Solutions for the Linear Waves in First-Order Spin Magneto-Hydrodynamics

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Zhe, Koichi, Jin, arXiv:2402.18601 (2024)

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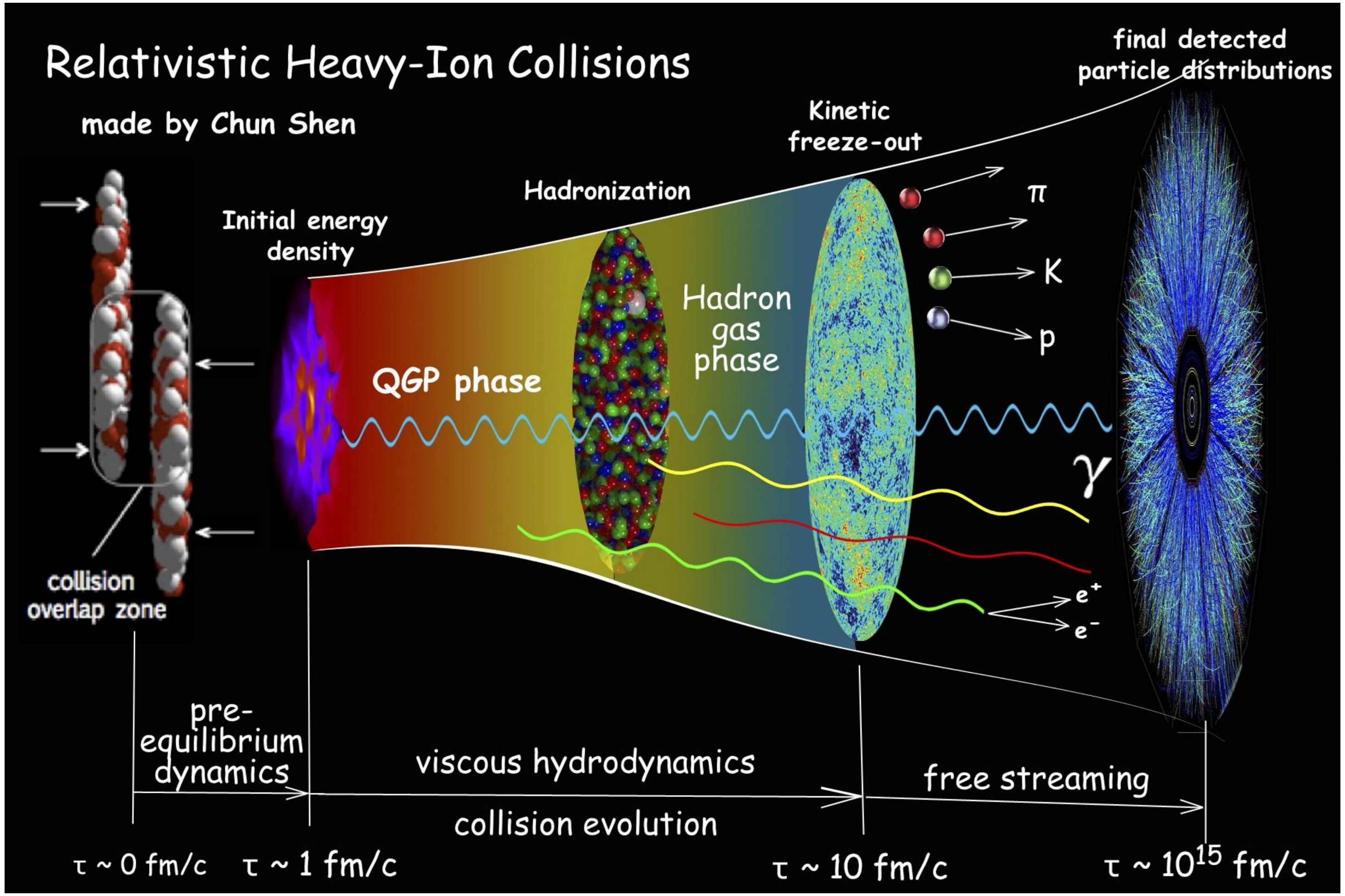
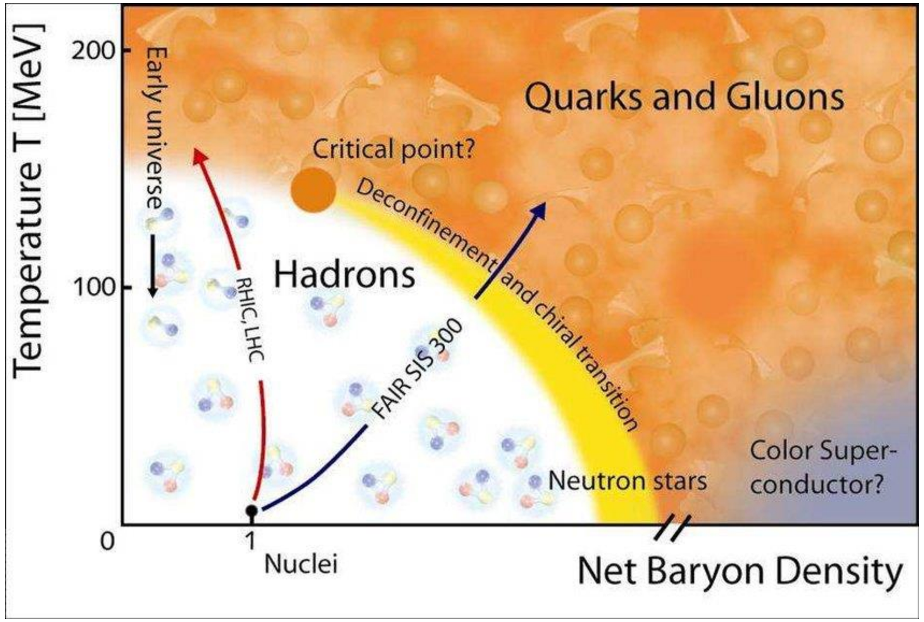
Background



phase diagram



QGP is a deconfined nuclear matter and can be created in heavy ion collisions. It is well described by hydrodynamic simulations, which is successful in describing the long wavelength modes.

