Elliptic Feynman Integrals in Normal Form Made Simple

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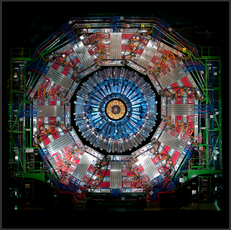
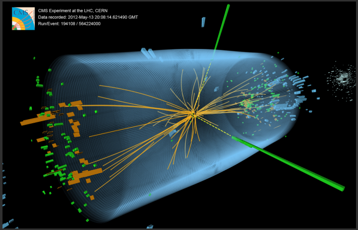
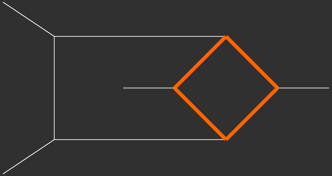
Feynman integrals

- Experiment: Data of observables from colliders (LHC, Future: HL-LHC, CEPC, FCC-ee, etc) in a high precision

- Theory: Standard Model, perturbative QFT, tasks boiled into the calculation of Feynman integrals (in dimensional regularization)



Differential equation method

Differential equation: Calculate families of integrals

- Linear system: **Master integrals** as the basis integrals, reduction with **IBP relations**, **intersection number**, etc

$$I_i = c_{ij} M_j \,. \tag{1}$$

- Closure under differentiation: **Gauss-Manin system**

$$\mathrm{d}\boldsymbol{M}(\boldsymbol{x},\varepsilon) = \boldsymbol{A}(\boldsymbol{x},\varepsilon)\boldsymbol{M}(\boldsymbol{x},\varepsilon)\,. \tag{2}$$

- **Special linear form**:

$$\mathrm{d}\boldsymbol{M}(\boldsymbol{x},\varepsilon) = \boldsymbol{A}(\boldsymbol{x},\varepsilon)\boldsymbol{M}(\boldsymbol{x},\varepsilon) = (\boldsymbol{A}^{(0)}(\boldsymbol{x}) + \varepsilon\boldsymbol{A}^{(1)}(\boldsymbol{x}))\boldsymbol{M}(\boldsymbol{x},\varepsilon)\,. \tag{3}$$

- **ε-factorised form**: Easy to expand in ε, **iterated integrals** as the coefficients

$$\mathrm{d}\boldsymbol{M}(\boldsymbol{x},\varepsilon) = \varepsilon\boldsymbol{A}(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x},\varepsilon)\,. \tag{4}$$

MPLs (Multiple polylogarithms)

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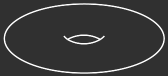
$$G(z_1, \cdots, z_n; y) = \int_0^y \frac{\mathrm{d}t_1}{t_1 - z_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{t_2 - z_2} \cdots \int_0^{t_{n-1}} \frac{\mathrm{d}t_n}{t_n - z_n} . \tag{5}$$

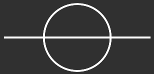
Iterated integrals on (the covering space of) the **moduli space** $\mathcal{M}_{0,n}$ where the points can be understood as the Riemann sphere (with **marked points** $z_1, \cdots, z_n$), and different points mean the marked points are not isomorphic.

Functions beyond MPLs

- The next-to-easiest class of function is the **elliptic Feynman integrals**, associated with **elliptic curves** whose Riemann surface is a **torus** of genus 1.



- The existence of such integrals can be traced back to the two-loop corrections to the electron self-energy by Sabry in 1962, while the first fully analytic calculation was done half a century later. [Hönemann *et al.* 2018]



- Other more complicated functions include
  - Higher genus: **Hyperelliptic**
  - Higher dimensions: **Calabi-Yau**

Elliptic curve

Elliptic curve:

A smooth, projective, algebraic curve in $\mathbb{CP}^2$ of genus 1 with one marked point.

