李政道研究所
TSUNG-DAO LEE INSTITUTE

Small Instantons and the Post-Inflationary QCD Axion in a Special Product GUT

Junxuan Xu (徐俊轩)

Tsung-Dao Lee Institute, Shanghai Jiao Tong University

2025.5.11

轴子暗物质：理论与观测@青岛

based on Hor, Nakai, Suzuki, JX, arXiv:2504.02033 [hep-ph]

Strong CP Problem & Axion Solution

Tsung-Dao Lee Institute

QCD Lagrangian allows the interaction which explicitly violates P and CP symmetry:

$$\mathcal{L}_\theta = \theta \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}$$

The physical strong CP Phase:

The upper bound on neutron EDM requires

$$\bar{\theta} = \theta + 2\Sigma(\alpha_i^u + \alpha_i^d) = \theta - \mathrm{Arg}\,\det(\mathcal{M}^u \mathcal{M}^d) \lesssim 10^{-10} \quad \textbf{!!}$$

Strong CP Problem & Axion Solution

QCD Lagrangian allows the interaction which explicitly violates P and CP symmetry:

$$\mathcal{L}_\theta = \theta \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}$$

The physical strong CP Phase:

The upper bound on neutron EDM requires

$$\bar{\theta} = \theta + 2\Sigma(\alpha_i^u + \alpha_i^d) = \theta - \mathrm{Arg}\,\det(\mathcal{M}^u \mathcal{M}^d) \lesssim 10^{-10} \;\; !!$$

The axion solution introduces an additional chiral $U(1)$ symmetry, known as $U(1)_{\mathrm{PQ}}$, which is spontaneously broken and the axion is the corresponding pseudo-Nambu-Goldstone boson.

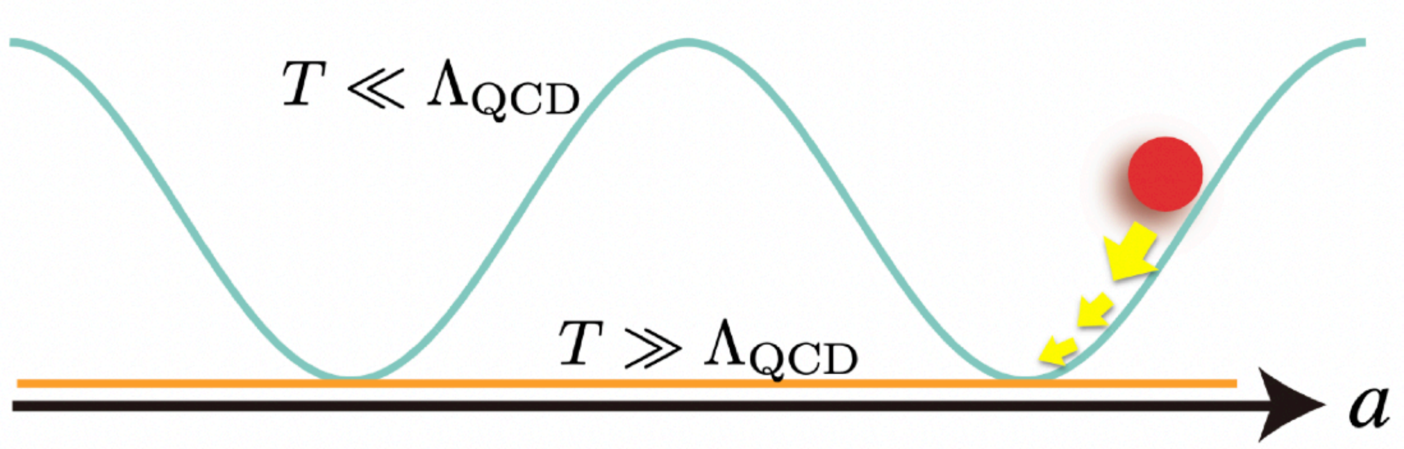
Peccei and Quinn (1977)

The strong CP phase is promoted to a dynamical field $\bar{\theta}_a(x)$

$$\mathcal{L} \supset \underbrace{\left(\bar{\theta} + N_{\mathrm{DW}} \frac{a(x)}{f_{\mathrm{PQ}}}\right)}_{\bar{\theta}_a(x)} \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}$$

After QCD confinement, the strong dynamics generates the axion potential and dynamically cancel the phase.

$$V(a) = m_a^2 f_a^2 \left(1 - \cos\left(\frac{a}{f_a}\right)\right)$$



Domain Wall Problem

Axion is a good **dark matter** candidate but suffers from cosmological threats:

★$U(1)_{PQ}$ breaking **before** inflation → Isocurvature Problem

★$U(1)_{PQ}$ breaking **after** inflation → Domain Wall Problem

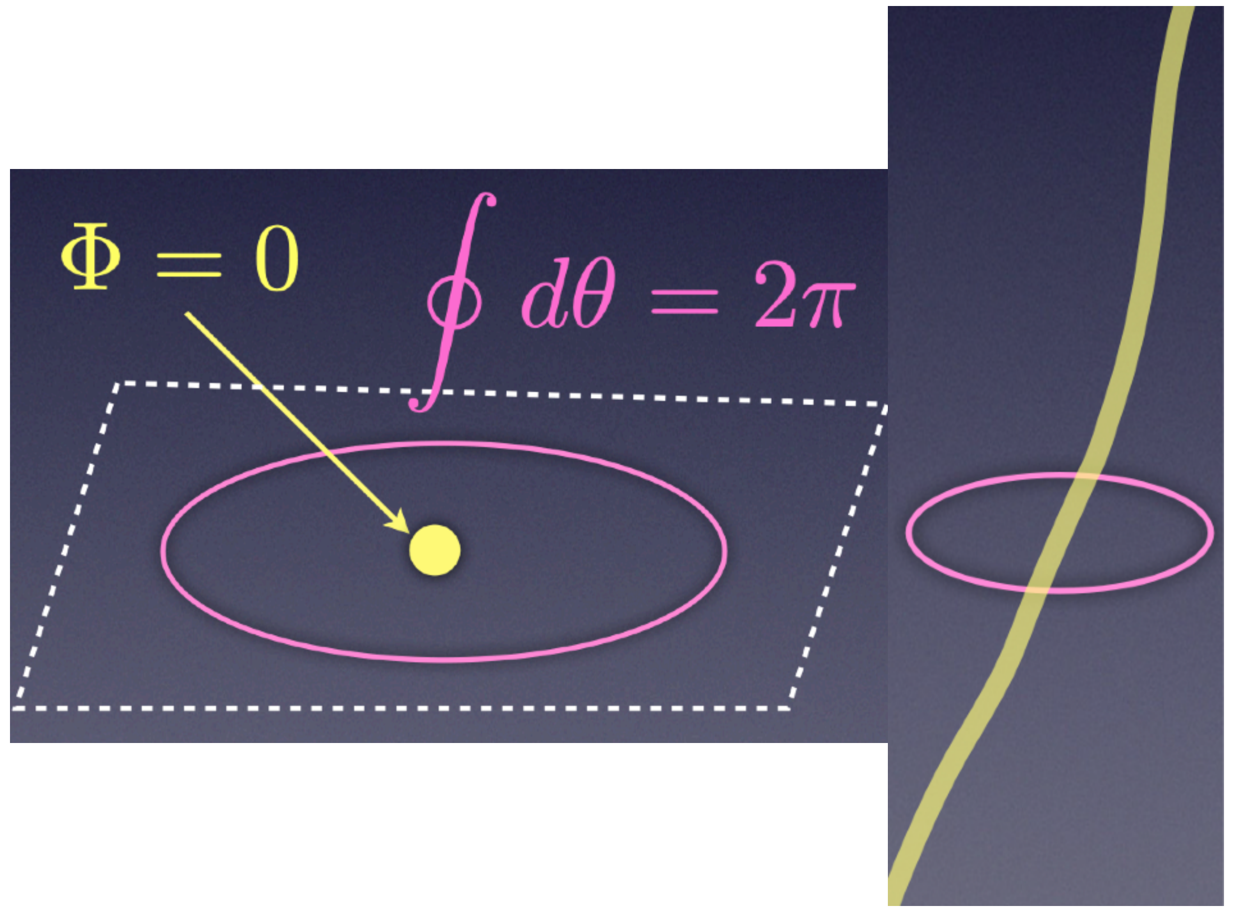
Domain Wall Problem

Axion is a good **dark matter** candidate but suffers from cosmological threats:

★$U(1)_{PQ}$ breaking **before** inflation → Isocurvature Problem

★$U(1)_{PQ}$ breaking **after** inflation → Domain Wall Problem

After SSB, cosmic strings are produced by Kibble mechanism



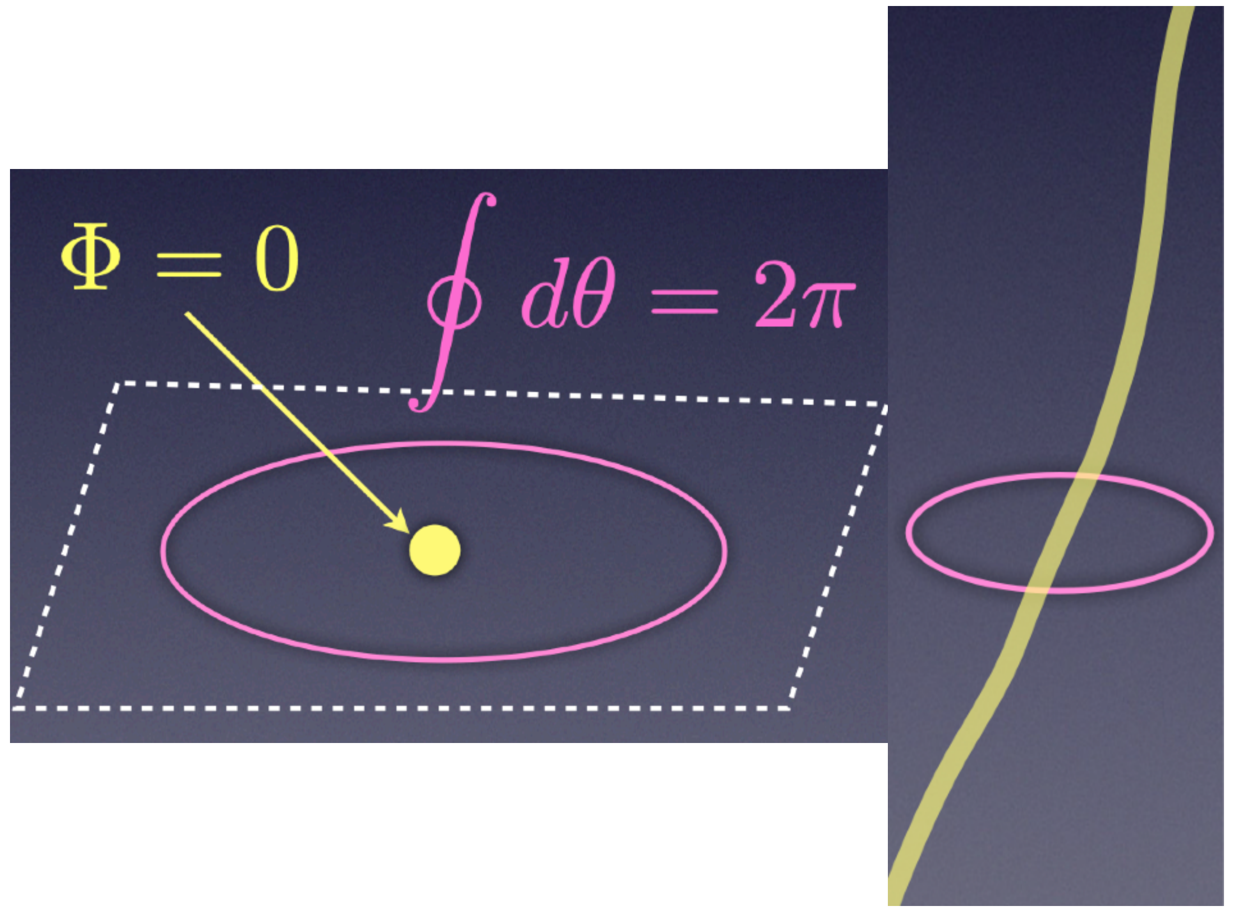
Domain Wall Problem

Axion is a good **dark matter** candidate but suffers from cosmological threats:

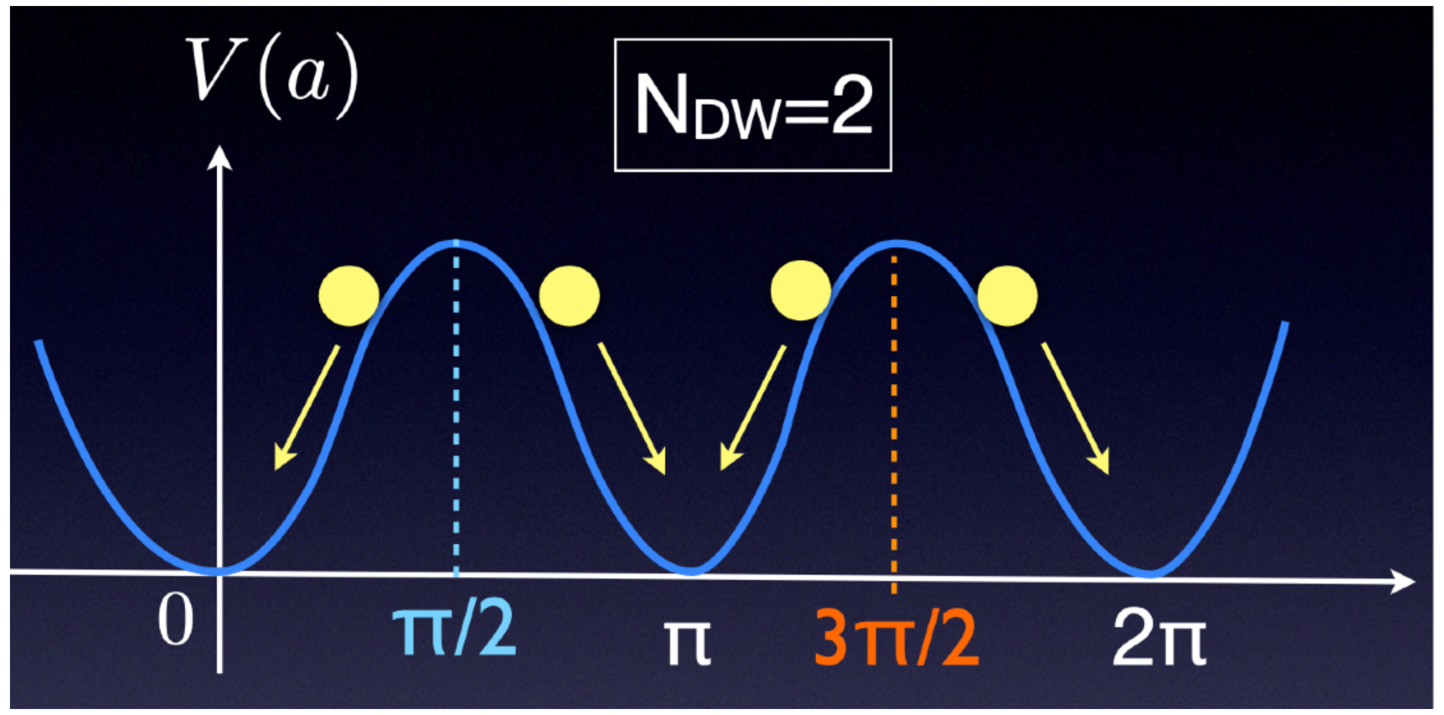
★$U(1)_{PQ}$ breaking **before** inflation → Isocurvature Problem