A introduction for the relativistic hydrodynamics

Conserved current:

$$\partial_\mu j^\mu = 0$$

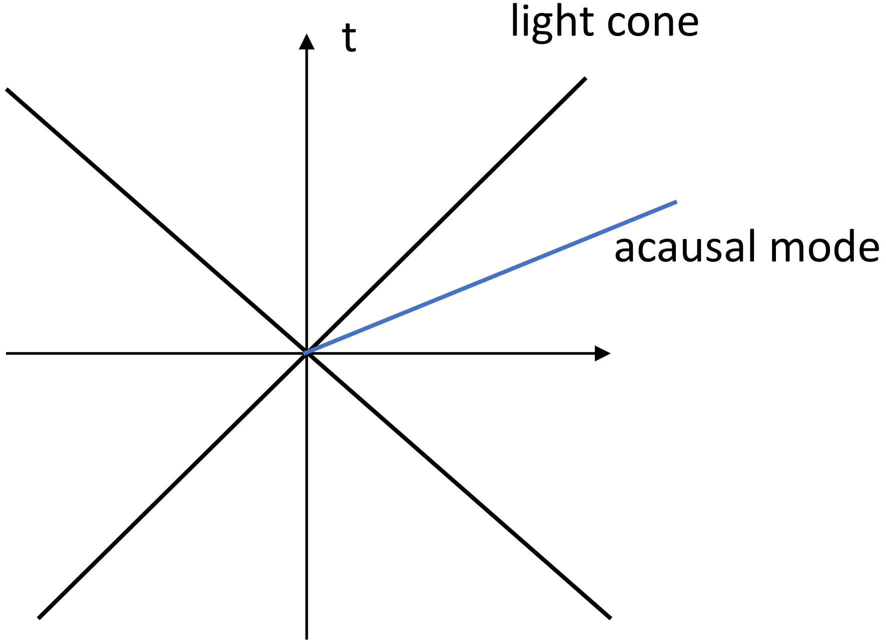
While j^μ could be constructed by some hydrodynamic variables:

$$j^\mu = j_0^\mu + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + \cdots$$

Considering a excitation as

$$u = u_0 e^{-i\omega t + ikx}$$

- A basic requirement to hydro simulation is the stability, i.e. $Im(\omega) \leq 0$.
- For a relativistic hydrodynamics, the causality promise the stability is protected in any frames



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Magnetic field and spin in Quark gluon plasma

- If the collision is non-central, there will be a large vorticity in the QGP.
- The vorticity may result in a magnetic field in non-central collision, which inspired the investigation of relativistic magnetic hydrodynamics (MHD) in high energy physics.

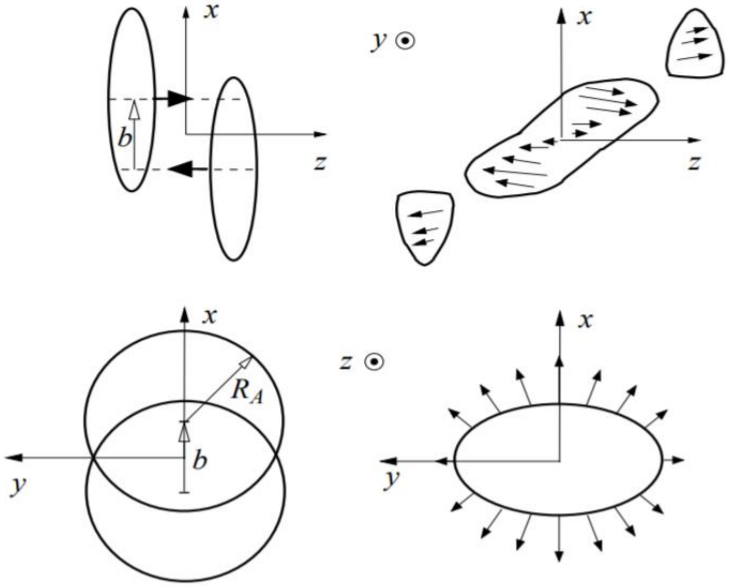


Figure 1: non-central collision of heavy ion Liang-Wang, et al. arXiv:0410079 (2004)

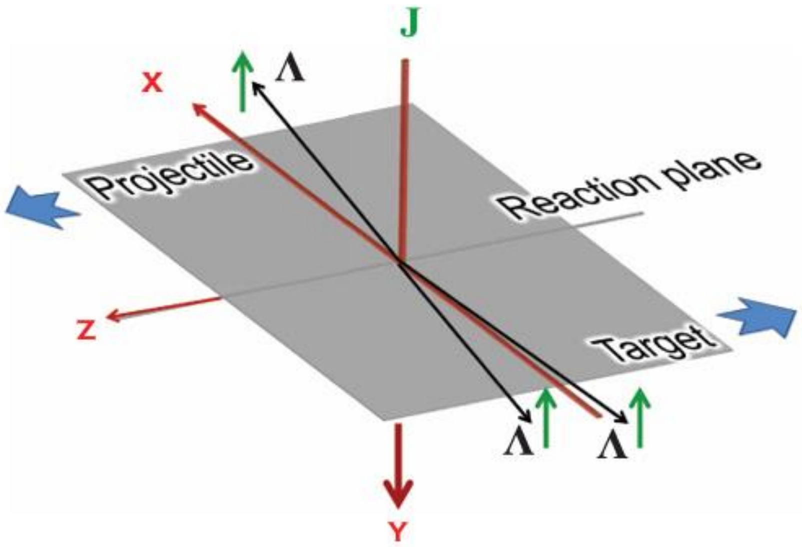
Spin polarization in QGP

The obit angular momentum in QGP can exchange with spin and cause the spin polarization. Recently, it draw a lot of interest to study the QGP system with both magnetic field and spin.

