SEIBERG-WITTEN CURVE OF QUIVER THEORY WITH ANTISYMMETRIC MATTER

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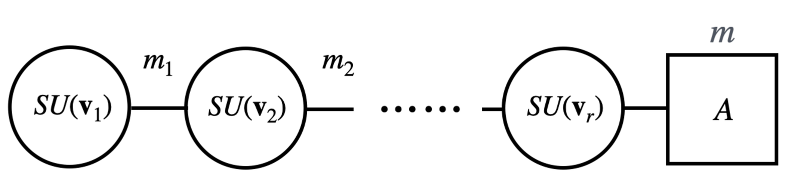
Introduction

Quantum field theory has been successfully applied in many areas of physics, but the standard perturbative approach is insufficient to describe many important phenomena. It is therefore inevitable to consider non-perturbative effects, such as instantons.

The 4D $\mathcal{N}=2$ supersymmetric gauge theories have been a fruitful field of study for theoretical physicists for decades. The solution to the low-energy dynamics can be encoded in the Seiberg-Witten geometry. This solution can be derived from an honest multi-instanton calculus.

Our aim is to apply the instanton counting approach to derive the Seiberg-Witten solution for linear SU quiver gauge theories decorated with antisymmetric matter on one or both sides:



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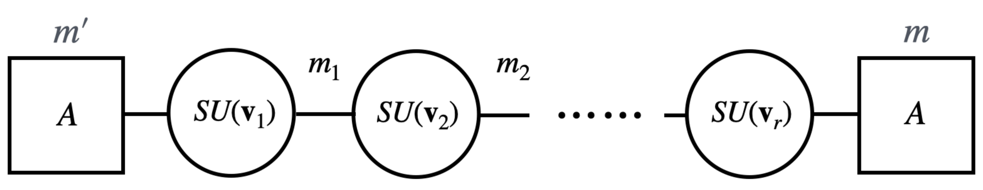
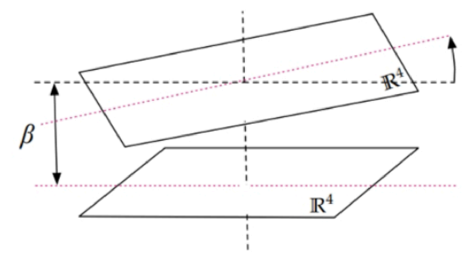


The partition function



$(z_1, z_2, t) \sim (e^{i\beta\varepsilon_1} z_1, e^{i\beta\varepsilon_2} z_2, t+\beta)$

Ω-background

The partition function Z of 4D $\mathcal{N}=2$ supersymmetric gauge theory in the Ω-background can be computed exactly, taking the form:

$$Z(a, m, \mathfrak{q}; \varepsilon_1, \varepsilon_2) = Z^{\text{pert}}(a, m; \varepsilon_1, \varepsilon_2) \times \sum_{k=0}^{\infty} \mathfrak{q}^k Z_k(a, m; \varepsilon_1, \varepsilon_2),$$

where Z^{pert} stands for perturbative contribution and Z_k is the non-perturbative contribution from the k-instanton sector, given by a statistical sum over Young diagrams.

Flat space limit and prepotential

- In the limit $\varepsilon_1, \varepsilon_2 \to 0$, the Ω-background goes back to the flat space $\mathbb{R}^4$, and the main contribution to Z comes from a limit shape configure with $k \sim \frac{1}{\varepsilon_1\varepsilon_2}$. The Wilsonian low energy effective prepotential $\mathcal{F}$ can then be extracted by
$$\mathcal{F} = -\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\varepsilon_2 \ln Z.$$
- Combining contributions from perturbative and non-perturbative parts, the expression for Z can be expressed by the continuous density function $\rho(x)$ as:
$$Z = \int \mathcal{D}\rho \, e^{-\frac{1}{\varepsilon_1\varepsilon_2}(\mathcal{F}^{\text{eff}}(\rho) + \mathcal{O}(\varepsilon_1,\varepsilon_2))}.$$

Derive the difference equation

Define the amplitude functions

$$\mathcal{Y}_i(x) = \exp \int_{\mathbb{R}} \mathrm{dx}\rho_i(\mathrm{x}) \ln(x - \mathrm{x}), \quad i \in \text{Vert}_\gamma \text{ labels the SU subgroups.}$$

The saddle point equation of $\mathcal{F}^{\text{eff}}(\rho)$ gives the difference equation serving exactly as constraints specifying the way $\mathcal{Y}_i$ is joined at the branch cut:

$$\mathcal{Y}_i^+(x)\mathcal{Y}_i^-(x) = \mathfrak{q}_i F(\mathcal{Y}(x)),$$

where $F(\mathcal{Y}(x))$ is a highly schematic notation, absorbing all possible polynomials of x and mass deformed parameter in $\mathcal{Y}_i(x)$.

