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Intersection theory in Feynman parametrization

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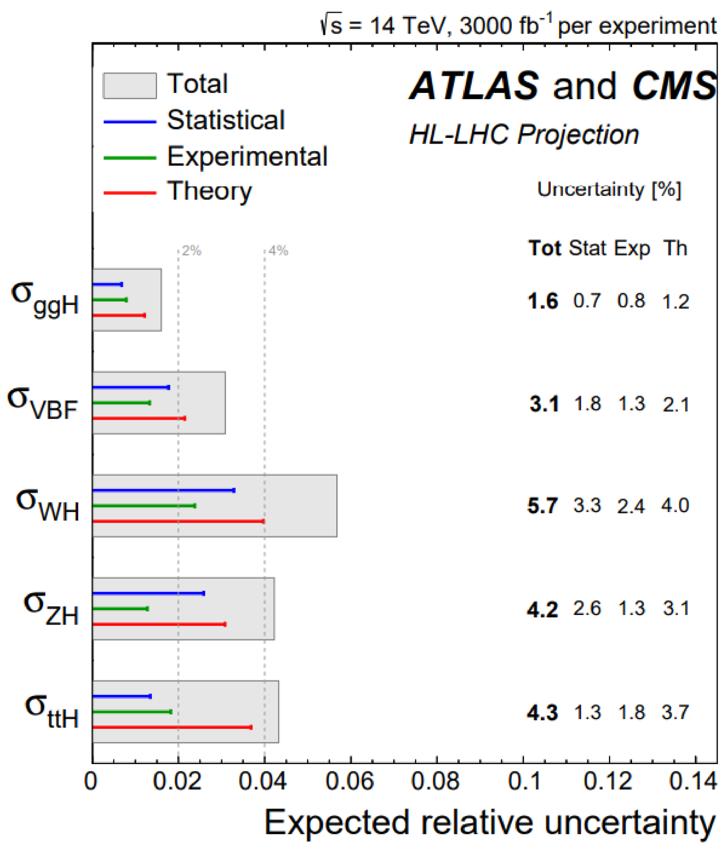
Background

Feynman integrals (FIs) are essential ingredients of perturbative quantum field theories, but a complicated multi-loop scattering amplitude may involve a gigantic amount of FIs. The high-precision data demands high-order calculation. To make full use of data, theoretical errors should be much smaller than experimental errors. Nowadays the computation involves two parts: 1) Reduce loop integrals to master integrals (MIs) via Integration-by-Parts Identities (IBPs) and 2) Calculate MIs. In general, reductions with IBPs for state-of-the-art problems are usually very expensive in terms of required CPU time and also main memory usage. The computational complexity of these integrals increases rapidly with the number of loops, external legs, and physical energy scales.



Problems	Massless double box	Massless double-pentagon	4-loop $g+g\to H$
Reduction time	18h	12d	860d

Brief introduction of intersection theory

By elucidating the vector space structure of FIs, intersection theory[1] provides an intriguing new perspective to formulate IBP relations, which can be articulated within the framework of twisted de Rham cohomology.

$$\int_{\mathcal{C}_R} u\varphi_L := \langle \varphi_L | \mathcal{C}_R] \,, \quad \varphi_L \sim \varphi_L + \nabla_\omega \xi \,, \quad \nabla_\omega := \mathrm{d} + \omega \wedge \,.$$

The twisted cohomology group can be endowed with an inner product structure called intersection number $\langle \bullet | \bullet \rangle : H^p(\nabla_\omega) \times H_c^{n-p}(\nabla_{-\omega}) \to \mathbb{C}$. One may introduce the well known Leray coboundary[2]

$$\delta_{x_1,\cdots,x_m} = \frac{u}{u(0)} \bigwedge_{i=1}^{m} \mathrm{d}\theta_{x_i,0}$$

to avoid using compactly supported differential forms. This method is known as relative cohomology, which has proven applicable across various reduction problems within the frame of Baikov representation.

Intersection theory in Feynman and LP parametrization

Feynman integrals can be expressed in the famous Feynman parametrization. We will work with a slightly different version of the Feynman parametrization proposed by Lee and Pomeransky (and so-called LP parametrization):

$$I(\nu_1,\cdots,\nu_n) = e^{\epsilon\gamma_E L} \frac{(-1)^\nu \Gamma(d/2)}{\Gamma((L+1)d/2-\nu)} \int_0^\infty \left(\prod_i \frac{x_i^{\nu_i-1}\mathrm{d}x_i}{\Gamma(\nu_i)} \right) (\mathcal{U}+\mathcal{F})^{-d/2} .$$

In case of $\nu_i = 0$, they are integrals belong to sub-sectors within the integral family. Equivalently, we can insert a $\delta(x_i)$ into the integration measure.

