Feynman Integral Reduction Without IBP

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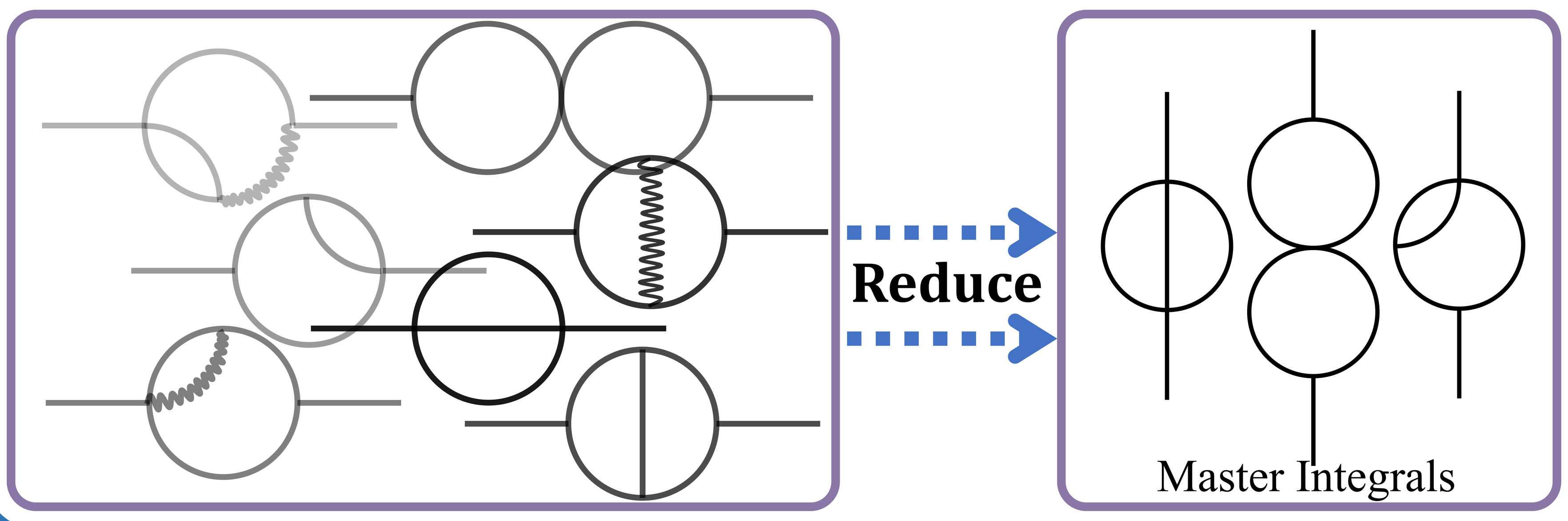
arXiv : [2412.15962]

Motivation



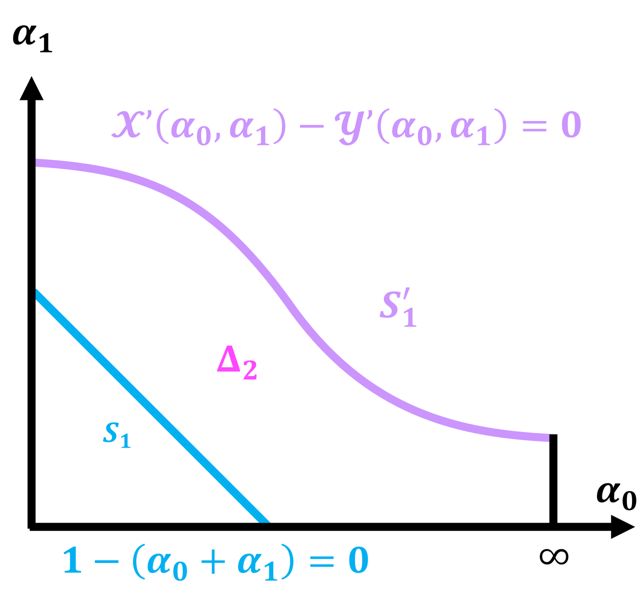
The reduction of Feynman integrals plays a crucial role in the study of quantum field theory. Currently, the standard method for integral reduction is the Laporta algorithm for solving the integration-by-parts (IBP) identities of Feynman integrals. However, when dealing with complex families of integrals, the IBP reduction process can become highly intricate and tedious, often requiring substantial computational time even with the use of supercomputers. Therefore, improving the computational efficiency of integral reduction is of great importance, necessitating the development of more effective methods.