Kiamari, et al. arXiv:2310.01874 (2023)
Bhadury, et al. arXiv:2204.01357 (2022)

Experimentally, Λ hyperons is a typical probe to detect the polarization in QGP

F. Becattini, et al. PHYSICAL REVIEWC88,034905 (2013)
STAR Collaboration M.I. Abdulhamid, et al. Phys.Rev.C 108 (2023)
STAR Collaboration Medina Ablikim, et al. Phys.Rev.D 109 (2024)



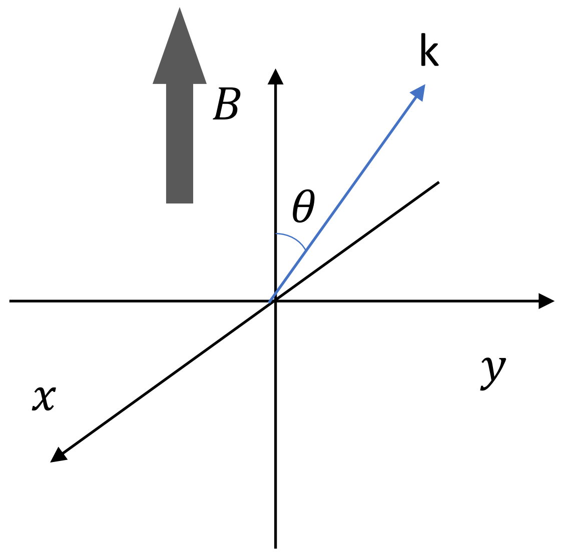
Previous works

- The solution for leading order MHD at arbitrary angle is already obtained. (where angle means the angle between the orient of magnetic field and the wave vector.)

Biswas, et al. arXiv:2007.05431 (2020)

- The solution for first order MHD at two specific angle θ=0 and $\pi/2$ have already been calculated. However, their solutions at $\pi/2$ don't agree with each other.

Grozdanov, et al. arXiv:1610.07392 (2017)
Armas, et al. arXiv:2201.06847 (2022)
Kovtun, et al. arxiv:1703.08757(2017)



Structure and Perturbation Method for spin-MHD

Conservation laws and Constitutive equation

The MHD stems from the conservation law for the energy-momentum tensor, magnetic flux and the total angular momentum

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0. \quad \partial_\mu J^{\mu\alpha\beta} = 0,$$

The equation for angular momentum can be casted into another form

$$\partial_\mu \Sigma^{\mu\alpha\beta} = -2\Theta^{[\alpha\beta]} \, .$$

Conserved current can be expressed with conserved charges and thermodynamic variable

$$\begin{aligned} \Theta^{\mu\nu} &= e u^\mu u^\nu + p_{\|} b^\mu b^\nu + p_\perp \Xi^{\mu\nu} + \Theta^{\mu\nu}_{(1)}, \\ \Sigma^{\mu\alpha\beta} &= u^\mu S^{\alpha\beta} - u^\alpha S^{\mu\beta} + u^\beta S^{\mu\alpha} + \Sigma^{\mu\alpha\beta}_{(1)}, \\ \tilde{F}^{\mu\nu} &= B^\mu u^\nu - B^\nu u^\mu + \tilde{F}^{\mu\nu}_{(1)}, \end{aligned}$$

$$b^\mu = B^\mu / \sqrt{B^\nu B_\nu}$$

$$\Xi^{\mu\nu} = \Delta^{\mu\nu} - b^\mu b^\nu$$

To derive the equations up to ∂^2 order, we need to write down the expansion of $\Theta^{\mu\nu}_{(1)}$ and $F^{\mu\nu}_{(1)}$:

$$\Theta^{\mu\nu}_{(1)} = -T\Bigg[\begin{pmatrix} b^\mu b^\nu & \Xi^{\mu\nu} \end{pmatrix}\begin{pmatrix} \zeta_\parallel & \zeta_\times \\ \zeta'_\times & \zeta_\perp \end{pmatrix}\begin{pmatrix} b^\rho b^\sigma \\ \Xi^{\rho\sigma} \end{pmatrix}$$

$$+ 2\eta_\parallel (b^\mu \Xi^{\nu(\rho} b^{\sigma)} + b^\nu \Xi^{\mu(\rho} b^{\sigma)}) + \eta_\perp (\Xi^{\mu\rho}\Xi^{\nu\sigma} + \Xi^{\mu\sigma}\Xi^{\nu\rho} + \Xi^{\mu\nu}\Xi^{\rho\sigma})\Bigg]\partial_{(\rho}\beta_{\sigma)},$$

$$F^{\mu\nu}_{(1)} = T\Big[2\rho_\perp (b^\mu \Xi^{\nu[\rho} b^{\sigma]} - b^\nu \Xi^{\mu[\rho} b^{\sigma]}) - 2\rho_\parallel \Xi^{\mu[\rho}\Xi^{\sigma]\nu}\Big]\partial_{[\rho}(\beta H_{\sigma]})$$

Linear mode analysis

We consider a perturbation around the equilibrium state, so the thermodynamic variables have a small shift from the value at equilibrium state:

$$e \to e + \delta e(x), \quad u^\mu \to u^\mu + \delta u^\mu(x), \quad B^\mu \to B^\mu + \delta B^\mu(x).$$

$$S^{\mu\nu} \to 0 + \delta S^{\mu\nu}$$

equilibrium state:

$$u^\mu = (1, 0, 0, 0)$$

$$B^\mu = (0, 0, 0, B)$$

The equations for spin-MHD reads

Spin-MHD

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$$M_{(0)}=\begin{pmatrix} 0 & i\omega & iBk_{\parallel} & 0 & 0 \\ -i\omega & \frac{ihv_A{}^2\sin}{B\cos}\omega & ihk_{\perp} & ih\left(1-v_A{}^2\right)k_{\parallel} & 0 \\ ics^2k_{\perp} & -\frac{ihv_A{}^2}{B\cos}k & -ih\omega & 0 & -\frac{4i(\gamma_{\parallel}-\xi_{\parallel})}{\chi}k_{\parallel} \\ ics^2k_{\parallel} & 0 & 0 & -ih\left(1-v_A{}^2\right)\omega & \frac{4i(\gamma_{\parallel}+\xi_{\parallel})}{\chi}k_{\perp} \\ 0 & 0 & 2i(\gamma_{\parallel}-\xi_{\parallel})k_{\parallel} & -2i(\gamma_{\parallel}+\xi_{\parallel})k_{\perp} & \frac{8\gamma_{\parallel}}{\chi}-i\omega \end{pmatrix};$$