Elliptic curve in different forms:

- From the **maximal cut**:

$$w^2 = P_4(z) \equiv (z - c_1)(z - c_2)(z - c_3)(z - c_4)\,, \tag{6}$$

  which is the most relevant one in the context of Feynman integrals.

- **Legendre normal form**:

$$y^2 = P_L(x) \equiv x(x-1)(x-\lambda)\,, \tag{7}$$

  with the geometry encoded in λ.

The different forms can be connected to each other with suitable **Möbius transformations**.

Abelian differentials

We define the **pre-canonical basis** with **Abelian differentials**

Abelian differentials

- First kind: holomorphic

$$\phi_1 = \frac{1}{2}\frac{\mathrm{d}x}{\sqrt{P_L(x)}} = \mathrm{d}F(\sin^{-1}\sqrt{x/\lambda}, \lambda)\,, \tag{8}$$

- Second kind: meromorphic with vanishing residues

$$\phi_2 = \frac{1}{2}\frac{x\,\mathrm{d}x}{\sqrt{P_L(x)}} = \mathrm{d}F(\sin^{-1}\sqrt{x/\lambda}, \lambda) - \mathrm{d}E(\sin^{-1}\sqrt{x/\lambda}, \lambda)\,, \tag{9}$$

- Third kind: meromorphic with non-vanishing residues

$$\phi_{i-2} = \frac{1}{2}\frac{\sqrt{P_L(e_i)}}{x-e_i}\frac{\mathrm{d}x}{\sqrt{P_L(x)}} = -\frac{\sqrt{P_L(e_i)}}{e_i}\mathrm{d}\Pi(\lambda/e_i, \sin^{-1}\sqrt{x/\lambda}, \lambda)\,. \tag{10}$$

Differential equations

The differential equations are in a special linear form, and the $\boldsymbol{A}^{(0)}$ part is

$$\boldsymbol{A}^{(0)} = \mathrm{d}\lambda \begin{pmatrix} \frac{1}{2(1-\lambda)} & \frac{1}{2(\lambda-1)\lambda} & 0 & \cdots & 0 \\ \frac{1}{2(1-\lambda)} & \frac{1}{2(\lambda-1)} & 0 & \cdots & 0 \\ \frac{(e_5-1)e_5}{2(\lambda-1)\sqrt{P_L(e_5)}} & -\frac{(e_5-1)e_5}{2(\lambda-1)\lambda\sqrt{P_L(e_5)}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(e_{n+1}-1)e_{n+1}}{2(\lambda-1)\sqrt{P_L(e_{n+1})}} & -\frac{(e_{n+1}-1)e_{n+1}}{2(\lambda-1)\lambda\sqrt{P_L(e_{n+1})}} & 0 & \cdots & 0 \end{pmatrix}$$

$$+ \sum_{i=5}^{n+1} \mathrm{d}e_i \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{e_i}{2\sqrt{P_L(e_i)}} & \frac{1}{2\sqrt{P_L(e_i)}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} . \tag{11}$$

ε-factorised basis with constant period matrix

- The transformation from the pre-canonical basis to the ε-factorised basis

$$\mathcal{T} = \begin{pmatrix} E(1-\lambda) & -K(1-\lambda) & 0 & 0 \\ E(\lambda)-K(\lambda) & K(\lambda) & 0 & 0 \\ E(\sin^{-1} a_5,\lambda) - F(\sin^{-1} a_5,\lambda) & F(\sin^{-1} a_5,\lambda) & 1 & 0 \\ E(\sin^{-1} a_\infty,\lambda) - F(\sin^{-1} a_\infty,\lambda) & F(\sin^{-1} a_\infty,\lambda) & 0 & 1 \end{pmatrix} . \quad (12)$$

- With such transformation, the **period matrix** turns out to be a constant matrix,

$$\mathcal{P} = \begin{pmatrix} \pi & 0 & 0 & 0 \\ 0 & -\pi i & 0 & 0 \\ \pi i & 0 & \pi i & 0 \\ \pi i & 0 & 0 & \pi i \end{pmatrix} , \quad (13)$$

from which we obtain special relations for elliptic integrals, e.g.