The Method

We found that Feynman integrals can be expressed as

$$I(\nu_n) = C(\nu_n)\int_0^\infty \left(\prod_{j=0}^{n} \alpha_j^{\nu_j-1}\mathrm{d}\alpha_j\right)(\alpha_0\mathcal{U}+\mathcal{F})^{\lambda_0}\ \mathcal{X}\delta(\mathcal{X}-\mathcal{Y})\,,$$

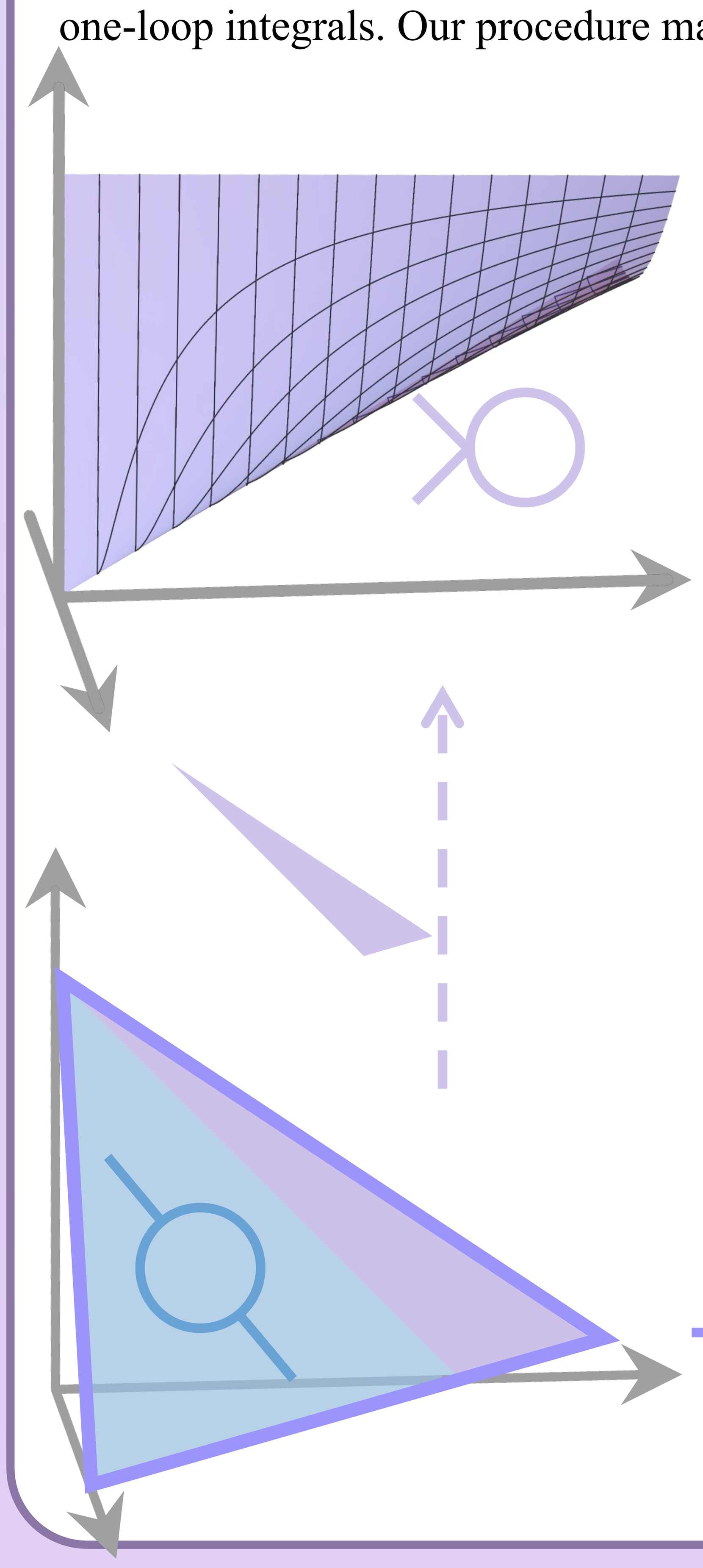
where we generalize the well-known Cheng-Wu theorem into a more general form. The $\mathcal{X}$ and $\mathcal{Y}$ in the delta function can be adjusted arbitrarily to some extent. This essentially provides equivalence classes of integration domains: when the n-form $f\omega$ is integrated over any hypersurface S_n defined by $\mathcal{X}-\mathcal{Y}=0$, the results are all equivalent.

