Level repulsion in QCD:
light-ray operators and DGLAP/BFKL mixing

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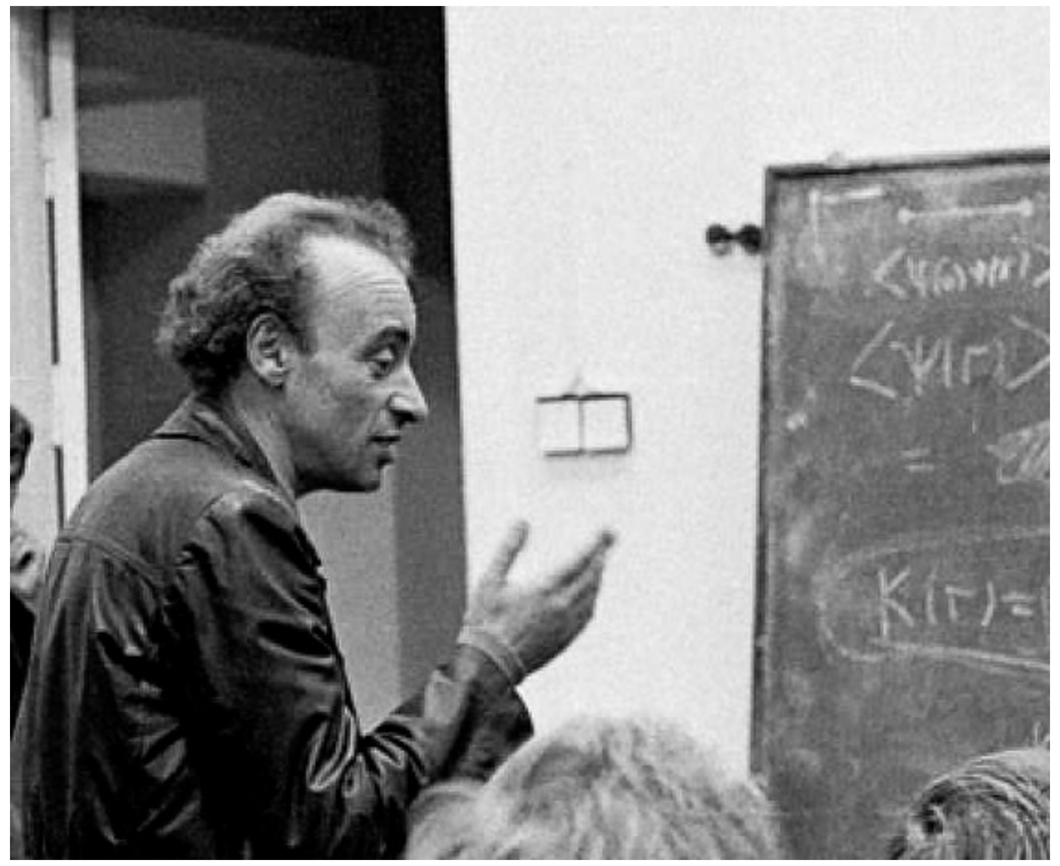
coming soon

New Opportunities in Particle and Nuclear Physics with Energy Correlators

May 09, 2025

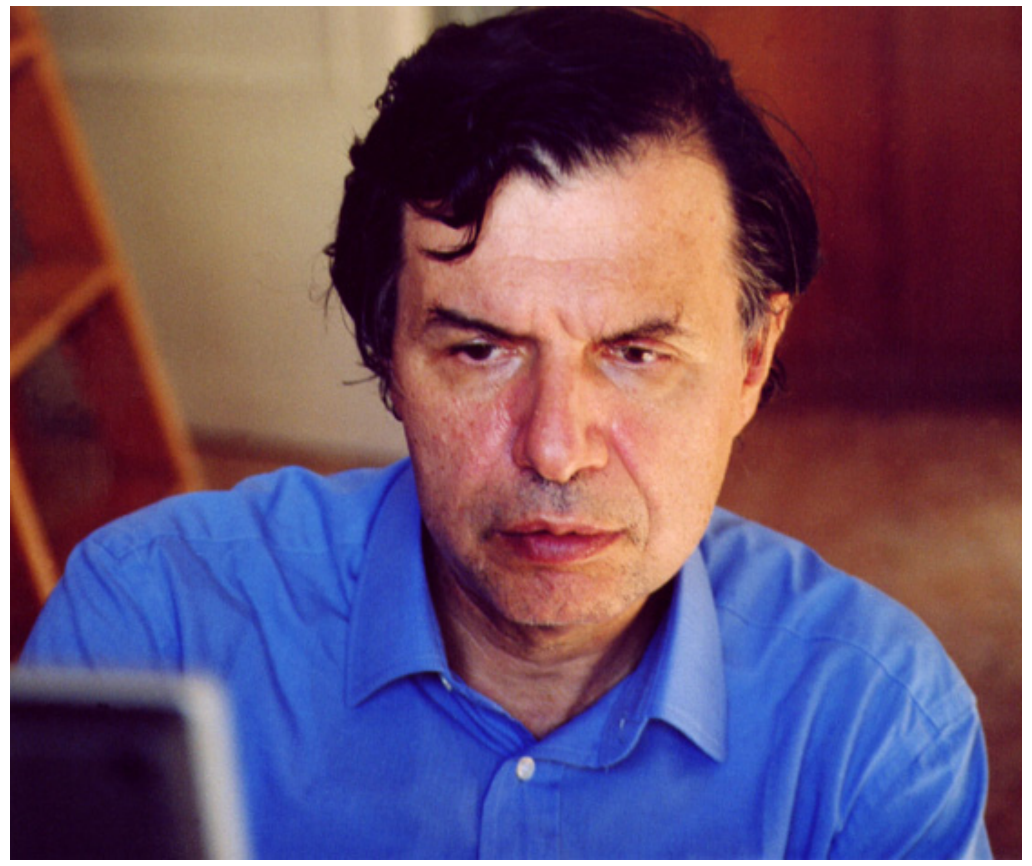
QCD Giants


Dokshitzer

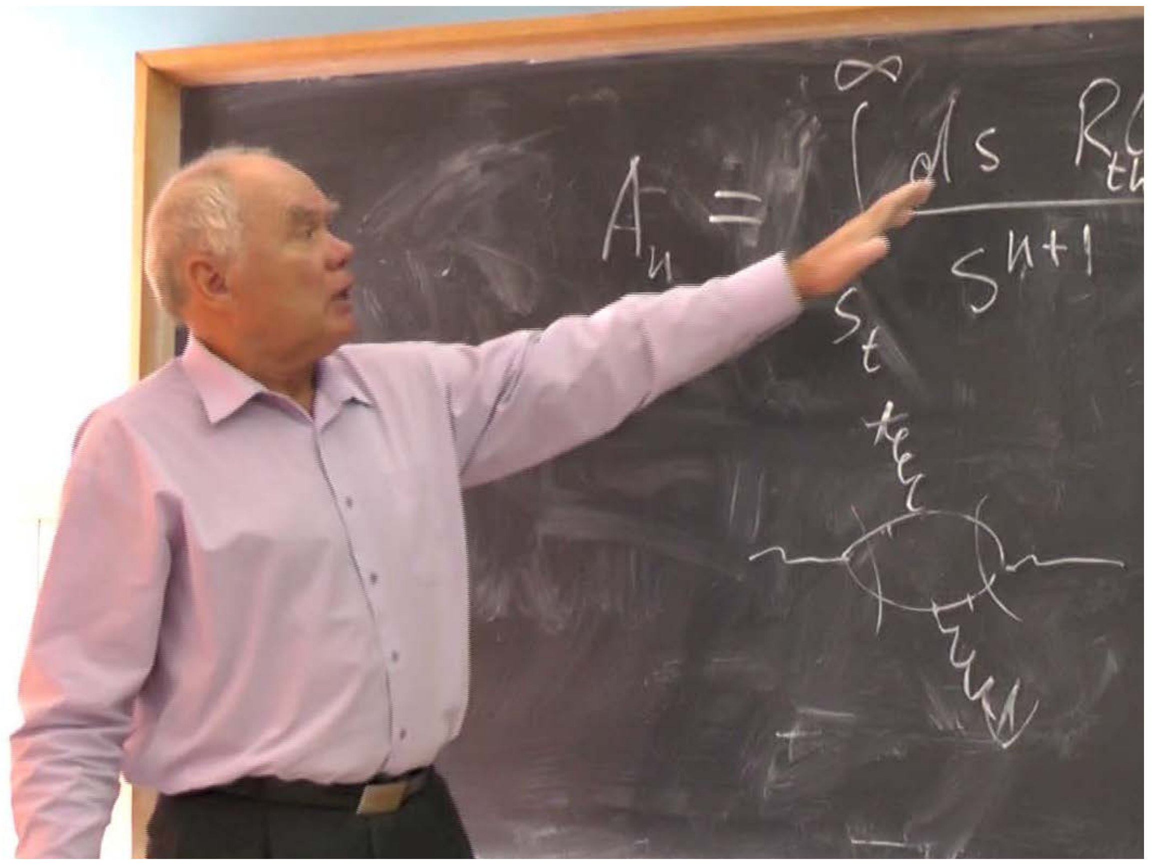

Gribov


Lipatov


Altarelli


Parisi


Balitsky


Fadin


Kuraev


Lipatov

Outline

- (Long) Introduction
  - an exercise in QM
  - introduction to DGLAP and BFKL equations
- (Relatively Short) Derivation of DGLAP/BFKL Mixing
  - a little bit technical
- Application in QCD Phenomenology
  - with many colorful figures

Physics in a Nutshell

In physics, our investigations often focus on two aspects:

kinematics	**dynamics**
↓	↓
phase space	evolution equation
states	Hamiltonian/Lagrangian
operators	spectrum and OPE coefficients
● ● ●	● ● ●

If not exactly solvable, we are likely to try perturbation theory for many cases …

[non-perturbative methods include Monte Carlo, bootstrap, duality, variational and mean field theory methods…]

Perturbation Theory in QM

Physicists are extremely good at doing perturbation theories.

Hamiltonian: $H = H_0 + \underset{\text{perturbation}}{\underline{\lambda V}}$ ⟶ Goal: solving equation $H|\Psi_n\rangle = E_n|\Psi_n\rangle$

Hilbert space: $H_0|n\rangle = E_n^{(0)}|n\rangle$ [assume no degeneracy]

Perturbative expansion:

$$E_n(\lambda) = E_n^{(0)} + \lambda\,\langle n|V|n\rangle + \lambda^2 \sum_{k\neq n} \frac{|\,\langle k|V|n\rangle\,|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$+\lambda^3\left(-\,\langle n|V|n\rangle \sum_{k\neq n} \frac{|\,\langle k|V|n\rangle\,|^2}{(E_n^{(0)} - E_k^{(0)})^2} + \sum_{k\neq n}\sum_{m\neq n} \frac{\langle n|V|m\rangle\ \langle m|V|k\rangle\ \langle k|V|n\rangle}{(E_n^{(0)} - E_k^{(0)})(E_n^{(0)} - E_m^{(0)})}\right) + \cdots$$

At each order in the expansion, we find pole structures when energy levels are very close.

⟶ Numerically, this approximation is not good when the energy gap is $\mathcal{O}(\lambda)$ [resummation is needed]

Two-level system example

If the first excited state is close to the ground state, while all other states are far-separated,

$\longrightarrow$ the leading approximation for lowest two states is a two-level system

Example: $H = \frac{B}{2}\sigma_z + \lambda(3\sigma_x + \sigma_z)$

$|B|$ is the energy gap for "free" Hamiltonian $H_0 = \frac{B}{2}\sigma_z$

Perturbative expansion for the ground state energy

$$E_g = -\frac{B}{2} - \lambda - \frac{9\lambda^2}{B} + \frac{18\lambda^3}{B^2} + \frac{45\lambda^4}{B^3} - \frac{414\lambda^5}{B^4} + \cdots \qquad B > 0$$

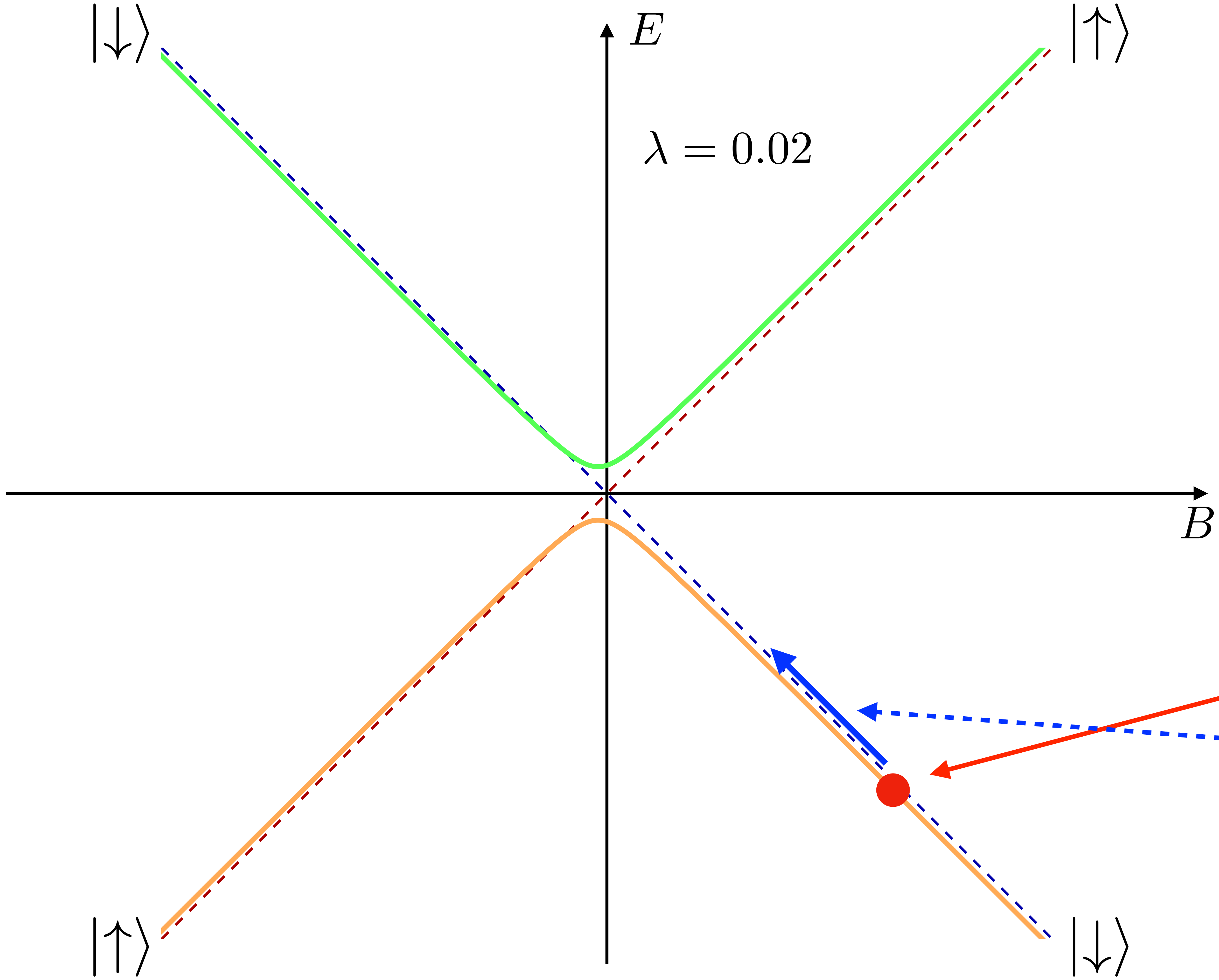
Not easy to resum if one does not recognize the pattern of coefficients

Hellmann-Feynman theorem $\frac{dE_g}{d\lambda} = \langle\psi_g|\frac{dH}{d\lambda}|\psi_g\rangle = \frac{a_1\lambda + a_2}{\sqrt{\lambda^2 + b_1\lambda + b_2}} \xrightarrow{\textbf{solution}} E_g = -\frac{1}{2}\sqrt{B^2 + 4B\lambda + 40\lambda^2}$

But everyone knows there is a straightforward way! [direct diagonalization]

$$\det(H - EI) = E^2 - (B^2/4 + B\lambda + 10\lambda^2) \longrightarrow E = \pm\frac{1}{2}\sqrt{B^2 + 4B\lambda + 40\lambda^2}$$

Avoided Level Crossing



Varying the <u>external field B</u>, we find avoided level crossing near $B \sim 0$.

The “free” Hamiltonian has degeneracy at $B = 0$, but is lifted by small perturbation.

Comparison btw two methods:

1. Perturbation + resummation

   [may not know the existence of the second level]

   Apply perturbation within the valid regime

   Resum the series near the intersection

2. The existence of the second level is known, the direct diagonalization is much simpler.

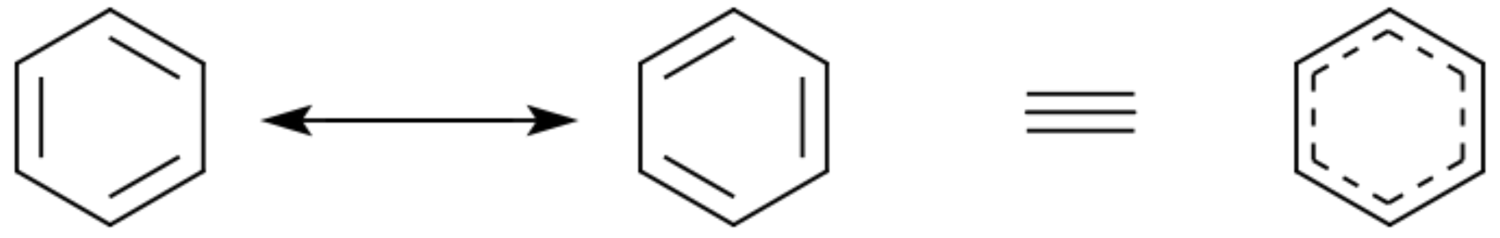
Avoided Level Crossing

Avoided level crossing is a general phenomenon in physics.

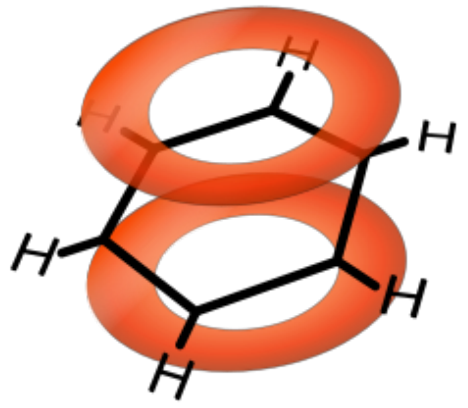
The immediate impact of avoided level crossing in a degenerate two state system is the emergence of a lowered energy eigenstate.

https://en.wikipedia.org/wiki/Avoided_crossing

Quantum Resonance: e.g. benzene



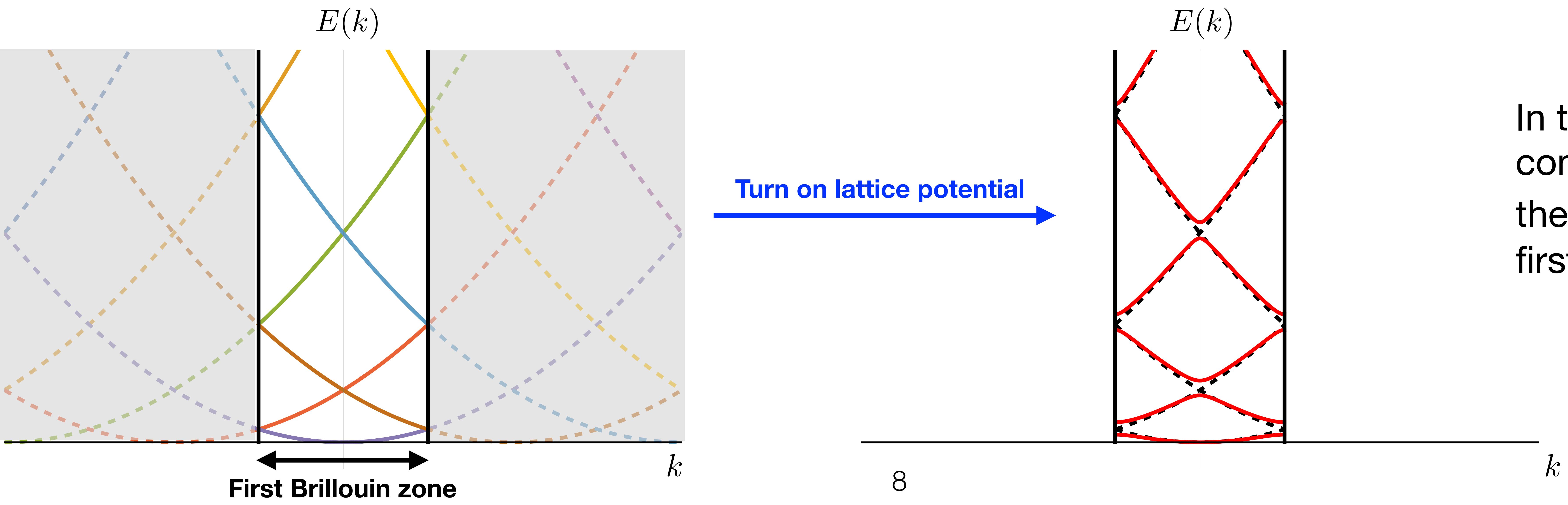
have the same e.v. $\langle H \rangle$



delocalized pi system

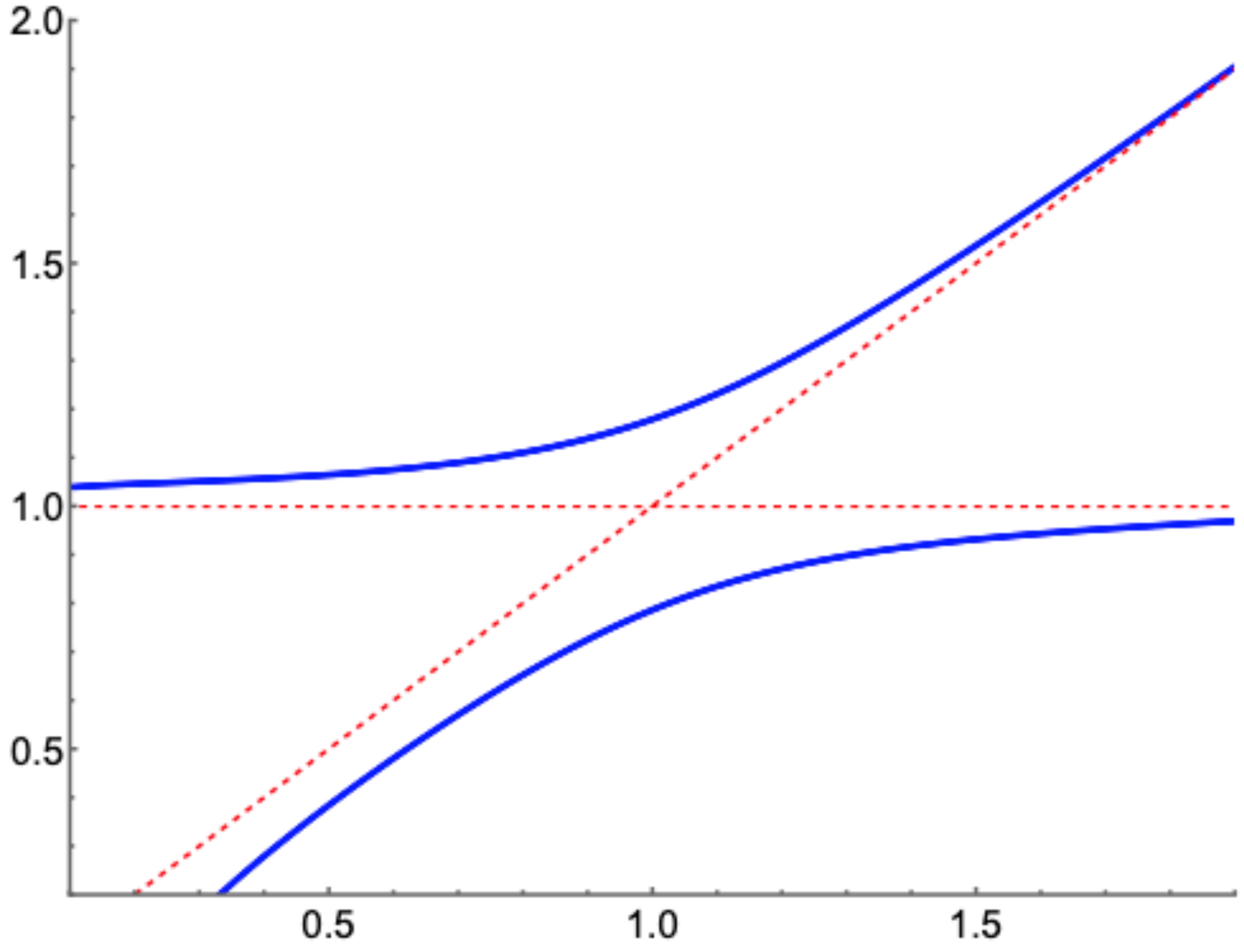
In solid physics, the avoided crossing can have an influence on the band structures.

Nearly free electron model: single electron spectrum



In this example, the continuous parameter is the momentum k in the first Brillouin zone.

A similar scenario occurs in QCD



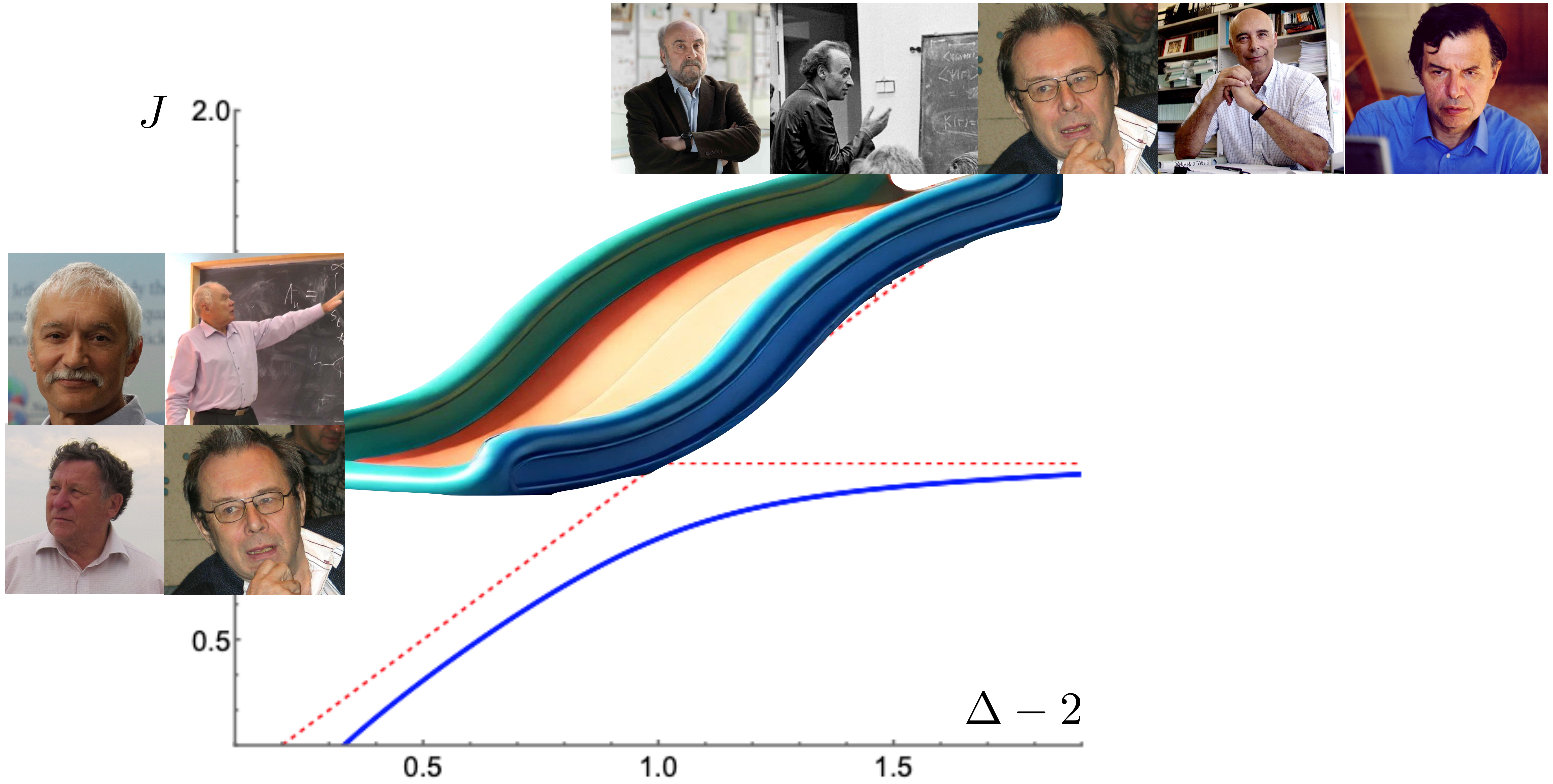
Chew-Frautschi Plot

What object are we going to discuss?

What is meaning of the spectrum?

What is the continuous parameter?

For those who may know relevant concepts



Chew-Frautschi Plot

Renormalization and Scale Dependence

Renormalization is one of the most important concepts in QFT.