$$E(\sin^{-1} a,\lambda)K(1-\lambda) + F(\sin^{-1} a,\lambda)\left[E(1-\lambda)-K(1-\lambda)\right] = \frac{1}{a}\sqrt{(1-a^2)(1-\lambda a^2)}\left[\frac{1}{a^2-1}\Pi\left(\frac{a^2(1-\lambda)}{a^2-1}, 1-\lambda\right) + K(1-\lambda)\right] . \quad (14)$$

Pros and cons

- Pros:
  - Remove non-ε-factorised mixing easier
  - Asymptotically $\mathrm{d}\log$-**forms** in the degenerated limit, **UT** boundary conditions
- Cons:
  - Logarithmic integration kernels around any degenerated points
  - Too many different kinds of elliptic integrals

Question

A **canonical basis**? $\Rightarrow$ Apply algorithms in [Görges *et al.* 2023], done!

Canonical basis

"I have discovered a truly marvelous proof of this, which this margin is too narrow to contain."

-- Pierre de Fermat

Conclusion

Procedure:

Construct basis in Legendre normal family, which is simpler, and with Möbius transformation we can project the basis back to the generic integral family without loss of generality.

- Method for systematizing the construction of ε-factorised (canonical) basis for (univariate) elliptic Feynman integrals (with kinematic marked points)
- Pre-canonical basis which enjoys nice properties, make subsequent procedures to remove the mixing with sub-sectors easier
- ε-factorised basis for the non-planar double box with an inner massive loop, useful for the calculations of phenomenological processes, e.g. dijet and diphoton production
- No new algorithms for constructing ε-factorised basis, however, we can apply any algorithms with the workflow here and offer the corresponding basis directly once the integral family is specified, without following algorithms case by case

Outlook

- Check if the intersection matrix is constant (the definition for canonical basis beyond MPLs? [Duhr *et al.* 2024])
- Prove the equivalence of the methods with Picard-Fuchs operator [Pögel *et al.* 2023] and with semi-simple unipotent decomposition [Görges *et al.* 2023]
- Express the connection matrices in modular variables and consider their modular properties, potential applications to faster numerical computation with modular transformations (both for the base and the fibre) [Weinzierl 2021]
- Further simplification for the case with $\mathrm{d}\log$-forms (when we will have $\mathrm{d}\log$-forms?) and special relations involved.
- Better interpretation (choice) for the canonical basis integral associated with Abelian differential of the second kind
- Apply the workflow to more complicated geometries (with marked points), e.g. hyperelliptic Feynman integrals [Duhr *et al.* 2024]

The End

The integral family

- We consider a generic **univariate** integral family defined by

$$I = \int_{\mathcal{C}} u\,\varphi\,, \tag{15}$$

where u is a **multi-valued** function

$$u = P_4(z)^{-1/2} \prod_{i=1}^{n} (z - c_i)^{-\beta_i \varepsilon}\,, \tag{16}$$

while φ is single-valued 1-form with potential poles at the branch points c_i's (we denote $c_{n+1} = c_\infty \equiv \infty$).

- We apply the Möbius transformation from $P_4(z)$ to $P_L(x)$, then

$$u_L = P_L(x)^{-1/2} \prod_{i=2}^{n+1} (x - e_i)^{-\beta_i \varepsilon}\,, \tag{17}$$

with $e_1 = \infty$, $e_2 = 0$, $e_3 = \lambda$ and $e_4 = 1$.