$$M_{(1)}=\begin{pmatrix} -\frac{Bcs^2\rho_{\perp}}{h(1-\mathrm{vA}^2)}k_{\parallel}k_{\perp} & -k^2\rho_{\perp} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{\parallel}^2(\eta_{\parallel}+\gamma_{\parallel}-2\xi_{\parallel})+k_{\perp}^2(\eta_{\perp}+\zeta_{\perp}) & k_{\parallel}k_{\perp}(\eta_{\parallel}-\gamma_{\parallel}+\zeta_{\times}) & 0 \\ 0 & 0 & k_{\parallel}k_{\perp}(\eta_{\parallel}-\gamma_{\parallel}+\zeta_{\times}) & k_{\perp}^2(\eta_{\parallel}+\gamma_{\parallel}+2\xi_{\parallel})+\zeta k_{\parallel}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

Solution for first order Spin-MHD

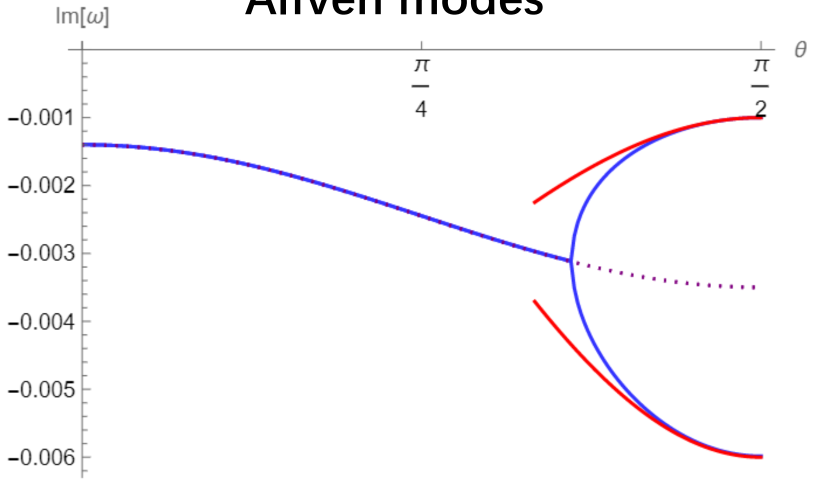
The solutions we obtained are

$$\omega = \pm v_A k\cos\theta - i\tilde{\omega}_{1,2}k^2$$
$$\omega = -i\Gamma_{\parallel} - i\tilde{\omega}_3 k^2$$
$$\omega = -i\Gamma_{\perp} - i\tilde{\omega}_4 k^2$$

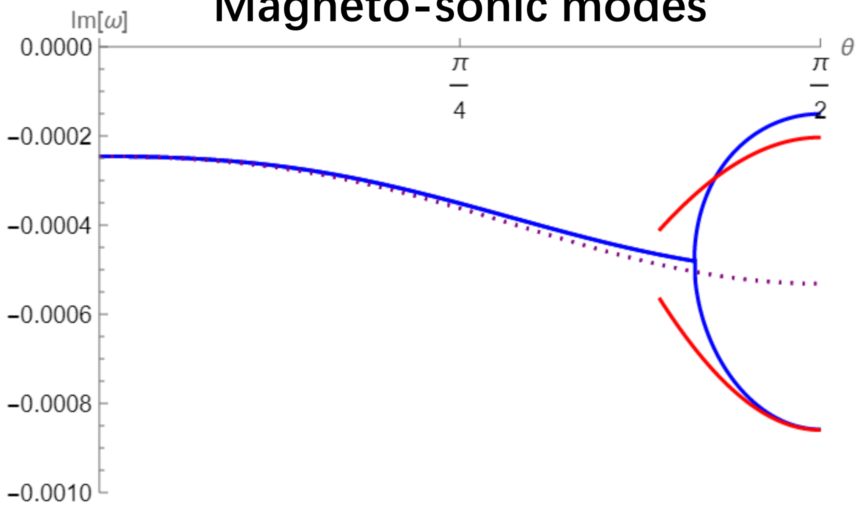
$$\omega = \pm v_1 k + ik^2\frac{W_1 - W_2 v_1^2}{2(v_1^2 - v_2^2)},$$
$$\omega = \pm v_2 k - ik^2\frac{W_1 - W_2 v_2^2}{2(v_1^2 - v_2^2)},$$
$$\omega = -i\Gamma_{\parallel} - ik^2\left(W_3 - W_2\right),$$

The issue arise in spin-MHD when we consider the $\theta \longrightarrow \frac{\pi}{2}$ limit. The result of k expansion is valid only when $\cos\theta$ is not too small

Alfven modes



Magneto-sonic modes



Explaining for the non-commutable

The limit $\frac{\pi}{2}$ is different from previous MHD works (if you ignore the spin). We conclude that the difference stems from the non-analytical point near the point $k=0, \cos=0$.

Grozdanov, et al. arXiv:1610.07392 (2017)
Armas, et al. arXiv:2201.06847 (2022)
Kovtun, et al. arxiv:1703.08757(2017)

$$\omega = \pm\sqrt{v_A^2 k^2 \cos^2 - \frac{1}{4}(\tilde{\rho}-\tilde{\eta})^2 k^4} - \frac{i}{2}(\tilde{\rho}+\tilde{\eta})k^2.$$

At $k=0, \cos=0$, dispersion relation is non-analytical.
If we assume that k is much smaller than $\cos\theta$, we can obtain the dispersion relation as

$$\omega = \pm v_A k_{\|} - \frac{i}{2}(\tilde{\rho}+\tilde{\eta})k^2 + \mathcal{O}(\frac{k}{|\cos\theta|}k^2), \quad \hat{k} < |\cos\theta|,$$

The cos appear in the denominator in the higher order. Thus if we take cos to 0, we will suffer from an infinity in higher order.