★$U(1)_{PQ}$ breaking **after** inflation → Domain Wall Problem

After SSB, cosmic strings are produced by Kibble mechanism



After confinement, axion gets the potential with $N_{\rm DW}$ minima



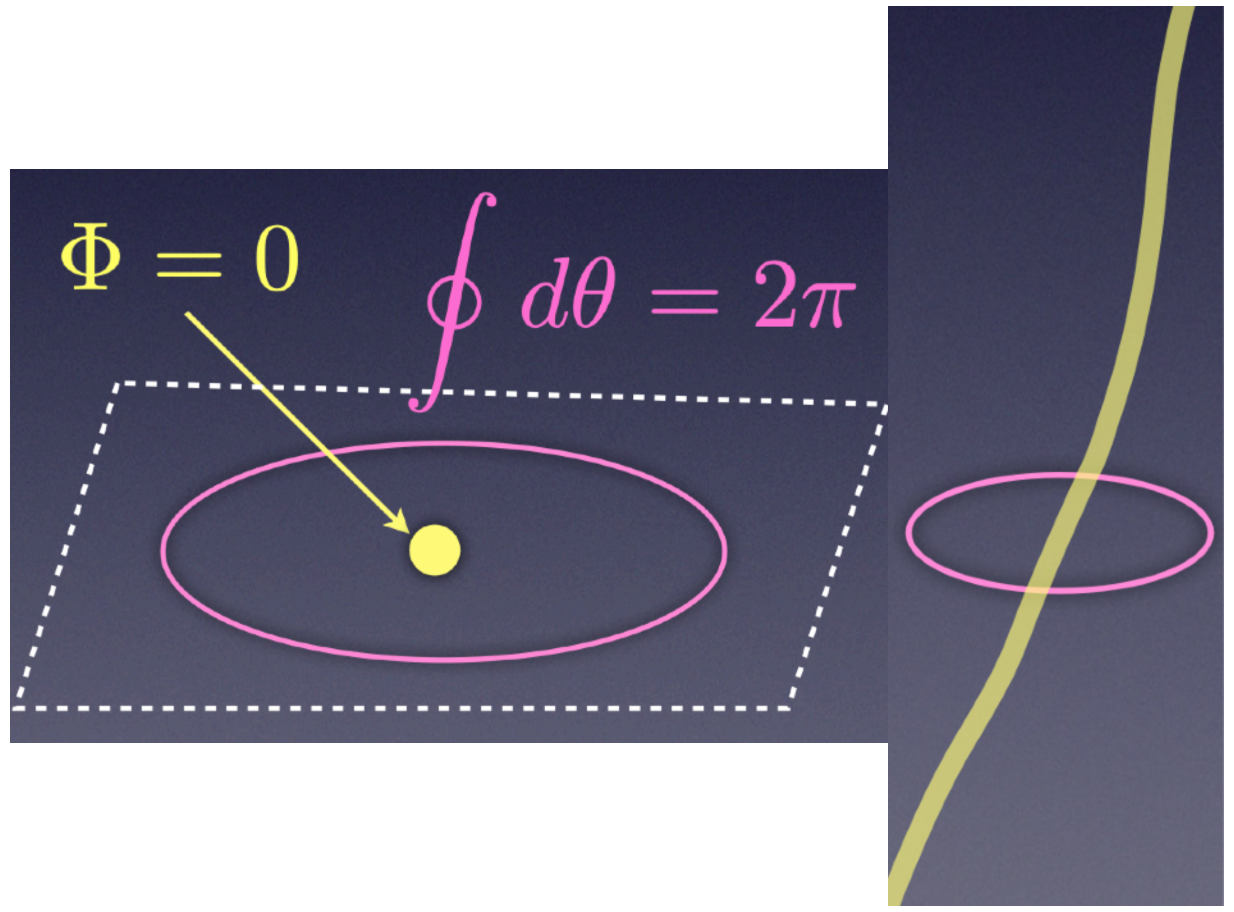
Domain Wall Problem

Axion is a good **dark matter** candidate but suffers from cosmological threats:

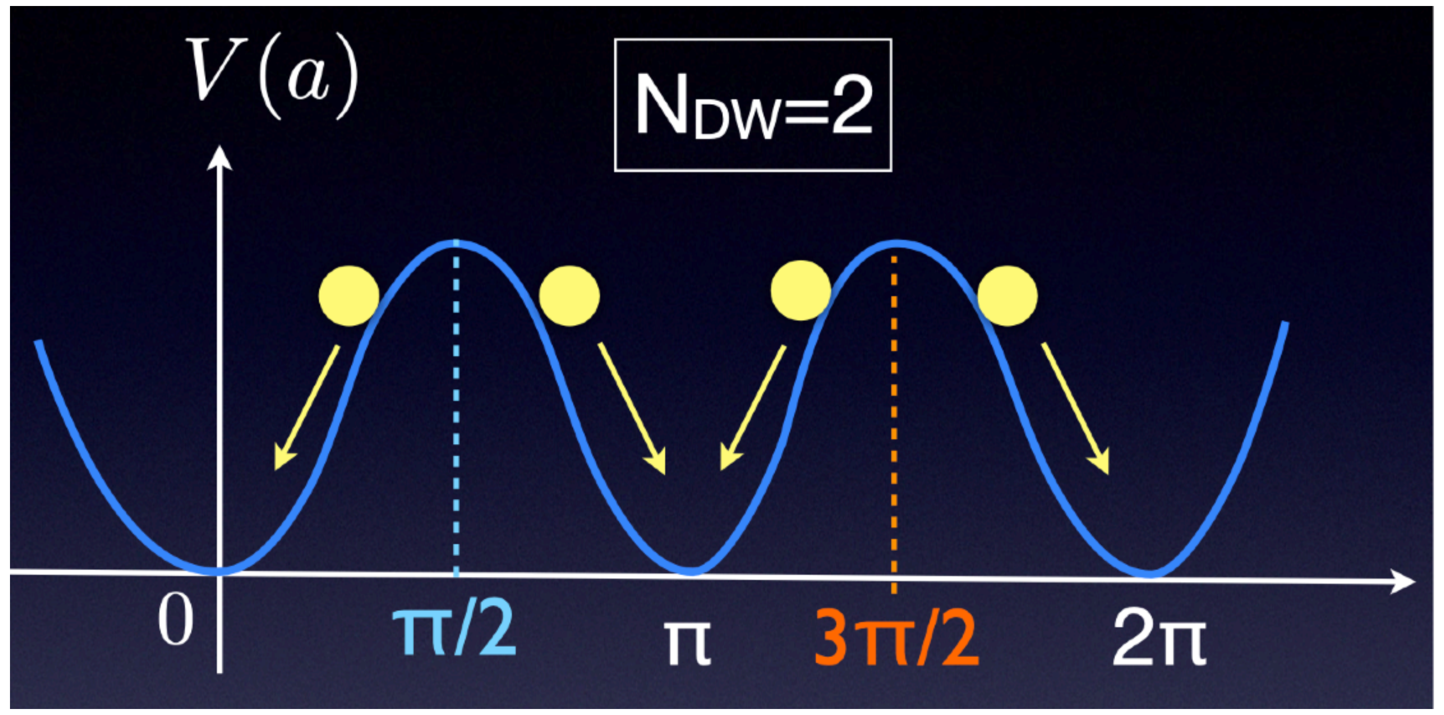
★$U(1)_{PQ}$ breaking **before** inflation → Isocurvature Problem

★$U(1)_{PQ}$ breaking **after** inflation → Domain Wall Problem

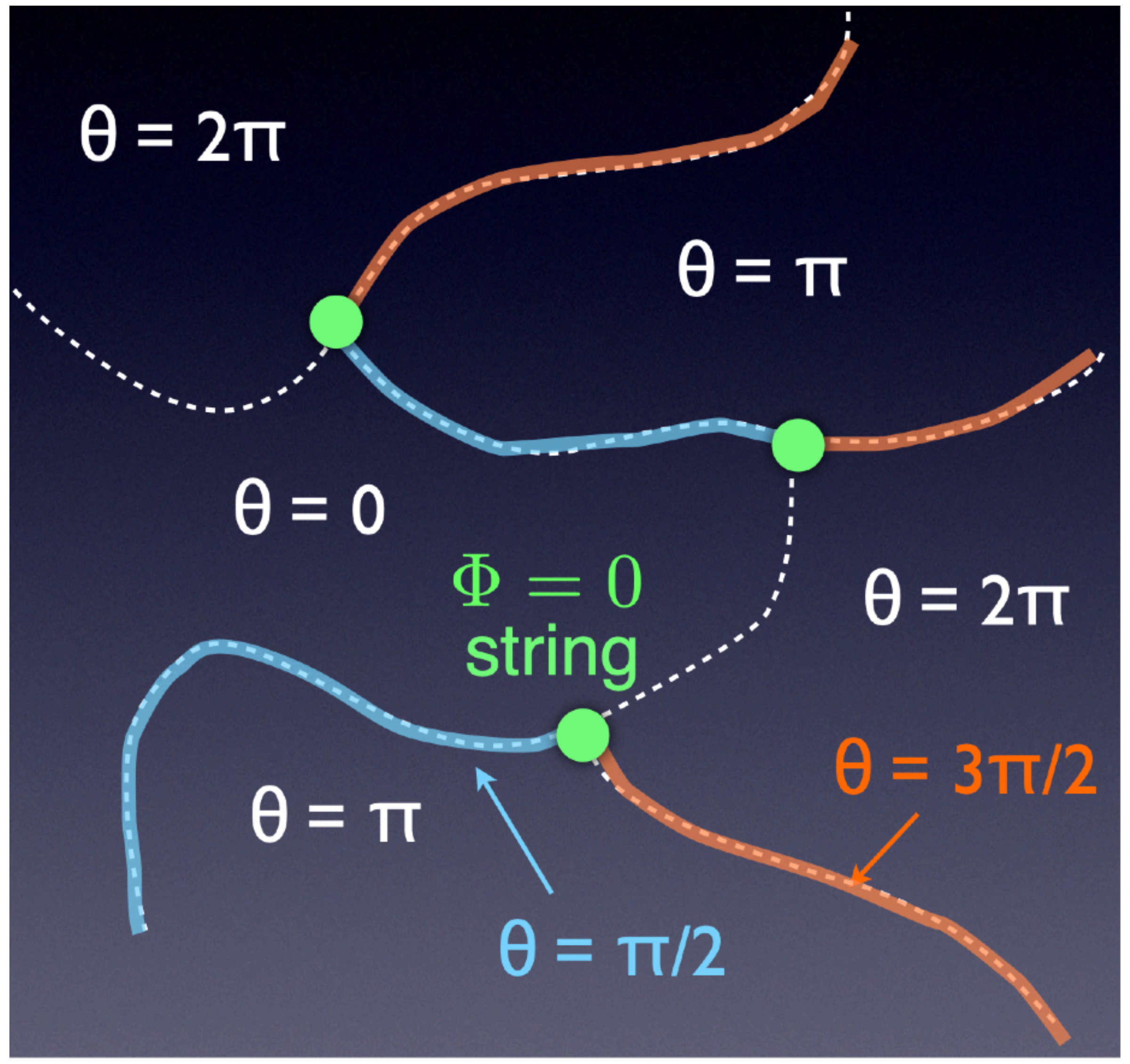
After SSB, cosmic strings are produced by Kibble mechanism



