}, \nabla_\omega) && \\
\text{Dual integrands} \quad & \in H^n(\mathbb{CP}^n \setminus \mathcal{P}_\omega \cup \{D=0\}, \nabla_{-\omega}) && \text{(alg./holo. forms)}
\end{aligned}$$

The dual forms contain factors such as $1/x_i$ that are singular at the relative boundaries. We can then calculate intersection numbers just as in Baikov representation with a slight change.

$$\delta_{x_{m+1},\cdots,x_n}\phi_L = \frac{u(x_1,\cdots,x_m,0,\cdots,0)}{u(x_1,\cdots,x_m,x_{m+1},\cdots,x_n)}\,\phi_L \wedge \mathrm{d}\theta_{x_{m+1},0} \wedge \cdots \wedge \mathrm{d}\theta_{x_n,0}\,,$$

while the intersection numbers being

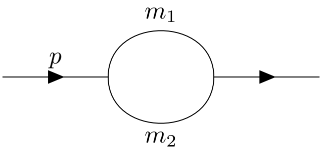
$$\left\langle \delta_{x_{m+1},\cdots,x_n}\phi_L \middle| h_i \right\rangle_\omega = \left\langle \phi_L \middle| \mathrm{Res}_{x_{m+1}=\cdots=x_n=0}\, \frac{u_0}{u} h_i \right\rangle_{\omega_0}\,.$$

Note that **we can always choose the dual basis** $\{|h_i\rangle\}$ **to have at most simple poles** at the relative boundaries so that the ratio u_0/u do not contribute here.

Part IV: Applications

Example 1: One-loop bubble

Consider bubble integrals whose kinematic variables are m_1^2, m_2^2 and $p^2 = s$.



$$G = \mathcal{U} + \mathcal{W} = x_1 + x_2 + m_1^2 x_1^2 + m_2^2 x_2^2 + m_1^2 x_1 x_2 + m_2^2 x_1 x_2 - s x_1 x_2 \,.$$

The dimension of each layer is obtained by considering all non-zero sectors defined by products of zero or more δ functions and summing the dimensions of them.

$$\hat{e}_i^{(\mathbf{1})} \in \{1, \delta(x_1)\}\,, \quad \hat{e}_i = \hat{e}_i^{(\mathbf{2})} \in \{1, \delta(x_1), \delta(x_2)\}\,, \quad \hat{h}_i^{(\mathbf{1})} \in \left\{1, \frac{1}{x_1}\right\}, \quad \hat{h}_i = \hat{h}_i^{(\mathbf{2})} \in \left\{1, \frac{1}{x_1}, \frac{1}{x_2}\right\}.$$

They correspond to the MIs $J_{\text{Bub}}(1,1)$, $J_{\text{Bub}}(0,1)$ and $J_{\text{Bub}}(1,0)$.

Let us decompose $I_{\text{Bub}}(1,2)$ (corresponding to $\varphi_L = x_2\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$) as

$$I_{\text{Bub}}(1,2) = c_1\,I_{\text{Bub}}(1,1) + c_2\,I_{\text{Bub}}(0,1) + c_3\,I_{\text{Bub}}(1,0)\,.$$

Intersection number gives

$$c_1 = -\frac{(d-3)(m_1^2 - m_2^2 + s)}{\lambda(m_1^2, m_2^2, s)}, \quad c_2 = -\frac{(d-2)(m_1^2 + m_2^2 - s)}{2m_2^2\lambda(m_1^2, m_2^2, s)}, \quad c_3 = \frac{d-2}{\lambda(m_1^2, m_2^2, s)}\,,$$

Another example is $I_{\text{Bub}}(-1,2)$, with

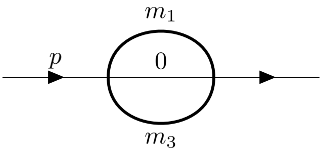
$$\varphi_L = \frac{d}{2}\frac{x_2\left(1 + m_1^2 x_2 + m_2^2 x_2 - s x_2\right)}{G}\,\delta(x_1)\mathrm{d}x_1 \wedge \mathrm{d}x_2\,.$$

Repeating the procedure as above, one can find

$$I_{\text{Bub}}(-1,2) = \frac{2m_2^2 - (d-2)\left(m_1^2 - m_2^2 - s\right)}{2m_2^2}\,I_{\text{Bub}}(0,1)\,.$$