Stability in Spin-MHD

We noticed that the solution is always damping in the second order. The stability at the two boundary are easy to check. For the general solution, we use the following restrictions for the transport coefficient and check the stability term by term.

$$
\begin{aligned}
&\zeta_{\perp} \geq 0, \quad \zeta_{\|} \geq 0, \quad \zeta_{\|}\zeta_{\perp} \geq \zeta_{\times}^{2}, \quad \eta_{\|} \geq 0, \quad \eta_{\perp} \geq 0, \\
&\gamma_{\|} \geq 0, \quad \gamma_{\perp} \geq 0, \quad \gamma_{\|}\eta_{\|} \geq \xi_{\|}^{2}, \\
&\rho_{\perp} \geq 0, \quad \rho_{\|} \geq 0.
\end{aligned} \tag{16}
$$

Summary:

- We have got the formulation for first order spin magneto-hydrodynamics
- We got the solution for first order MHD and spin-MHD for an anisotropic case.
- We faced a challenge when we take the limit to $\frac{\pi}{2}$, and we realized it stems from the expansion at an inappropriate point.

Thanks

Valid Expansion at $\frac{\pi}{2}$ limit

Since we are consider the limit $\cos\theta \longrightarrow 0$, we assume that k is non-zero and use a cos expansion. The leading order should be the result we got at $\theta = \frac{\pi}{2}$. The expansion reads

$$\omega = -ik^2\rho'_{\parallel} - i\left((\rho'_{\perp} - \rho'_{\parallel})k^2 + \frac{hv_A^2}{\eta_{\perp} - h\rho'_{\parallel}}\right)\cos^2\theta + \mathcal{O}(\cos^4\theta), \quad (49b)$$

$$-\frac{ik^2}{h}\eta_{\perp} - i\left(\frac{1}{h}(\eta_{\parallel} - \eta_{\perp})k^2 - \frac{hv_A^2}{\eta_{\perp} - h\rho'_{\parallel}}\right)\cos^2\theta + \mathcal{O}(\cos^4\theta),$$

$$|\cos\theta| < \hat{k}. \quad (49c)$$

The k will appear in the denominator in higher order expansion, but if cos is much smaller than k, the effective of the higher order will be depressed. On the other hand, this expansion is valid only when cos is smaller than k.

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Algorithm for Solution Search

The solution for the 2*2 equations is already obtained as

$$\omega = \pm v_A k_{\parallel} - \frac{i}{2}(\tilde{\rho} + \tilde{\eta})k^2$$

$$\tilde{\rho} = \rho'_{\perp} \cos^2\theta + \rho'_{\parallel} \sin^2\theta$$
$$\tilde{\eta} = \eta'_{\parallel} \cos^2\theta + \eta'_{\perp} \sin^2\theta$$

As for the 4*4 part, the solution should be propagating at leading order, and damping at next to leading order, so we get another form of the solution as

$$\begin{aligned} f_{\text{ans}}(\omega) = &\{\omega - (v_1 k - i w_1 k^2)\}\{\omega - (-v_1 k - i w_1 k^2)\} \\ &\times\{\omega - (v_2 k - i w_2 k^2)\}\{\omega - (-v_2 k - i w_2 k^2)\} \\ &+\mathcal{O}(\omega^3 k^3) + \mathcal{O}(\omega^2 k^4) + \mathcal{O}(\omega^1 k^5) + \mathcal{O}(\omega^0 k^6). \end{aligned}$$

By comparing the terms of order $\omega^i k^j$ with $\det\left(M_0^{(4)} + M_1^{(4)}\right)$, we can get the coefficient w_i, which is just the coefficient for k^2 order.

Israel-Stewart structure

Such hydro mode is already formulated under the IS frame. (cite) What we are doing is to construct a hydro with magnetic field and also the spin.

Spin polarization induced by rotation and B

- Vorticity – Mechanical effect → Does not depend on electric charges
- Magnetic field → Depends on electric charges via magnetic moment

spin of different particles. L is the angular momentum, B is magnetic field. '+' refer to particle, while '-' refer to anti-particle.

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Solutions for the Quartic Equation

$$\omega = \pm v_1 k - i w_1 k^2, \quad \omega = \pm v_2 k - i w_2 k^2,$$

$$w_1 = -\frac{W_1 - v_1^2 W_2}{2(v_1^2 - v_2^2)}, \quad w_2 = +\frac{W_1 - v_2^2 W_2}{2(v_1^2 - v_2^2)}.$$

$$W_1 = \eta'_{\parallel}\left(c_s^2 \cos^2(2\theta) + \frac{v_A^2 \sin^2\theta}{1 - v_A^2}\right) + \rho'_{\perp} c_s^2 \frac{1 - v_A^2 \cos^2\theta}{1 - v_A^2}$$
$$+\zeta'_{\parallel}\frac{v_A^2}{1 - v_A^2}\cos^2\theta + \{\zeta'_{\parallel} - 2\zeta'_{\times} + (\zeta'_{\perp} + \eta'_{\perp})\} c_s^2 \sin^2\theta \cos^2\theta,$$

$$W_2 = \eta'_{\parallel}\left(1 + \frac{v_A^2}{1 - v_A^2}\sin^2\theta\right) + \rho'_{\perp}\left(1 + \frac{v_A^2 c_s^2}{1 - v_A^2}\sin^2\theta\right)$$
$$+\frac{\zeta'_{\parallel}}{1 - v_A^2}\cos^2\theta + (\zeta'_{\perp} + \eta'_{\perp})\sin^2\theta,$$

This result is consistent with Kovtun's work (arXiv: 1703.08757) at $\pi/2$ limit , but is inconsistent with Armas's work (2201.06847)