After confinement, axion gets the potential with $N_{\rm DW}$ minima



String-domain wall system dominates the universe if $N_{\rm DW} > 1$



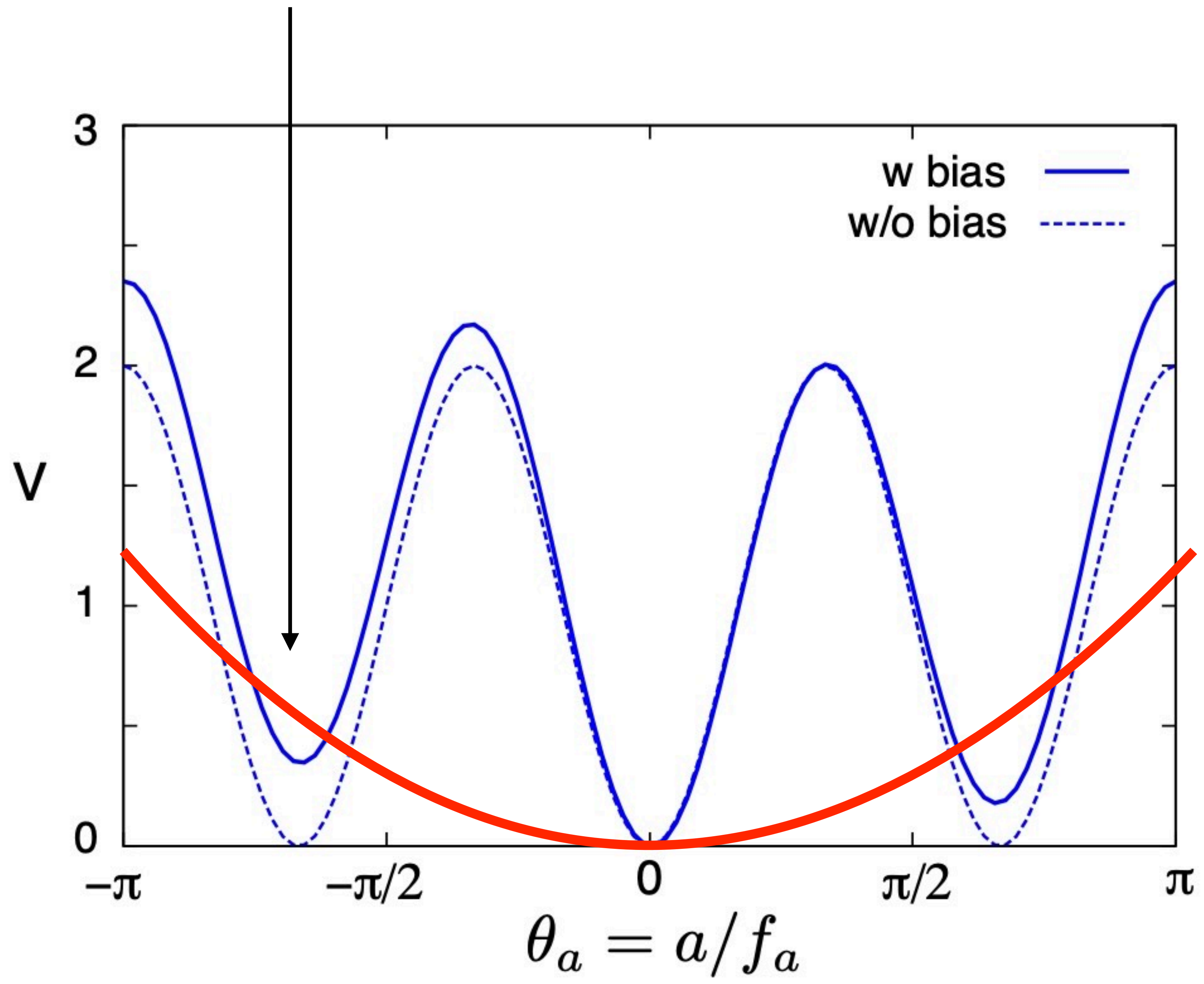
Masahiro Kawasaki slide

Domain Wall Problem

A potential solution to the domain wall problem is to introduce a small explicit breaking of the $U(1)_{PQ}$ symmetry (**Bias term**).

Bias term lifts the vacuum degeneracy !

→ **Domain walls collapse before they dominate the Universe.**



★ Axion DM abundance is changed.

However …

★ **Ad hoc introduction** of the bias term is unsatisfactory.

★ The minimum of the extra potential contribution needs to be **aligned with** that of the QCD potential to solve the strong CP problem.

Small Instanton

李政道研究所
TSUNG-DAO LEE INSTITUTE

Instanton : localized object in Euclidean spacetime, satisfying Euclidean EOM with non-trivial topology and minimizing the Euclidean action

SU(2) BPST instanton solution with $Q = 1$:

$$A_\mu^a(x)\Big|_{1-\mathrm{inst.}} = 2\eta_{a\mu\nu}\frac{(x-x_0)_\nu}{(x-\underline{x_0})^2+\underline{\rho^2}}$$

$$\frac{g^2}{32\pi^2}\int d^4x\, F^{a\mu\nu}\widetilde{F}^a_{\mu\nu}\Big|_{\mathrm{inst.}} = Q \quad (Q\in\mathbb{Z})$$

Position

Instanton size

Instantons contribute to the axion potential

$$\propto \exp\left(-\frac{8\pi^2}{g^2(1/\rho)}\right)$$

In QCD, large-size instantons dominate due to asymptotic freedom.

A hidden gauge sector beyond QCD

↓

Small instanton effects → A possible origin of **the bias term** !

GUT is a natural candidate.

However, a naive embedding into SU(5) GUT does not work because the resulting small instanton effects **do not** lift the vacuum degeneracy.

Special Product GUT

李政道研究所 Tsung-Dao Lee Institute

Our GUT model :

$$\underline{SU(10)} \times SU(5)_1 \to \underline{SU(5)_V} \supset SU(3)_C \times SU(2)_L \times U(1)_Y$$

$\supset SU(5)_2$: **Special embedding** **Diagonal subgroup**

- All SM matter fields are charged under $SU(5)_1$.
- **A vector-like pair of PQ-charged fermions** transform as (anti-)fundamental reps. under SU(10), so that $N_{\mathrm{DW}} = 1$.
- After GUT breaking, # of vector-like pairs of PQ-charged quarks is larger than one, due to special embedding.

  Domain wall problem ??
- The apparent vacuum degeneracy is lifted by **small instanton effects** on the axion potential that operates as a PQ-violating bias term.

Special Embedding

We focus on a gauge symmetry breaking : $SU(2N) \to SU(N)$. Consider **a Weyl fermion** that transforms as the **fund** rep. of $SU(2N)$ but behaves as the **r** rep. of the $SU(N)$ subgroup :

$$\mathcal{D}_\mu \psi = \partial_\mu \psi - i g_{\rm UV} A^m_{{\rm UV},\mu} (T^m_{\rm UV})\psi$$
$$\supset \partial_\mu \psi - i g_{\rm IR} A^a_{{\rm IR},\mu} (T^a_{\rm IR})\psi$$

with

$T^m_{\rm UV}$ $(m = 1, \ldots, (2N)^2 - 1)$: $SU(2N)$ generators (**fund** rep.)

$T^a_{\rm IR}$ $(a = 1, \ldots, N^2 - 1)$: $SU(N)$ generators (**r** rep.)

Special Embedding

We focus on a gauge symmetry breaking : $SU(2N) \to SU(N)$. Consider **a Weyl fermion** that transforms as the **fund** rep. of $SU(2N)$ but behaves as the **r** rep. of the $SU(N)$ subgroup :

$$\mathcal{D}_\mu \psi = \partial_\mu \psi - i g_{\mathrm{UV}} A^m_{\mathrm{UV},\mu} (T^m_{\mathrm{UV}}) \psi$$
$$\supset \partial_\mu \psi - i g_{\mathrm{IR}} A^a_{\mathrm{IR},\mu} (T^a_{\mathrm{IR}}) \psi$$

with

T^m_{UV} $(m = 1, \ldots, (2N)^2 - 1)$: $SU(2N)$ generators (**fund** rep.)