Möbius transformation

- As promised, the pre-canonical basis constructed in Legendre family can be transformed back to the original generic family with a Möbius transformation

$$\phi_1 = \frac{\sqrt{c_{13}c_{24}}}{2}\frac{\mathrm{d}z}{\sqrt{P_4(z)}}\,, \tag{18}$$

$$\phi_2 = \frac{1}{2}c_{41}\sqrt{\frac{c_{13}}{c_{24}}}\frac{z-c_2}{z-c_1}\frac{\mathrm{d}z}{\sqrt{P_4(z)}}\,, \tag{19}$$

$$\phi_{i-2} = \frac{1}{2}\sqrt{P_4(c_i)}\left(\frac{1}{z-c_i}-\frac{1}{c_{1i}}\right)\frac{\mathrm{d}z}{\sqrt{P_4(z)}}\,, \tag{20}$$

$$\phi_{n-1} = -\frac{1}{2}(z-c_1)\frac{\mathrm{d}z}{\sqrt{P_4(z)}}\,. \tag{21}$$

- The pre-canonical basis integrals are asymptotically $\mathrm{d}\log$-forms in the degenerated limit $c_1 \to c_2$, which indicates a UT boundary conditions there.

Semi-simple unipotent decomposition

- For simplicity, we reorder pre-canonical basis defined before as

$$\widetilde{\phi}_1 = \phi_1 \,, \quad \widetilde{\phi}_{i-3} = \phi_{i-2} \,, \quad \widetilde{\phi}_{n-1} = \phi_2 \,. \tag{22}$$

- Decompose the period matrix into the **semi-simple** and the **unipotent** part and remove the semi-simple part.

$$\widetilde{\mathcal{P}}_{\mathrm{pre}} = \widetilde{\mathcal{P}}^{\mathrm{ss}}_{\mathrm{pre}} \cdot \widetilde{\mathcal{P}}^{\mathrm{u}}_{\mathrm{pre}} \,, \tag{23}$$

where

$$\widetilde{\mathcal{P}}^{\mathrm{u}}_{\mathrm{pre}} = \begin{pmatrix} 1 & 0 & 0 & \frac{\varpi_1(\lambda)}{\varpi_0(\lambda)} \\ 0 & 1 & 0 & \frac{\varpi_0(\lambda)\vartheta_1(e_5,\lambda)-\varpi_1(\lambda)\vartheta_0(e_5,\lambda)}{\varpi_0(\lambda)} \\ 0 & 0 & 1 & \frac{\varpi_0(\lambda)\vartheta_1(e_6,\lambda)-\varpi_1(\lambda)\vartheta_0(e_6,\lambda)}{\varpi_0(\lambda)} \\ 0 & 0 & 0 & 1 \end{pmatrix} \,. \tag{24}$$

- Rescale the last integrals.
- Integrate out the non-ε-factorised terms.

Canonical basis

We denote the reordered pre-canonical basis as $\widetilde{I}_i$ and the canonical basis as M_i.
The basis integrals associated to Abelian differentials of the first and the third kind are

$$M_1 = \frac{1}{\varpi_0}\widetilde{I}_1\,, \quad M_{i-3} = \widetilde{I}_{i-3} - \vartheta_0(e_i,\lambda)M_1\,. \tag{25}$$

The basis integral associated to Abelian differentials of the second kind is the most special one

$$M_{n-1} = \left\{\frac{1}{\pi\varepsilon}\frac{\partial}{\partial\tau} - \frac{1}{8}\left[\sum_{i=2}^{n+1}\beta_i\left(\lambda + e_i - 1\right)\varpi_0(\lambda)^2 + \sum_{i=5}^{n+1}\beta_i\vartheta_0(e_i,\lambda)^2\right]\right\}M_1\,, \tag{26}$$

where we choose the variables for the base manifold (moduli space $\mathcal{M}_{1,n-2}$) as the **modular variables** $\frac{\varpi_1(\lambda)}{\varpi_0(\lambda)} = -i\tau$, $\frac{\varpi_0(\lambda)\vartheta_1(e_i,\lambda)-\varpi_1(\lambda)\vartheta_0(e_i,\lambda)}{\varpi_0(\lambda)} = -2(z_i+\tau)$.