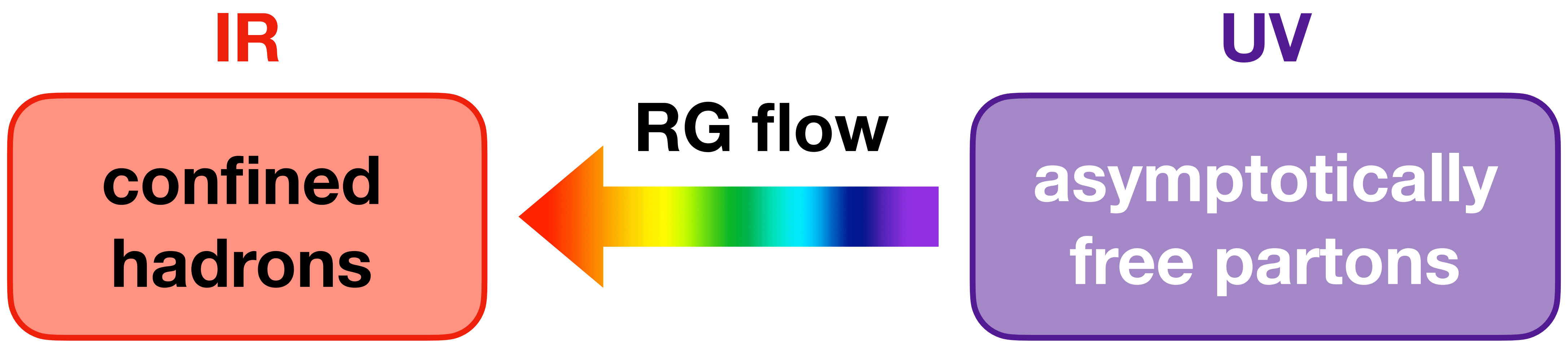
It is closely related to UV or IR behavior (divergences).

Wilsonian RG picture

- Flows from UV to IR by integrating out high-energy d.o.f. .

Physics at different scales might look completely different.

Example: QCD



IR

confined hadrons

RG flow

UV

asymptotically free partons

In high-energy scattering, parton distribution function of is an important example that connects these two concepts.

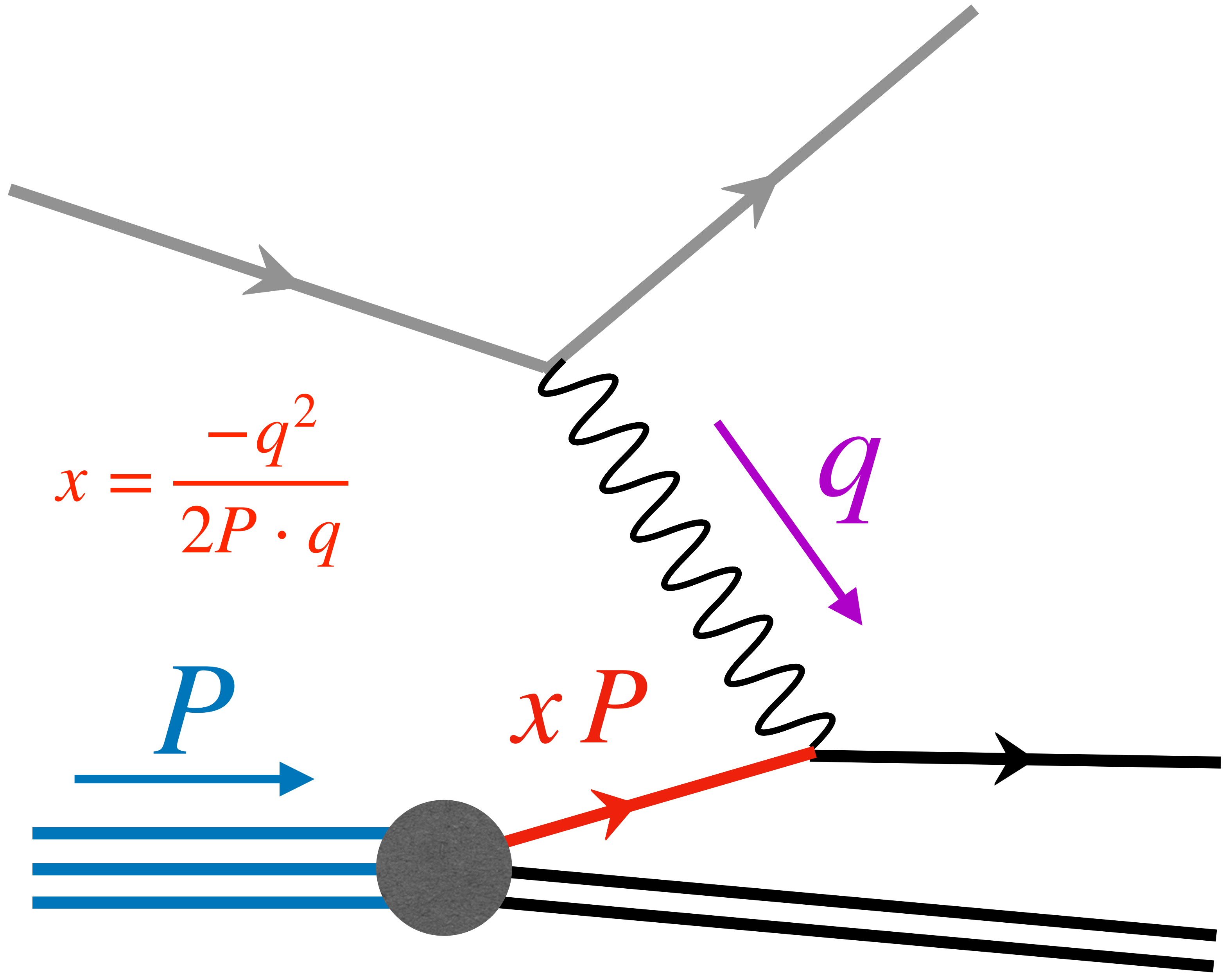
Review of Deep Inelastic Scattering

In DIS, we have the notion of Parton Distribution Functions $f_a(x;\mu)$

[philosophy of factorization]

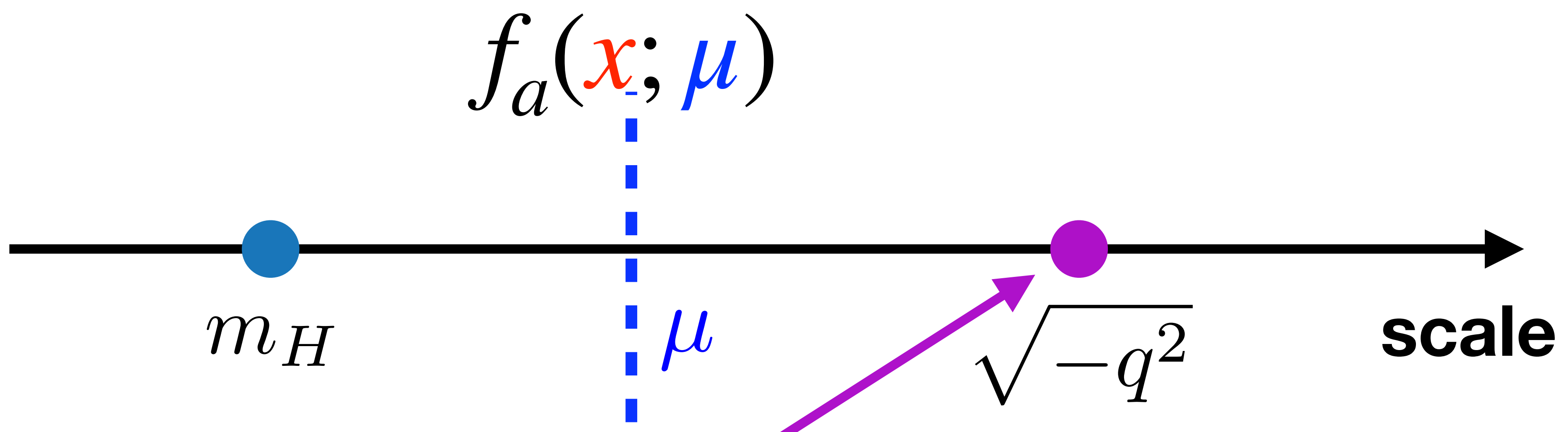
momentum fraction

PDFs are scale dependent
[factorization scale]



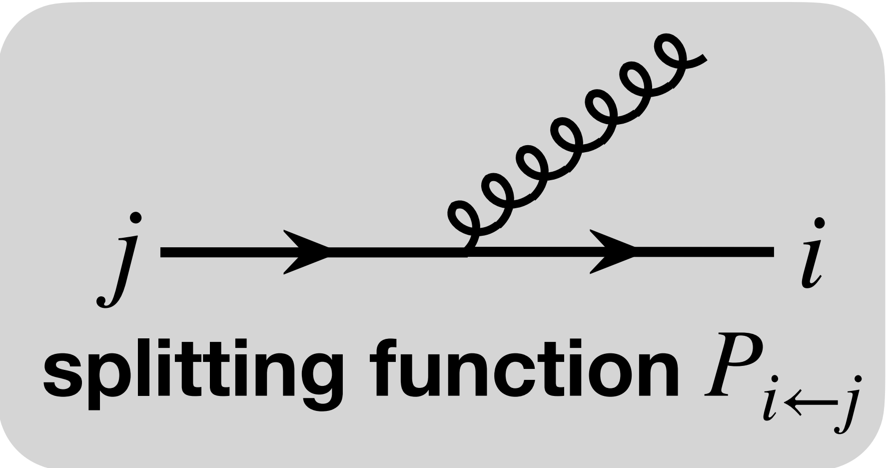
Matrix element definition (hadronic part):
$\langle P \,|\, J_\mu(q) J_\nu(-q) \,|\, P \rangle$

The scale μ can be regarded as off-shellness/virtuality or k_T .



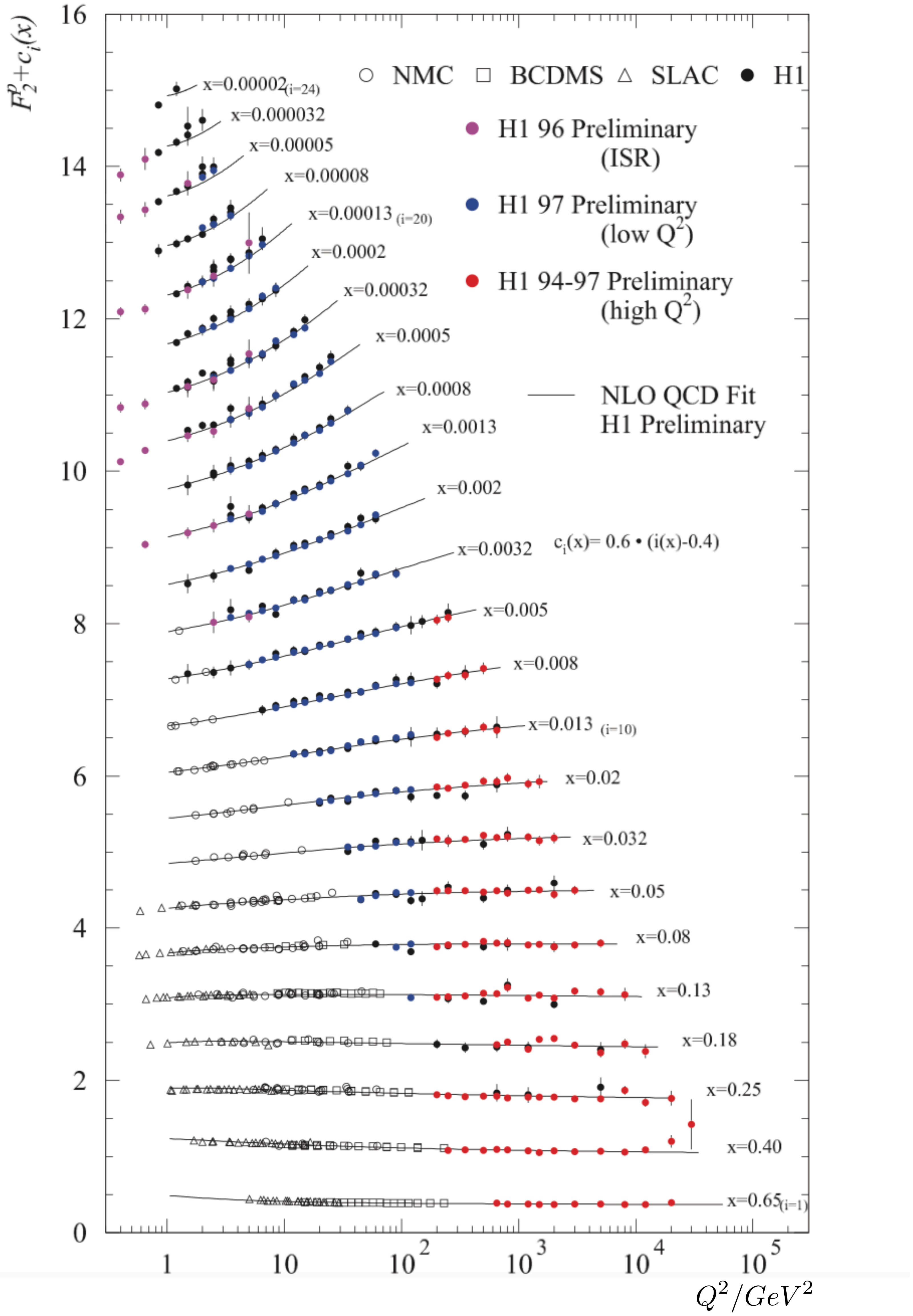
The natural high-energy scale in DIS is determined by the momentum transfer $\sim \sqrt{-q^2}$

Radiation from partons can increase virtuality.



splitting function $P_{i\leftarrow j}$

Parton Distribution Functions



Bjorken Scaling and Violation

Parton Distribution Functions $f_a(x;\mu)$

- PDFs are non-perturbative functions

  [from experiment data or lattice calculation]

- Their evolution is perturbatively calculable, called DGLAP equation Dokshitzer-Gribov-Lipatov-Altarelli-Parisi

$$\mu\frac{d}{d\mu}f_a(x;\mu) = \int_x^1 \frac{dz}{z} P_{ab}(z,\alpha_s(\mu))f_b(x/z;\mu)$$

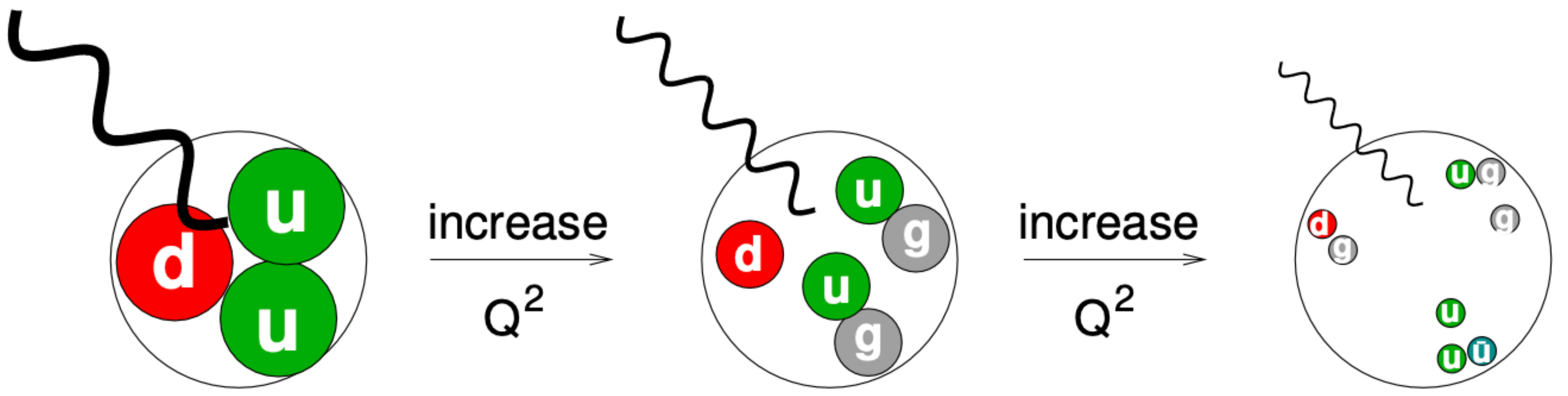


illustration from Gavin Salam's QCD lectures

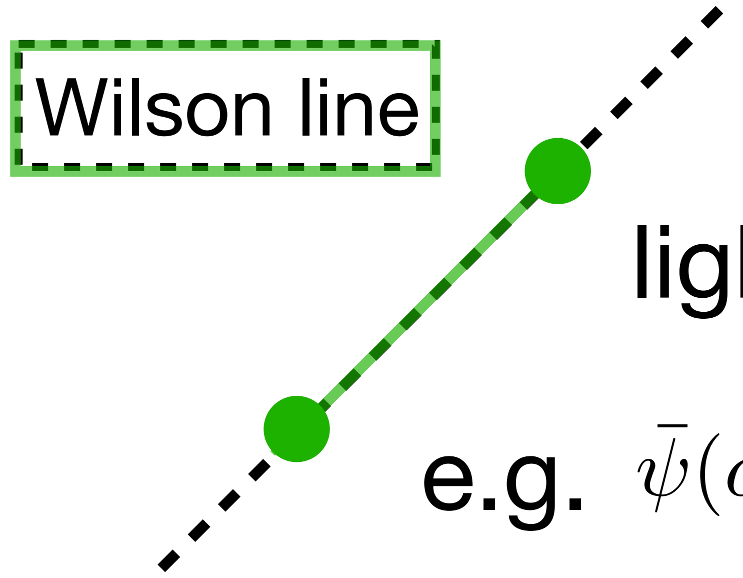
From OPE perspective, they are the evolution of local twist-2 operators

$$\langle P|J_\mu(q)J_\mu(-q)|P\rangle \sim \frac{q_{\mu_1}\cdots q_{\mu_J}}{(-q^2)^{(\Delta+J-2)/2}}\boxed{\langle P|O^{\mu_1\cdots\mu_J}|P\rangle}$$

Moments of PDF

Dictionary for PDF

parton distribution function $f_a(x;\mu)$ $\xrightarrow[\int_0^1 dx\, x^{J-1}]{\textbf{integer moment}}$ matrix element of twist-2 spin-J operator $\langle P|\mathcal{O}_a^J(0;n)|P\rangle_\mu$



light-ray operators

e.g. $\bar\psi(\alpha_1 n)\not{n} W[\alpha_1,\alpha_2]\psi(\alpha_2 n)$

e.g. [Balitsky, Braun, 1988]

analytic continuation ←

Local (integer spin) operators

e.g. $\bar\psi\not{n}(n\cdot D)^{J-1}\psi$

PDFs are FT of matrix elements of light-ray operators

$$\int d\alpha\, e^{ix\alpha n\cdot P}\langle P|\bar\psi(\alpha n)\not{n} W[\alpha,0]\psi(0)|P\rangle$$

Later we will discuss another form of light-ray operators.

Evolution kernels:

(space-like) splitting functions $\xrightarrow{\textbf{Mellin moment}}$ (space-like) anomalous dimensions

$$P_{ab}(z;\alpha_s)$$

$$\gamma_{ab}(J;\alpha_s)$$

[Gross, Wilczek, 1973]

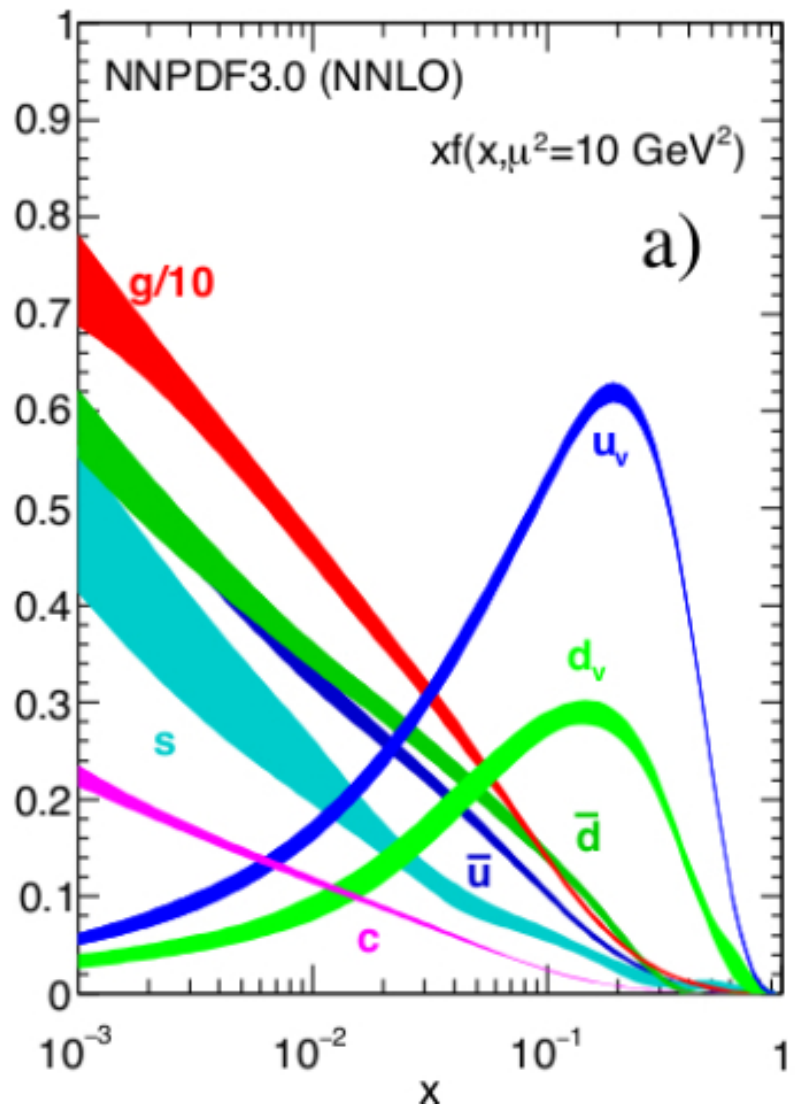
$$P_{gg}^{\mathrm{LO}}(z)\sim 6\left[\frac{1-z}{z}+\frac{z}{(1-z)_+}+z(1-z)+\left(\frac{11}{12}-\frac{n_f}{18}\right)\delta(1-z)\right]$$

state-of-the-art calculations:
@ 3 loop [Moch,Vermaseren,Vogt, 2004]
partially available @ 4 loop

$$\gamma_{gg}^{\mathrm{LO}}(J)\sim -6\left[4H_J-\frac{4}{J(J-1)}-\frac{4}{(J+1)(J+2)}-\frac{11}{3}+\frac{2}{9}n_f\right]$$

Different Physics at Small Momentum Fraction

small-x physics

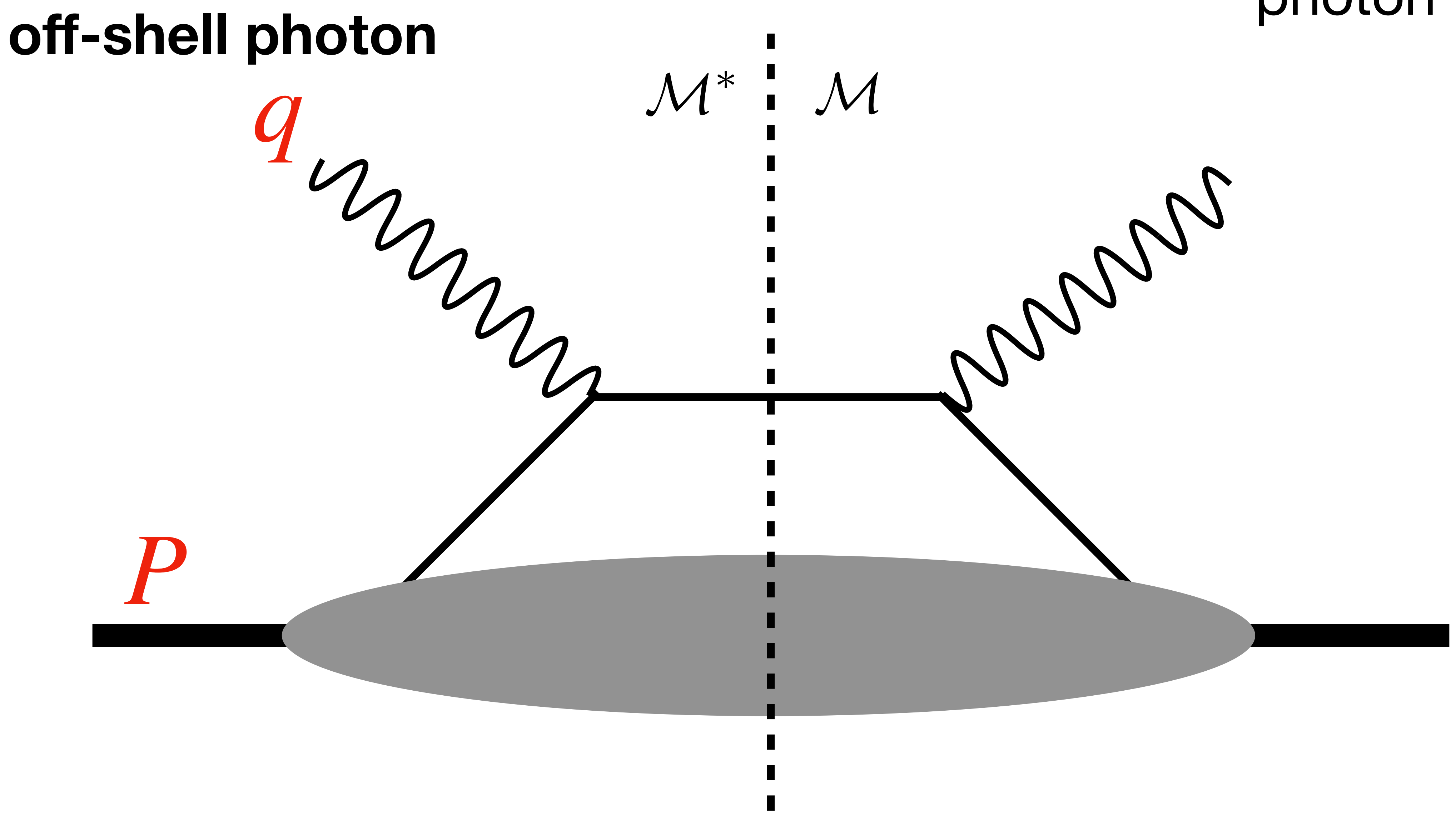


- Gluon and sea quarks dominate in PDF
- The DGLAP evolution kernels have divergences

e.g. @ leading order $P_{gg}(z) \sim \frac{\alpha_s}{z}$ or $\gamma_{gg}(J) \sim \frac{\alpha_s}{J-1}$

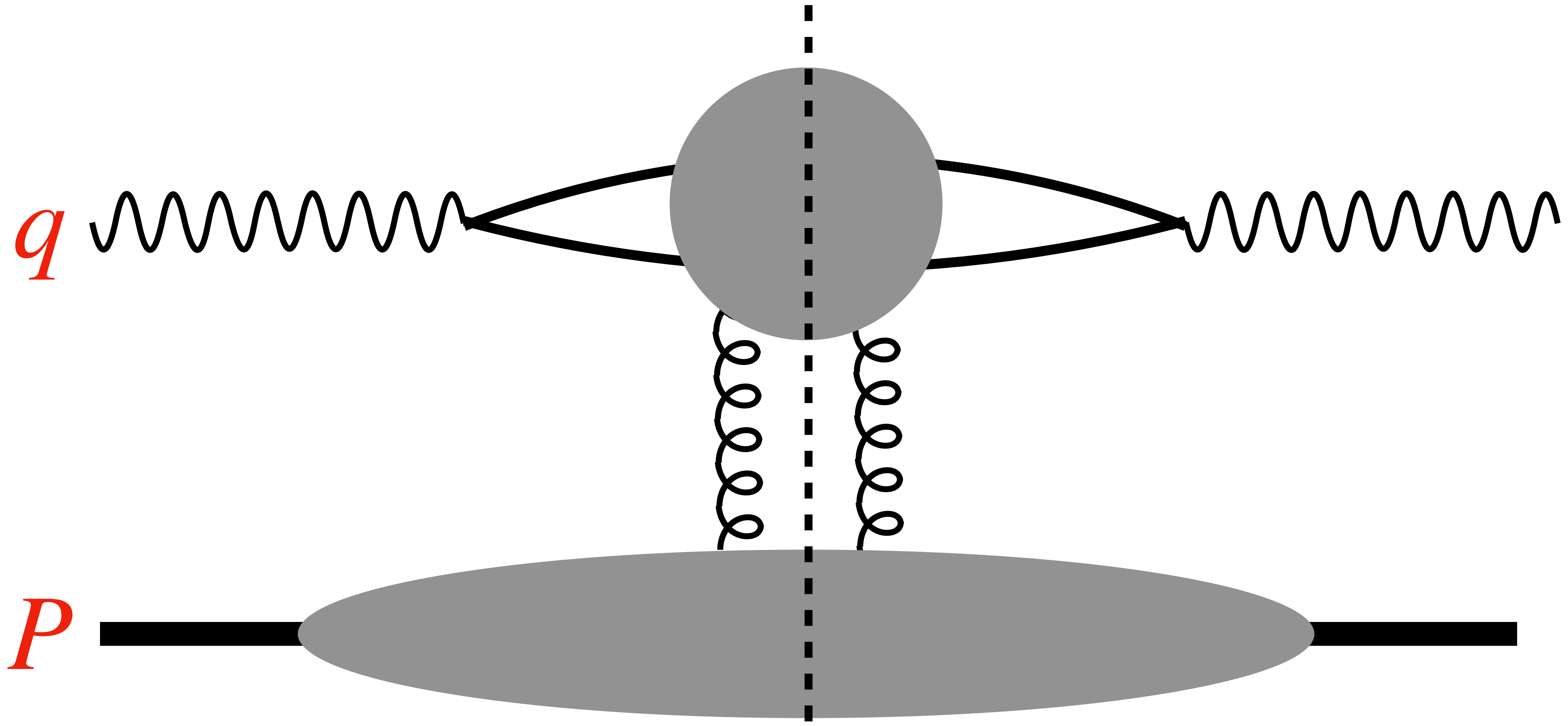
- Different kinematics/dynamics at small-x $x = \frac{-q^2}{2P \cdot q}$