T^a_{IR} $(a = 1, \ldots, N^2 - 1)$: $SU(N)$ generators (**r** rep.)

Special Subalgebras

r ≠ fund

$$\boxed{T^a_{\mathrm{IR}} = \underbrace{\mathcal{O}^{am}}_{\text{Coefficients}} T^m_{\mathrm{UV}}}$$

$$\mathrm{tr}(T^m_{\mathrm{UV}} T^n_{\mathrm{UV}}) = \frac{1}{2}\delta^{mn}$$
$$\mathrm{tr}(T^a_{\mathrm{IR}} T^b_{\mathrm{IR}}) = \underline{T_{\mathrm{IR}}(\mathbf{r})}\delta^{ab}$$

$$\mathcal{O}^{am}\mathcal{O}^{bn}\delta_{mn} = c\delta^{ab}$$

Embedding index

$$\boxed{c \equiv \frac{T_{\mathrm{IR}}(\mathbf{r})}{1/2}}$$

Special embedding corresponds to c > 1.

Special Embedding

李政道研究所
TSUNG-DAO LEE INSTITUTE

We focus on a gauge symmetry breaking : $SU(2N) \to SU(N)$. Consider **a Weyl fermion** that transforms as the **fund** rep. of $SU(2N)$ but behaves as the **r** rep. of the $SU(N)$ subgroup :

$$\mathcal{D}_\mu \psi = \partial_\mu \psi - i g_{\mathrm{UV}} A^m_{\mathrm{UV},\mu}(T^m_{\mathrm{UV}})\psi$$
$$\supset \partial_\mu \psi - i g_{\mathrm{IR}} A^a_{\mathrm{IR},\mu}(T^a_{\mathrm{IR}})\psi$$

with

T^m_{UV} $(m = 1, \ldots, (2N)^2 - 1)$: $SU(2N)$ generators (**fund** rep.)

T^a_{IR} $(a = 1, \ldots, N^2 - 1)$: $SU(N)$ generators (**r** rep.)

Special Subalgebras

r $\neq$ fund

$$\boxed{T^a_{\mathrm{IR}} = \underbrace{\mathcal{O}^{am}}_{\text{Coefficients}} T^m_{\mathrm{UV}}}$$

$$\mathrm{tr}(T^m_{\mathrm{UV}} T^n_{\mathrm{UV}}) = \frac{1}{2}\delta^{mn}$$
$$\mathrm{tr}(T^a_{\mathrm{IR}} T^b_{\mathrm{IR}}) = T_{\mathrm{IR}}(\mathbf{r})\delta^{ab}$$

Embedding index

$$\boxed{c \equiv \frac{T_{\mathrm{IR}}(\mathbf{r})}{1/2}}$$

$$\mathcal{O}^{am}\mathcal{O}^{bn}\delta_{mn} = c\delta^{ab}$$

Special embedding corresponds to c > 1.

A part of $SU(2N)$ gauge field is expressed in terms of SU(N) gauge field :

$$A^l_{\mathrm{UV},\mu} = \frac{g_{\mathrm{IR}}}{g_{\mathrm{UV}}} A^a_{\mathrm{IR},\mu}(\mathcal{O})^{al} + \cdots$$

Canonically normalized kinetic term

$$\boxed{g_{\mathrm{IR}} = g_{\mathrm{UV}}/\sqrt{c}}$$

In our model, **r** = **10** rep. of $SU(5) \subset SU(10)$ leading to **c = 3.**

Theta term : $\displaystyle\int \frac{g^2_{\mathrm{UV}}}{8\pi^2}\mathrm{tr}(F_{\mathrm{UV}} \wedge F_{\mathrm{UV}}) = \int \frac{c g^2_{\mathrm{IR}}}{8\pi^2}\mathrm{tr}(F_{\mathrm{IR}} \wedge F_{\mathrm{IR}})$

Special Product GUT

To achieve $SU(10) \times SU(5)_1 \to SU(5)_V$ we introduce a Higgs field :

Φ_a^{ij} : $(\mathbf{10},\ \overline{\mathbf{10}})$ $a\,(=1-10)$: SU(10) index $i, j\,(=1-5)$: SU(5)$_1$ indices

VEV of Φ is described by the embedding of the $\overline{\mathbf{10}}$ rep. of SU(5)$_1$ into the anti-fundamental rep. of SU(10) :

$$\langle \Phi \rangle = v \mathbf{1}_{10\times 10}$$

GUT Sector

Field	Spin	$SU(10)$	$SU(5)_1$	$U(1)_{\rm PQ}$	$U(1)_\eta$
Φ	0	$\mathbf{10}$	$\overline{\mathbf{10}}$	0	0
$\mathbf{24}_H^{(1)}$	0	$\mathbf{1}$	$\mathbf{24}$	0	0

Field	Spin	$SU(10)$	$SU(5)_1$	$U(1)_{\rm PQ}$	$U(1)_\eta$
$\mathbf{10}_f^{(1)} (f=1-3)$	$1/2$	$\mathbf{1}$	$\mathbf{10}$	0	0
$\bar{\mathbf{5}}_f^{(1)} (f=1-3)$	$1/2$	$\mathbf{1}$	$\bar{\mathbf{5}}$	0	0
$\mathbf{5}_H^{(1)}$	0	$\mathbf{1}$	$\mathbf{5}$	0	0

Axion Sector

Field	Spin	$SU(10)$	$SU(5)_1$	$U(1)_{\rm PQ}$	$U(1)_\eta$
ψ	$1/2$	$\mathbf{10}$	$\mathbf{1}$	$+1$	-1
$\bar{\psi}$	$1/2$	$\overline{\mathbf{10}}$	$\mathbf{1}$	0	$+1$
$\Phi_{\rm PQ}$	0	$\mathbf{1}$	$\mathbf{1}$	-1	0

The PQ mechanism is implemented by a PQ breaking field and **a vector-like pair of PQ-charged fermions** (**KSVZ fermions**) that transform as (anti-)fundamental reps. under SU(10).

→ **N$_{\rm DW}$ = 1**

Axion Potential

李政道研究所 TSUNG-DAO LEE INSTITUTE

- **QCD effects**

Theta terms @ UV : $$\int \frac{\theta_{10} g_{10}^2}{8\pi^2} \mathrm{tr}(F_{10} \wedge F_{10}) + \frac{\theta_5 g_5^2}{8\pi^2} \mathrm{tr}(F_5 \wedge F_5)$$

After the breaking of $SU(10) \times SU(5)_1 \to SU(5)_V$

Theta term @ IR : $$\int \frac{\theta_c g^2}{8\pi^2} \mathrm{tr}(F \wedge \underline{F})$$ $SU(5)_V$ gauge field

Special embedding $$\frac{1}{g^2} = \frac{\boxed{3}}{g_{10}^2} + \frac{1}{g_5^2} \qquad \theta_c = \underline{3}\theta_{10} + \theta_5$$

Axion coupling to gluons : $$\int (\boxed{3}\theta_a + \theta_c) \frac{g^2}{8\pi^2} \mathrm{tr}(G \wedge G)$$

<u>A vector-like pair of SU(2)L doublet KSVZ (anti-)quarks</u> and <u>a pair of SU(2)L singlet (anti-)quarks</u> appear after GUT breaking.

Pion mass and decay constant

$$\Rightarrow \quad V_{\mathrm{QCD}} \approx -\frac{m_u m_d}{(m_u + m_d)^2} \underline{m_\pi^2 f_\pi^2} \cos\left(\boxed{3}\theta_a + \underline{\bar{\theta}_c}\right)$$

Physical phase

$\bar{\theta}_c \ll 10^{-2}$

due to the **SCPV**

Axion potential seems to correspond to the case with **N_{DW} = 3.**

Axion Potential

- **Small instanton effects**

The instanton effects can be captured by **a local fermion operator**.