Stability of the solution

We noticed that the solution is always damping in the second order. The stability at the two boundary are easy to check. For the general solution, we use the following restrictions for the transport coefficient and check the stability term by term.

$$\zeta_\perp \geq 0, \quad \zeta_{||} \geq 0, \quad \zeta_{||}\zeta_\perp \geq \zeta_\times^2,$$
$$\eta_{||} \geq 0, \quad \eta_\perp \geq 0, \quad \rho_\perp \geq 0, \quad \rho_{||} \geq 0.$$

Zhe, Koichi, Jin, arXiv:2402.18601 (2024)

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Solution at $\theta = \frac{\pi}{2}$

Equations at $\frac{\pi}{2}$ limit:

$$\begin{pmatrix} \omega + i\rho'_{\parallel} k_{\perp}^2 & 0 \\ 0 & \omega + i\eta'_{\perp} k_{\perp}^2 \end{pmatrix} \begin{pmatrix} \delta B_y \\ \delta u_y \end{pmatrix} = 0,$$

$$\begin{pmatrix} -i\frac{Bc_s^2}{h(1-v_A^2)}\rho'_{\perp} k_{\perp}^2 & -k_{\perp} B & \omega + i\rho'_{\perp} k_{\perp}^2 & 0 \\ \omega & -hk_{\perp} & h\frac{v_A^2}{B}\omega & 0 \\ -c_s^2 k_{\perp} & h\omega + i(\zeta_{\perp} + \eta_{\perp})k_{\perp}^2 & -h\frac{v_A^2}{B}k_{\perp} & 0 \\ 0 & 0 & 0 & h(1-v_A^2)\omega + i\eta_{\parallel} k_{\perp}^2 \end{pmatrix} \begin{pmatrix} \delta\epsilon \\ \delta u_x \\ \delta B_z \\ \delta u_z \end{pmatrix} = 0,$$

Comparing the Solution with previous one

Solution for $\theta = \frac{\pi}{2}$:

(Take $\theta \longrightarrow \frac{\pi}{2}$ before expand the solution)

$$\omega_{1,2}(\theta = \frac{\pi}{2}) = \pm v_1 k_\perp - \frac{i}{2}\left(\eta'_\perp + \zeta'_\perp + \rho'_\perp (1 - c_s^2)^2 \frac{v_A^2}{v_1^2}\right) k_\perp^2 + \mathcal{O}(k_\perp^3),$$

$$\omega_3(\theta = \frac{\pi}{2}) = -\frac{i\rho'_\perp c_s^2}{v_1^2(1 - v_A^2)} k_\perp^2 + \mathcal{O}(k_\perp^4),$$

$$\omega_4(\theta = \frac{\pi}{2}) = -\frac{i\eta'_\parallel}{1 - v_A^2} k_\perp^2 + \mathcal{O}(k_\perp^4).$$

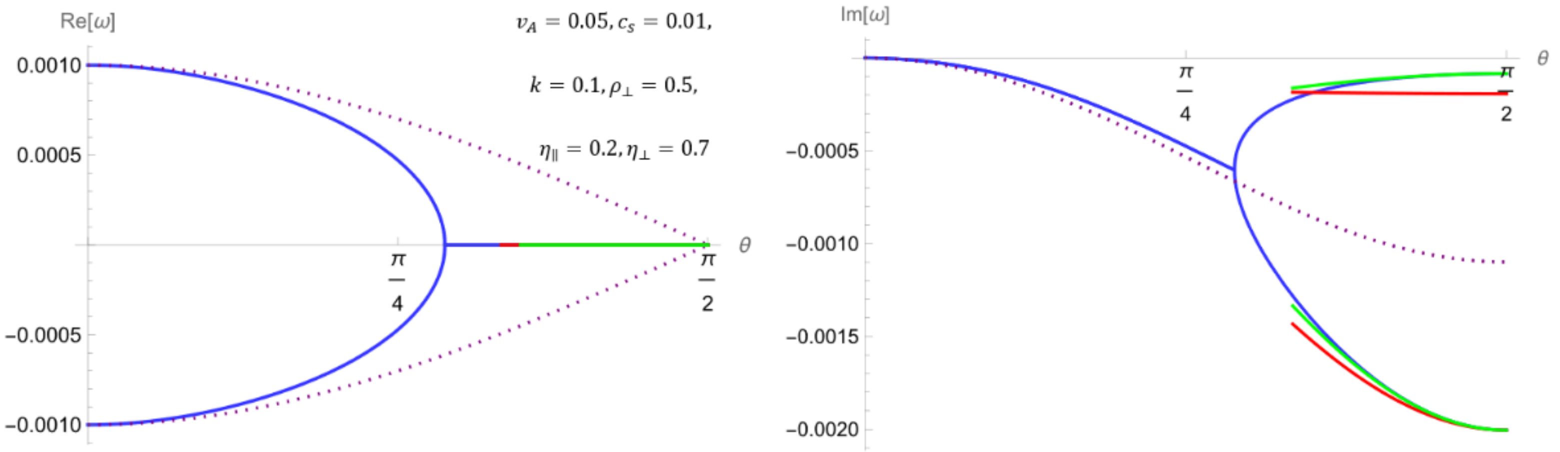
(Take limit $\theta \longrightarrow \frac{\pi}{2}$ for solution up to k^2 order)

$$\omega = \pm v_1 k - \frac{ik^2}{2}(\eta'_\perp + \zeta'_\perp + \rho'_\perp (1 - c_s^2)^2 \frac{v_A^2}{v_1^2}),$$

$$\omega = \pm v_2 k - ik^2 [\frac{\eta'_\parallel}{2(1 - v_A^2)} + \frac{\rho'_\perp c_s^2}{2v_1^2(1 - v_A^2)}]$$

Illustration for Different Expansion

Let's compare these expansions with the analytic result



One can notice that the k expansion correspond well with the analytic solution before the critical point.