“generic”-x



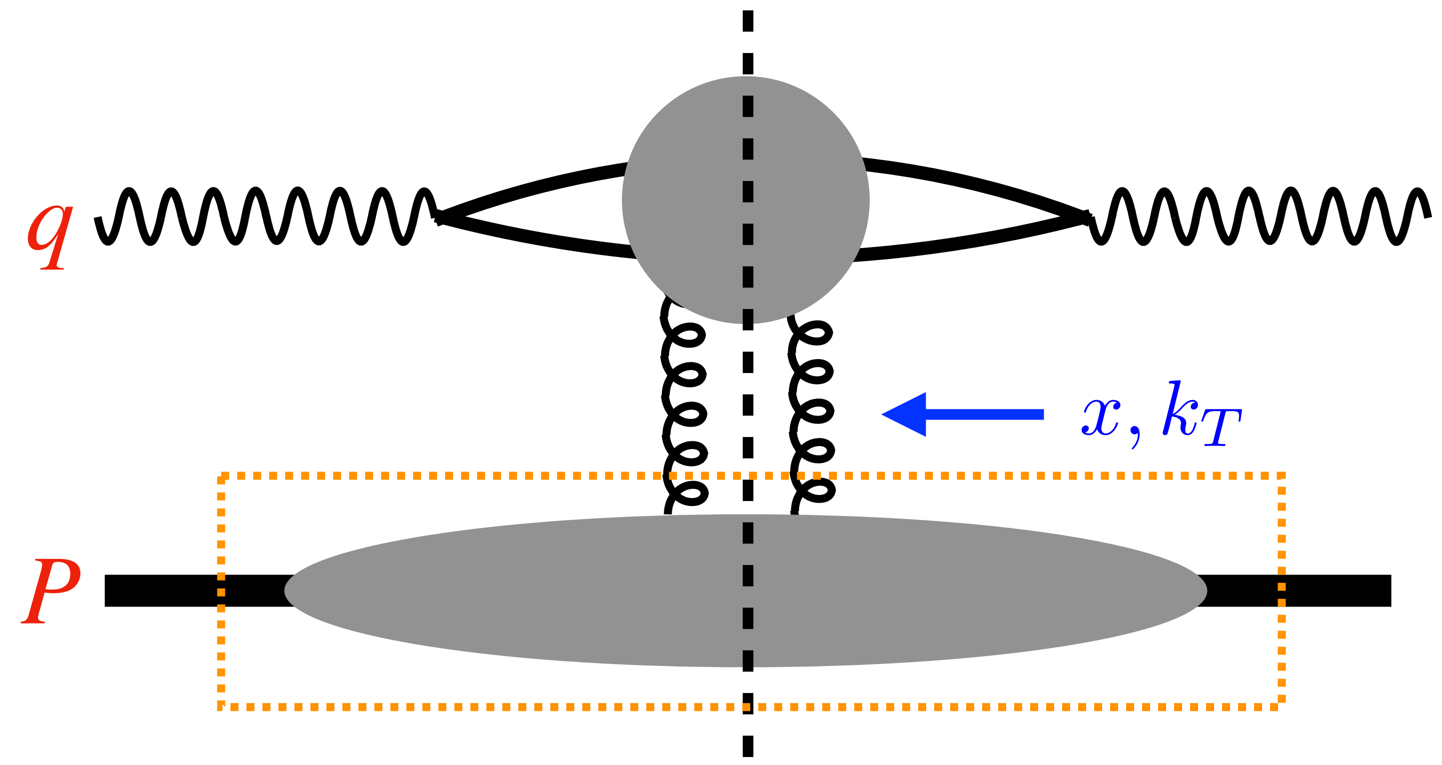
photon becomes nearly on-shell

small-x



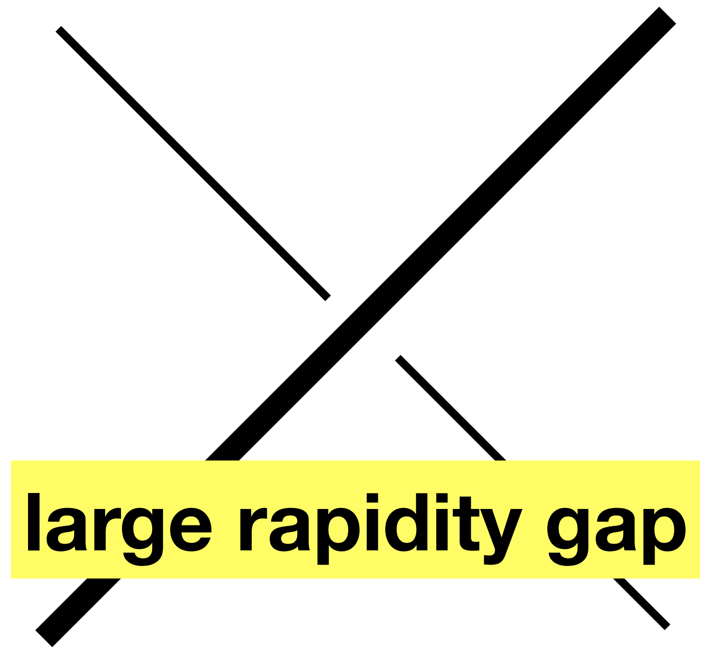
related to forward scattering configuration

Small-x Resummation and BFKL equation

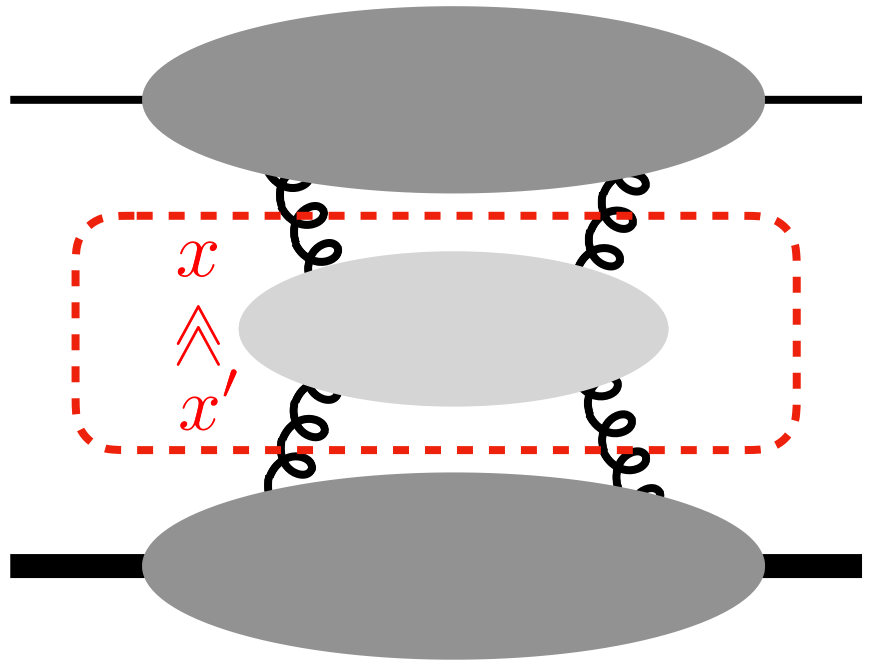


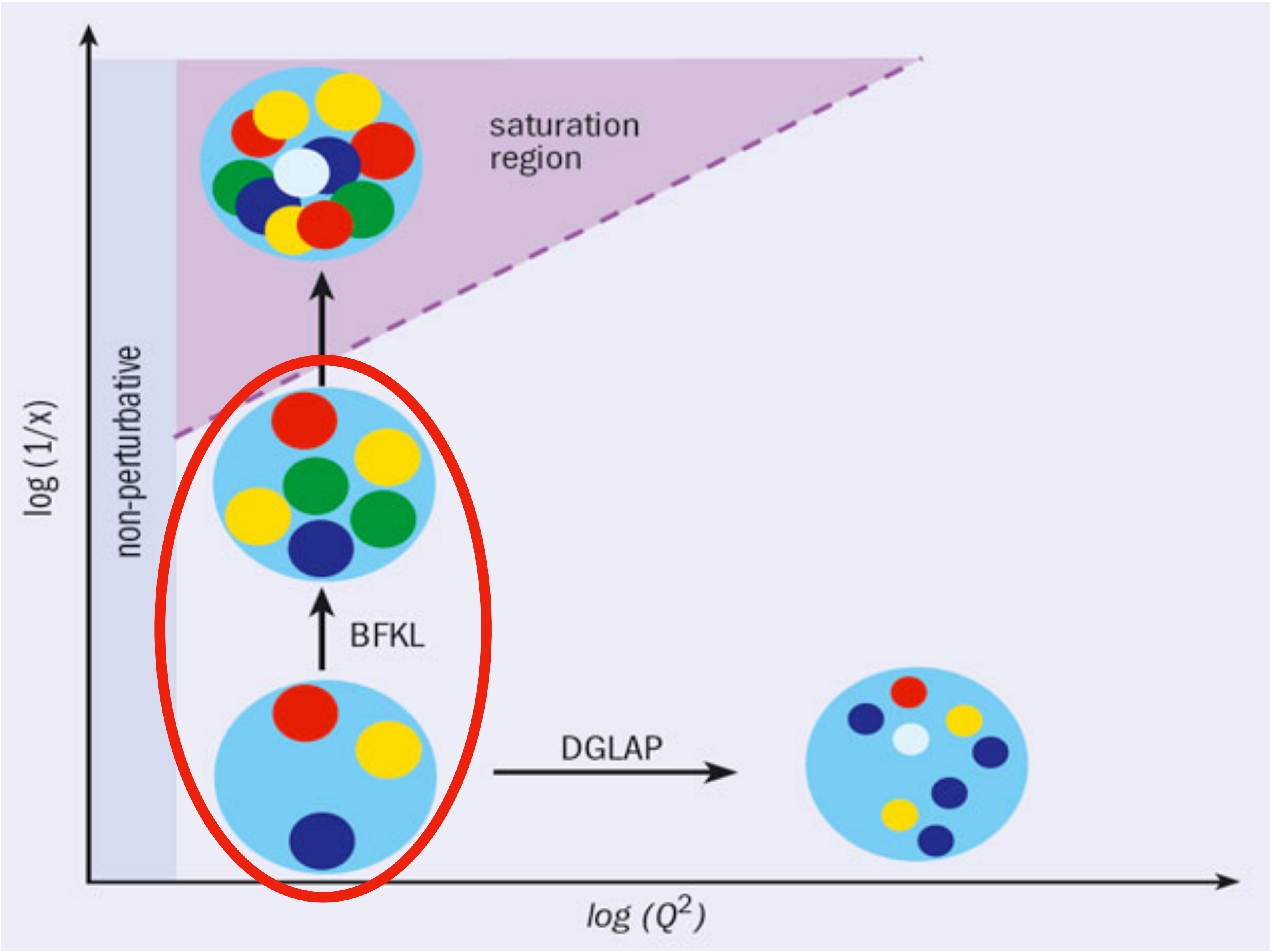
Generalized PDF $\mathcal{F}(x, k_T)$

transverse momentum



Regge limit/forward scattering limit





BFKL equation (LO)

$$x\frac{\partial}{\partial x}\mathcal{F}(x,k_T) = -\frac{\alpha_s N_c}{2\pi^2}\int d^2k'_T\,\frac{k_T^2}{k_T'^2(k_T-k'_T)^2}\left[2\mathcal{F}(x,k'_T)-\mathcal{F}(x,k_T)\right]$$

eigenfunction $\mathcal{F}_\delta(x,k_T) = x^{-\omega(\delta)}\left(\frac{k_T^2}{\mu^2}\right)^{\delta-1/2}$

BFKL eigenvalue $\omega(\delta) = -\frac{\alpha_s N_c}{\pi}(\psi(\delta+1/2)+\psi(1/2-\delta)-2\psi(1))$

- NLO kernel has been calculated in [Fadin, Lipatov, 1998]
- At very small value of x, non-linear evolution equation — BK/JIMWLK.

[Balitsky, 1995; Kovchegov, 1999; JKMW, 1996; JKLW 1997; ILM, 2000…]

DGLAP vs. BFKL

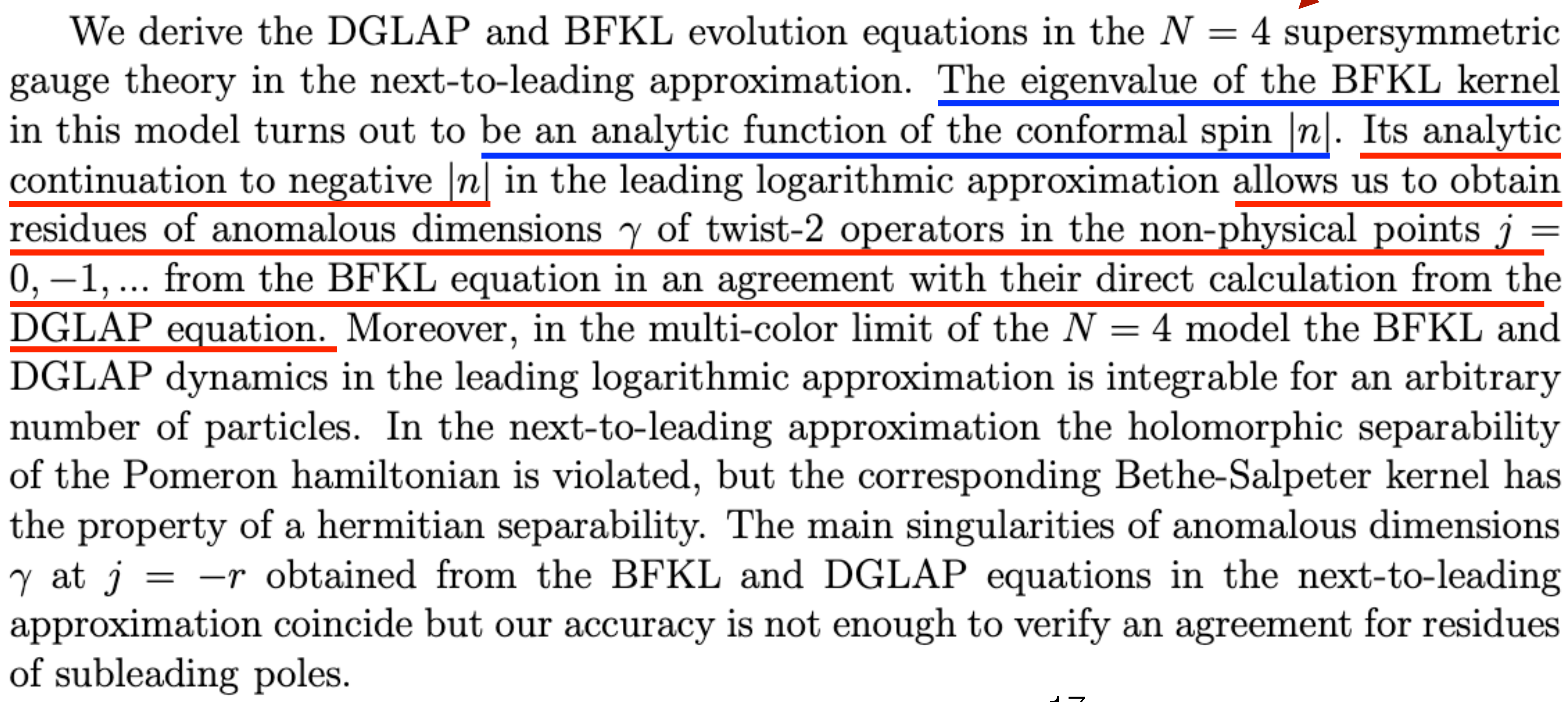
From *50 Years of Quantum Chromodynamics*

The meaning of evolution in the two cases is essentially different.

DGLAP: action $d/d\ln k_T^2$, dynamics in x; … Eigenvalues are anomalous dimensions; …

BFKL: action $d/d\ln(1/x)$, dynamics in $\vec{k}_T$; … the spectrum of BFKL is Regge trajectories; …

In spite of all the difference the two are intimately related [Kotikov, Lipatov, 2002].



We derive the DGLAP and BFKL evolution equations in the $N = 4$ supersymmetric gauge theory in the next-to-leading approximation. The eigenvalue of the BFKL kernel in this model turns out to be an analytic function of the conformal spin $|n|$. Its analytic continuation to negative $|n|$ in the leading logarithmic approximation allows us to obtain residues of anomalous dimensions γ of twist-2 operators in the non-physical points $j = 0, -1, ...$ from the BFKL equation in an agreement with their direct calculation from the DGLAP equation. Moreover, in the multi-color limit of the $N = 4$ model the BFKL and DGLAP dynamics in the leading logarithmic approximation is integrable for an arbitrary number of particles. In the next-to-leading approximation the holomorphic separability of the Pomeron hamiltonian is violated, but the corresponding Bethe-Salpeter kernel has the property of a hermitian separability. The main singularities of anomalous dimensions γ at $j = -r$ obtained from the BFKL and DGLAP equations in the next-to-leading approximation coincide but our accuracy is not enough to verify an agreement for residues of subleading poles.

pole structures are related after analytic continuation

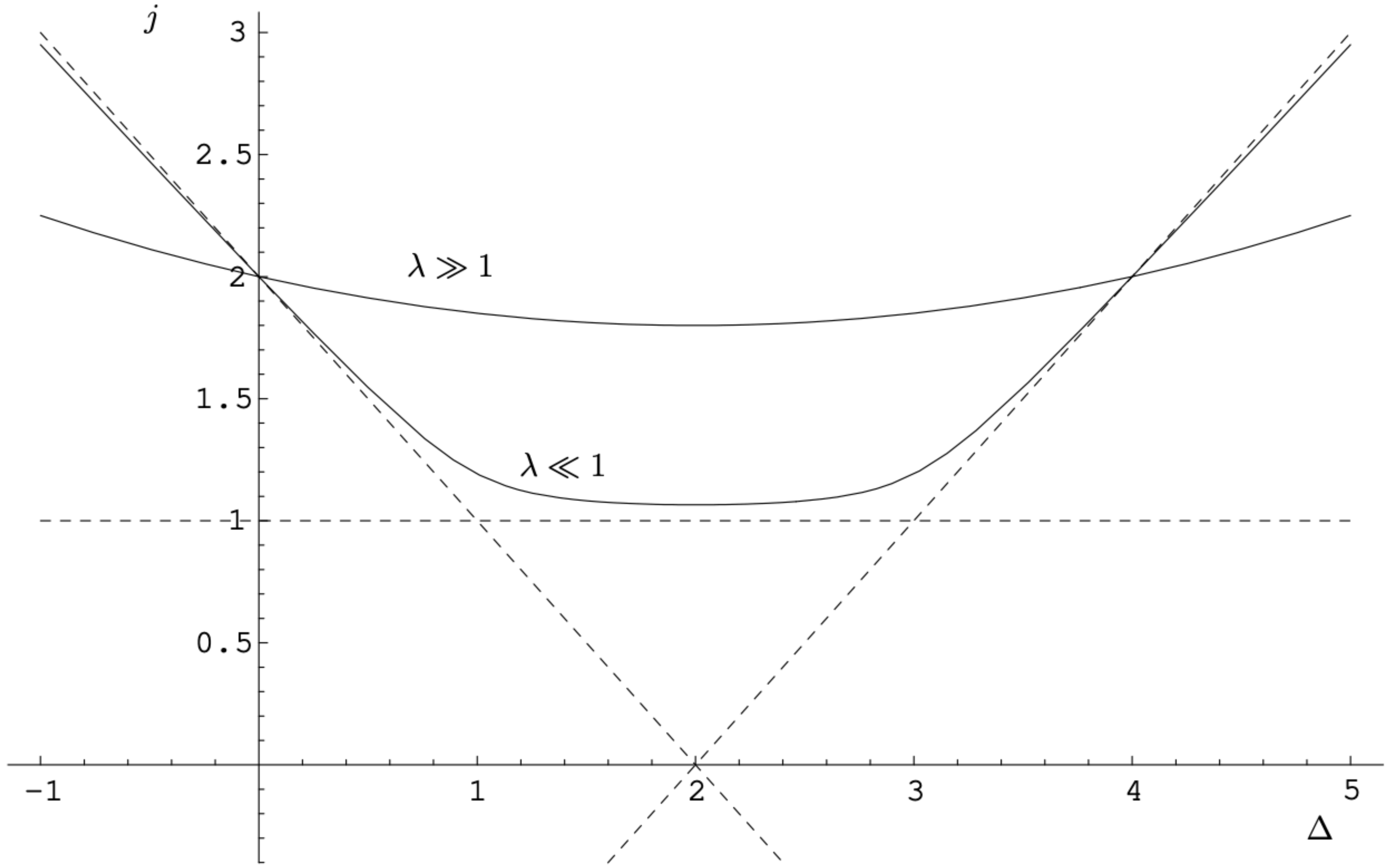
e.g. LO BFKL $\longrightarrow \gamma_{gg}(J) \sim \dfrac{\alpha_s}{J-1}$

see also [Jaroszewicz,1982; Lipatov,1996; Kotikov, Lipatov, 2000;…]

More Analytic Structures

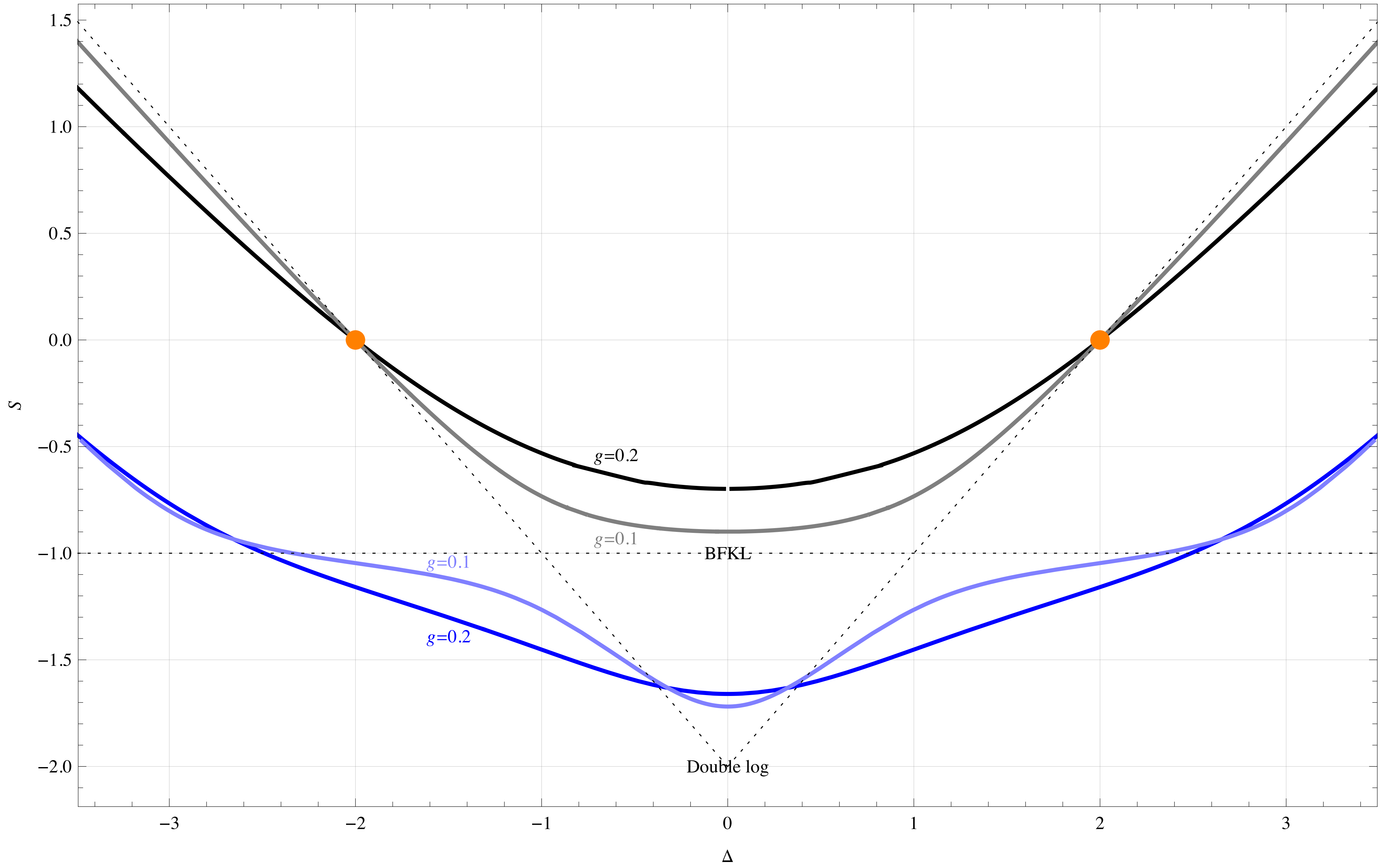
Gauge/String Duality

[Brower, Polchinski, Strassler, Tan, 2006]



Integrability of planar $\mathcal{N} = 4$ SYM

[Gromov, Levkovich-Maslyuk, Sizov, 2015]



- No divergence in dimensions

$\longrightarrow$ $(J-1)$ pole in DGLAP anom. dim. is not "physical"

- DGLAP and BFKL are recombined near intersection

The spectrum of local operators is discrete, what is the meaning of the analytic curves?

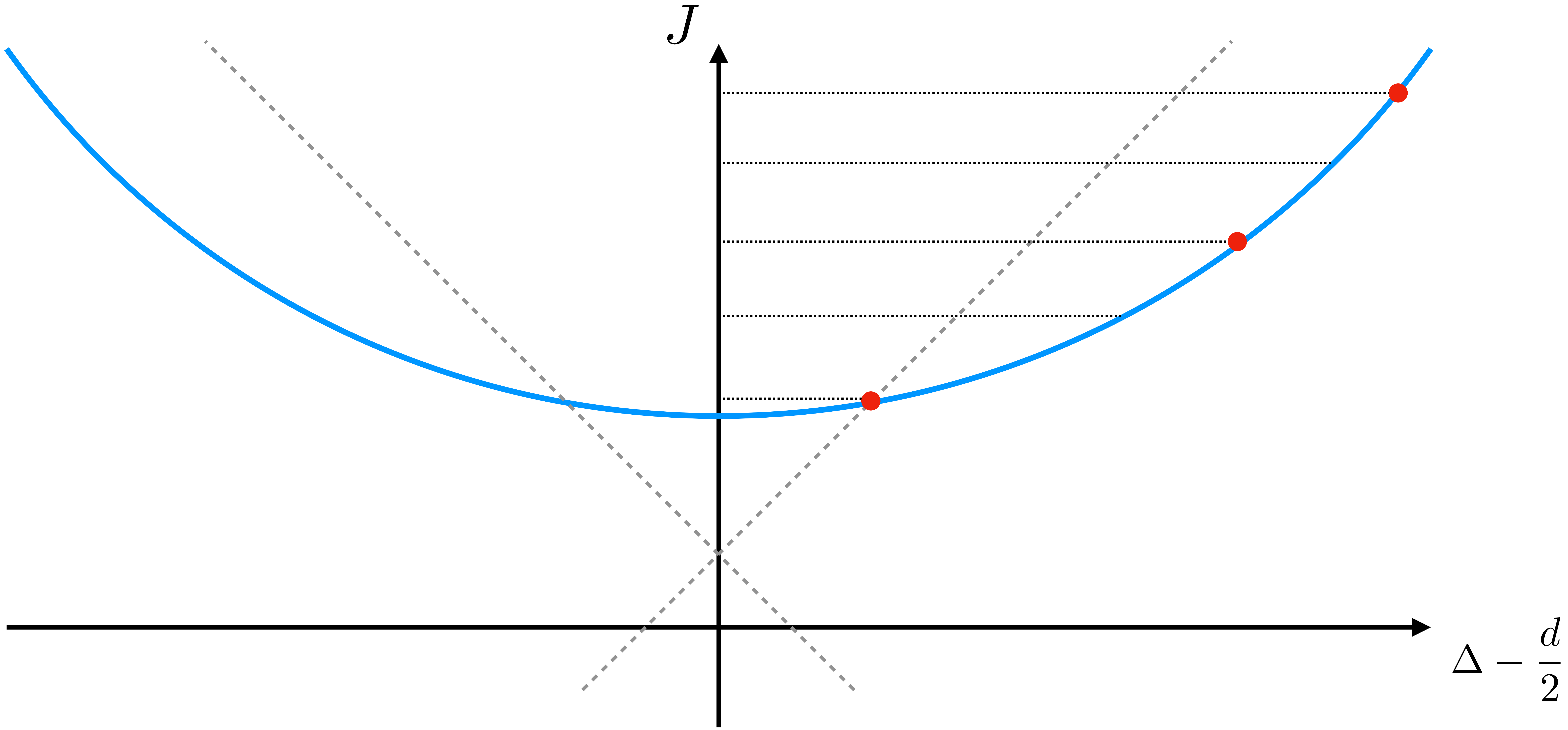
Any operator interpretation or just analytic continuation of functions?

Analyticity in Spin

[S-matrix: Gribov-Froissart formula;
CFT: Caron-Huot, 2017]

Light-ray operators are expected to be the **analytic continuation** of local operators.

[Kravchuk, Simmons-Duffin, 2018]



Light-ray Operators

- The most important example of light-ray operator is energy flow operator/calorimeter/ANEC operator.