★ One flavor of KSVZ fermions in the (anti-)fundamental reps. of SU(10)

★ Four flavors of Weyl fermions in the **99** rep. of SU(10)

Field	Spin	$SU(10)$	$SU(5)_1$	$U(1)_{\rm PQ}$	$U(1)_\eta$
$\Psi_{99,f'}$ $(f'=1-4)$	$1/2$	$\mathbf{99}$	$\mathbf{1}$	0	0
$\Psi^{(1)}_{75,f'}$ $(f'=1-4)$	$1/2$	$\mathbf{1}$	$\mathbf{75}$	0	0

We assume an approximate chiral symmetry :

$$\Psi_{99} \to \Psi_{99} e^{i\beta}, \quad \Psi^{(1)}_{75} \to \Psi^{(1)}_{75} e^{-i\beta}$$

Under $SU(5) \subset SU(10)$ $\mathbf{99} = \mathbf{75} \oplus \underline{\mathbf{24}}$

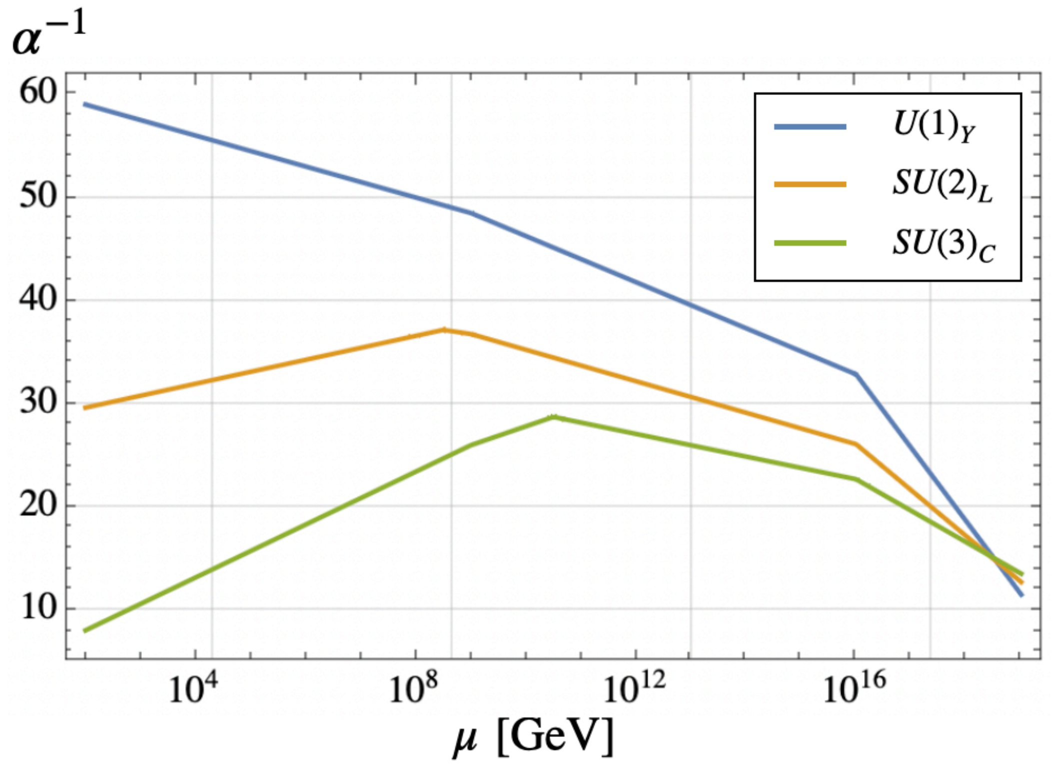
Mass terms are suppressed by a small parameter κ ≪ 1

$$\mathcal{L} \sim \kappa^2 M (\Psi_{99})^a_b (\Psi_{99})^b_a + \kappa^{\dagger 2} \underline{M} (\Psi^{(1)}_{75})^{kl}_{ij} (\Psi^{(1)}_{75})^{ij}_{kl}$$

GUT scale $M \sim M_{\rm Pl}$

All components except the **SU(2)$_L$ triplet and SU(3)$_C$ octet** acquire masses near the Planck scale.

Gauge coupling unification in non-SUSY model



One-loop RGE

- Suppress the instanton effect
- **24** multiplet within **99** acquires a mass of $\mathcal{O}(\kappa^2 M)$

Axion Potential

- **Small instanton effects**

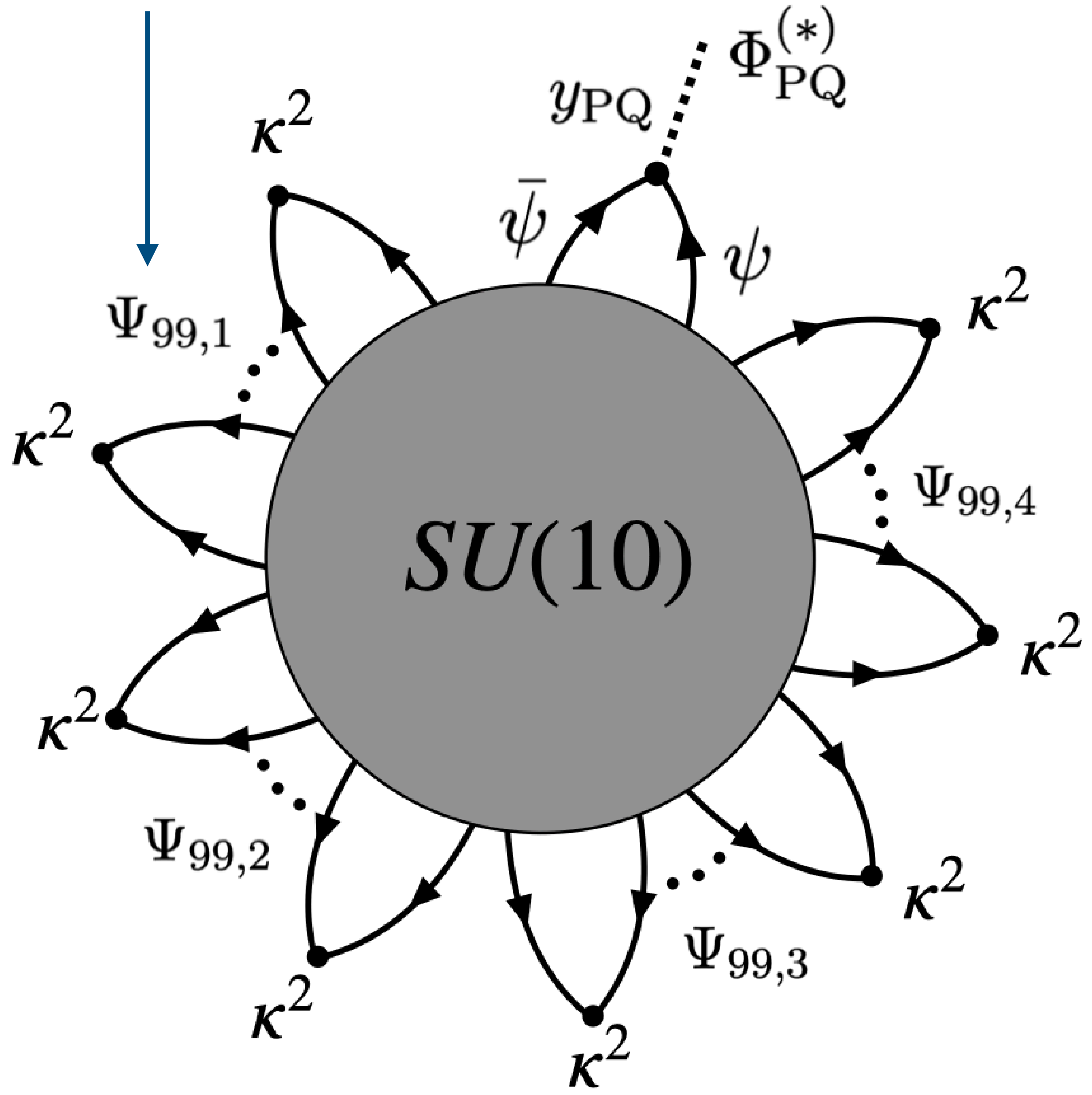
Instanton NDA

Csaki, D'Agnolo, Kuflik, Ruhdorfer (2024)

$$V_{\rm bias} \approx C_{10}\left(\frac{2\pi}{\alpha_{\rm UV}(M)}\right)^{2\times 10}(\Phi_{\rm PQ}+\Phi^*_{\rm PQ})$$
$$\times\int\frac{d\rho}{\rho^5}(\Lambda_{SU(10)}\rho)^{b_0}\,\underline{e^{-2\pi^2\rho^2M^2}}\,y_{\rm PQ}(\kappa^2M\rho)^{10N_F}\rho$$
$$\approx(\kappa^2)^{10N_F}C_{10}\left(\frac{2\pi}{\alpha_{\rm UV}(M)}\right)^{2\times 10}e^{-2\pi^2}$$
$$\times\frac{\Phi_{\rm PQ}}{M}M^4e^{-2\pi/\alpha_{\rm UV}(M)}+c.c.$$

Suppression originating from SU(10) breaking

Each flavor of Ψ_{99} has 2T(Adj) = 20 legs closed by 10 mass vertices.