Construct the iWeyl invariant

The transformations above generate the iWeyl group ${}^i\mathcal{W}$ of the theory. By analytically continue $\mathcal{Y}_i(x)$ over $C_{\langle x\rangle}$, one constructs the holomorphic iWeyl invariants $\mathcal{X}_i(\mathcal{Y}_i(x))$ that are invariant under the action of ${}^i\mathcal{W}$:

$$\mathcal{X}_i(\mathcal{Y}(x)) = \mathcal{Y}_i(x) + \Psi(\mathcal{Y}(x)) = T_i(x) \in \text{Poly}[x],$$

where the schematic Laurent polynomial $\Psi(\mathcal{Y}(x))$ stands for higher order terms in $\mathfrak{q}_i$ which are generated by applying iWeyl transformations on $\mathcal{Y}_i(x)$.

The above set of equations gives the cameral curve $\mathcal{C}_u$ over $\mathbb{CP}^1_x = C_{\langle x\rangle} \cup \{\infty\}$. The cameral curve together with the vector-valued Seiberg-Witten differential which is schematically given by

$$\mathrm{d}\mathbb{S} = x \sum_i (\alpha_i^\vee \mathrm{d}\ln \mathcal{Y}_i - \lambda_i^\vee \mathrm{d}\ln \mathcal{P}_i(x)),$$

forms a geometric interpretation of the theory, containing all information of Seiberg-Witten data.

Extract Seiberg-Witten data

Eliminating all but one of the $\mathcal{Y}_i(x)$'s in the above equations formally gives the cameral curve $\mathcal{C}_u$, which is a ${}^i\mathcal{W}$-cover of $\mathbb{CP}^1_x$.

For our case of the SU linear quiver gauge theory, it's much easier to construct the equivalent but much "smaller" spectral curve, obtained by organizing the iWeyl invariants $\mathcal{X}_i$ into a generating polynomial

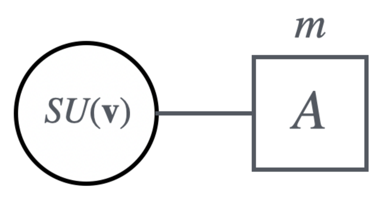
$$\det(t \cdot \mathbb{I}_{r+1} - g_{\lambda_i}(x)) = 0,$$

and replacing all occurrences of $\mathcal{X}_i$ in its coefficients by $T_i(x)$. This is precisely the Seiberg-Witten curve that we are trying to derive, and the associated Seiberg-Witten differential is:

$$\lambda = x\frac{dt}{t}.$$

A simple case: SU theory with antisymmetric matter

As the simplest example, consider the 4D $\mathcal{N}=2$ theory with gauge group $SU(\mathbf{v})$, containing a single antisymmetric hypermultiplet of mass m in addition to N_f fundamental ones of mass $m_f, f = 1, \cdots, N_f$.



$\mathcal{F}^{\text{eff}}(\rho)$ of the theory writes:

$$\begin{aligned}\mathcal{F}^{\text{eff}}(\rho) = &-\iint \mathrm{d}x'\mathrm{d}x''\rho(x')\rho(x'')\left(\mathcal{K}(x'-x'') - \frac{1}{2}\mathcal{K}(x'+x''-m)\right) \\ &+ \int dx'\rho(x')\left(\frac{x^2}{2}\ln\mathfrak{q} - 2\mathcal{K}\left(x - \frac{m}{2}\right) + \sum_{f=1}^{N_f}\mathcal{K}(x'-m_f)\right) \\ &+ \sum_l \mathfrak{b}_l\left(1 - \int_{I_l}\rho(x')\mathrm{d}x'\right) + \sum_l \mathfrak{a}_l^D\left(\mathfrak{a}_l - \int_{I_l} x'\rho(x')\mathrm{d}x'\right).\end{aligned}$$

Taking the variation of $\mathcal{F}^{\text{eff}}$ with respect to x twice gives the the difference equation for $\mathcal{Y}(x)$:

$$\mathcal{Y}^+(x)\mathcal{Y}^-(x) = \mathcal{P}(x)\mathcal{Y}(m-x), \qquad \mathcal{P}(x) = \mathfrak{q}\frac{\prod_{f=1}^{N_f}(x-m_f)}{(x-\frac{m}{2})^2}.$$

By constructing the iWeyl invariant $\mathcal{X}$, its master equation is given by:

$$\mathcal{X}(\mathcal{Y}(x)) = \mathcal{Y}(x) + \mathcal{P}(x)\frac{\mathcal{Y}(m-x)}{\mathcal{Y}(x)} + \mathcal{P}(x)\mathcal{P}(m-x)\frac{1}{\mathcal{Y}(m-x)} = T(x).$$

After eliminating $\mathcal{Y}(m-x)$ from the above equation one immediately obtains a cubic Seiberg-Witten curve:

$$\mathcal{Y}^3 + P(x)\mathcal{Y} + Q(x) = 0,$$

which is consistent with the physical interpretation of the additional antisymmetric part.

General case

The similar procedure can be applied to the general case of SU linear quiver theory decorated by antisymmetric matter on one or both sides, where we derive their Seiberg-Witten curve using the pure field theoretical techniques described above and compare the results with those obtained from the brane construction method in string theory.

References

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