- The energy flow operator is a non-local operator defined on a light-ray located at future null infinity

$$\mathcal{E}(\vec{n}) = \lim_{r\to\infty} r^2 \int_0^\infty dt\ \vec{n}_i T^{0i}(t, r\vec{n})$$

[Sveshnikov, Tkachov, 1996; Hofman, Maldacena, 2008;…]

- Generalization to other local operators

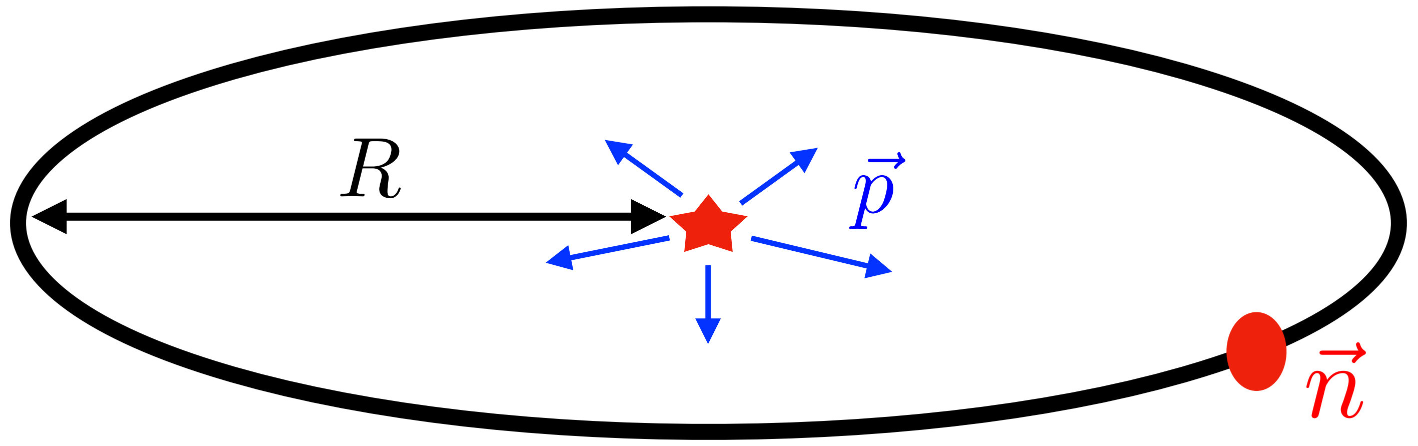
direction

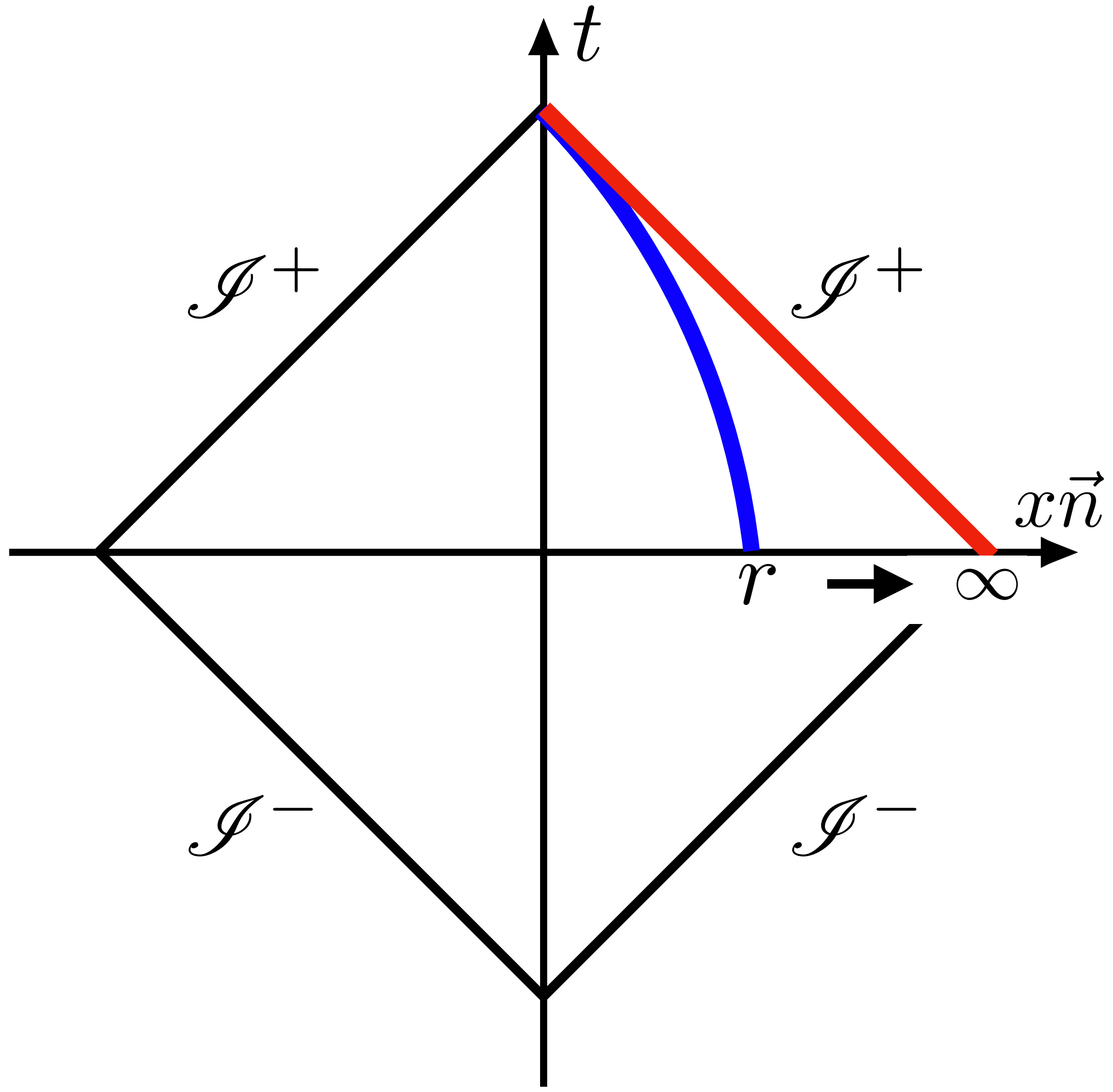
$$\mathbf{L}[\mathcal{O}](x,n) = \int_{-\infty}^{\infty} d\alpha (-\alpha)^{-\Delta-J} \mathcal{O}\left(x - \frac{n}{\alpha}, n\right)$$

starting point

[Kravchuk, Simmons-Duffin, 2018]

Examples of more general light-ray operators, see [Chang, Kologlu, Kravchuk, Simmons-Duffin, Zhiboedov, 2020; Caron-Huot, Kologlu, Kravchuk, Meltzer, Simmons-Duffin, 2022;…]





Interesting Objects for studying Lorentzian Dynamics

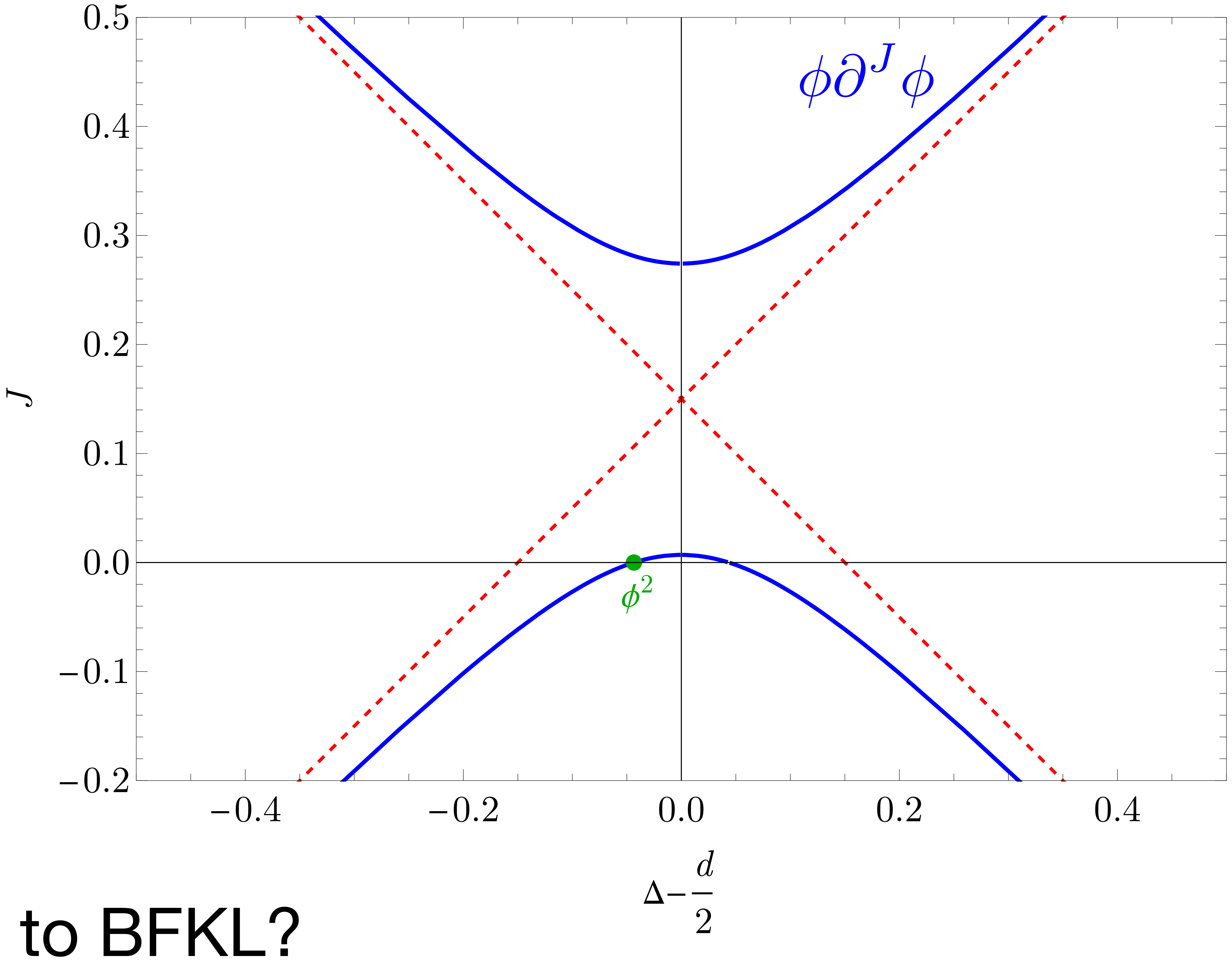
Recombination of Light-ray Operators in Scalar Theory

[Caron-Huot, Kologlu, Kravchuk, Meltzer , Simmons-Duffin, 2022]

Provides an operator interpretation for level repulsion in a scalar theory.

In Wilson-Fisher theory, resolving mixing between leading twist trajectory with its celestial shadow predicts the level crossing near Regge intercept.

$\phi\partial^J\phi$ is located at upper branch when $J > 0$, while ϕ^2 is on the lower branch.



- What about gauge theories?
- How to construct the operator related to BFKL?

DGLAP/BFKL Mixing

in the detector language

DGLAP Operators in QCD

For unpolarized cases, there are only two kinds of twist-2 operators in QCD

Local Operators

$$\mathcal{O}_q^{[J]} = \frac{1}{2^J}\bar{\psi}\gamma^+(iD^+)^{J-1}\psi$$

$$\mathcal{O}_g^{[J]} = -\frac{1}{2^J}F_a^{\mu+}(iD^+)^{J-2}F_a^{\mu+}$$

$$\xrightarrow[\text{Light Transform}]{\lim_{r\to\infty} r^2 \int_0^\infty dt}$$

Light-ray Operators

$$\mathbb{O}_q^{[J]}(\vec{n}) = \mathcal{D}^q_{J_L}(n)$$

$$\mathbb{O}_g^{[J]}(\vec{n}) = \mathcal{D}^g_{J_L}(n)$$

J_L is the spin of light-ray operator, which is more useful than label J in the mixing problem. For bare DGLAP detectors, $J_L = -1 - J$.

The analytic continuation of even spin branch is

Physics Interpretation
[in free theory]
Measuring E^{J-1}

$$\mathbb{O}_q^{[J]}(\vec{n}) = \sum_s \int \frac{d^3p}{(2\pi)^3 2E}\delta^{(2)}(\hat{p}-\vec{n})\, E^{J-1} (b^\dagger_{\vec{p},s}b_{\vec{p},s} + d^\dagger_{\vec{p},s}d_{\vec{p},s})$$

$$\mathbb{O}_g^{[J]}(\vec{n}) = \sum_{\lambda,c} \int \frac{d^3p}{(2\pi)^3 2E}\delta^{(2)}(\hat{p}-\vec{n})\, E^{J-1}\, a^\dagger_{\vec{p},\lambda,c}a_{\vec{p},\lambda,c}$$

Not IR-safe measurement

[HC, Moult, Zhu, 2021]

Renormalization of Light-ray Operators

In perturbation theory, the light-ray operators have divergences.

⟶ require renormalization [Caron-Huot, Kologlu, Kravchuk, Meltzer, Simmons-Duffin, 2022]

bare twist-2 local operators

$$\mathbb{O}^{[J]}_{a;\mathrm{bare}}(n) = \lim_{\bar{n}\cdot x\to\infty} \left(\frac{\bar{n}\cdot x}{2}\right)^2 \int_{-\infty}^{\infty} d(n\cdot x)\, \mathcal{O}^{[J]}_{a;\mathrm{bare}}(x;\bar{n})$$

Introducing a renormalization factor and define renormalized light-ray operators

$$\mathbb{O}^{[J]}_{a;\mathrm{bare}} = \mathbb{Z}^{[J]}_{ab}\mathbb{O}^{[J]}_{b;\mathrm{ren}}$$

[IR behaviors of detectors]

contains scale dependence

Only $\mathbb{O}^{[J]}_{q}$ and $\mathbb{O}^{[J]}_{g}$ can mix, as long as J is large enough.

Time-like anomalous dimension

different from the space-like anomalous dimension for the local operators. But they are related by **reciprocity relation.**

RG equation: $$\frac{d}{d\ln\mu^2}\mathbb{O}^{[J]}_{a;\mathrm{ren}}(n;\mu) = \underline{\gamma^{T}_{ab}(J;\alpha_s(\mu))}\mathbb{O}^{[J]}_{b;\mathrm{ren}}(n;\mu)$$

This describes the DGLAP evolution of final-state fragmentation.

spin of detectors

From now on, we will switch from J to J_L

$J = -1 - J_L$

$$\mathbb{O}_q^{[J]}(\vec{n}) = \mathcal{D}^q_{J_L}(n)$$

$$\mathbb{O}_g^{[J]}(\vec{n}) = \mathcal{D}^g_{J_L}(n)$$

We will work with dimensional regularization $d = 4 - 2\epsilon$

Also for simplicity, let us consider pure gluon theory first

Key question: how to construct BFKL detector?

One-Loop Divergences (DGLAP)

We insert the gluon detector into a matrix element

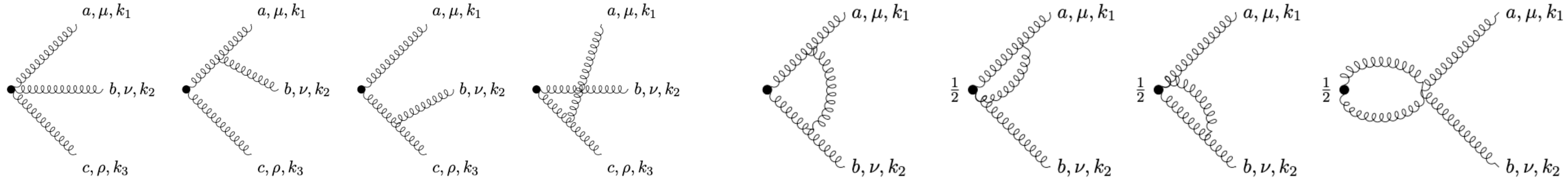
$$\mathcal{O} = \frac{1}{4N_c}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

$$\mathcal{D}^{g,\mathrm{bare}}_{J_L}(z) = \sum_{\lambda,c}\int \frac{E^{-J_L}dE}{(2\pi)^{d-1}2E}\left[a^\dagger_{\lambda,c}(p)a_{\lambda,c}(p)\right]_{p=Ez}$$

$$\langle\Omega|\mathcal{O}(-q)\mathcal{D}^{g,\mathrm{bare}}_{J_L}(z)\mathcal{O}(q)|\Omega\rangle = \boxed{(N_c^2-1)\frac{d-2}{2^{d+1}\pi^{d-2}}(2z\cdot q)^{J_L}(q^2)^{1-J_L}} + \cdots$$

tree-level

One loop calculation:



$\langle\Omega|\mathcal{O}_{\mathrm{ren}}(-q)\mathcal{D}^{g,\mathrm{bare}}_{J_L}(z)\mathcal{O}_{\mathrm{ren}}(q)|\Omega\rangle_{\text{1-loop}}$ has divergence in ϵ [soft and collinear divergences]

$\epsilon - \mathrm{pole}$

DGLAP γ_{gg}

$$\frac{g^2}{(4\pi)^2\epsilon}\boxed{\left[4N_c\left(\psi^{(0)}(-J_L)+\gamma_E-\frac{1}{J_L(J_L-1)}-\frac{1}{(J_L+2)(J_L+1)}\right)-\beta_0\right]}\langle\mathbb{O}^{\mathrm{bare}}_{J_L}(z)\rangle_{\mathrm{tree},d=4}$$

but it also has divergences at $J_L = -2 + \mathbb{N}$

Leading pole

$$\frac{1}{J_L+2}\left(\frac{g^2N_c}{(4\pi)^2}e^{\epsilon\gamma_E}\mu^{2\epsilon}\frac{N_c^2-1}{(2\pi)^{2-2\epsilon}}\frac{\Gamma(2-\epsilon)}{\epsilon\Gamma(1-2\epsilon)}(2z\cdot q)^{-2}(q^2)^{3-\epsilon}\right).$$

Soft Theorem and J_L Pole

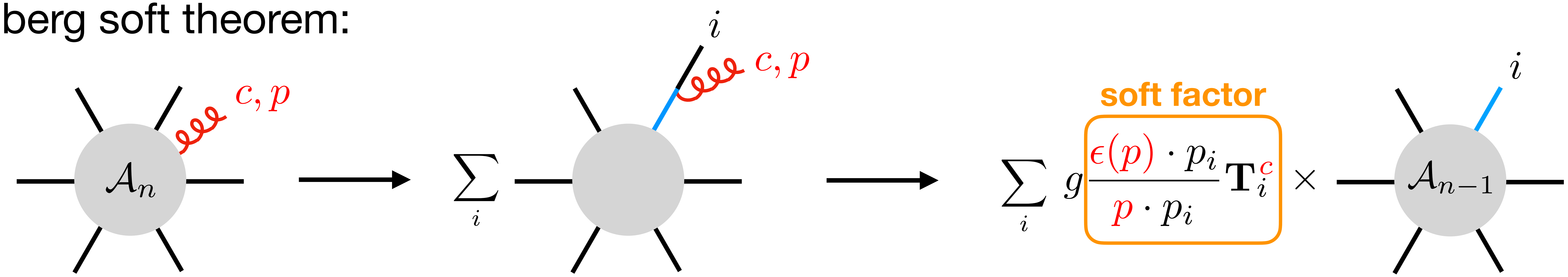
Origin of the pole J_L+2

$$\mathcal{D}^{g,\text{bare}}_{J_L}(z) = \sum_{\lambda,c}\int_0^\infty \frac{E^{-J_L}dE}{(2\pi)^{d-1}2E}\boxed{\left[a^\dagger_{\lambda,c}(p)a_{\lambda,c}(p)\right]_{p=Ez}}$$

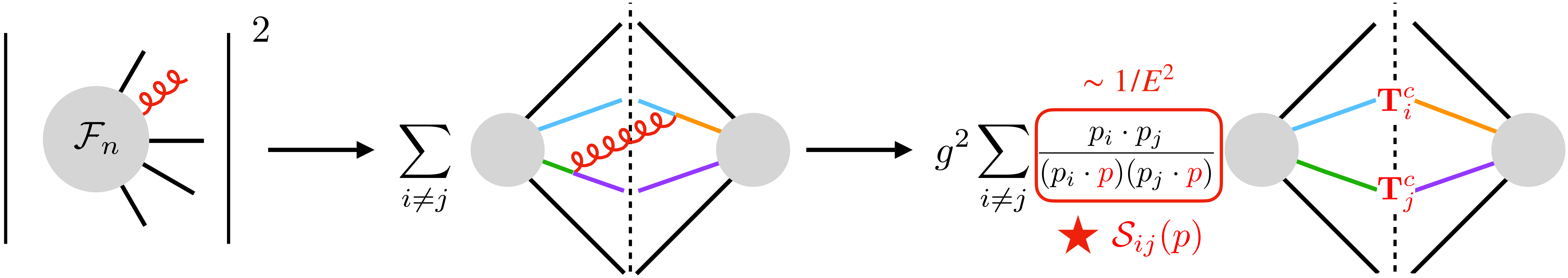
$\sim 1/E^2$ in the $E \to 0$ limit (soft limit)

Steven Weinberg

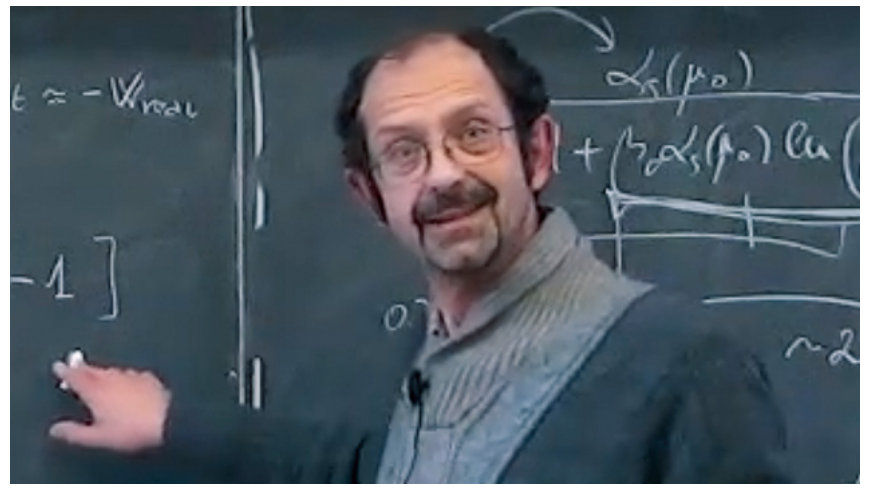
Weinberg soft theorem:



Soft theorem at cross section level (see [Catani, Grazzini, 1999], here we use **form factor** for illustration):



key to construct BFKL detector



Stefano Catani

BFKL Detector

Apply DGLAP measurement and extract its leading J_L pole from soft theorem

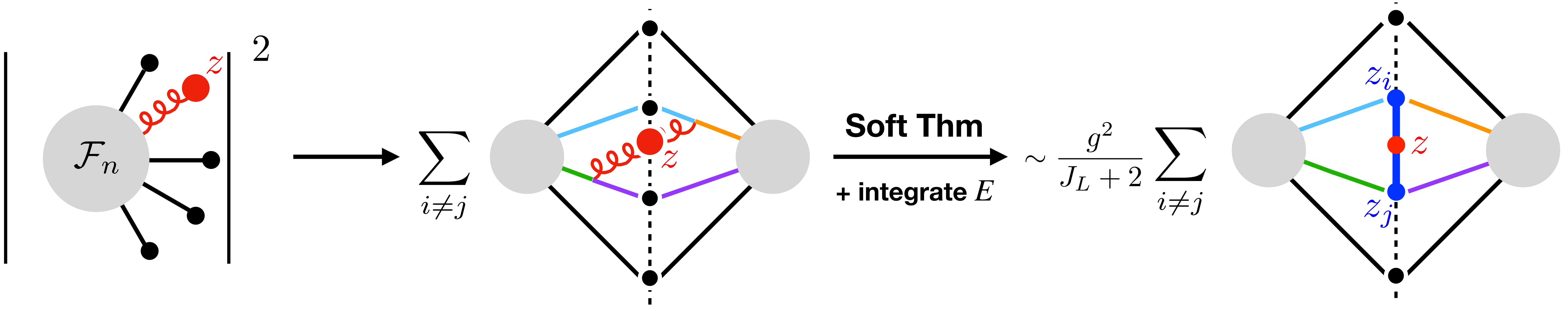
- full phase space

$$\int \frac{d^{d-1}\vec{p}_i}{(2\pi)^{d-1}2E_i}$$

- **DGLAP detector**

$$\int \frac{E^{-J_L}dE}{(2\pi)^{d-1}2E}\int d^dp\,\delta(p-Ez)$$

[constrained P.S.]



Integrating out soft gluon → Measurement is transferred

BFKL Detector

Apply DGLAP measurement and extract its leading J_L pole from soft theorem

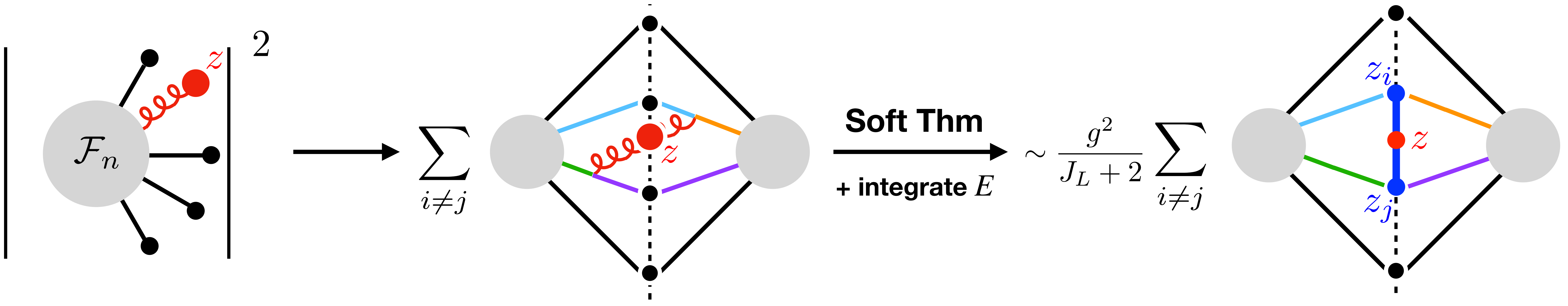
- full phase space

$$\int \frac{d^{d-1}\vec{p}_i}{(2\pi)^{d-1}2E_i}$$

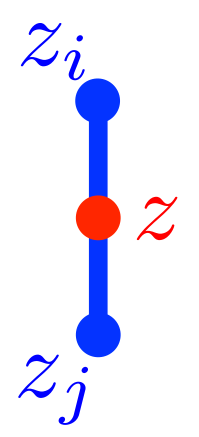
- **DGLAP detector**

$$\int \frac{E^{-J_L}dE}{(2\pi)^{d-1}2E} \int d^d p\, \delta(p - Ez)$$

[constrained P.S.]



New “measurement function” — BFKL detector



$$\longrightarrow \int d^{d-2}z_i d^{d-2}z_j \frac{z_i \cdot z_j}{(z \cdot z_i)(z \cdot z_j)} \mathcal{N}^c(z_i)\mathcal{N}^c(z_j)$$

color-interference number detector $\mathcal{N}^c(z_i) \leftrightarrow \mathbf{T}_i^c \int \frac{E_i^{d-2}dE_i}{(2\pi)^{d-1}2E_i}\int d^d p_i\, \delta(p_i - E_i z_i)$

BFKL Detector

Apply DGLAP measurement and extract its leading J_L pole from soft theorem

- full phase space

$$\int \frac{d^{d-1}\vec{p}_i}{(2\pi)^{d-1}2E_i}$$

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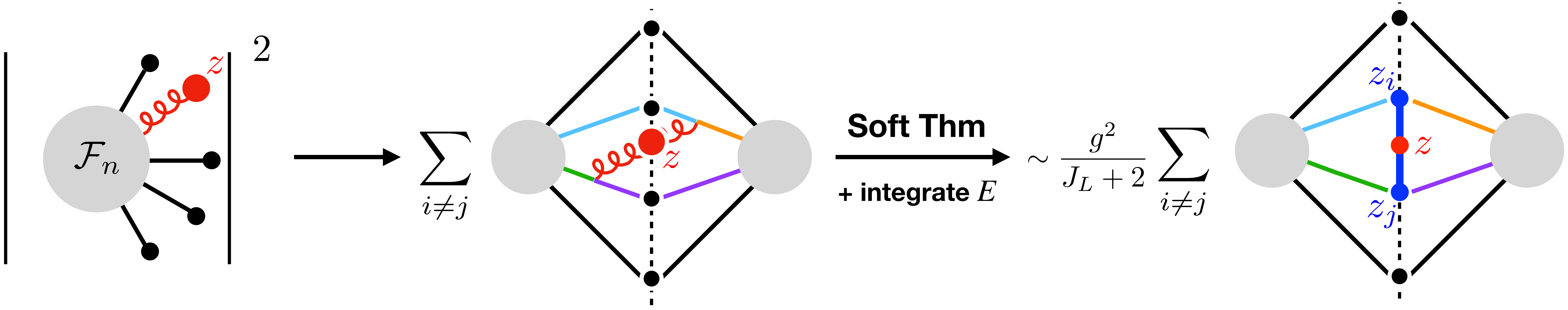
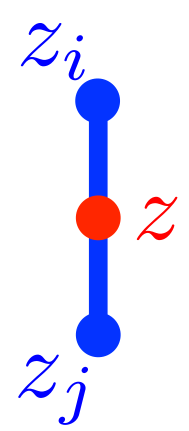


$$\left|\mathcal{F}_n\right|^2 \longrightarrow \sum_{i\neq j} \xrightarrow[\text{+ integrate } E]{\text{Soft Thm}} \sim \frac{g^2}{J_L+2}\sum_{i\neq j}$$

New “measurement” function — BFKL detector



detector at $J_L = -2$

$$\int d^{d-2}z_i d^{d-2}z_j \frac{z_i\cdot z_j}{(z\cdot z_i)(z\cdot z_j)}\,\mathcal{N}^c(z_i)\mathcal{N}^c(z_j)$$

BFKL detector

$$\mathcal{D}^{\mathrm{BFKL}}_{J_L}(z) = \frac{\Gamma(J_L+d-2)}{\Gamma(\frac{J_L+d-2}{2})^2}\int d^{d-2}z_i d^{d-2}z_j \left(\frac{2z_i\cdot z_j}{(2z\cdot z_i)(2z\cdot z_j)}\right)^{-J_L/2}\mathcal{N}^c(z_i)\mathcal{N}^c(z_j)$$

color-interference number detector $\mathcal{N}^c(z_i) \leftrightarrow \mathbf{T}_i^c \int \frac{E_i^{d-2}dE_i}{(2\pi)^{d-1}2E_i}\int d^dp_i\,\delta(p_i - E_i z_i)$

BFKL Detector

Apply DGLAP measurement and extract its leading J_L pole from soft theorem

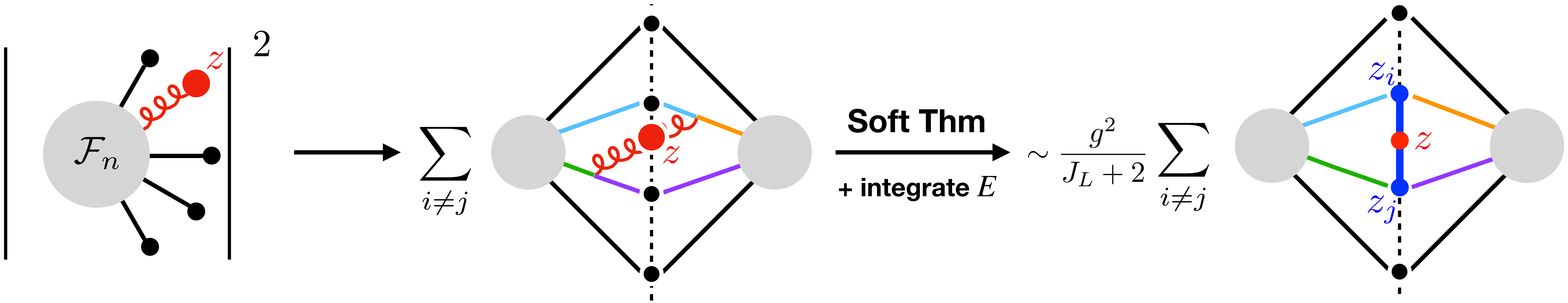
- full phase space

$$\int \frac{d^{d-1}\vec{p}_i}{(2\pi)^{d-1}2E_i}$$

- **DGLAP detector**

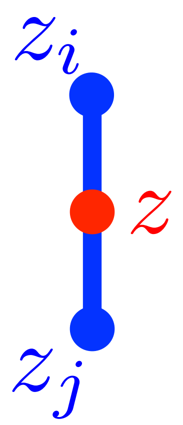
$$\int \frac{E^{-J_L}dE}{(2\pi)^{d-1}2E}\int d^dp\,\delta(p-Ez)$$

[constrained P.S.]



$$\left|\mathcal{F}_n\right|^2 \longrightarrow \sum_{i\neq j} \xrightarrow[\text{+ integrate } E]{\text{Soft Thm}} \sim \frac{g^2}{J_L+2}\sum_{i\neq j}$$

New “measurement” function — BFKL detector



detector at $J_L = -2$

$$\longrightarrow \int d^{d-2}z_i d^{d-2}z_j \frac{z_i\cdot z_j}{(z\cdot z_i)(z\cdot z_j)}\,\mathcal{N}^c(z_i)\mathcal{N}^c(z_j) \longrightarrow$$

BFKL detector

$$\mathcal{D}^{\rm BFKL}_{J_L}(z) = \frac{\Gamma(J_L+d-2)}{\Gamma(\frac{J_L+d-2}{2})^2}\int d^{d-2}z_i d^{d-2}z_j \left(\frac{2z_i\cdot z_j}{(2z\cdot z_i)(2z\cdot z_j)}\right)^{-J_L/2}\mathcal{N}^c(z_i)\mathcal{N}^c(z_j)$$

color-interference number detector

$$\mathcal{N}^c(z_i) \leftrightarrow \mathbf{T}_i^c \int \frac{E_i^{d-2}dE_i}{(2\pi)^{d-1}2E_i}\int d^dp_i\,\delta(p_i-E_iz_i)$$

Why this name?