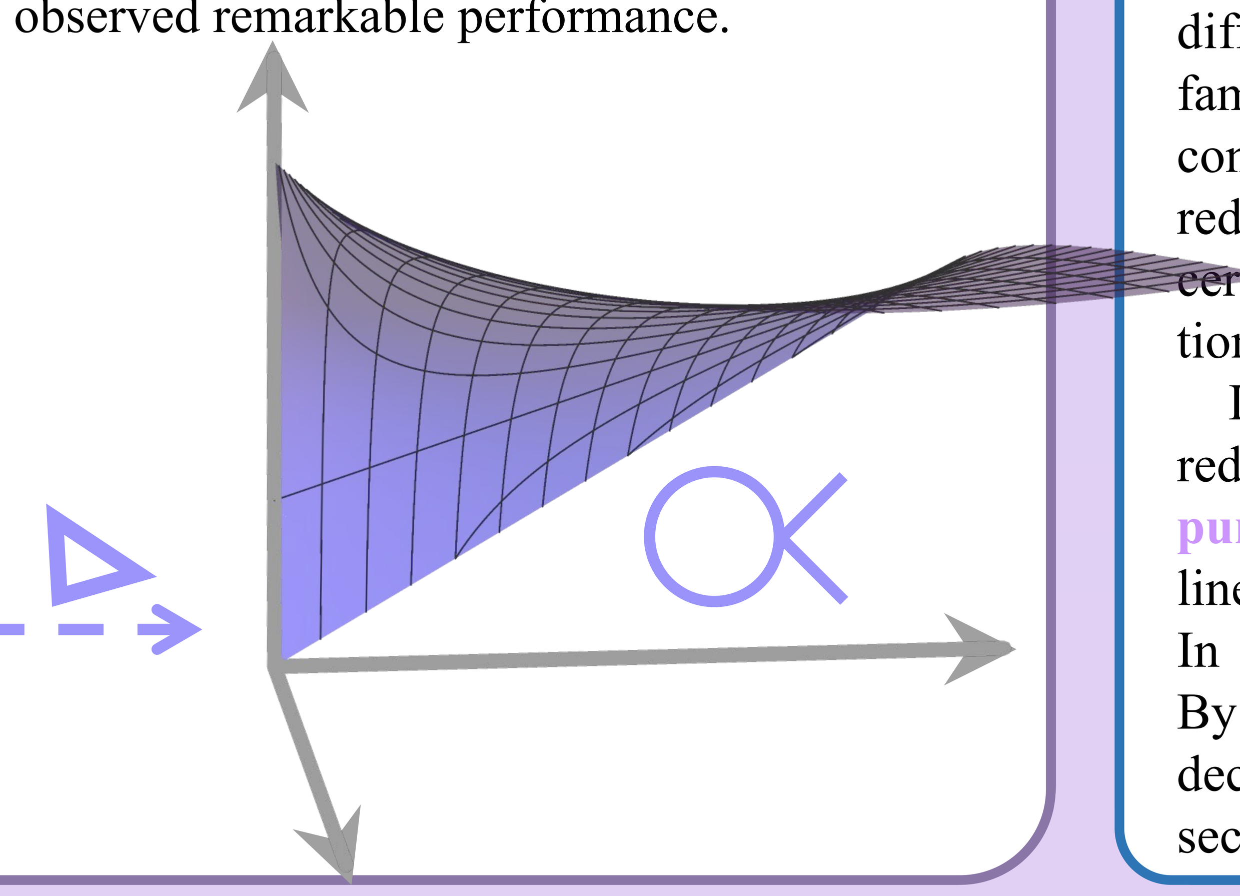


In the language of twisted homology, the equivalence classes of integration contours are elements (cycles) of a twisted homology group determined by the polynomial $\alpha_0\mathcal{U}+\mathcal{F}$. This homology group is dual to the twisted cohomology group of the integrands. In the literature [1711.00469] and [1810.03818], there have been extensive discussions on how to perform integral reduction using the vector-space structure of the cohomology groups. This can be done using the techniques of intersection theory. The homology groups of integration contours also admit a vector-space structure, and in principle can be used for integral reduction as well. However, this path has not been followed in the literature to the best of our knowledge. Now, we will exploit the equivalence of integration domains to reduce one-loop integrals. Our procedure mainly consists of two steps:

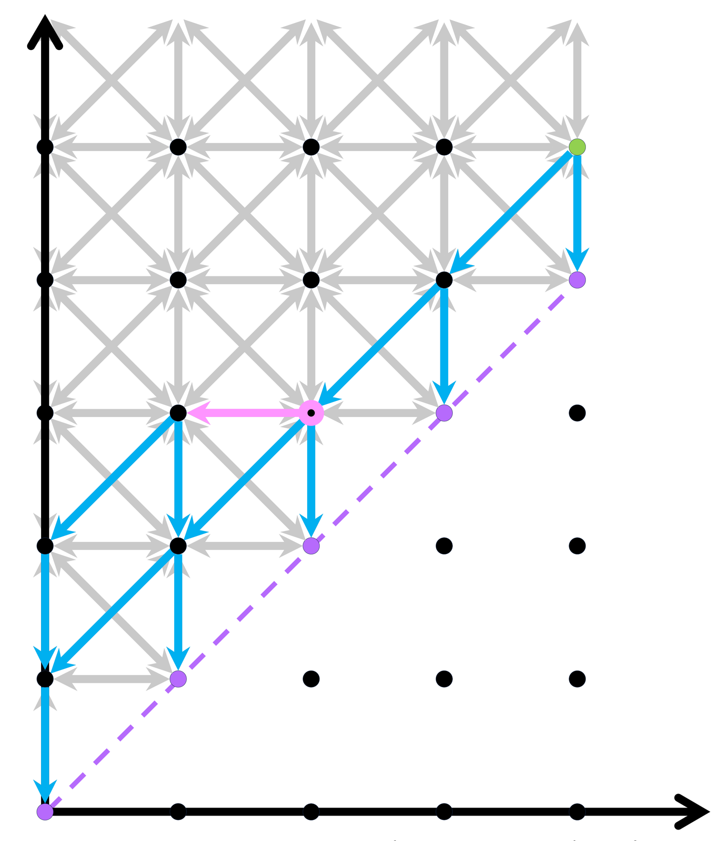
a) transform of the integration contours either by explicit choices of the $\mathcal{X}$ and $\mathcal{Y}$ functions, or by appropriate variable changes;
b) split the integration contour into several parts, and identify each part manifestly as a Feynman integral.

The following diagram illustrates how this method is used to achieve the reduction of the bubble family. As a result, we have derived universal recursive formulas for the reduction of one-loop integrals in both irreducible and reducible sectors. The reduction relations can be easily implemented in a computer program. We have applied them to various examples and observed remarkable performance.





