Thermodynamic geometry of fluctuations in small heat engines

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Abstract

Minimizing the dissipative loss and the fluctuation of the dissipation are important for the improvement of the performance for small heat engines, since the fluctuations become non-negligible in small systems. We explore the simultaneous optimization of the dissipation and its fluctuation employing some relations on the thermodynamic geometry of the parameter space of a micro scale heat engine within linear response regime.

Introduction

Thermodynamic geometry [1,2] is a suitable framework to address the issue of optimization of small heat engines. The thermodynamic geometric expression of dissipation is known for micro scales [3-6]. The fluctuation of the dissipation is also expressed similarly [8]. Although there are optimization schemes with minimum dissipation [9] or the trade-off approach between work fluctuation and efficiency, up to our knowledge, the simultaneous minimization of dissipation and its fluctuation is still an open problem.

In this work, we state the simultaneous optimization of the dissipation and its fluctuation for **small cycles** employing some properties of geometry [10].

Motivation

To obtain a bound on a *thermodynamical quantity* in terms of a *geometric quantity* which is **independent of the protocol**.

Framework

Linear response regime, slow driving

Termodynamic Geometry

Dissipation: $\langle A \rangle = \langle U - W \rangle$ [7]

$(\lambda_w,\ \lambda_u) = (k(t),\ T(t))$: control parameters,

$(X_w,\ X_u) = P,\ S$: conjugate generalized forces (random variables)

$$\langle A \rangle = \int_0^\tau dt\ \lambda_u\ \langle \dot{X}_u \rangle - \int_0^\tau dt\ \langle X_w \rangle \dot{\lambda}_w \qquad (1)$$

Linear response of the system

$$\langle A \rangle = \oint dt\ g^{(1)}_{\mu\nu}(t)\ \dot{\lambda}_\mu(t)\ \dot{\lambda}_\nu(t) \qquad (2)$$

Its fluctuation: [8]

$$\langle (\Delta A)^2 \rangle = \int_0^\tau dt\ g^{(2)}_{\mu\nu}(t)\ \dot{\lambda}_\mu(t)\ \dot{\lambda}_\nu(t) \qquad (3)$$

where,

$$g^{(1)}_{\mu\nu}(t) = \beta(t) \int_0^\infty dt' \langle \Delta X_j(0) \Delta X_i(t') \rangle \qquad (4)$$

and for the fluctuations:

$$g^{(2)}_{\mu\nu}(t) = 2T(t)\ g^{(1)}_{\mu\nu}(t)\ . \qquad (5)$$

Thermodynamic length:

$$\mathcal{L}^{(i)} = \int_0^\tau dt \sqrt{\mathbf{g}^{(i)}_{\mu\nu}(t)\ \dot{\lambda}_\mu(t)\ \dot{\lambda}_\nu(t)} = \oint \sqrt{g^{(1)}_{\mu\nu}\ d\lambda_\mu\ d\lambda_\nu}\ . \qquad (6)$$

Small cycle ($k(t) = k_0 + \delta k$ and $T(t) = T_0 + \delta T$) approximation on (5):

$$\mathcal{L}^{(2)} = \sqrt{2T_0}\ \mathcal{L}^{(1)}\ . \qquad (7)$$

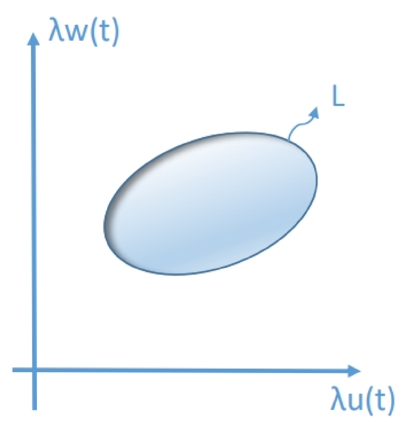

Figure 1. Parameter space

Optimization with a Small Cycle

Using Cauchy-Schwartz inequality in (2) and (3)

$$\langle A \rangle \geq {\mathcal{L}^{(1)}}^2/\tau\ , \quad \langle (\Delta A)^2 \rangle \geq {\mathcal{L}^{(2)}}^2/\tau \qquad (8)$$

Equalities hold : $\mathcal{L}^{(i)}$ integrand is *constant in time*.