$$\frac{x_i^{\rho-1}}{\Gamma(\rho)} = \delta(x_i) + \mathcal{O}(\rho^1) \,.$$

We can also incorporate the situations where some $\nu_i < 0$ by simply integrating by parts

$$\int_0^\infty \mathrm{d}x_i \frac{x_i^{\rho+\nu_i-1}}{\Gamma(\rho+\nu_i)} \mathcal{G}^{-d/2} = \int_0^\infty \mathrm{d}x_i\, \delta(x_i) \left(-\frac{\partial}{\partial x_i} \right)^{-\nu_i} \mathcal{G}^{-d/2} + \mathcal{O}(\rho^1) \,.$$

By introducing these $\delta(x_i)$, we can express integrals in all sectors of the family using a unified LP polynomial $\mathcal{G} := \mathcal{U} + \mathcal{F}$. We then propose an innovative approach to reduce Feynman integrals in LP parametrization by reckoning these δ-forms as elements in relative cohomology group $H^p(\mathbb{CP}^n \setminus \mathcal{P}_\omega, \mathcal{D}, \nabla_\omega)$, where $\mathcal{P}_\omega$ is set of singularities of $\omega = \mathrm{d}\log(u)$ and $\mathcal{D}$ is the zero locus of Feynman parameters $\mathcal{D} = \bigcup_i \{x_i = 0\}$.

Applications

In our method, we can straightforwardly obtain

$$\langle \delta_{x_1,\cdots,x_m} \phi_L | h_i \rangle = \left\langle \phi_L \middle| \mathrm{Res}_{x_1=\cdots=x_m=0} \frac{u(0)}{u} h_i \right\rangle \,.$$

And the remaining part of intersection number can be calculated in usual way which has been well developed. A two-loop example is the sunrise integral family shown as below:

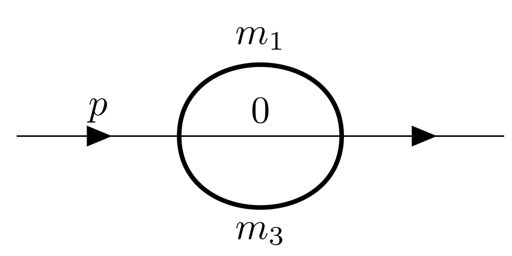


Fig. 1: Sunrise diagram with two internal masses.

By calculating the intersection number in LP parametrization, one can find the decomposition of any target integral into chosen MIs. The basis and dual basis are chosen as

$$\hat{e}_i^{(\mathbf{1})} \in \{1, \delta(x_1)\} \,, \quad \hat{e}_i^{(\mathbf{2})} \in \{1, x_2, \delta(x_2)\} \,, \hat{e}_i = \hat{e}_i^{(\mathbf{3})} \in \{1, x_1, x_3, \delta(x_2)\} \,.$$

$$\hat{h}_i^{(\mathbf{1})} \in \left\{1, \frac{1}{x_1}\right\}, \quad \hat{h}_i^{(\mathbf{2})} \in \left\{1, x_2, \frac{1}{x_2}\right\}, \quad \hat{h}_i = h_i^{(\mathbf{3})} \in \left\{1, x_1, x_3, \frac{1}{x_2}\right\}.$$

The corresponding result are in full agreement with common IBP reduction software.

The method of relative cohomology has subtleties in certain degenerate limits. These occur when Feynman diagrams contain equal internal masses or involve light-like external momentum. In the degenerate limits, the basis may shrink or need to be modified.

Summary

We have proposed a novel approach to reduce loop integrals in the Feynman parametrization with the help of intersection theory and relative cohomology. In practice, this approach has many advantages:

- In the Feynman parametrization, the recursive structure of representations in different sub-sectors of an integral family is manifest.
- The spacetime dimension shift cause by partial derivatives in usual IBP reduction methods does not take into consideration, since integrands with extra factors of G in the denominator are treated automatically in a unified way.
- Compared with usual Baikov parametrization, the Symanzik polynomials are homogeneous in the Feynman parameters, and are usually simpler than the Baikov polynomial.
- In the Feynman parametrization, one does not need to introduce ISPs unless they appear in the numerator. When that happens, the integral can be easily represented by integrands with Symanzik polynomials in the denominator.

All the above lead to the fact that performing integral reduction using intersection theory in the Feynman parametrization can be simpler than in the Baikov representation.

References

[1] Pierpaolo Mastrolia and Sebastian Mizera. "Feynman Integrals and Intersection Theory". In: *JHEP* 02 (2019), p. 139. DOI: 10.1007/JHEP02(2019)139. arXiv: 1810.03818 [hep-th].

[2] Keiji Matsumoto. "Relative twisted homology and cohomology groups associated with Lauricella's F_D". In: (Apr. 2018). arXiv: 1804.00366 [math.AG].