I have little to explain the name… Let's follow the advice from great physicists:

Shut up and Calculate!

Richard Feynman

Nathaniel David Mermin

Though it is often misattributed to Richard Feynman, Mermin coined the phrase "shut up and calculate!" to characterize the views of many physicists regarding the interpretation of quantum mechanics.

https://en.wikipedia.org/wiki/N._David_Mermin

This spirit may date back to

Radiation Lab at MIT

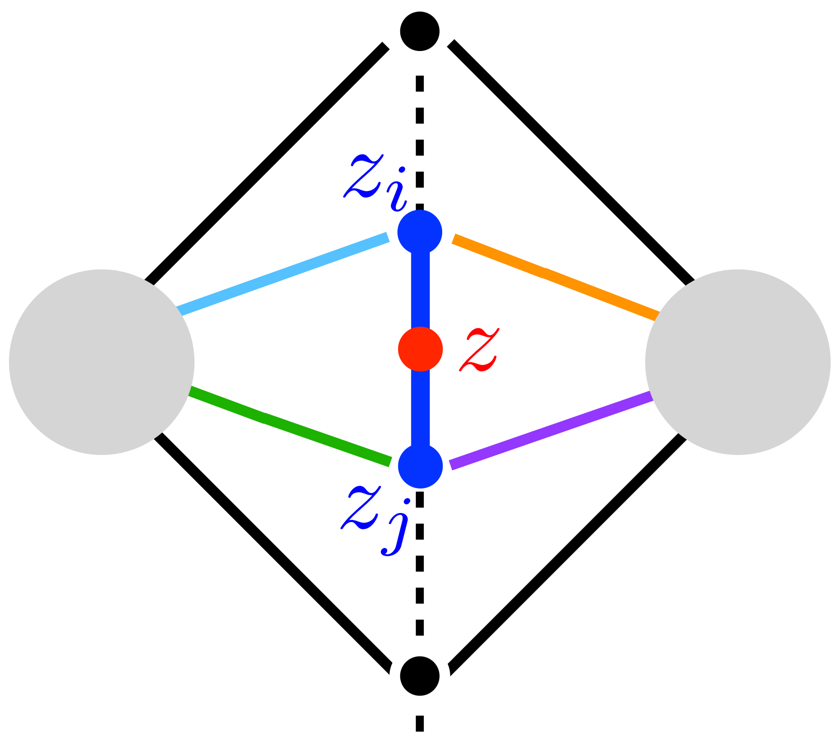
The Rad. Lab rallying cry of "Get the numbers out" shaded into "Shut up and calculate!"

https://www.nature.com/articles/505153a#/b10

Julian Schwinger (standing) with colleagues at MIT's Radiation Laboratory during the Second World War. Credit: MIT MUSEUM

One-Loop Divergences (BFKL)

Matrix element calculation via amplitude/form factor



$$\left\langle \mathcal{D}^{\rm BFKL}_{J_L,\rm bare}(z)\right\rangle = \frac{\Gamma(J_L+d-2)}{\Gamma(\frac{J_L+d-2}{2})^2}\sum_{X_n}\int d\mathrm{LIPS}_n \sum_{\substack{i,j\in X_n\\ i\neq j}}\left(\frac{2p_i\cdot p_j}{(2z\cdot p_i)(2z\cdot p_j)}\right)^{-J_L/2}\left\langle \mathcal{F}^*_{X_n}\middle|\mathbf{T}^c_i\otimes\mathbf{T}^c_j\middle|\mathcal{F}_{X_n}\right\rangle$$

- same one-loop diagrams as DGLAP case
- the “measurement” is different
  - color interference
  - non-trivial kernel in the phase space integral

One-loop divergence in ϵ

$$\longrightarrow \quad \left\langle \mathcal{D}^{\rm BFKL}_{J_L,\rm bare}(z)\right\rangle_{\text{1-loop}} = \frac{g^2 N_c}{8\pi^2\epsilon}\boxed{\left[2\gamma_E+\psi^{(0)}(1+J_L/2)+\psi^{(0)}(-J_L/2)\right]}\left\langle \mathcal{D}^{\rm BFKL}_{J_L,\rm bare}(z)\right\rangle_{\rm tree}+\mathcal{O}(\epsilon^0)$$

BFKL eigenvalue!

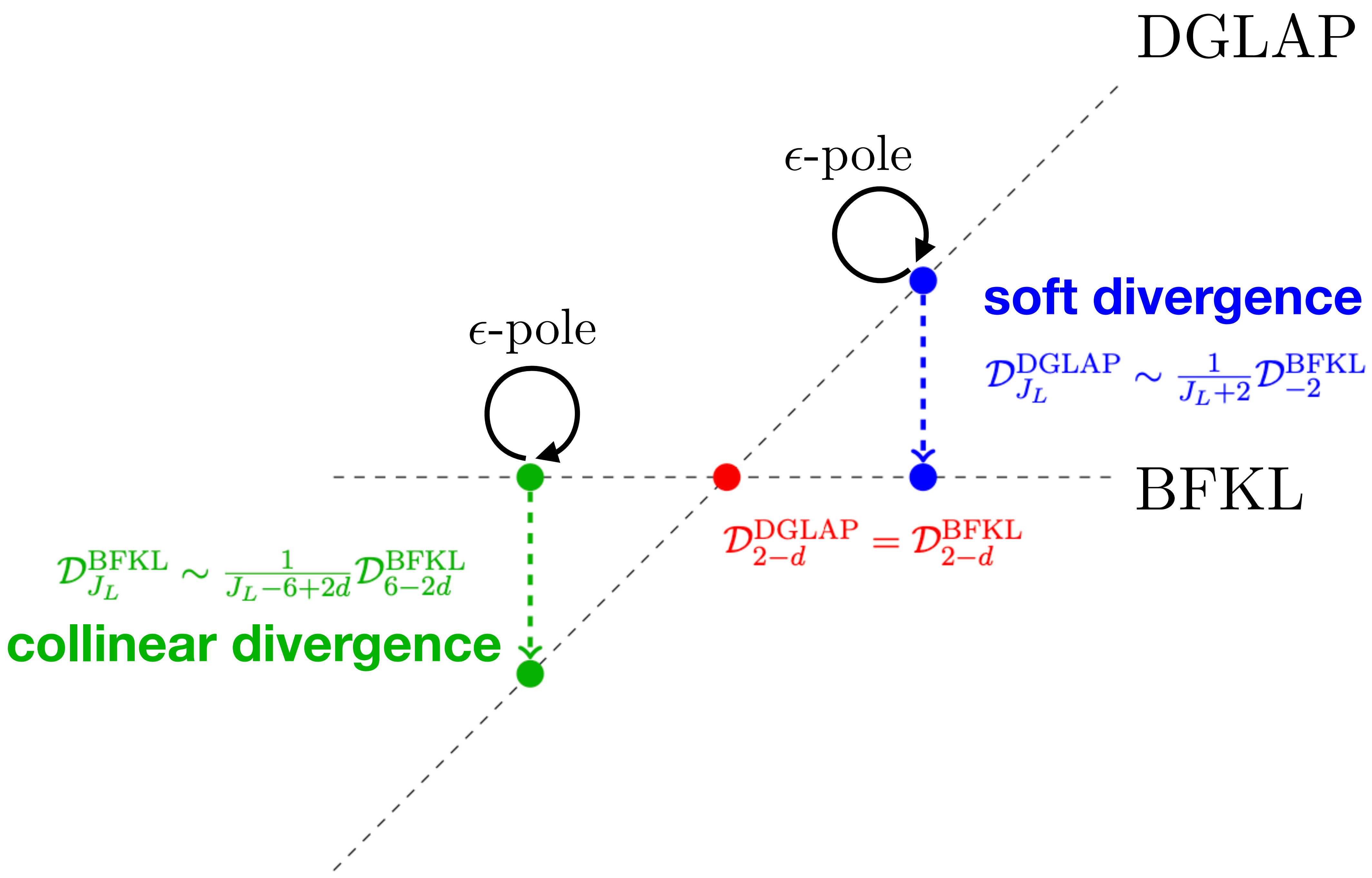
One-loop divergence near $J_L\sim-2$

$$\left\langle \mathcal{D}^{\rm BFKL}_{J_L,\rm bare}(z)\right\rangle_{\text{1-loop}} = \frac{g^2\tilde{\mu}^{2\epsilon}N_c(N_c^2-1)}{J_L-(-2+4\epsilon)}\mathcal{R}(\epsilon)\left\langle \mathcal{D}^{g,\rm bare}_{J_L}(z)\right\rangle_{\rm tree}+\cdots$$

- proportional to DGLAP detector
- comes from collinear divergence

$z_1, z_2\sim z$

Structures near the Intersection



This kernel has pole at $J_L = 2 - d$

example $\left(\frac{1}{2z\cdot z_i}\right)^{-J_L/2} = \Omega_{d-2}\frac{\delta^{d-2}(\hat{z}-\hat{z}_i)}{J_L-(2-d)} + \cdots$

Degeneracy at $J_L = 2 - d$

$$\left\langle \mathcal{D}^{\rm BFKL}_{J_L,{\rm bare}}(z)\right\rangle = \frac{\Gamma(J_L+d-2)}{\Gamma(\frac{J_L+d-2}{2})^2}\sum_{X_n}\int d{\rm LIPS}_n \sum_{\substack{i,j\in X_n\\ i\neq j}}\left(\frac{2p_i\cdot p_j}{(2z\cdot p_i)(2z\cdot p_j)}\right)^{-J_L/2}\left\langle \mathcal{F}^*_{X_n}\middle|\mathbf{T}^c_i\otimes\mathbf{T}^c_j\middle|\mathcal{F}_{X_n}\right\rangle$$

$$\xrightarrow{J_L = 2-d} \frac{\Omega_{d-2}}{4}\sum_{X_n}\int d{\rm LIPS}_n\sum_{\substack{i,j\in X_n\\ i\neq j}}\delta^{d-2}(\hat{z}-\hat{z}_i)\left\langle \mathcal{F}^*_{X_n}\middle|\mathbf{T}^c_i\otimes\mathbf{T}^c_j\middle|\mathcal{F}_{X_n}\right\rangle + (i\leftrightarrow j)$$

$$\xrightarrow{\text{color conservation}} -\frac{\Omega_{d-2}}{2}\sum_{X_n}\int d{\rm LIPS}_n\sum_{i\in X_n}\delta^{d-2}(\hat{z}-\hat{z}_i)\left\langle \mathcal{F}^*_{X_n}\middle|\mathbf{T}^c_i\mathbf{T}^c_i\middle|\mathcal{F}_{X_n}\right\rangle \xrightarrow{\text{color identity}} -\frac{\Omega_{d-2}}{2}C_i\left\langle \mathcal{D}^{i,{\rm bare}}_{2-d}(z)\right\rangle$$

Quark: $C_i = C_F$
Gluon: $C_i = C_A$

Pole Subtraction and Renormalization

Philosophy: near the intersection, we define renormalized detectors by subtracting all poles. (simultaneously subtract poles in ϵ and J_L)

First, we need to construct a non-degenerate basis $\mathbb{D}^{\mathrm{bare}}_{J_L} = \begin{pmatrix} \frac{\Omega_{d-2}}{2} C_A \mathcal{D}^{g,\mathrm{bare}}_{J_L} \\ \frac{\mu^{-J_L-2+2\epsilon}\mathcal{D}^{\mathrm{BFKL}}_{J_L,\mathrm{bare}} + \frac{\Omega_{d-2}}{2} C_A \mathcal{D}^{g,\mathrm{bare}}_{J_L}}{J_L+2-2\epsilon} \end{pmatrix}$

classical dimensions

$$\hat{D}_0 \mathcal{D}^{g,\mathrm{bare}}_{J_L} = (2-d-J_L)\mathcal{D}^{g,\mathrm{bare}}_{J_L}$$

$$\hat{D}_0 \mathcal{D}^{\mathrm{BFKL}}_{J_L,\mathrm{bare}} = 0$$

$$\Rightarrow \quad \hat{D}_0 \mathbb{D}^{\mathrm{bare}}_{J_L} = \begin{pmatrix} 2-d-J_L & 0 \\ -1 & 0 \end{pmatrix} \mathbb{D}^{\mathrm{bare}}_{J_L}$$

a log-multiplet

$\mathscr{D}_0$ [a Jordan matrix]

Define renormalized operators with the minimal subtraction scheme ($\overline{\mathrm{MS}}$-like)

$$\mathbb{D}^{\mathrm{bare}}_{J_L} = \mathcal{Z}_{J_L} \mathbb{D}^{\mathrm{ren}}_{J_L}$$

polynomials in $\frac{1}{\epsilon}, \frac{1}{J_L+2}, \frac{1}{J_L+2-4\epsilon}$

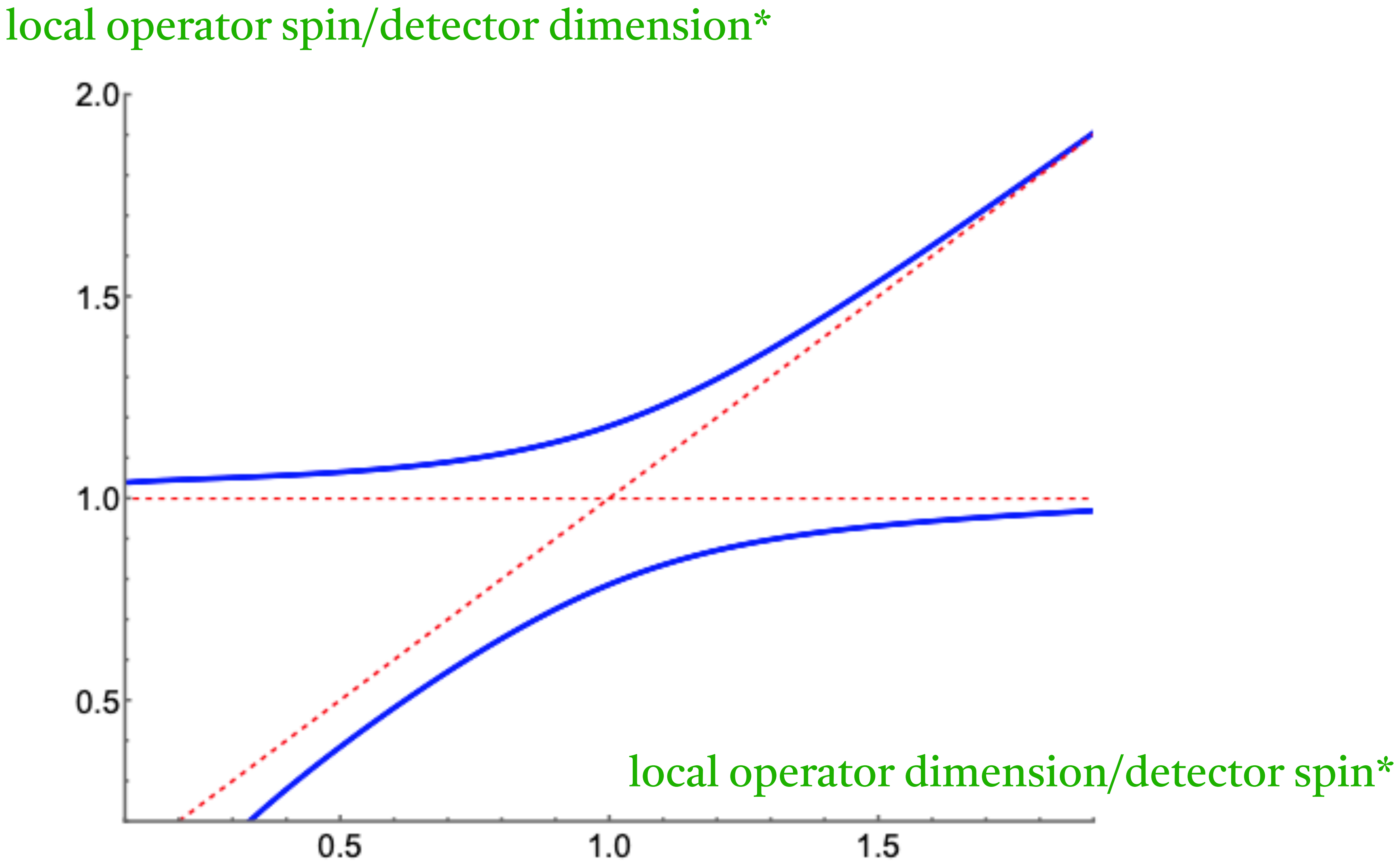
The matrix of dimension (or spectrum) can be extracted from the renormalization factor

$$\mathscr{D} = \mathcal{Z}^{-1}_{J_L}\left(\mathscr{D}_0 + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right)\mathcal{Z}_{J_L} \qquad \beta(\alpha_s) = -2\epsilon\alpha_s + \mathcal{O}(\alpha_s)$$

Avoided Crossing in QCD

The perturbative operator spectrum can be obtained by diagonalizing $\mathscr{D}$

pure gluon Chew-Frautschi Plot

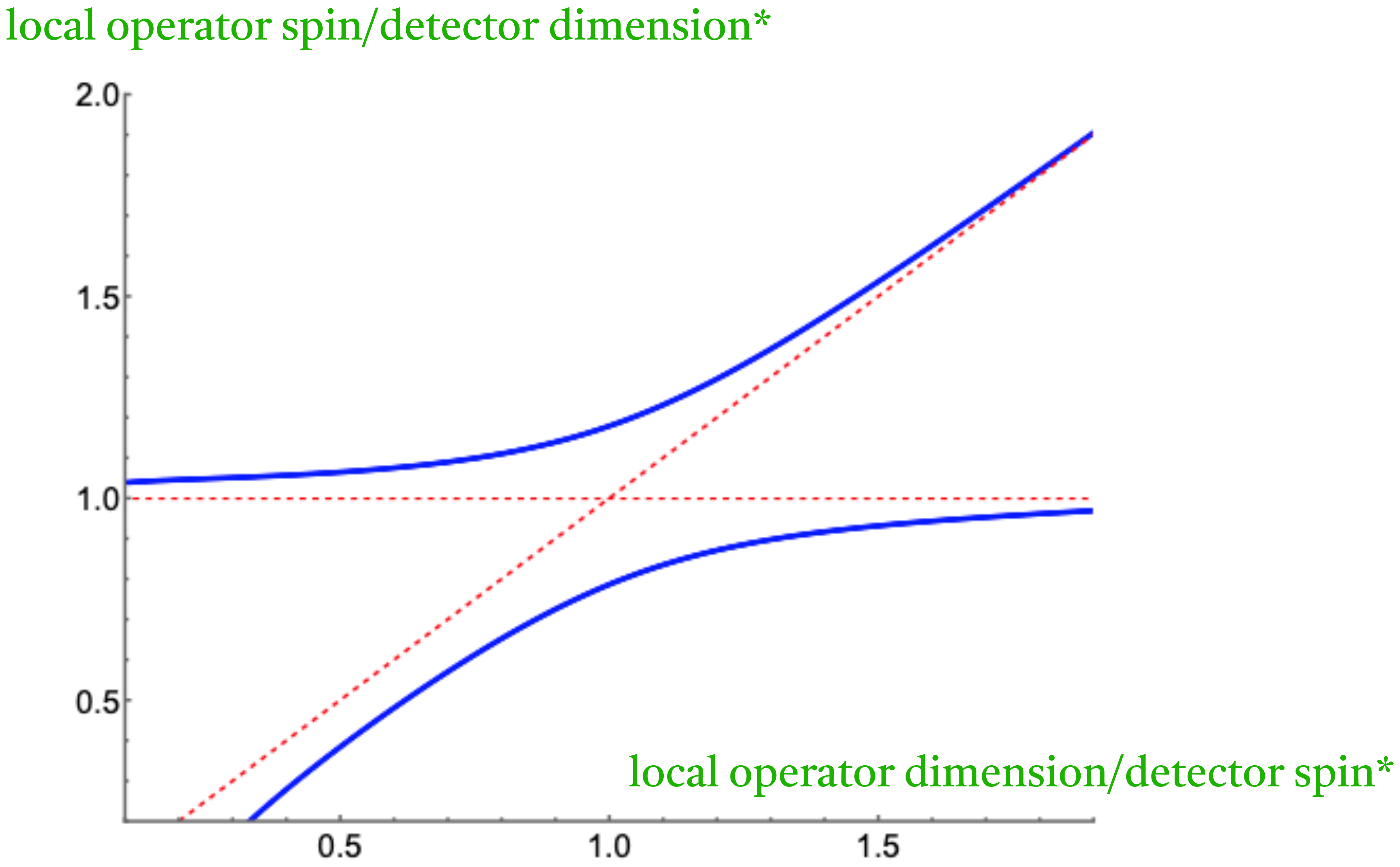


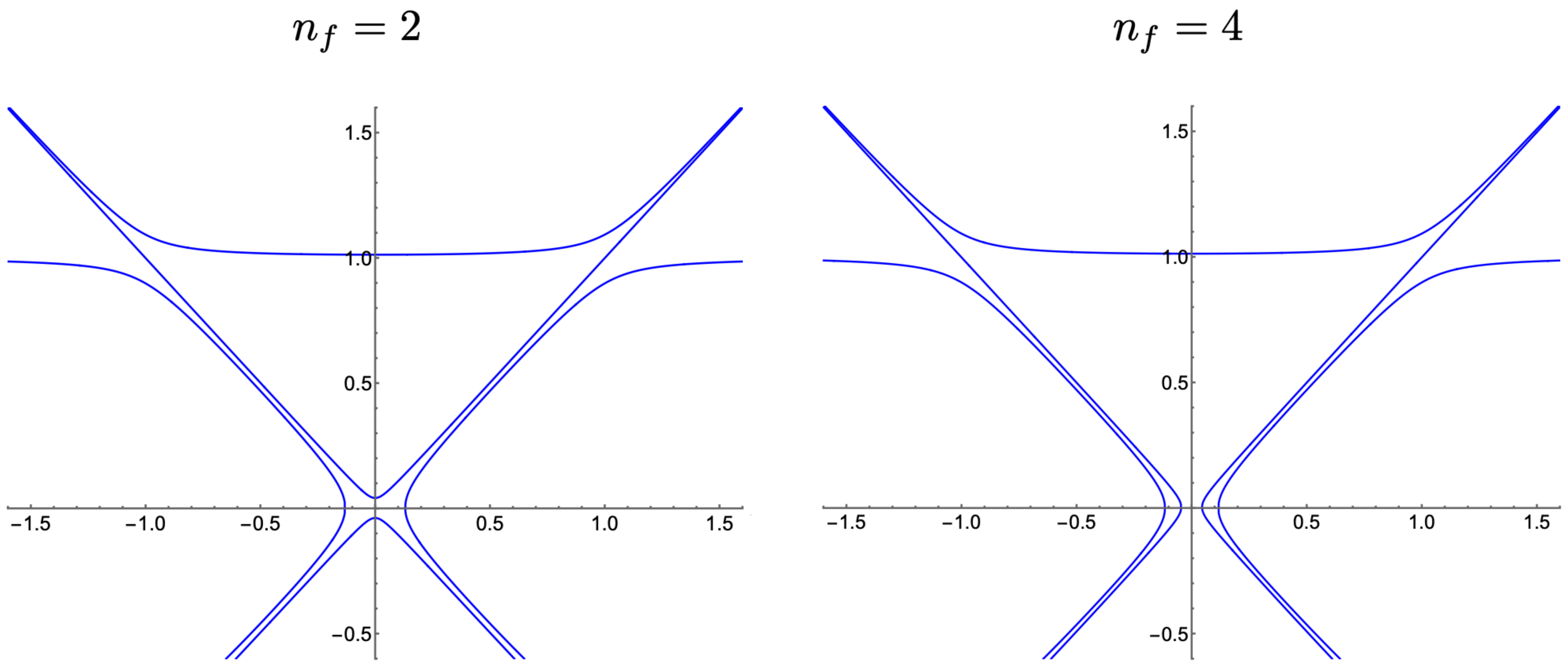
* there may be a possible constant shift and a minus sign.

Avoided Crossing in QCD

The perturbative operator spectrum can be obtained by diagonalizing $\mathscr{D}$

pure gluon Chew-Frautschi Plot $\xrightarrow{\text{add quarks}}$ QCD Chew-Frautschi Plot




$n_f = 2$

$n_f = 4$

- We also include the mixing with the celestial shadow
- The lower intersection is related to subleading soft theorem

* there may be a possible constant shift and a minus sign.

Application

The mixing story may sound nice theoretically…

but in a practical sense,

the BFKL detector is not measurable in a real-world experiment!

- final states are hadrons, impossible to impose color interference

Why should one care about this weird detector?

The mixing story may sound nice theoretically…

but in a practical sense,

the BFKL detector is not measurable in a real-world experiment!

Why should one care about this weird detector?

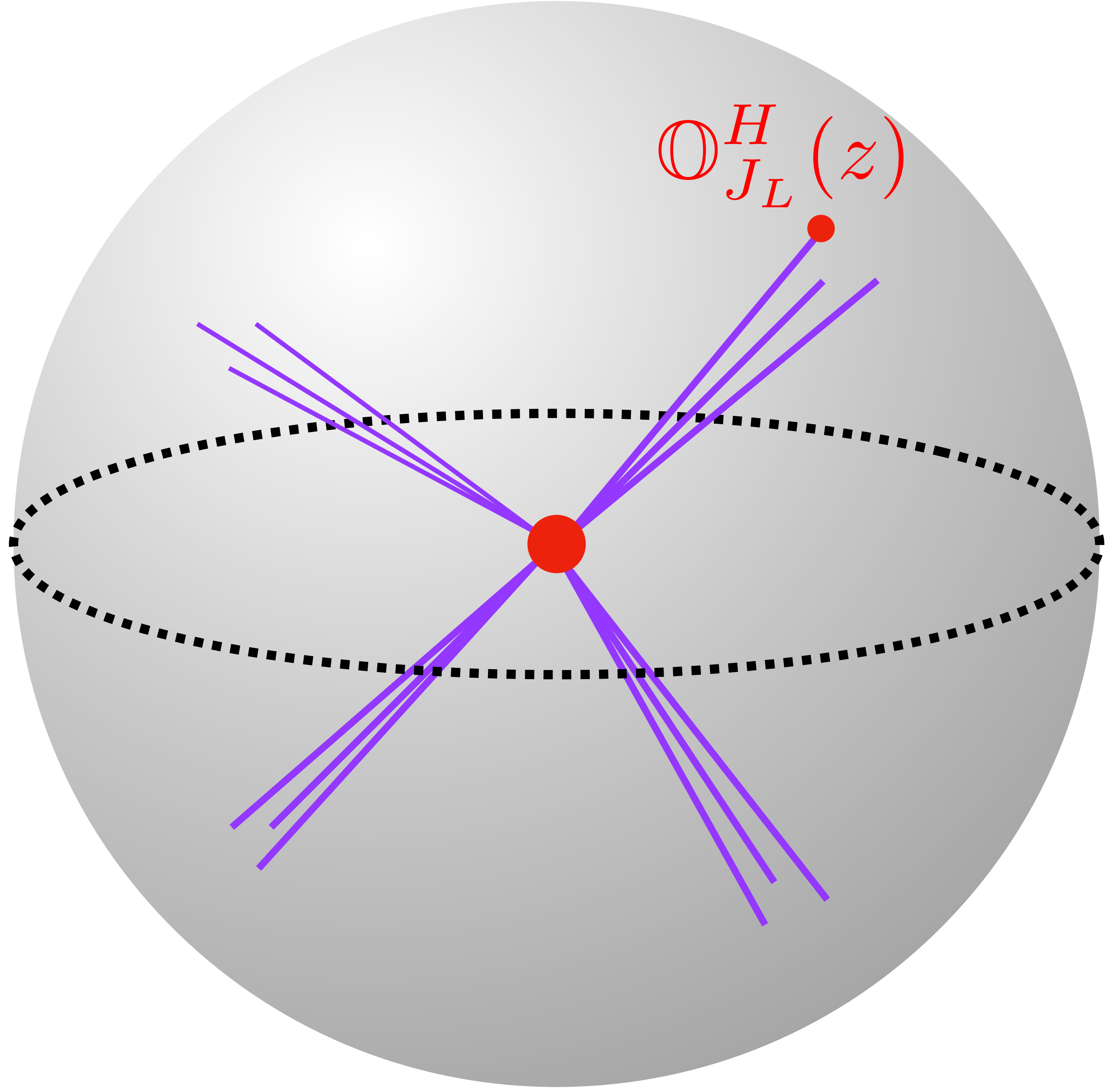
Philosophy: These perturbative detectors show up as intermediate states.