Because of the relation (5), the two metrics are different only by an overall factor for a sufficiently small cycle. Therefore simultaneous optimization of $\langle A \rangle$ and $\langle \Delta A^2 \rangle$ is possible for any small closed path with an appropriately chosen protocol $\equiv$ for given $k_0,\ T_0,\ \tau$ and W_{qs} .

Demonstration: Isoperimetric Inequality and Harmonic Oscillator Potential

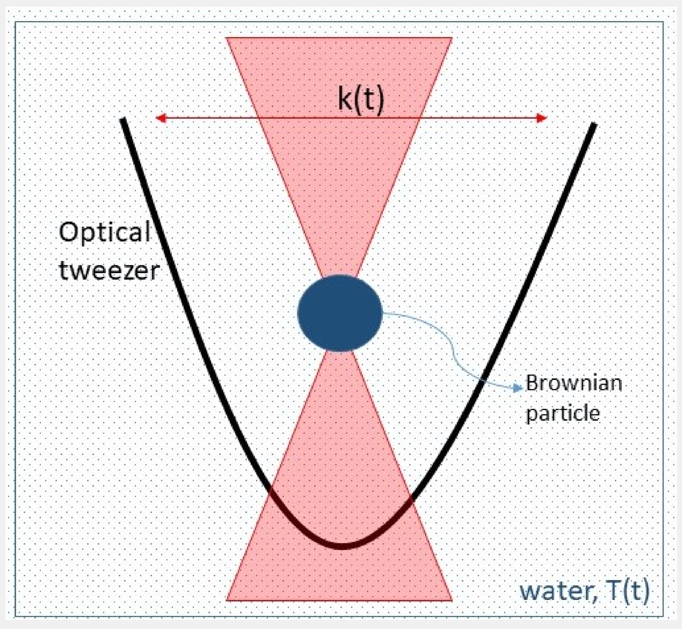

Figure 2. Brownian particle trapped in a harmonic oscillator potential, in water. (k, T) are control parameters

Isoperimetric inequality [10]:

$$ {\mathcal{L}^{(i)}}^2 \geq 4\pi \mathbf{A}^{(i)} - \kappa_i {\mathbf{A}^{(i)}}^2 \simeq 4\pi \mathbf{A}^{(i)} \qquad (9)$$

where $\mathbf{A}^{(i)} \simeq \sqrt{det[\mathbf{g}^{(i)}]_{k_0,p_0}}\ W_{qs}$ is the thermodynamic area with constant κ_i. Equality: the boundary curve is a **geodesic circle**.

A geodesic circle with a small radius r on the tangent space of $\mathbf{g}^{(1)}$ is **the same as** the one with a small radius $\sqrt{2T_0}\ r$ on the tangent space of $\mathbf{g}^{(2)}$.

$$\therefore \qquad \langle A \rangle_{opt} \simeq 2\pi \sqrt{\frac{m}{k_0}} \frac{W_{qs}}{\tau} \qquad (10)$$

$$\langle \Delta A^2 \rangle_{opt} \simeq 4\pi T_0 \sqrt{\frac{m}{k_0}} \frac{W_{qs}}{\tau} \qquad (11)$$

for harmonic oscillator potential, given $k_0,\ T_0,\ \tau$ and W_{qs} .

Concluding Remarks

Employing the thermodynamic geometry for **a small cycle** we can minimize the dissipation and its fluctuation simultaneously. Using isoperimetric inequality, we showed that the optimum values of dissipation and its fluctuation can be expressed in terms of given $k_0,\ T_0$ and W_{qs} values.

In our proceeding work, we compared our results with the well-known cycles such as Stirling, Otto and Carnot and observed that our optimum choice of the cycle is actually giving the minimum value of the dissipation and its fluctuation.

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