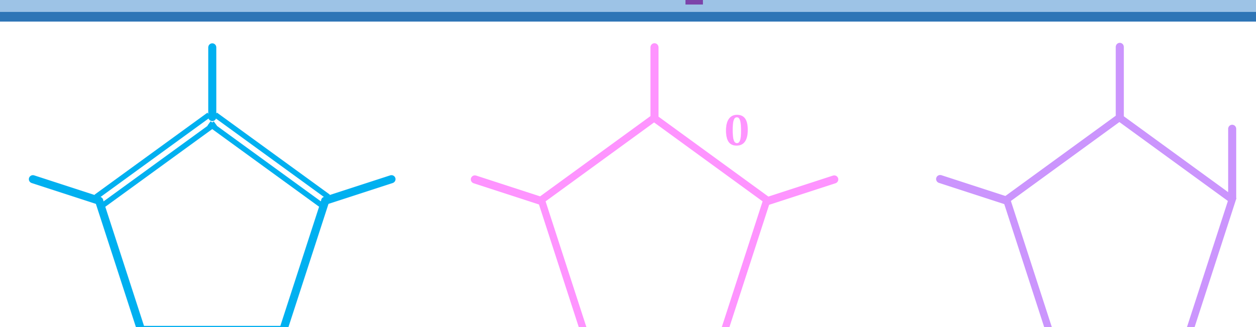
Comparison with IBP



As shown in the figure, each **point** on the coordinate plane represents a class of Feynman integrals with the same level of complexity. The horizontal axis denotes the number of propagators with nonzero indices, while the vertical axis represents the sum of the integral indices. For one-loop integrals, each point on the purple line corresponds to master integrals.

As an example, we depict the reduction process for the integral $I(2,1,1,1,1)$(**green point**) in the figure. In our method, the reduction can be carried out directly and directionally along the **colored lines** to the master integrals (the **pink line** represents the reduction of reducible sectors). In contrast, for the IBP method, the reduction cannot proceed directly or directionally. Instead, one must generate relationships between various integrals represented by the **gray lines** and solve a complex system of equations. Therefore, our new method is significantly more efficient than the IBP method.

Examples



As a relatively complex example, we consider the reduction of the pentagon family. As shown in the figure, the **blue diagram** represents a pentagon diagram with three internal lines of different masses, and its external lines are light-like. This family can be represented using a 5×5 symmetric matrix $\boldsymbol{Z}$, constructed from kinematic invariants. Subsequently, with the reduction formulas we provide, one only needs to compute certain cofactors of $\boldsymbol{Z}$ to achieve a direct and directional reduction of the integrals

In certain cases, the top sector of the pentagon diagram is reducible. For instance, the **pink diagram** ($p_1\cdot p_2=0$) and the **purple diagram** ($p_1=p_2$) represent cases where all internal lines are massless. The peculiarity of these cases lies in $\boldsymbol{Z}=0$. In our method, these cases can also be directionally reduced. By leveraging the null space of the matrix $\boldsymbol{Z}$, one can directly decompose the integrals of the top sector into those of lower sectors.

Outlook

Given the success in one-loop problems, it is natural to extend our approach to multi-loop integrals. A new ingredient at higher loops is that there can be more than one master integrals within a sector, and it is no longer guaranteed that all indices can be reduced to 1 or 0. Correspondingly, in our approach, we find that when we perform the variable transformations, certain variables appear in the denominators of the transformed integrands. Such transformations are therefore only valid if the corresponding indices are greater than 1. We have made initial attempts in the sunrise families. Finally, in this work we have only exploited the equivalence of integration contours in a simple way. There can be deeper mathematical structures behind these equivalence relations. For example, it is possible to employ the concept of intersection numbers between two integration contours to directly compute the reduction coefficients. This provides a particularly intriguing future perspective.