't Hooft vertex

$\Lambda^{b_0}_{SU(10)} = M^{b_0}e^{-\frac{8\pi^2}{g^2_{\rm UV}(M)}}$ b_0 : one-loop beta function coefficient

C_{10} : SU(10) instanton density $y_{\rm PQ}$: Yukawa coupling of $\Phi_{\rm PQ}$ and KSVZ fermions

$$\Rightarrow V_{\rm bias} = 3\times 10^2(\kappa^2)^{40}\frac{e^{-2\pi/\alpha_{\rm UV}}}{\alpha_{\rm UV}^{20}}e^{-2\pi^2}M^3\Phi_{\rm PQ}+c.c.$$
$$= 6\times 10^2\epsilon\frac{e^{-2\pi/\alpha_{\rm UV}}}{\alpha_{\rm UV}^{20}}e^{-2\pi^2}M^3v_{\rm PQ}\cos(\theta_a) \qquad \epsilon\equiv(\kappa^2)^{40}$$

NDW = 1

Axion potential from small instanton effects provides a bias term !

Axion Dark Matter

PQ symmetry is spontaneously broken after reheating.

We focus on the scenario where the axion field starts to oscillate due to the axion mass originated from **the bias term (before QCD phase transition)** :

$$m_{\rm bias}^2 \equiv \frac{\partial^2 V_{\rm bias}/\partial\theta_a^2}{v_{\rm PQ}^2}$$

Temperature when the oscillation starts :

$$m_a(T_1) \approx m_{\rm bias} = 3H(T_1)$$

$$H(T_1)^2 \approx \frac{\pi^2}{90 M_{\rm Pl}^2} g_* T_1^4$$

$$T_1 > 0.98\,{\rm GeV}\left(\frac{v_{\rm PQ}/3}{10^{12}\,{\rm GeV}}\right)^{-0.19} \equiv \underline{T_{1,{\rm QCD}}}$$

Temperature when the axion would start to oscillate with non-perturbative QCD effects if there was no bias term

Domain walls decay in a similar way as the standard $N_{\rm DW} = 1$ case.

Axion Dark Matter

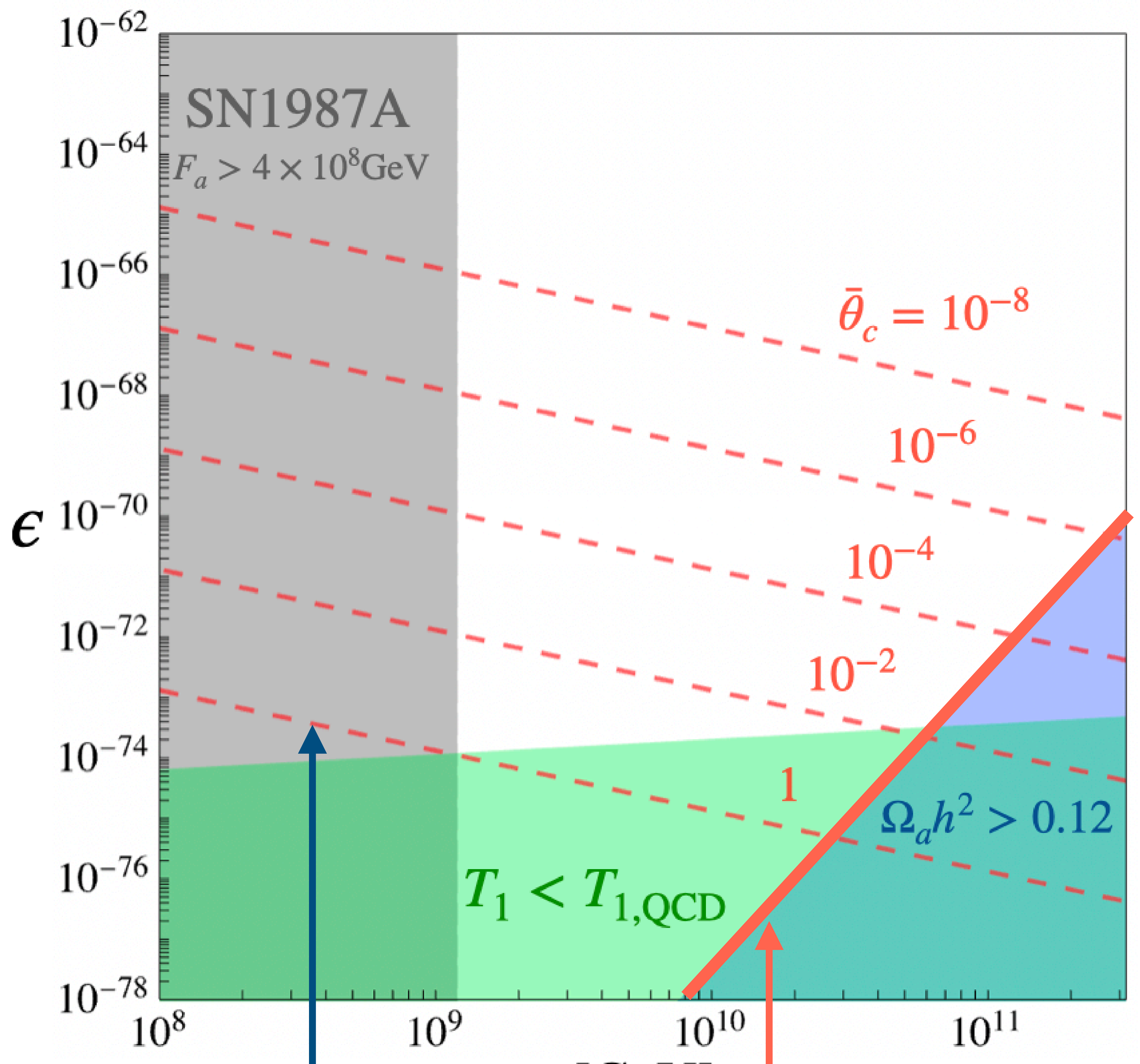
Axion abundance :

$$\Omega_a h^2 \approx 2 \times 10^{-12} \frac{v_{\rm PQ}}{T_1}$$

A large bias term shifts the axion potential minimum :

$$\Delta\bar{\theta} \equiv \frac{m_{\rm bias}^2 v_{\rm PQ}^2}{m_a^2 F_a^2} \bar{\theta}_c$$

Neutron EDM ➡ $\Delta\bar{\theta} \lesssim 10^{-10}$



$\Delta\bar{\theta} = 10^{-10}$

Correct axion DM abundance

A regime explored by ongoing and future experiments ! → $v_{\rm PQ} \gtrsim 6 \times 10^{10}\,{\rm GeV}$

Summary

- We have presented **a special product GUT** model equipped with a viable post-inflationary QCD axion.
- We have achieved a domain-wall-free UV completion for an IR model where N_{DW} appears larger than one.
- **Small instanton effects** on the axion potential operate as a PQ-violating bias term and allow the decay of domain walls.
- The model gives a prediction for a dark matter axion window different from the ordinary N_{DW} = 1 case.
- Future directions:

Supersymmetric version

⬇

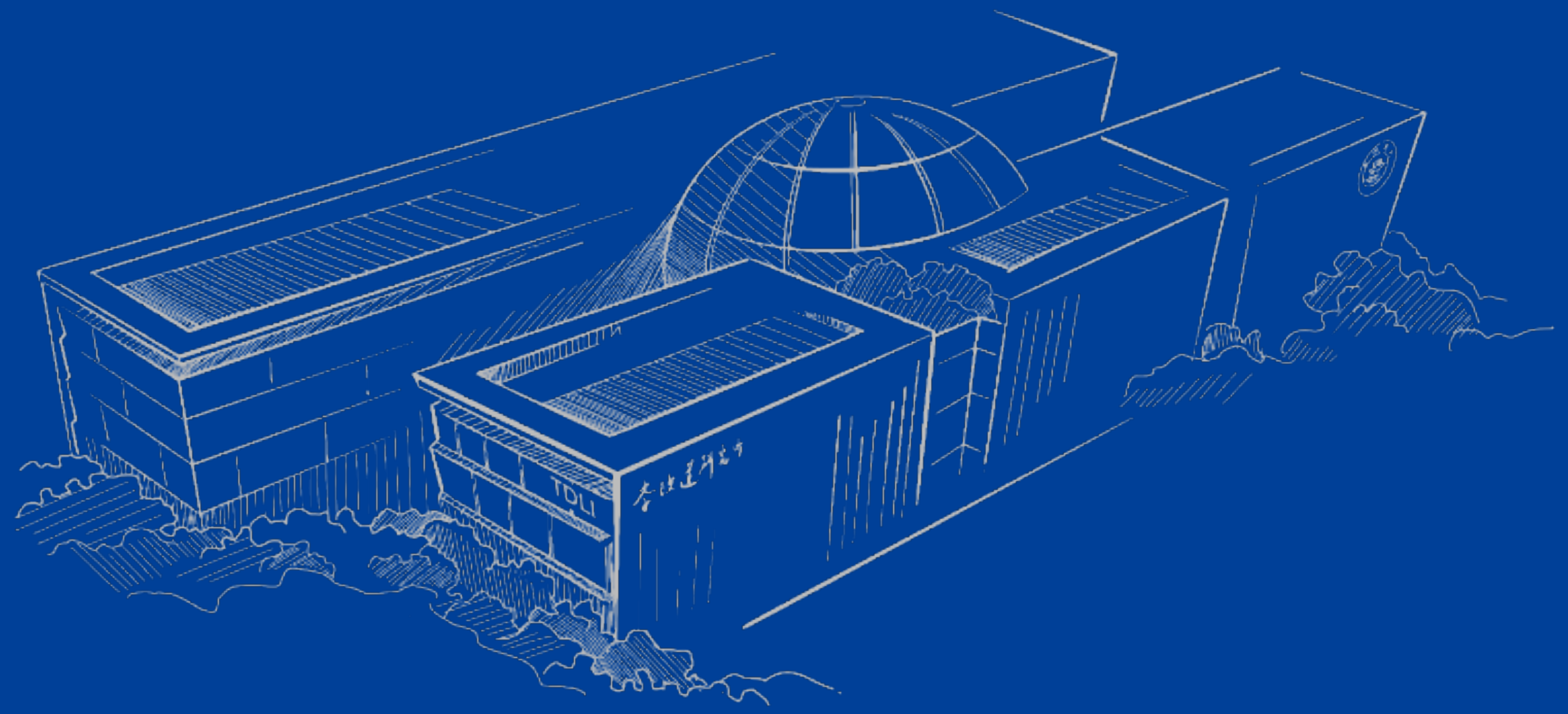
Automatic gauge coupling unification

Extra-dimensional version

⬇

Axion quality problem

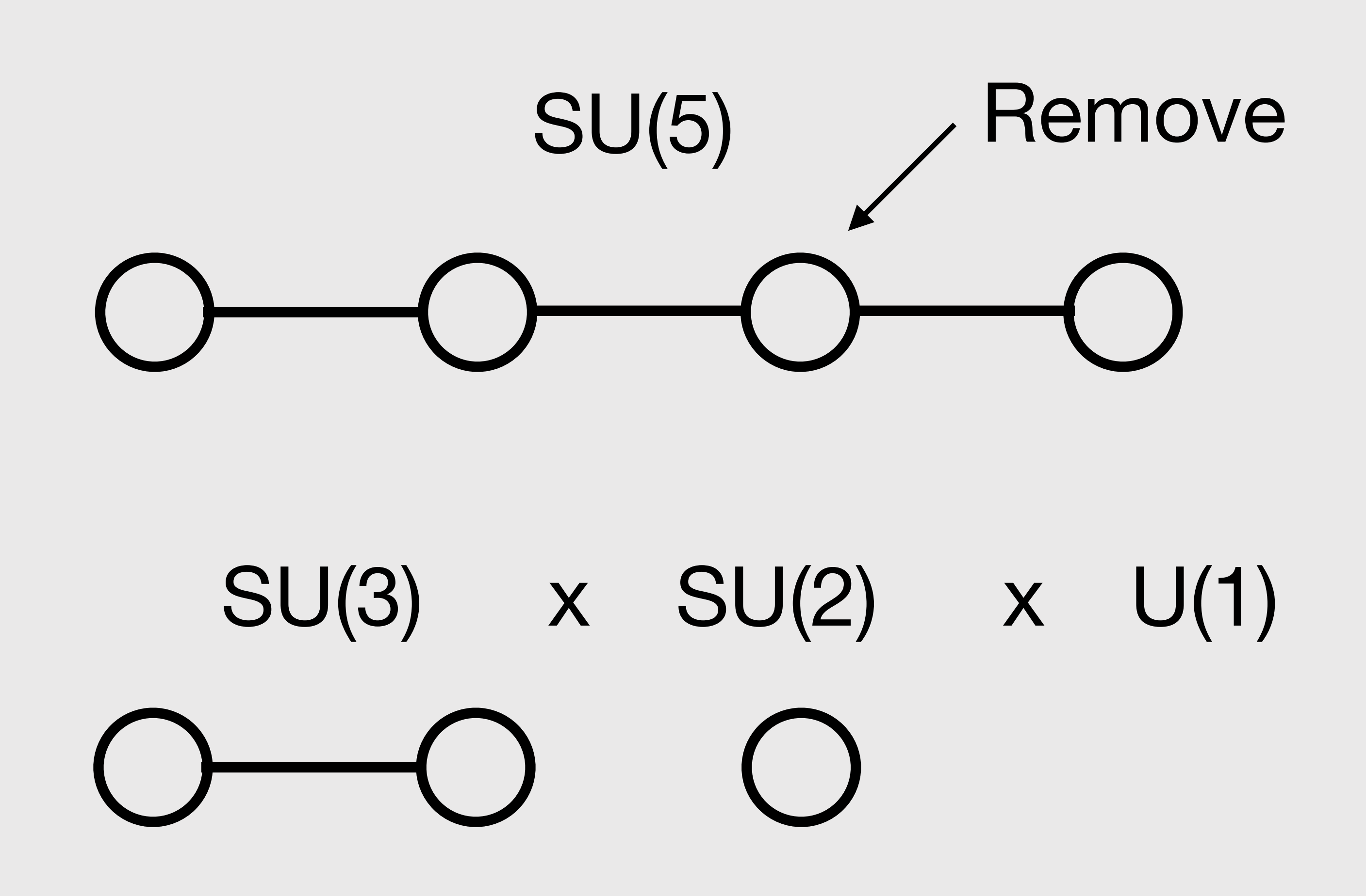
Thanks



Special Subalgebra

<u>Simple Lie algebras</u> possess not only **regular** subalgebras but also **special** subalgebras.

Regular subalgebras : systematically obtained by removing nodes from Dynkin diagrams.



Special subalgebras :

do not follow this scheme !

<u>To identify the SM gauge group as such a special subgroup of a UV gauge group</u> is essential to obtain **small instanton effects** resolving the vacuum degeneracy of the axion potential.

Spontaneous CP Violation

李政道研究所 TSUNG-DAO LEE INSTITUTE

The minimum of the axion potential generated by small instanton effects needs to be aligned with that of non-perturbative QCD effects.

Spontaneous CP violation

We introduce complex scalar fields with $\arg(\langle \eta_\alpha \rangle) = \mathcal{O}(1)$

Field	Spin	$SU(10)$	$SU(5)_1$	$U(1)_{\rm PQ}$	$U(1)_\eta$
$\eta_\alpha(a=1,2)$	0	$\mathbf{1}$	$\mathbf{1}$	0	-1

Forbid dangerous terms at the classical level.

To reproduce the CKM phase, η couple to the mixing term between the KSVZ fermion sector and the SM sector :

$$\mathcal{L} \sim \Phi_{\rm PQ}\psi\bar{\psi} + \sum_{f=1-3,\alpha=1,2} a^u_{\alpha f}\eta_\alpha \bar{\psi}^a (\Phi)^{ij}_a \mathbf{10}^{(1)}_{ij,f}$$

$$+ y^u_{ff'}\mathbf{10}^{(1)}_f \mathbf{10}^{(1)}_{f'} \mathbf{5}_H + y^d_{ff'}\mathbf{10}^{(1)}_f \bar{\mathbf{5}}^{(1)}_{f'} \mathbf{5}^\dagger_H$$

All coefficients are **real**.

Spontaneous CP Violation

The setup is similar to the **Nelson-Barr mechanism**.

Up-type quark mass matrix :

$$\mathcal{L} \sim (\underbrace{q_{uf}}_{\mathbf{10}_f^{(1)}} \ \underbrace{U}_{\bar{\psi}} \ \underbrace{Q_u}_{\psi})\mathcal{M}_u \begin{pmatrix} \bar{u}_{f'} \\ \bar{U} \\ \bar{Q}_u \end{pmatrix} \qquad \mathcal{M}_u = \begin{pmatrix} (m_u)_{ff'} & 0 & A^* \\ A^\dagger & v_{\mathrm{PQ}} & 0 \\ 0 & 0 & v_{\mathrm{PQ}} \end{pmatrix}$$

$$A^* = \sum_\alpha a^u_{\alpha f}\eta_\alpha \qquad (m_u)_{ff'} \equiv y^u_{ff'} v_{\mathrm{SM}}$$

O(1) CKM phase is properly generated when $\left(a^u \langle \eta \rangle\right)_f \gtrsim v_{\mathrm{PQ}}$

Since determinant is real, the physical θ-parameters of SU(10), SU(5)$_1$ or SU(3)$_C$ vanish at the tree-level.

Radiative corrections can still generate nonzero corrections.