Simplest Family of Observables in Experiments

One-point event shapes/DGLAP-type “hadron” detectors*

*Here “hadron” is to emphasize all particles after hadronization.

In the high-energy scattering, we assume the hadrons are almost massless.



$\nu = 2:$ energy
$\nu = 1:$ multiplicity

The simplest detectors do not distinguish the particle species

$\nu = -1 - J_L$

$$\mathbb{O}^H_{J_L}(z) = \sum_h \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \delta^{(2)}(\hat{p} - \hat{z})\, E^{\nu-1} a^\dagger_h(p) a_h(p)$$

sum over all particles

For example, we can measure the observables in the e^+e^- collider

measurement function

$$f(\nu, Q) = \frac{1}{\sigma_{\text{tot}}} \sum_X \int d\sigma_{e^+e^-\to X} \boxed{\sum_{h\in X} \left(\frac{E_a}{Q}\right)^{\nu-1}} = \frac{4\pi}{\sigma_{\text{tot}}} \frac{\langle \mathbb{O}^H_{J_L}(z)\rangle_Q}{Q^{\nu-1}}$$

c.o.m. energy

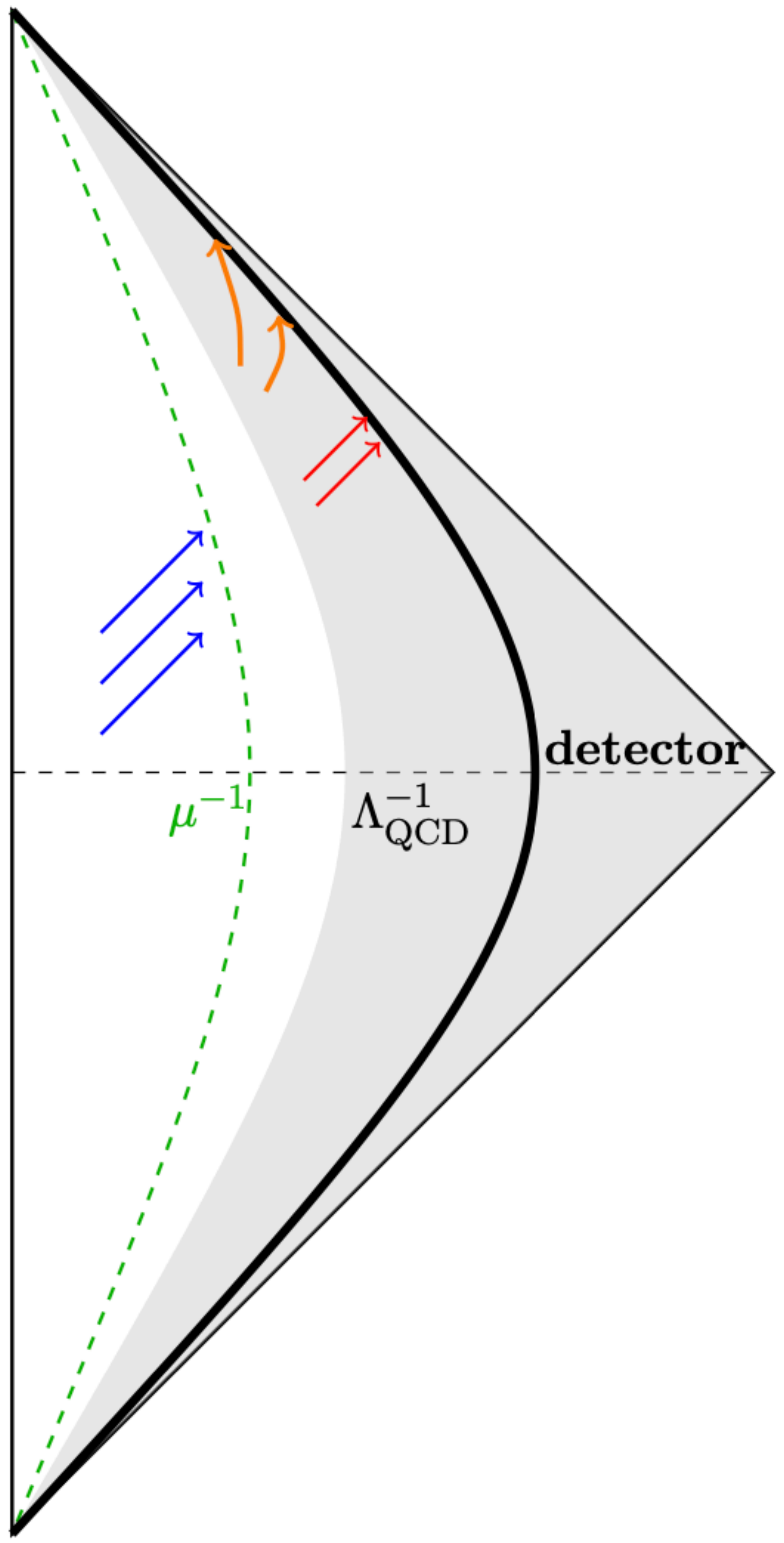
These are not IR-safe observables! (except $\nu = 2$)

Factorization Picture in QCD

In the high-energy limit, we can **factorize** QCD observables into

perturbative part and non-perturbative part.

[asymptotic freedom] [confinement]



We can find a factorization scale μ s.t. $Q \gg \mu \gg \Lambda_{\mathrm{QCD}}$

For hard scattering length scale $\frac{1}{Q}, \frac{1}{\mu}$ can be approximated as pert. infinity.

In our case, the factorization is

the matching of hadronic detectors onto parton detectors.

$$\mathbb{O}^{H}_{J_L}(z) \sim \sum_k \underline{C_k(J_L,\mu)}\,\boxed{\mathcal{D}^{\mathrm{ren}}_{J_L,k}(z;\mu)}$$

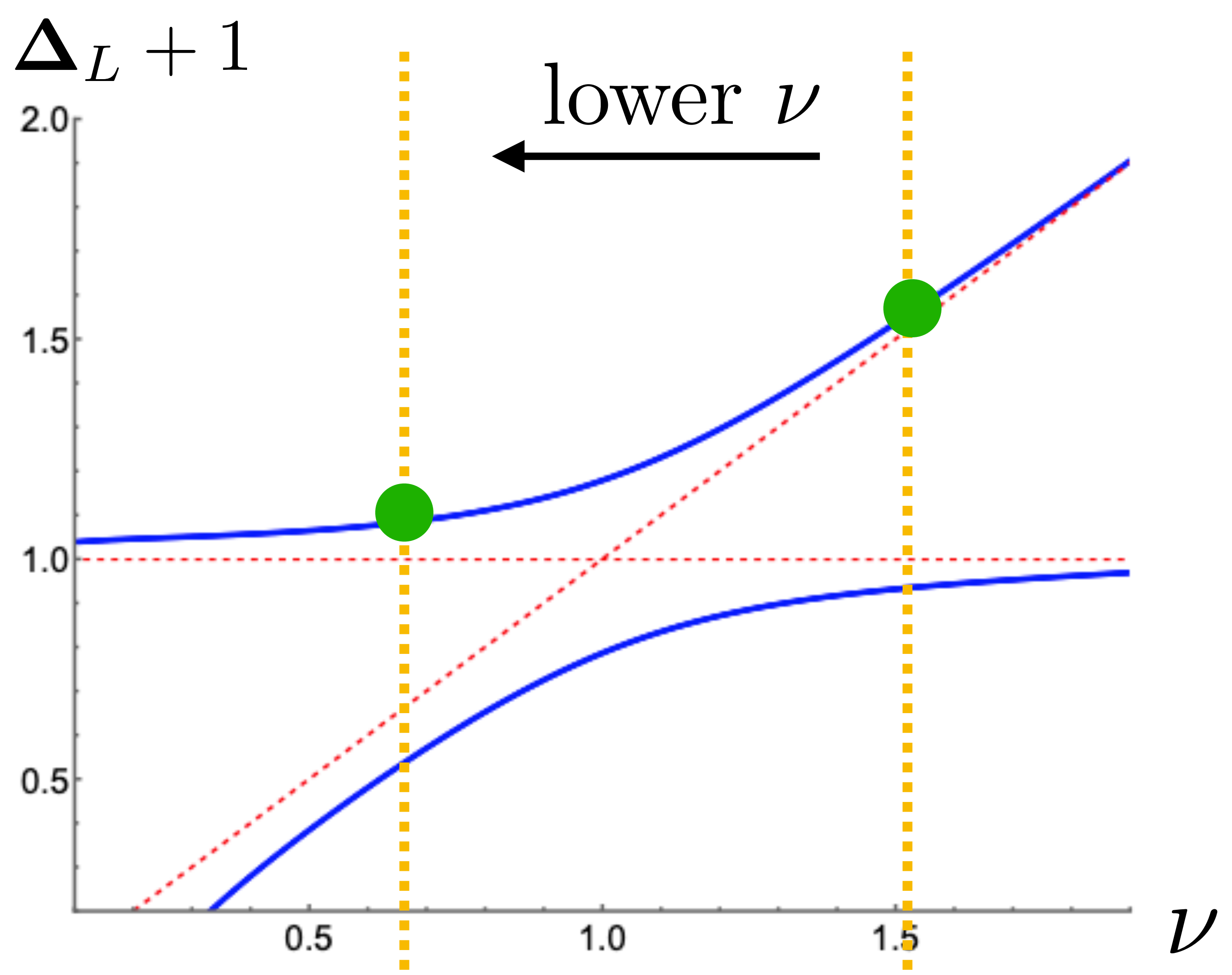
Wilson coefficients that contain hadronization information

Properties of Detector Matching

$$\langle \mathbb{O}^{H}_{J_L}(z)\rangle_Q \sim \sum_k C_k(J_L,\mu)\langle \mathcal{D}^{\text{ren}}_{J_L,k}(z;\mu)\rangle_Q$$

Dimensional analysis	$[\nu-1]$	$[\nu-1-\mathbf{\Delta}_L]$ $\leftarrow$	$[\mathbf{\Delta}_L]$	
		Λ_{QCD}	Q	**Typical energy scale**
		$\Lambda_{\text{QCD}}^{\nu-1-\mathbf{\Delta}_L}$	$Q^{\mathbf{\Delta}_L}$	**Typical size**

Largest dimension detector dominates the detector matching.



For $\nu > 1$, the dominant operator is DGLAP operators. The corresponding matching coefficients are related to the moments of the fragmentation functions.

leading approx. for 1-pt event shapes

$$\boxed{f(\nu,Q) \sim Q^{\mathbf{\Delta}_L^{\max}-\nu+1}}$$

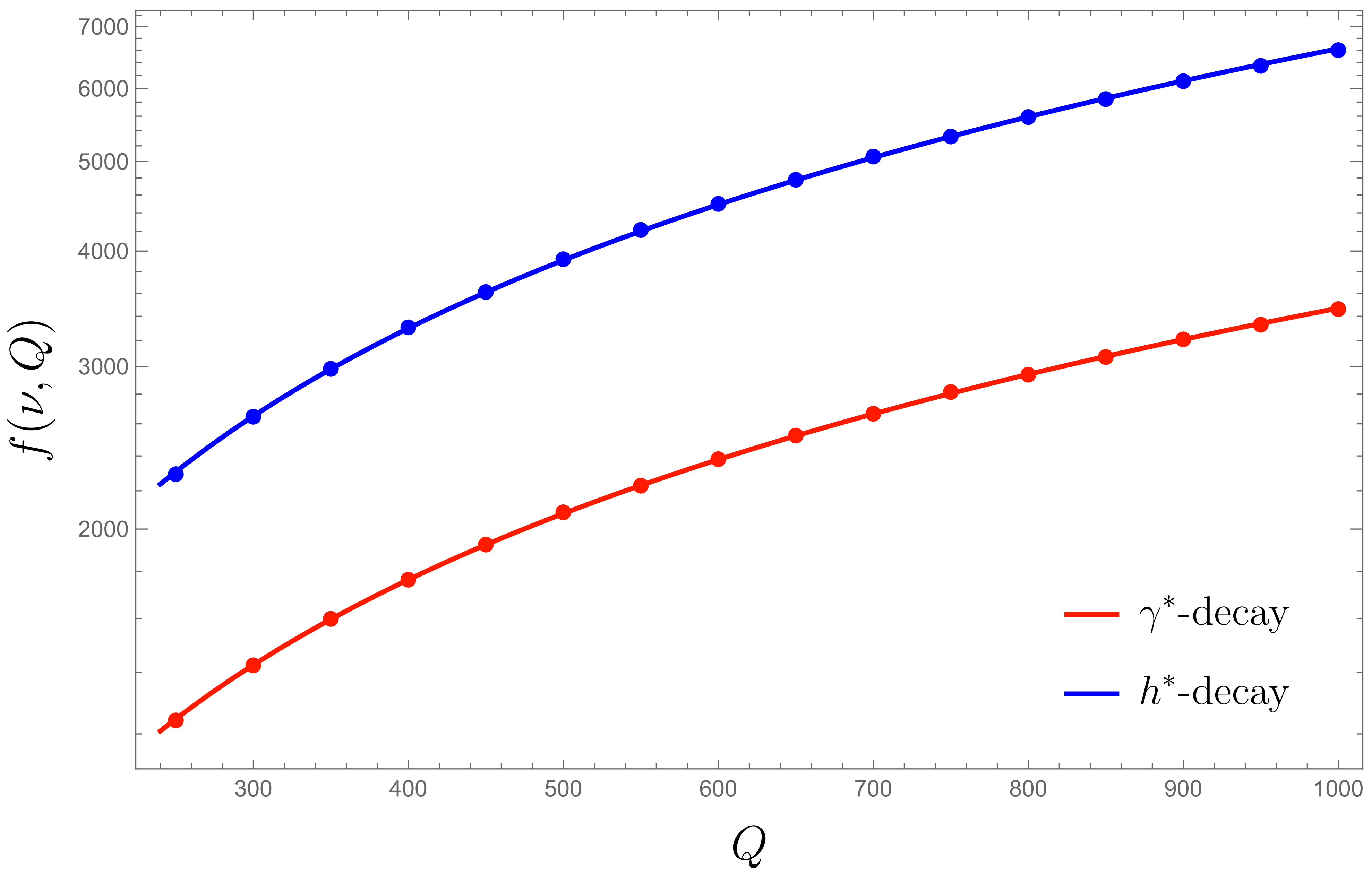
Monte Carlo Simulation Data (Pythia)

We generate events from γ^*- and h^*-decay respectively in Pythia at center of mass energy $Q = 250, 300, 350, \ldots, 1000\,\mathrm{GeV}$

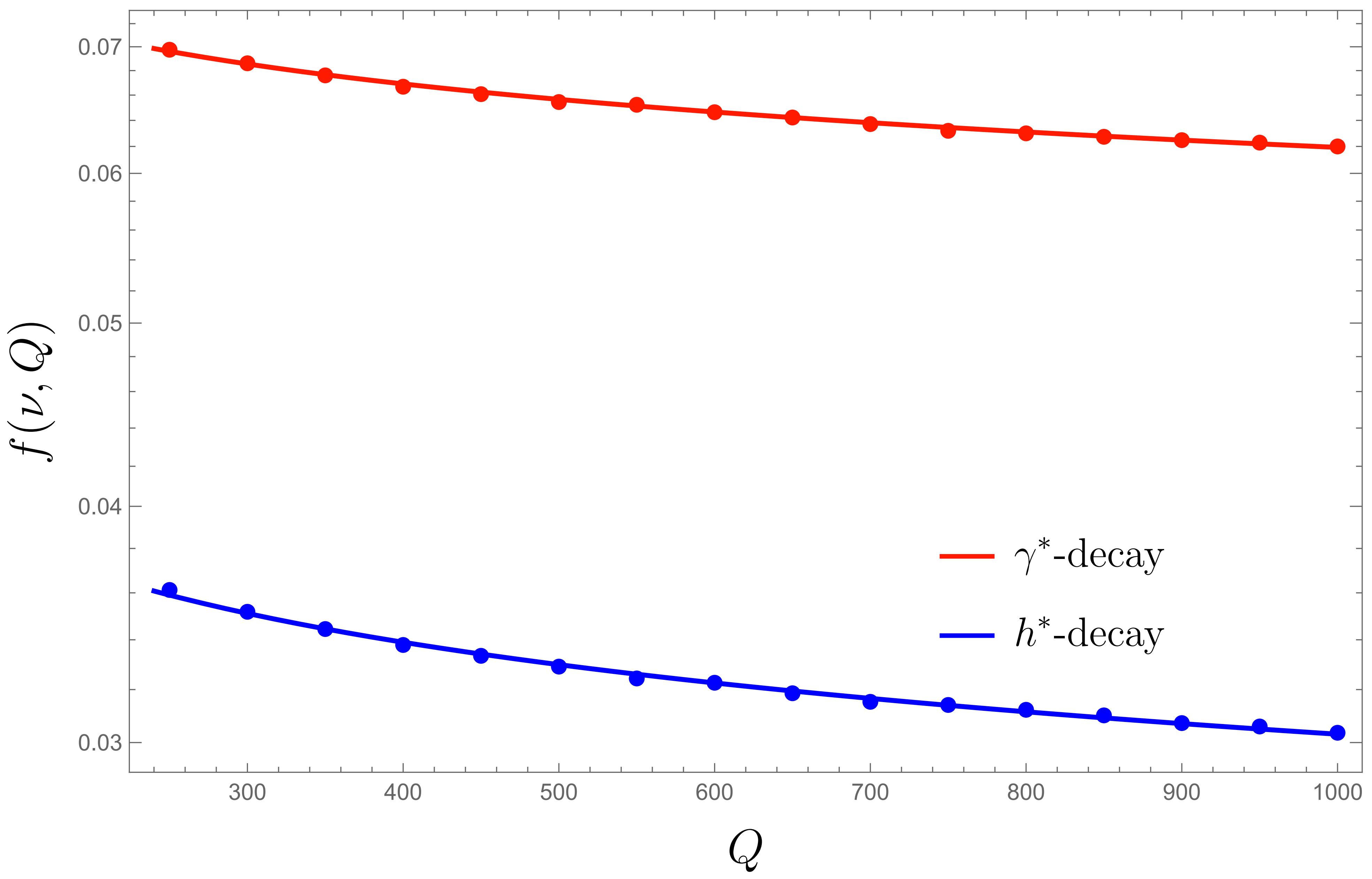
and fit the simulation data to the ansatz

$$\boxed{f(\nu, Q) \sim Q^{\boldsymbol{\Delta}_L^{\max} - \nu + 1}}$$

$\nu = 0.495$



$\nu = 2.99$



In the high energy limit, power law is a good approximation.

Monte Carlo Simulation Data (Pythia)

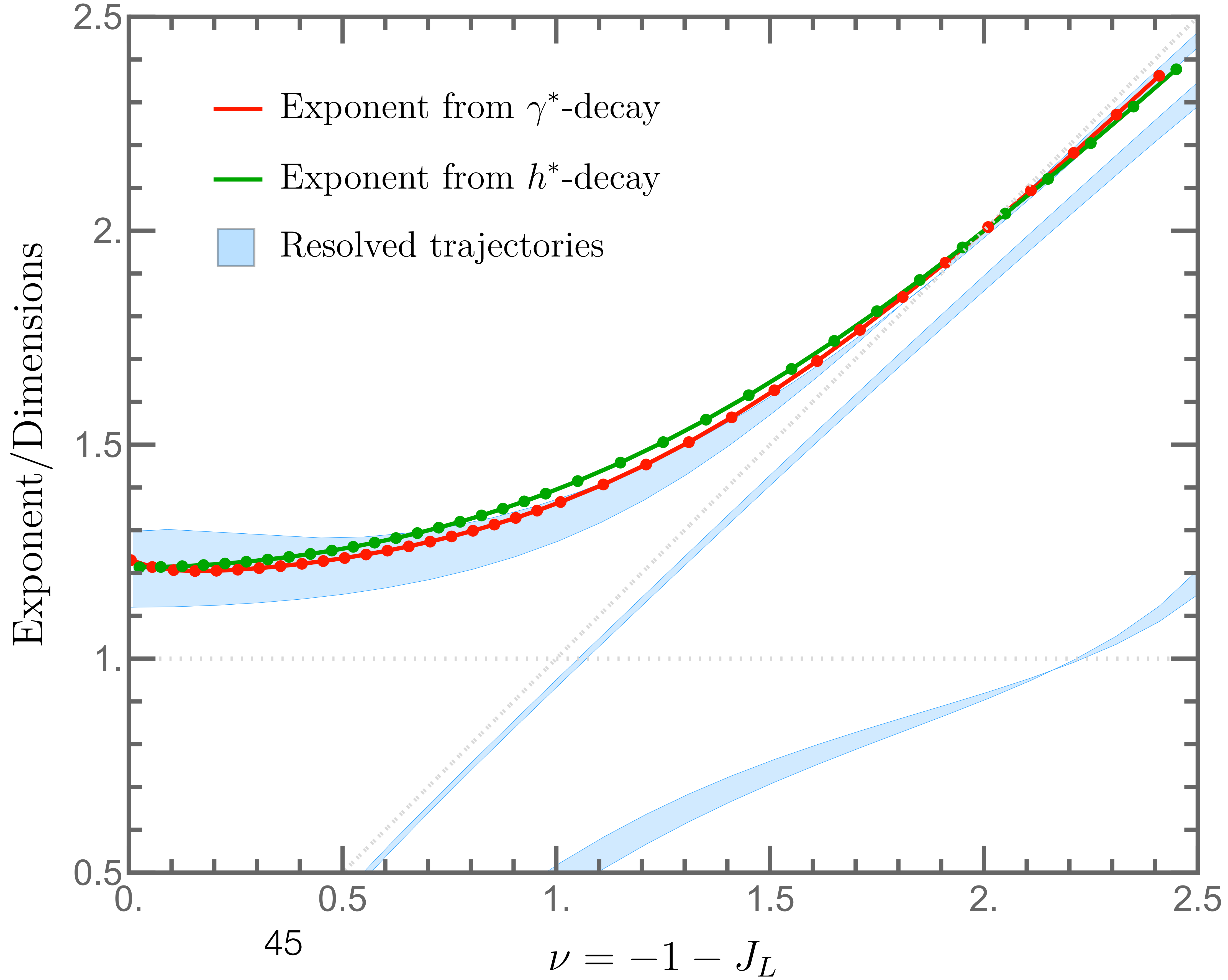
We generate events from γ^*- and h^*-decay respectively in Pythia at center of mass energy $Q = 250, 300, 350, \ldots, 1000\,\text{GeV}$

and fit the simulation data to the ansatz

$$\boxed{f(\nu, Q) \sim Q^{\Delta_L^{\max} - \nu + 1}}$$

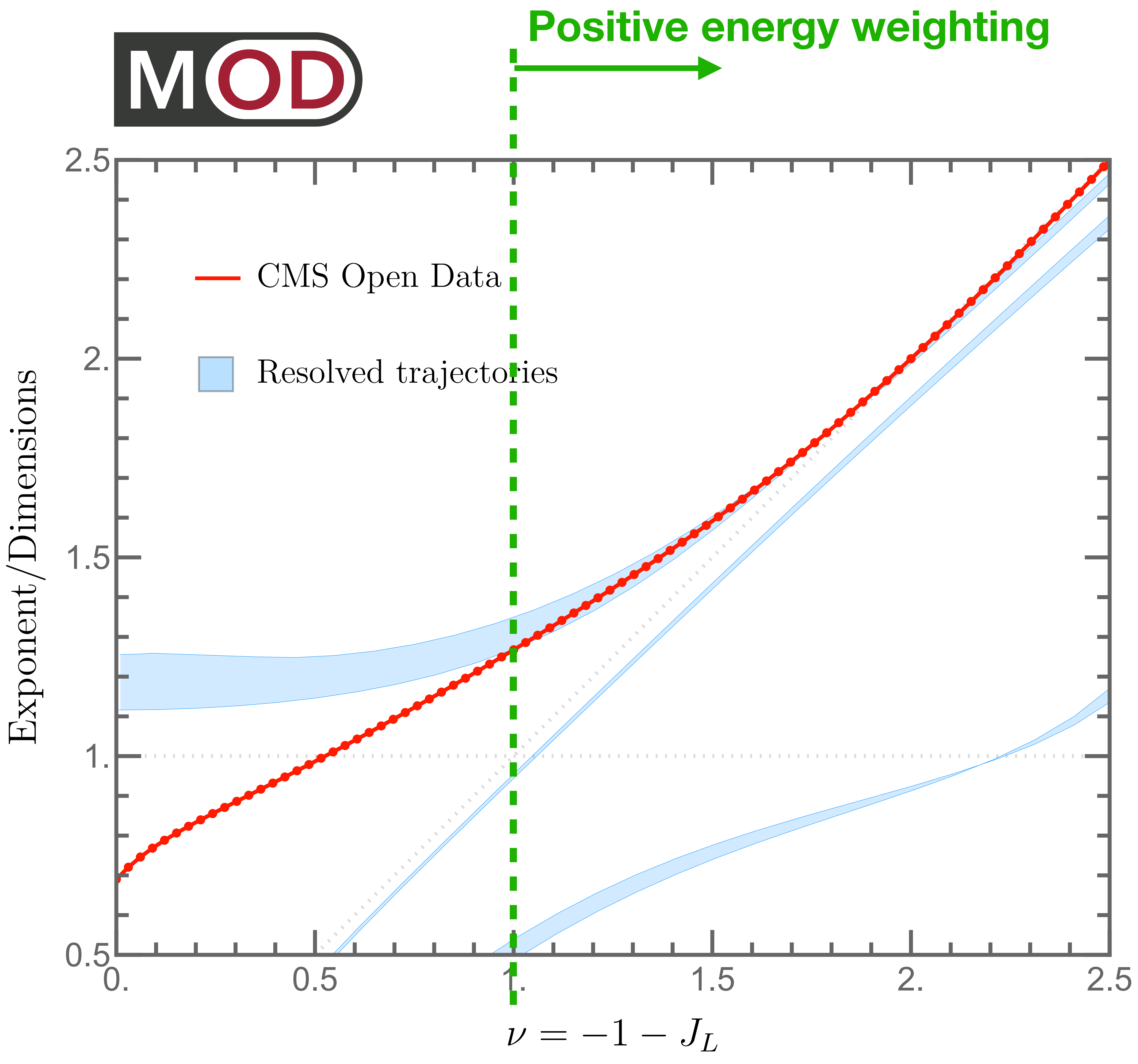
Extract $\Delta_L^{\max}$

as a function of ν →



CMS Open Data

We also used the CMS Open Data with jet energy in the range [375,1125] GeV.