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Constitutive equation for spin-MHD

In the last part, we reviewed the basic approaches for hydro and derived the dispersion relation for the first order MHD, and we discussed when the k expansion is valid. Then let's step forward to the spin-MHD.

In spin-MHD, the total angular momentum is to be considered

$$\partial_\mu J^{\mu\alpha\beta} = 0,$$

or it can be cast into another form

$$\partial_\mu \Sigma^{\mu\alpha\beta} = -2\Theta^{[\alpha\beta]} .$$

This equation indicates there is no symmetry that guarantees the conservation of spin in relativistic systems.

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Equations for spin-MHD

We will have an quartic equation and a quantic equation in spin-MHD, but the method is almost the same with MHD. The coefficient matrix for spin-MHD is obtained as

$$M_{(0)} = \begin{pmatrix} 0 & i\omega & iBk_{\parallel} & 0 & 0 \\ -i\omega & \frac{ihv_A{}^2 \sin}{B\cos}\omega & ihk_{\perp} & ih\left(1-v_A{}^2\right)k_{\parallel} & 0 \\ ics^2 k_{\perp} & -\frac{ihv_A{}^2}{B\cos}k & -ih\omega & 0 & -\frac{4i(\gamma_{\parallel}-\xi_{\parallel})}{\chi}k_{\parallel} \\ ics^2 k_{\parallel} & 0 & 0 & -ih\left(1-v_A{}^2\right)\omega & \frac{4i(\gamma_{\parallel}+\xi_{\parallel})}{\chi}k_{\perp} \\ 0 & 0 & 2i(\gamma_{\parallel}-\xi_{\parallel})k_{\parallel} & -2i(\gamma_{\parallel}+\xi_{\parallel})k_{\perp} & \frac{8\gamma_{\parallel}}{\chi}-i\omega \end{pmatrix};$$

$$M_{(1)} = \begin{pmatrix} -\frac{Bcs^2\rho_{\perp}}{h(1-\mathrm{vA}^2)}k_{\parallel}k_{\perp} & -k^2\rho_{\perp} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{\parallel}^2(\eta_{\parallel}+\gamma_{\parallel}-2\xi_{\parallel})+k_{\perp}^2(\eta_{\perp}+\zeta_{\perp}) & k_{\parallel}k_{\perp}(\eta_{\parallel}-\gamma_{\parallel}+\zeta_{\times}) & 0 \\ 0 & 0 & k_{\parallel}k_{\perp}(\eta_{\parallel}-\gamma_{\parallel}+\zeta_{\times}) & k_{\perp}^2(\eta_{\parallel}+\gamma_{\parallel}+2\xi_{\parallel})+\zeta k_{\parallel}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

Back up

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The solutions we obtained are

$$\omega = \pm v_A k \cos\theta - i\tilde{\omega}_{1,2} k^2$$

$$\omega = -i\Gamma_{\parallel} - i\tilde{\omega}_3 k^2$$

$$\omega = -i\Gamma_{\perp} - i\tilde{\omega}_4 k^2$$

$$\tilde{\omega}_1 = \tilde{\omega}_2 = \frac{1}{2h}\Big[\Big(\eta_{\parallel} - \frac{\xi_{\parallel}^2}{\gamma_{\parallel}} + h\rho'_{\perp}\Big)\cos^2\theta + (\eta_{\perp} + h\rho'_{\parallel})\sin^2\theta\Big],$$

$$\tilde{\omega}_{3,4} = \frac{(\gamma_{\parallel} - \xi_{\parallel})^2}{h\gamma_{\parallel}}\cos^2\theta,\ \frac{\gamma_{\perp}}{h}\sin^2\theta.$$

$$\omega = \pm v_1 k + ik^2 \frac{W_1 - W_2 v_1^2}{2(v_1^2 - v_2^2)},$$

$$\omega = \pm v_2 k - ik^2 \frac{W_1 - W_2 v_2^2}{2(v_1^2 - v_2^2)},$$

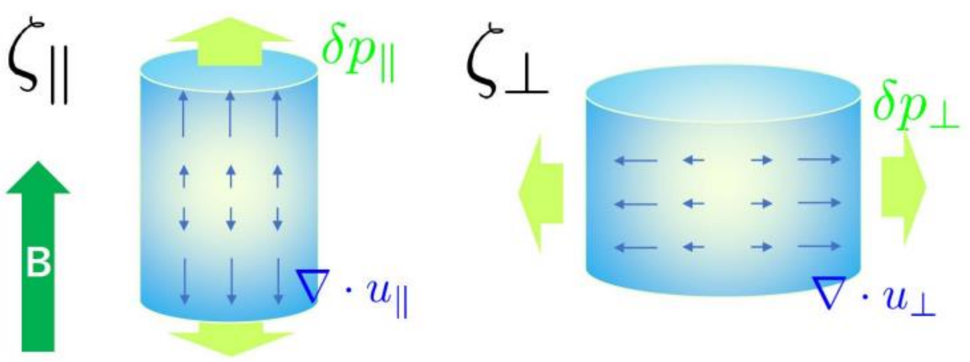
$$\omega = -i\Gamma_{\parallel} - ik^2\,(W_3 - W_2),$$