Example 2: Two-loop sunrise

Consider sunrise integrals whose kinematic variables are m_1^2, m_3^2 and $p^2 = s$.



$$G = x_1 x_2 + x_1 x_3 + x_2 x_3 + m_1^2 x_1^2 (x_2 + x_3) + (m_1^2 + m_3^2 - s) x_1 x_2 x_3 + m_3^2 x_3^2 (x_1 + x_2)\,.$$

We find the basis are

$$\hat{e}_i^{(\mathbf{1})} \in \{1, \delta(x_1)\}\,, \quad \hat{e}_i^{(\mathbf{2})} \in \{1, x_2, \delta(x_2)\}\,, \quad \hat{e}_i = e_i^{(\mathbf{3})} \in \{1, x_1, x_3, \delta(x_2)\}\,.$$

$$\hat{h}_i^{(\mathbf{1})} \in \left\{1, \frac{1}{x_1}\right\}, \quad \hat{h}_i^{(\mathbf{2})} \in \left\{1, x_2, \frac{1}{x_2}\right\}, \quad \hat{h}_i = h_i^{(\mathbf{3})} \in \left\{1, x_1, x_3, \frac{1}{x_2}\right\}.$$

Again, the basis of outermost layer correspond to the MIs $I_{\text{Sun}}(1,1,1)$, $I_{\text{Sun}}(2,1,1)$, $I_{\text{Sun}}(1,1,2)$ and $I_{\text{Sun}}(1,0,1)$.

We can now reduce a target integral, e.g., $I_{\text{Sun}}(1,2,1)$ into the 4 MIs:

$$I_{\text{Sun}}(1,2,1) = c_1 I_{\text{Sun}}(1,1,1) + c_2 I_{\text{Sun}}(2,1,1) + c_3 I_{\text{Sun}}(1,1,2) + c_4 I_{\text{Sun}}(1,0,1)\,.$$

Repeating the procedures as in the one-loop example, we find

$$\begin{aligned}
c_1 &= \frac{(d-3)(3d-8)(m_1^2+m_3^2-s)}{(d-4)\lambda(m_1^2,m_3^2,s)}\,,\\
c_2 &= -\frac{4(d-3)m_1^2(m_1^2-s)}{(d-4)\lambda(m_1^2,m_3^2,s)}\,,\\
c_3 &= -\frac{4(d-3)m_3^2(m_3^2-s)}{(d-4)\lambda(m_1^2,m_3^2,s)}\,,\\
c_4 &= -\frac{(d-2)^2}{(d-4)\lambda(m_1^2,m_3^2,s)}\,.
\end{aligned}$$

Degenerate limits

Subtleties in relative cohomology

The relative cohomology method involves certain subtleties in specific degenerate limits. These occur when Feynman diagrams contain **equal internal masses** or involve **light-like external momentum**. [arXiv: 2104.06898]

In the massless external limit, one may find some forms become exact on their respective cuts, such as the bubble integral $I_{\text{Bub}}(1,1)$. For this case the basis may shrink to a smaller set of basis.

$$\hat{e}^{(\mathbf{2})} = \{1, \delta(x_1), \delta(x_2)\} \xrightarrow{p^2 \to 0} \hat{e}^{(\mathbf{2})} = \{\delta(x_1), \delta(x_2)\}$$

An even trickier degenerate limit is when $m_1 = m_2 = m$, in addition to $p^2 = 0$. In this case, the two tadpoles $I_{\text{Bub}}(1,0)$ and $I_{\text{Bub}}(0,1)$ are equivalent by IBP relations alone, without invoking the symmetry relations. The reduction can still be done with

$$\hat{e} = \delta(x_2)\,, \quad \hat{h} = 1/x_1 + 1/x_2\,.$$

Summary

A novel approach to reduce loop integrals in the Feynman parametrization with the help of intersection theory and relative cohomology.

- In the Feynman parametrization, the recursive structure of representations in different sub-sectors of an integral family is manifest.
- The Symanzik polynomials are homogeneous in the Feynman parameters, and are usually simpler than the Baikov polynomial.
- In the Feynman parametrization, one does not need to introduce ISPs unless they appear in the numerator. When that happens, the integral can be easily represented by integrands with Symanzik polynomials in the denominator.

Outlooks:

- Improve the efficiency further and to tackle more difficult problems.

Thanks for listening!