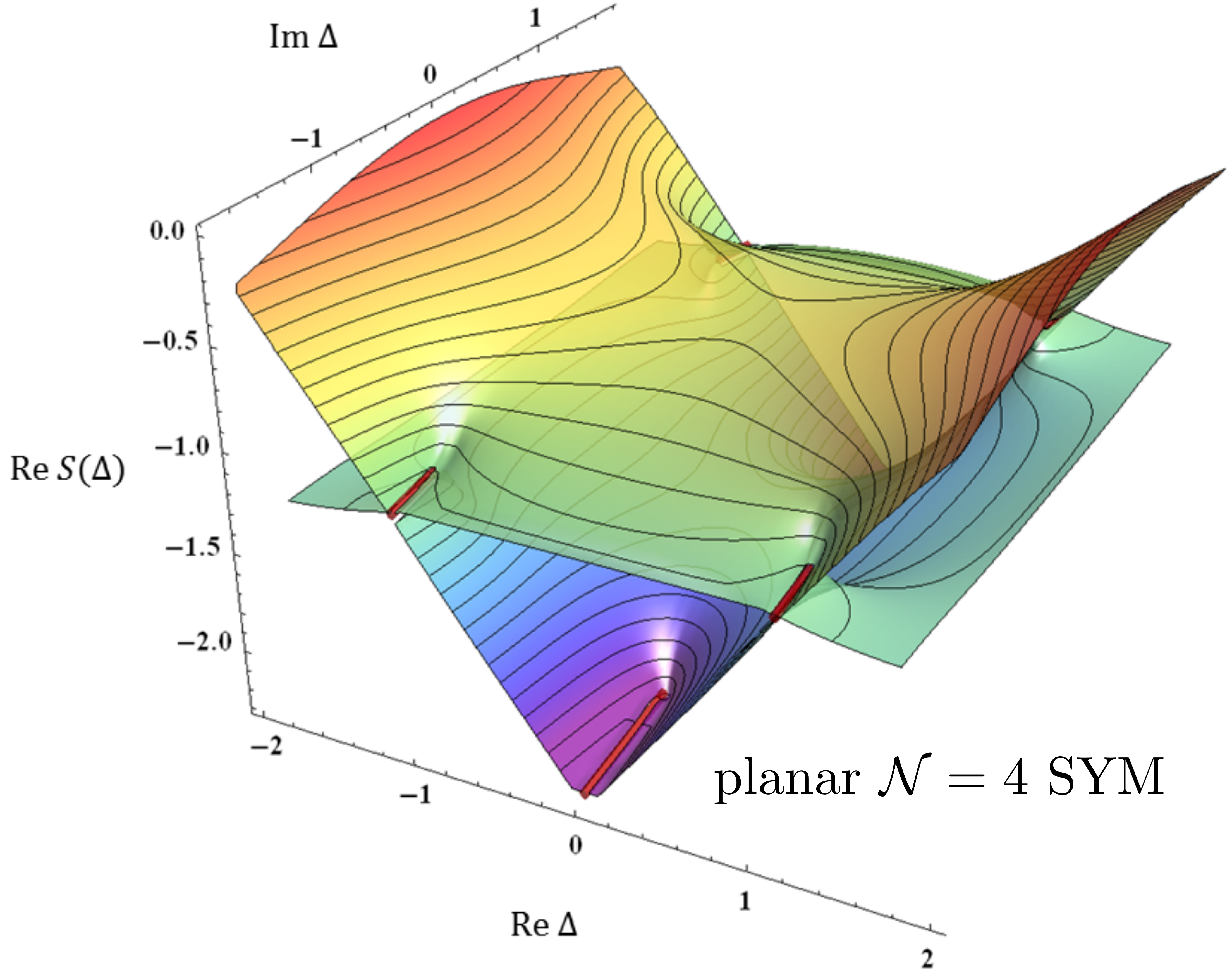


Positive energy weighting

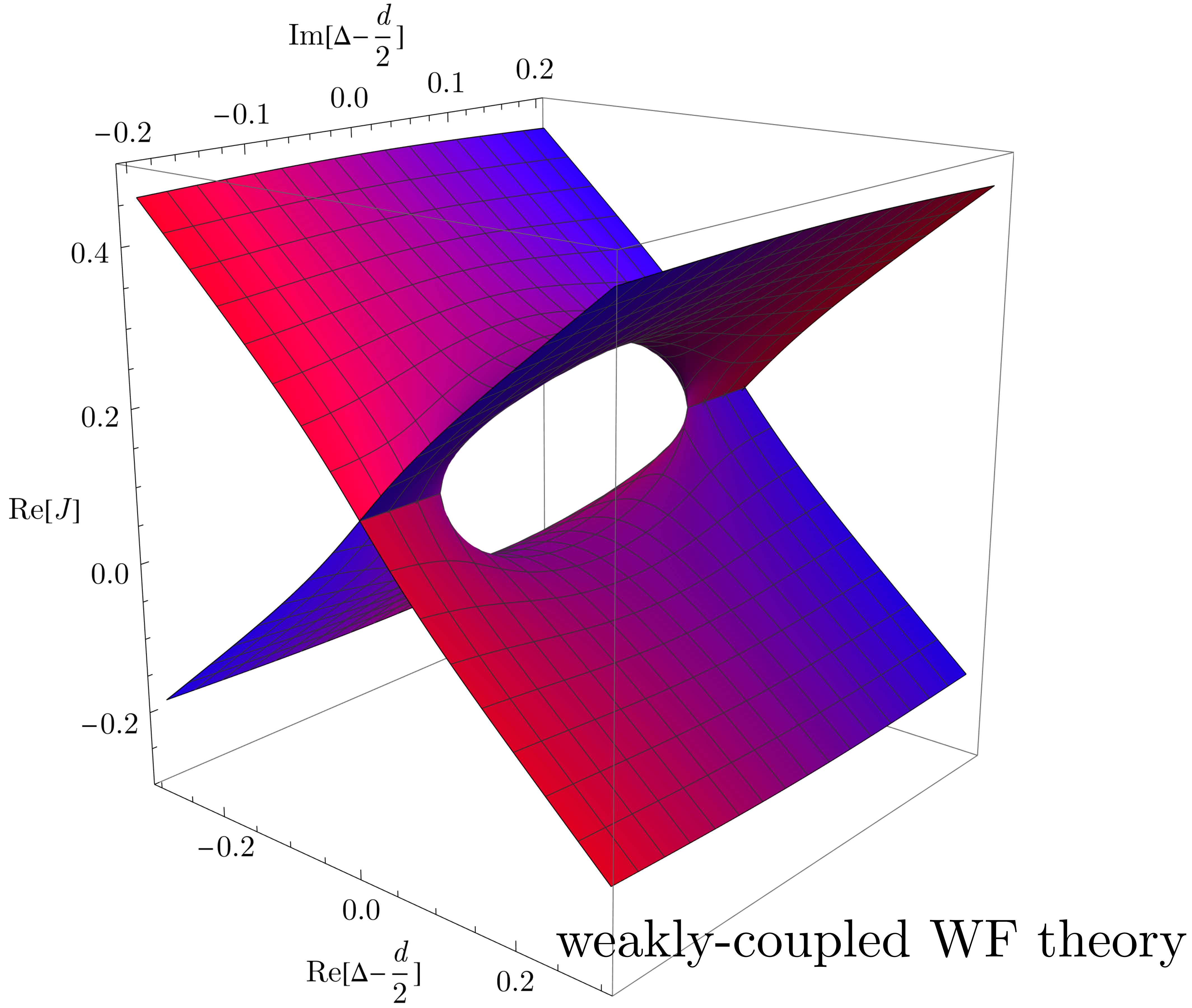
Due to the jet algorithm and other potential cuts, the negative energy weighting region might be affected.

Analytic Structure in Complex Space

When there is a level repulsion phenomenon, we can see branch cuts in complex space.



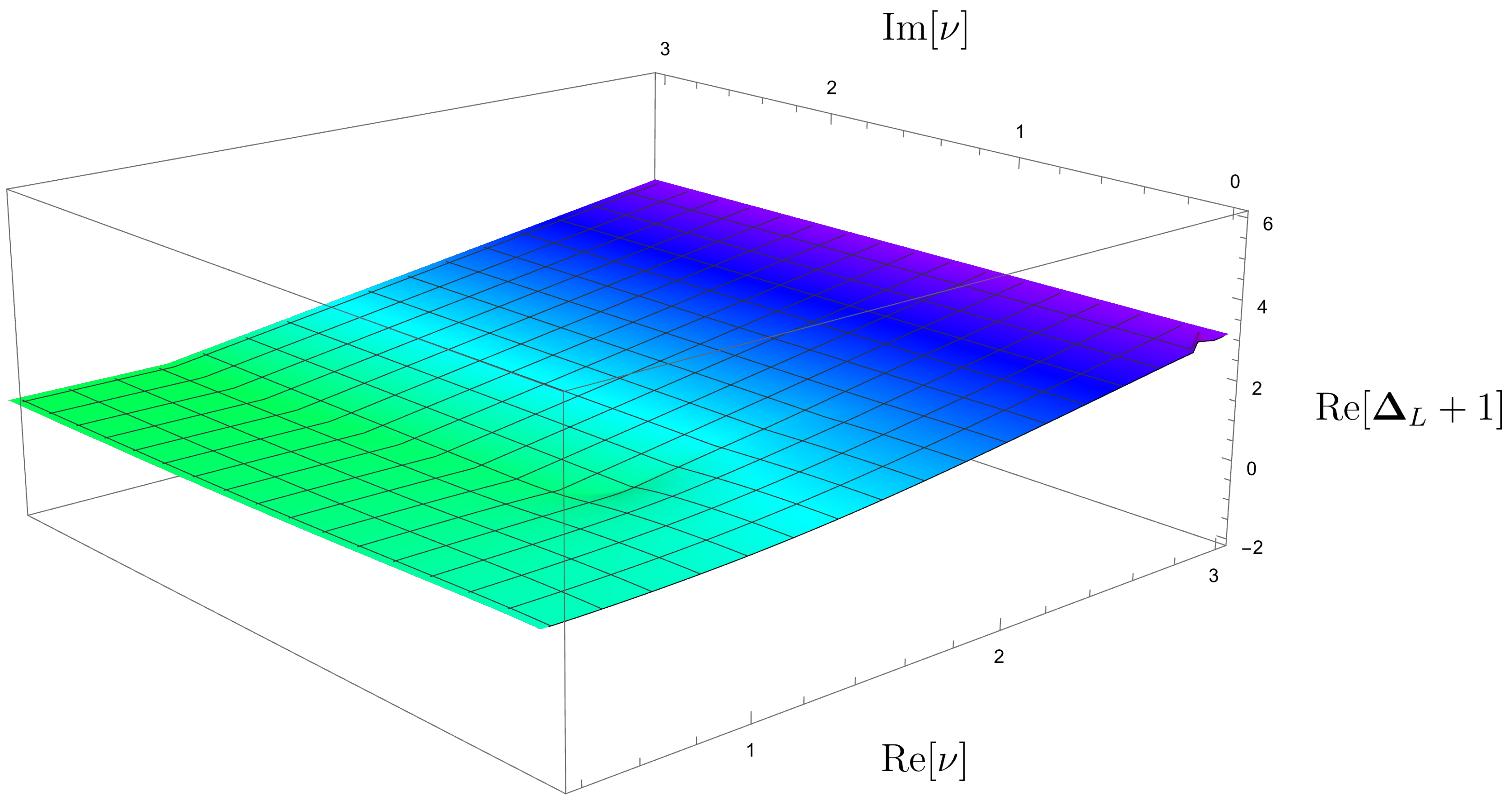
[Gromov, Levkovich-Maslyuk, Sizov, 2015]



[Caron-Huot, Kologlu, Kravchuk, Meltzer, Simmons-Duffin, 2022]

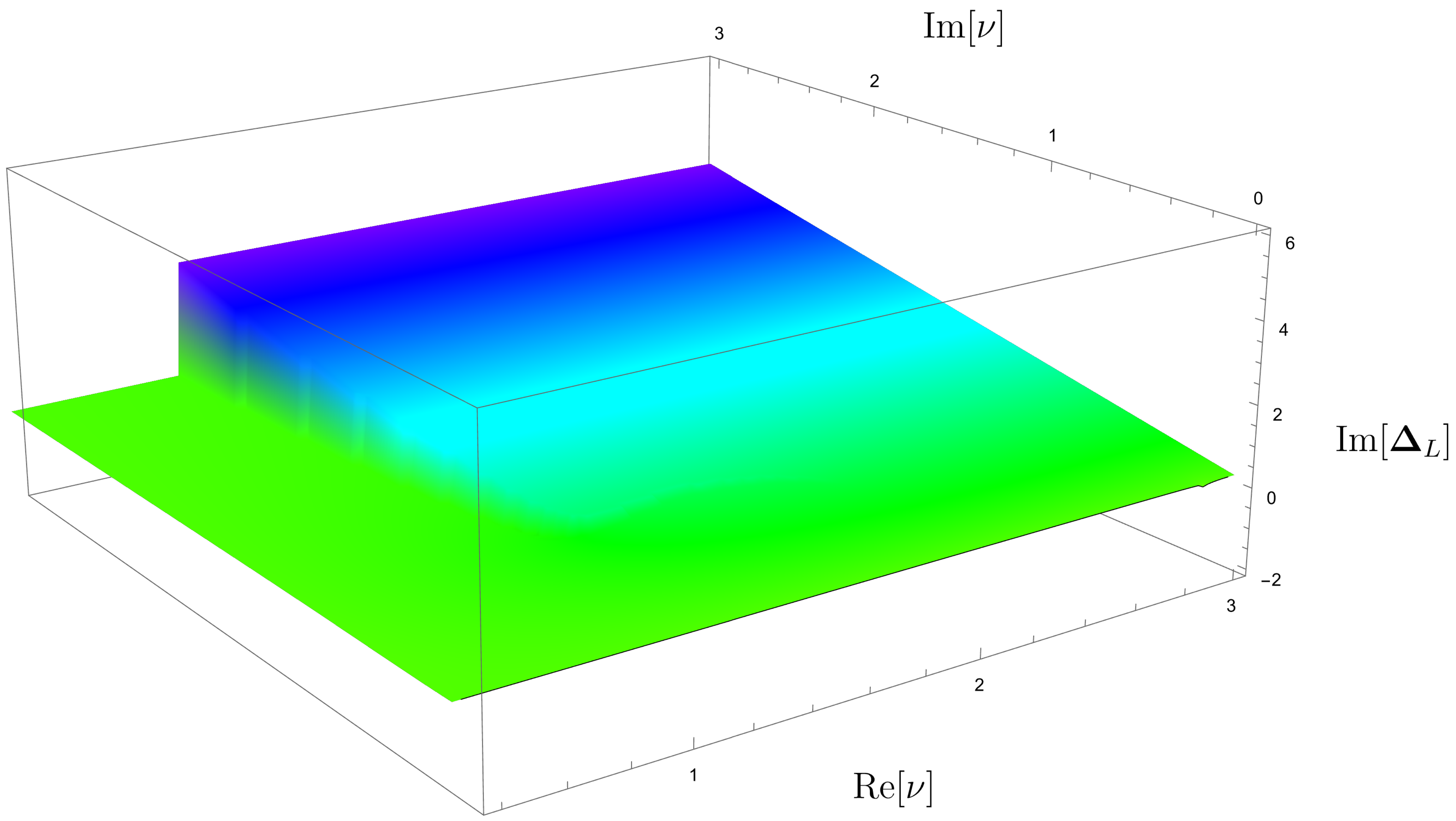
LO Top Trajectory in Complex Space

In complex space, we can easily see the branch cut from the discontinuity in the imaginary part.



Real Part

$\alpha_s = 0.09$

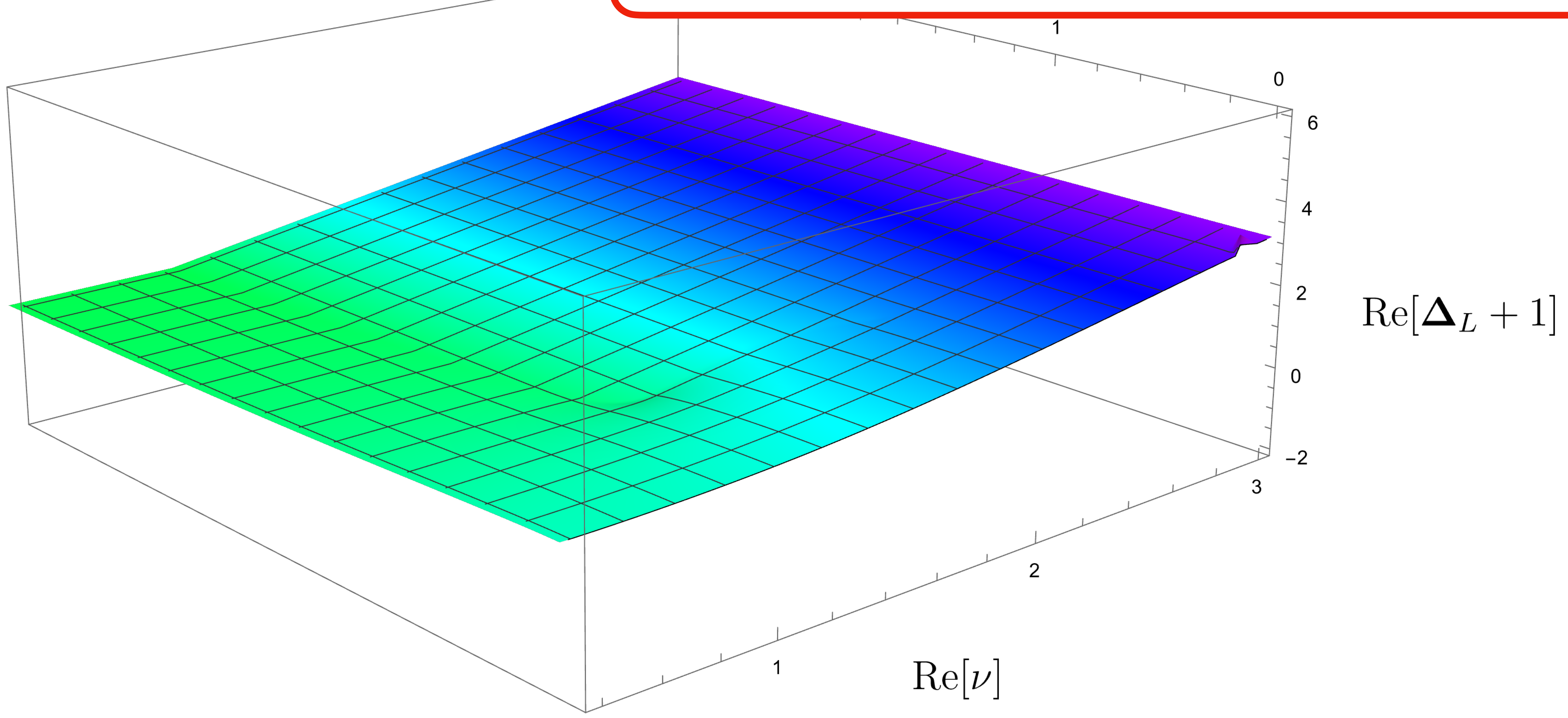


Imaginary Part

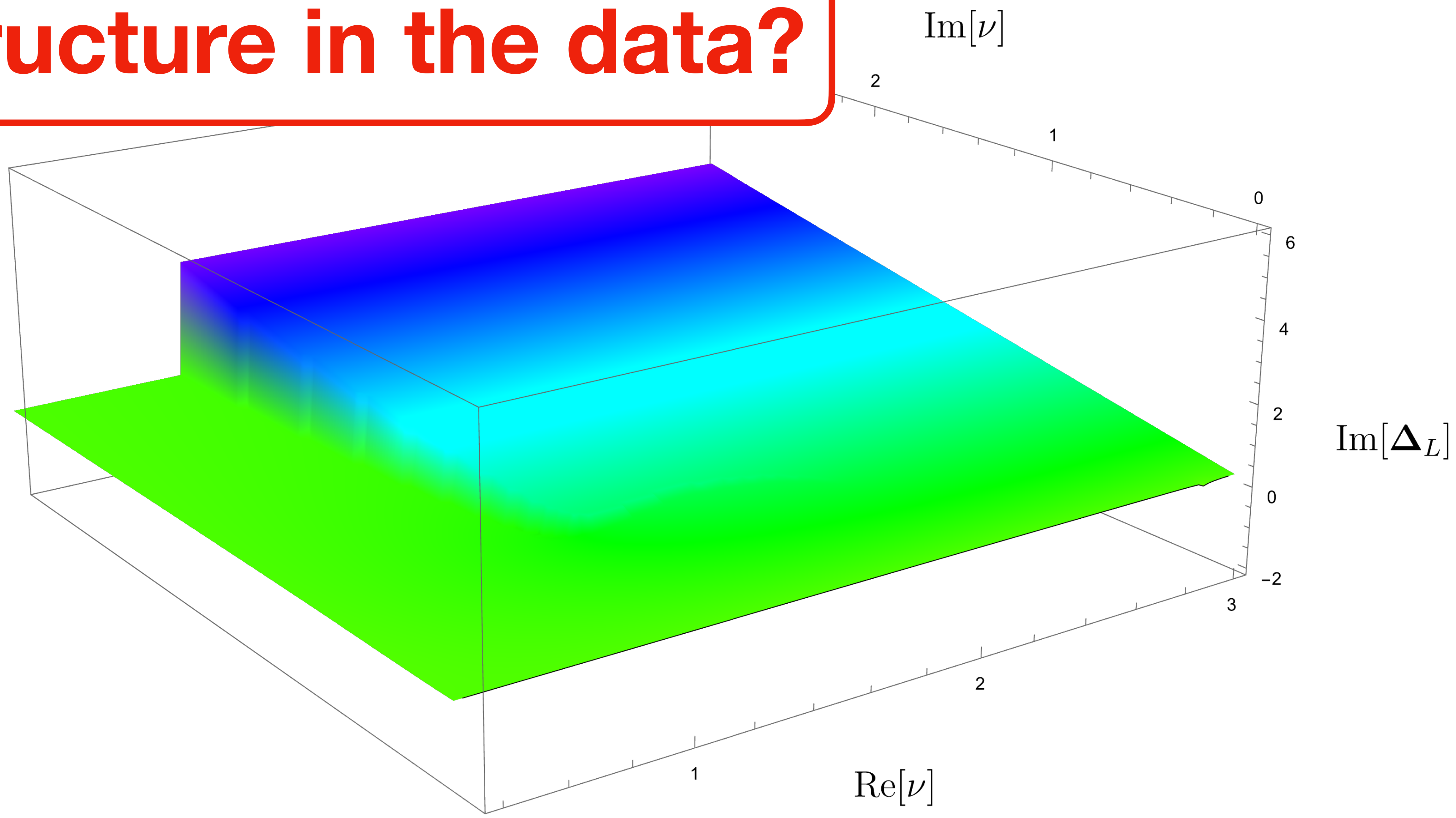
LO Top Trajectory in Complex Space

In complex space, we can easily see the branch cut from the discontinuity in the imaginary part.

Can we find similar structure in the data?



Real Part



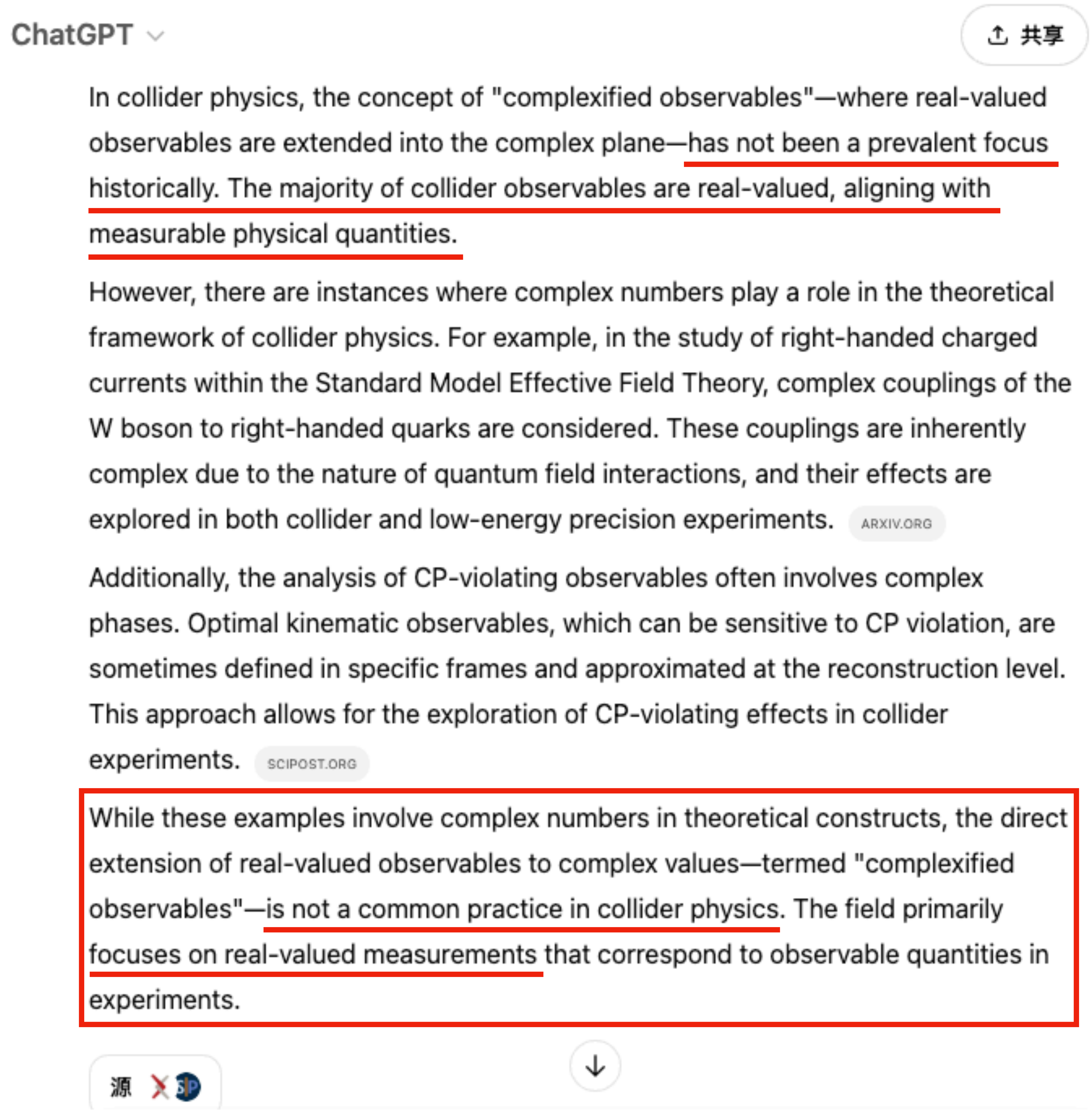
Imaginary Part

$$\alpha_s = 0.09$$

Let’s measure complexified observables!

$$f(\nu,Q) = \frac{1}{\sigma_{\text{tot}}}\sum_{X}\int d\sigma_{e^+e^-\to X}\sum_{h\in X}\left(\frac{E_a}{Q}\right)^{\nu-1}$$

To the best of my knowledge , this might be the first and should be the simplest non-trivial complexified observable in collider physics.



In collider physics, the concept of "complexified observables"—where real-valued observables are extended into the complex plane—has not been a prevalent focus historically. The majority of collider observables are real-valued, aligning with measurable physical quantities.

However, there are instances where complex numbers play a role in the theoretical framework of collider physics. For example, in the study of right-handed charged currents within the Standard Model Effective Field Theory, complex couplings of the W boson to right-handed quarks are considered. These couplings are inherently complex due to the nature of quantum field interactions, and their effects are explored in both collider and low-energy precision experiments. ARXIV.ORG

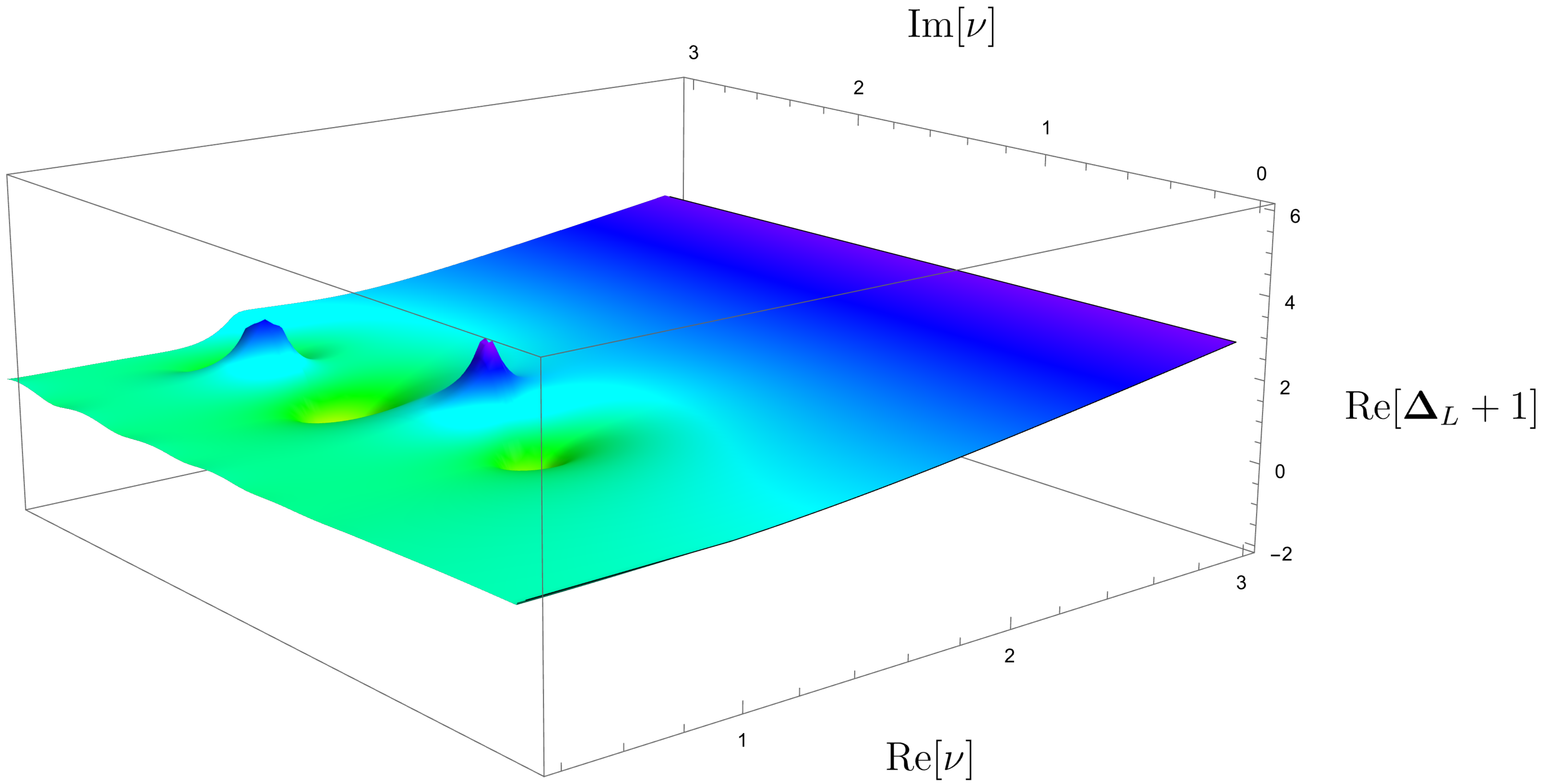
Additionally, the analysis of CP-violating observables often involves complex phases. Optimal kinematic observables, which can be sensitive to CP violation, are sometimes defined in specific frames and approximated at the reconstruction level. This approach allows for the exploration of CP-violating effects in collider experiments. SCIPOST.ORG

While these examples involve complex numbers in theoretical constructs, the direct extension of real-valued observables to complex values—termed "complexified observables"—is not a common practice in collider physics. The field primarily focuses on real-valued measurements that correspond to observable quantities in experiments.