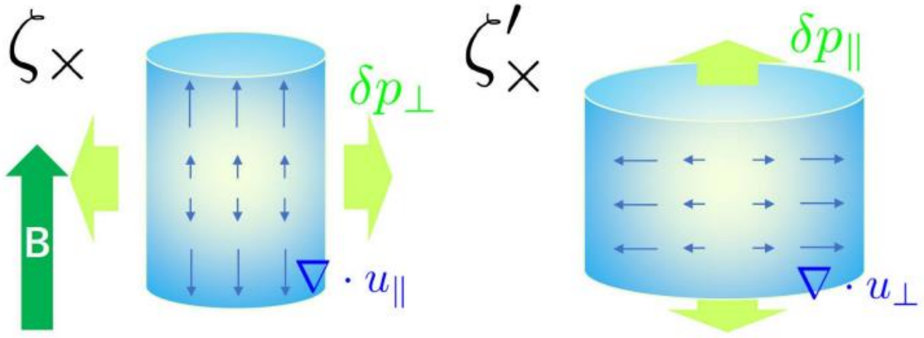
$$\begin{aligned}
W_1 = \frac{1}{h}\Big[&\Big(\eta_{\parallel} - \frac{\xi_{\parallel}^2}{\gamma_{\parallel}}\Big)\Big(c_s^2\cos^2(2\theta) + \frac{v_A^2\sin^2\theta}{1-v_A^2}\Big) \\
&+ h\rho'_{\perp} c_s^2 \frac{1 - v_A^2\cos^2\theta}{1-v_A^2} + \zeta_{\parallel}\frac{v_A^2}{1-v_A^2}\cos^2\theta \\
&+ \{\zeta_{\parallel} + \zeta_{\perp} - 2\zeta_{\times} + \eta_{\perp}\} c_s^2 \sin^2\theta\cos^2\theta\Big],
\end{aligned}$$

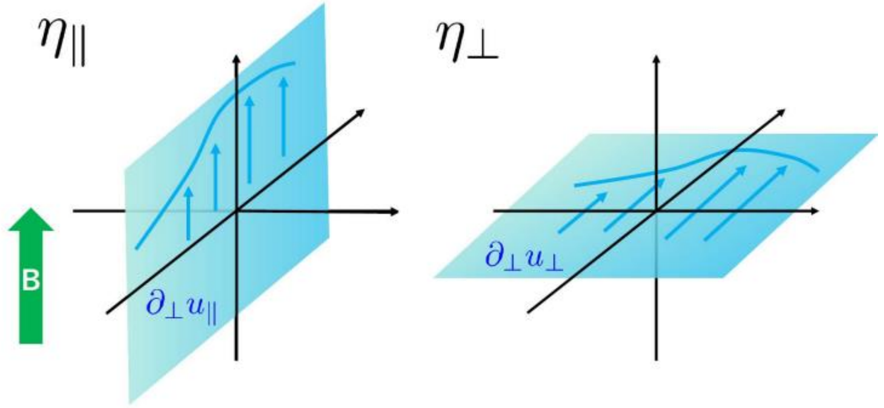
$$\begin{aligned}
W_2 = \frac{1}{h}\Big[&\Big(\eta_{\parallel} - \frac{\xi_{\parallel}^2}{\gamma_{\parallel}}\Big)\Big(1 + \frac{v_A^2}{1-v_A^2}\sin^2\theta\Big) \\
&+ h\rho'_{\perp}\Big(1 + \frac{v_A^2 c_s^2}{1-v_A^2}\sin^2\theta\Big) \\
&+ \frac{\zeta_{\parallel}}{1-v_A^2}\cos^2\theta + (\zeta_{\perp} + \eta_{\perp})\sin^2\theta\Big],
\end{aligned}$$

$$\begin{aligned}
W_3 = \frac{1}{h}\Big[&\cos^2\theta(\eta_{\parallel} + \gamma_{\parallel} - 2\xi_{\parallel}) \\
&+ \frac{\sin^2\theta}{1-v_A^2}(\eta_{\parallel} + \gamma_{\parallel} + 2\xi_{\parallel}) + \sin^2\theta(\zeta_{\perp} + \eta_{\perp}) \\
&+ \frac{\cos^2\theta}{1-v_A^2}\zeta_{\parallel} + \Big(1 + \frac{v_A^2 c_s^2}{1-v_A^2}\sin^2\theta\Big)h\rho_{\perp}\Big],
\end{aligned}$$

Transport coefficient







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General form for spin-MHD in first order

$$\Theta^{\mu\nu}_{(1s)} = -T\eta^{\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)} - T\xi^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} - 2\beta\omega_{\rho\sigma})$$
$$\Theta^{\mu\nu}_{(1a)} = -T\xi'^{\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)} - T\gamma^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} - 2\beta\omega_{\rho\sigma}).$$

$$\eta^{\mu\nu\rho\sigma} = \zeta_\perp \Xi^{\mu\nu}\Xi^{\rho\sigma} + \zeta_\| b^\mu b^\nu b^\rho b^\sigma$$
$$+\zeta_\times(b^\mu b^\nu \Xi^{\rho\sigma} + \Xi^{\mu\nu} b^\rho b^\sigma) + 2\eta_\|(b^\mu \Xi^{\nu(\rho} b^{\sigma)} + b^\nu \Xi^{\mu(\rho} b^{\sigma)})$$
$$+\eta_\perp(\Xi^{\mu\rho}\Xi^{\nu\sigma} + \Xi^{\mu\sigma}\Xi^{\nu\rho} + \Xi^{\mu\nu}\Xi^{\rho\sigma})$$

$$\gamma^{\mu\nu\rho\sigma}$$
$$= \gamma_\perp(\Xi^{\mu\rho}\Xi^{\nu\sigma} - \Xi^{\mu\sigma}\Xi^{\nu\rho}) - 2\gamma_\|(b^\mu \Xi^{\nu[\rho} b^{\sigma]} - b^\nu \Xi^{\mu[\rho} b^{\sigma]})$$

$$\xi^{\mu\nu\rho\sigma}$$
$$= 2\xi_\|(b^\mu \Xi^{\nu[\rho} b^{\sigma]} + b^\nu \Xi^{\mu[\rho} b^{\sigma]} + b^\mu \Xi^{\nu(\rho} b^{\sigma)} - b^\nu \Xi^{\mu(\rho} b^{\sigma)}).$$

$$F^{\mu\nu}_{(1)} = [2\rho_\perp(b^\mu \Xi^{\nu[\rho} b^{\sigma]} - b^\nu \Xi^{\mu[\rho} b^{\sigma]}) - 2\rho_\| \Xi^{\mu[\rho}\Xi^{\sigma]\nu}]\partial_{[\rho}\beta H_{\sigma]}$$

These are the general form of the first order. One can found the similar term in the paper [10,Cao,2022].