Pythia Data

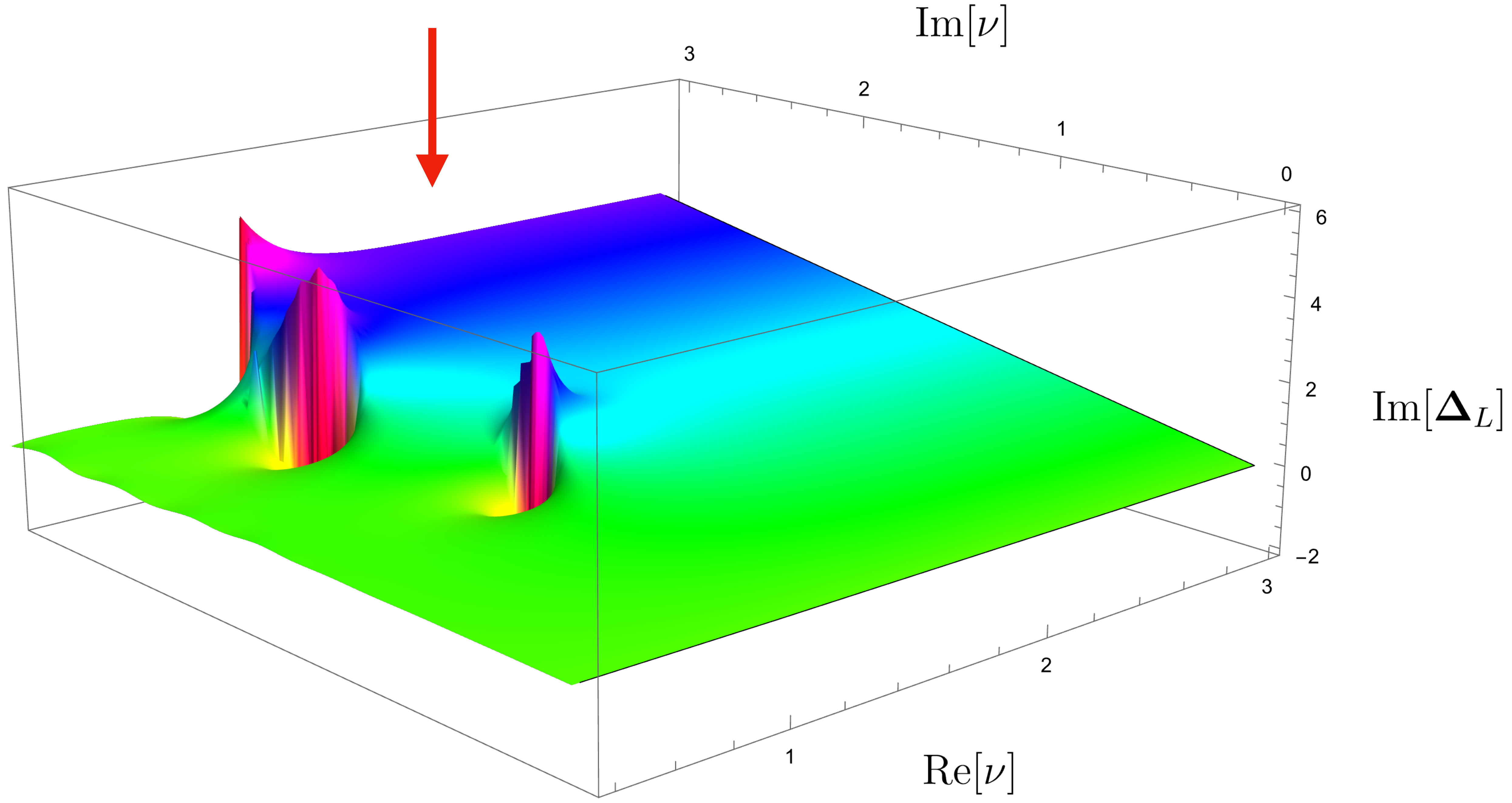
γ^*-decay events

Near the location of anticipated branch cuts, we see violent changes — a series of peaks and dips!



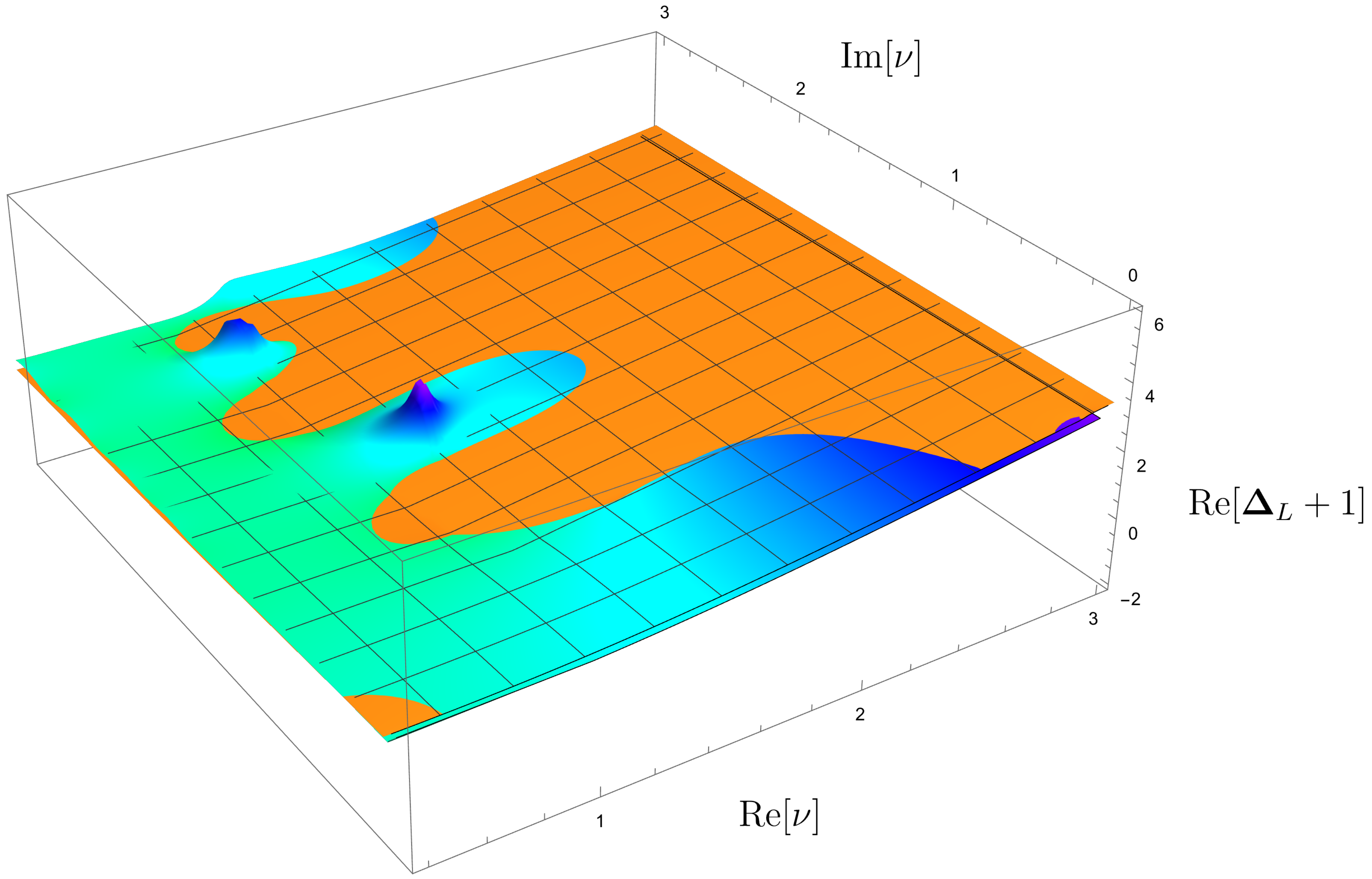
Real Part

We can clearly see the discontinuity!

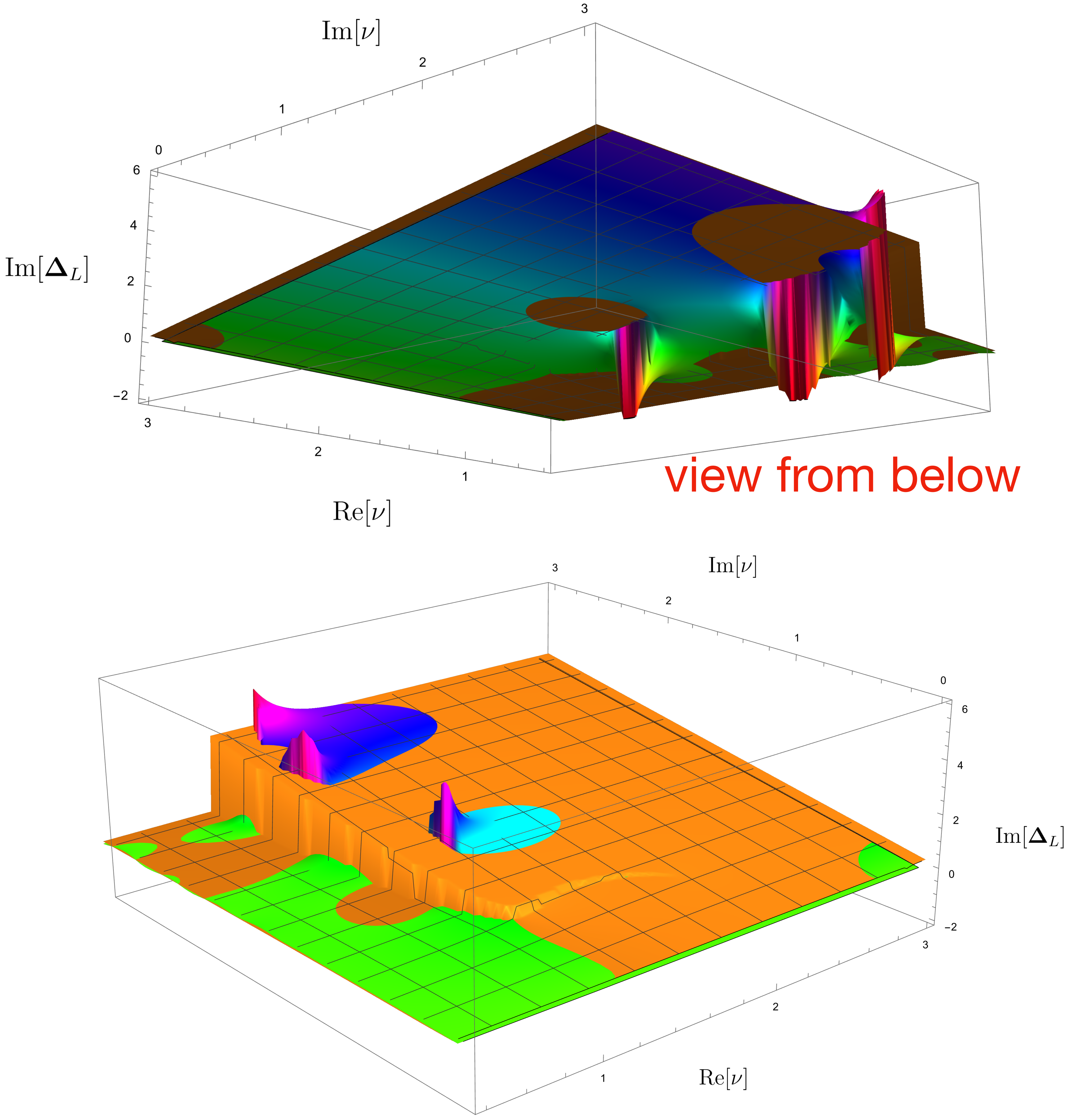


Imaginary Part

Pythia Data vs LO prediction



Real Part

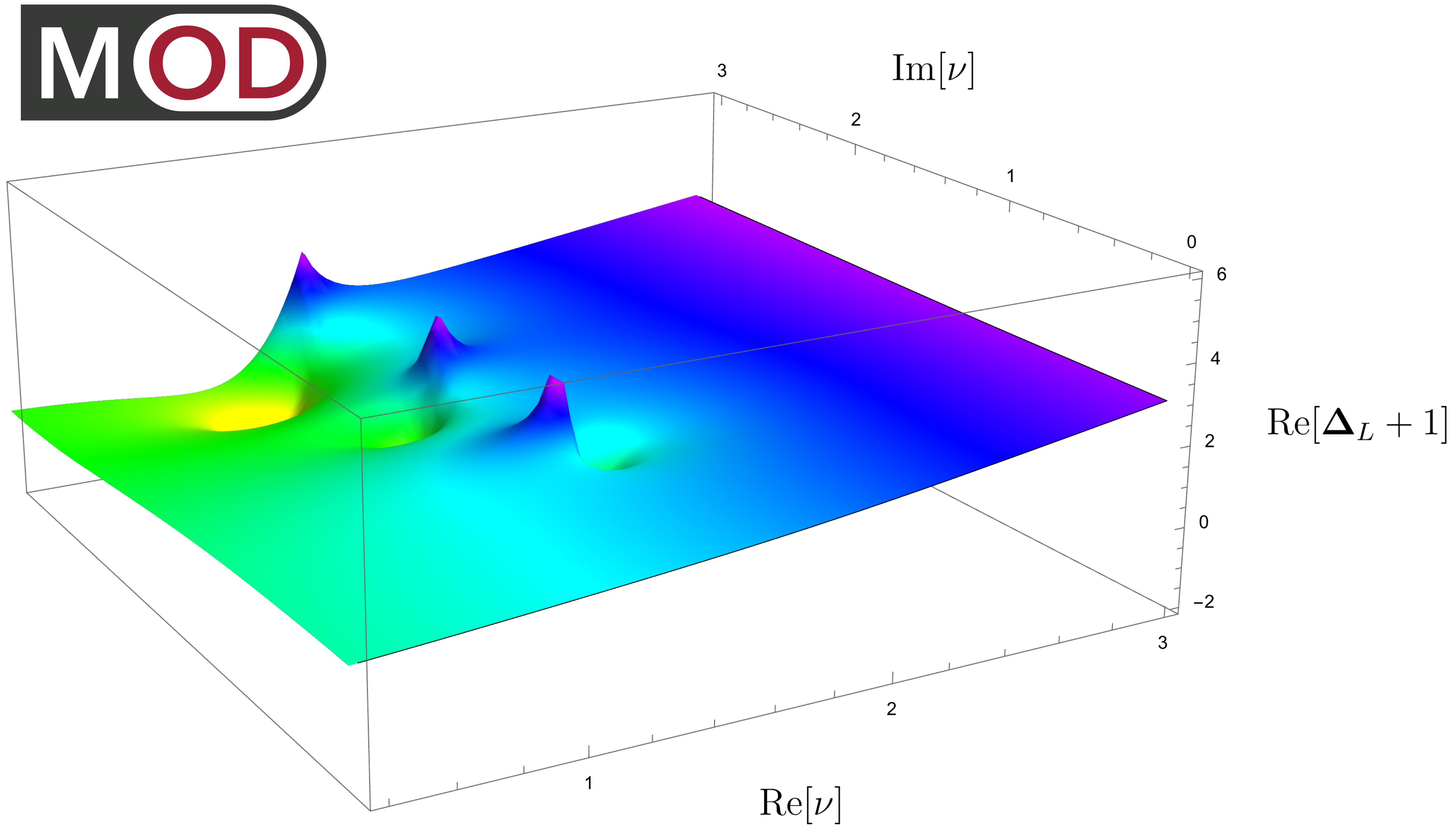


view from below

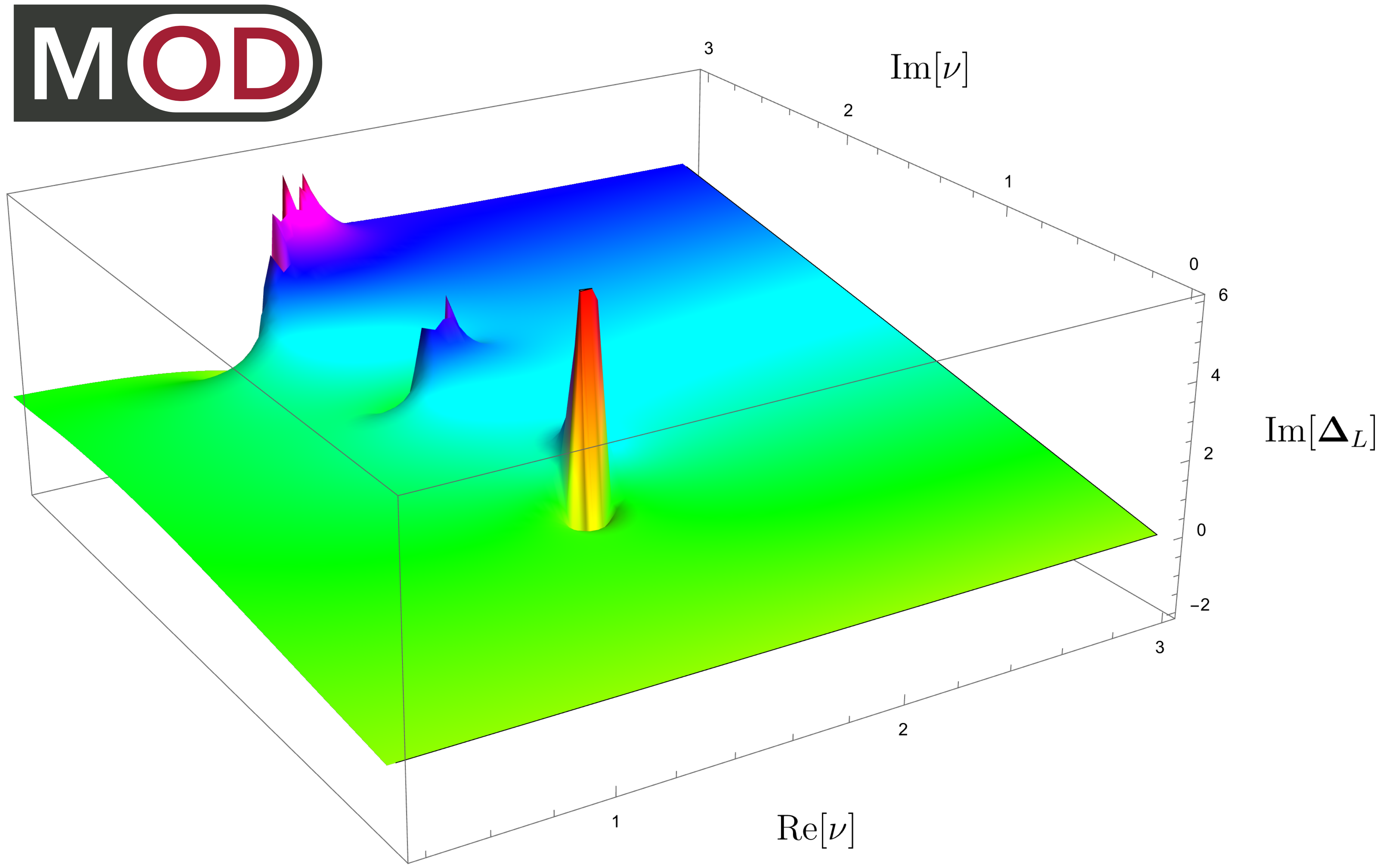
Imaginary Part

CMS Open Data

In complex space, the structure of branch cut seems to be robust to the effects of jet algorithms and other reasonable experimental cuts.



Real Part

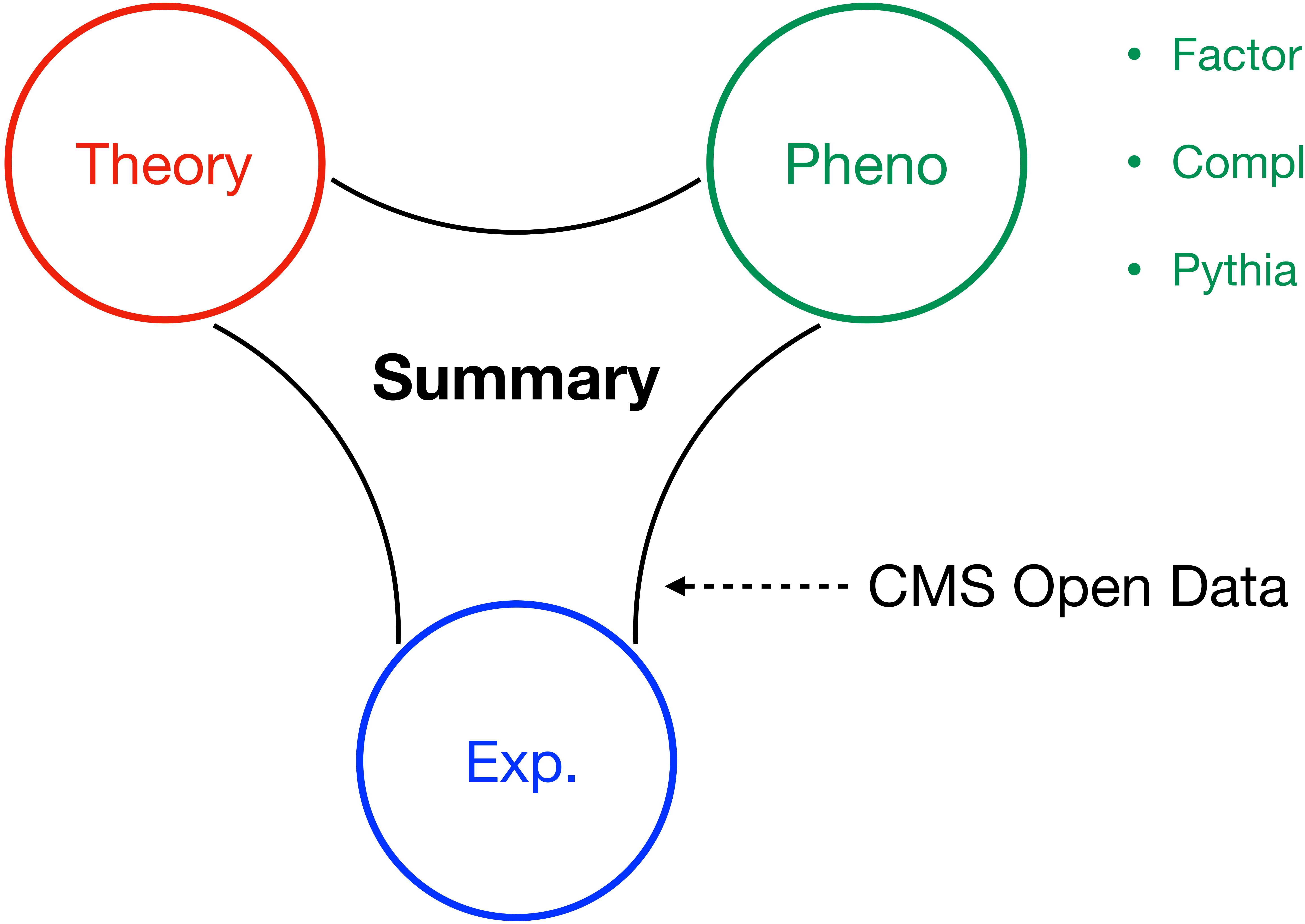


Imaginary Part

Summary

light-ray operators
BFKL/DGLAP mixing

- renormalization
- level repulsion
- analyticity



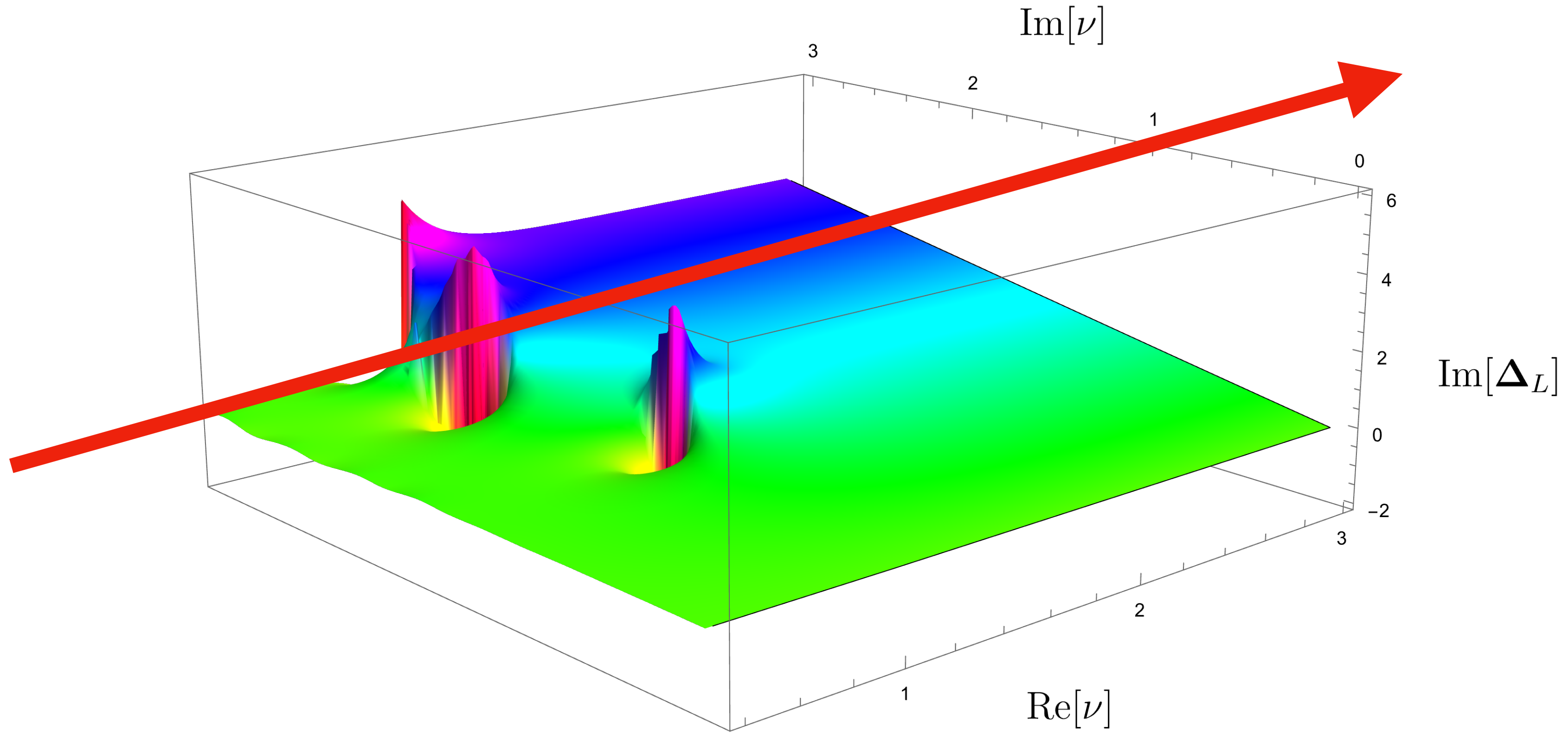
One-point event shape

- Factorization
- Complexification
- Pythia simulation

Thanks!

need our experimental colleagues

Behaviors Near Branch Cut



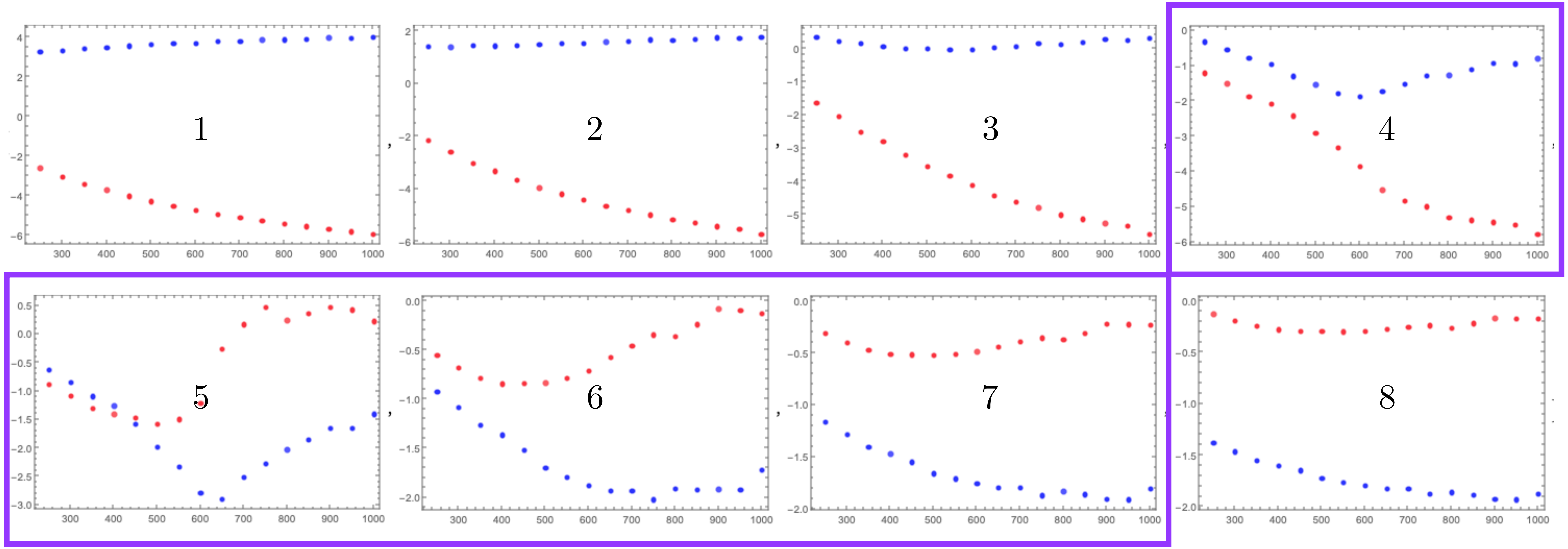
Choose a fixed $\mathrm{Im}\,\nu$ slice and increase $\mathrm{Re}\,\nu$ to cross the branch cut

$x-$axis: Q

$y-$axis: $\ln(f(\nu,Q))$

● **real part**

● **imaginary part**



Near the branch cut, the ansatz $\ln(f) = \gamma \ln Q + c$ is not a good approximation