Differential Equations and A Boostrap Approach of Energy Correlators

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Base on work: with Jingwen Lin, Rourou Ma, Kai Yan, Gang Yang and Yang Zhang, arxiv:2505.xxxxx
with Andrzej Pokraka, Kai Yan, Xiaoyuan Zhang, arxiv:2506.xxxxx

Outline

- Differential Equations Method of Any Angle Scattering ENC in N=4 sYM

  Algorithms of Differential Equations

  Analytic Information of the Master Integrals

  Symbol Calculations

- Bootstrap Method of Energy Correlators in the Collinear Limit in N=4 sYM and QCD

  Highest Weight Calculations

  Lower weight ansatz

  Squeeze Limit Informations

  Extra Information from Light-Ray OPE

- Summary and Outlooks

N-Point Energy Correlators (ENC) in Any Angle Scattering

$$\mathrm{E^nC}(\theta_{ij}) = \int \prod_{i=1}^{n} \mathrm{d}\Omega_{\vec{n}_i} \prod_{i\neq j} \delta(\vec{n}_i \cdot \vec{n}_j - \cos\theta_{ij}) \frac{\int d^4x e^{iqx} \langle 0|\mathcal{O}^+(x)\mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_n)\mathcal{O}(0)|0\rangle}{(q^0)^n \int d^4x e^{iqx}\langle 0|\mathcal{O}^+(x)\mathcal{O}(0)|0\rangle},$$

$$\int \mathrm{d}^4x\, e^{iqx}\langle X|\mathcal{O}(x)|0\rangle \equiv (2\pi)^4\delta^4(q-q_X)\mathcal{F}_X.$$

Form Factor Depends on p_i

$$\mathrm{E^nC} = \frac{1}{\sigma_{\mathrm{tot}}} \sum_{(n_1,\cdots,n_n)\in X} \int \mathrm{d}\Pi_X \left(\prod_{i=1}^{n} \delta^2(\vec{n}_i - \hat{p}_{n_i})\frac{E_i}{q^0}\right) |\mathcal{F}_X|^2,$$

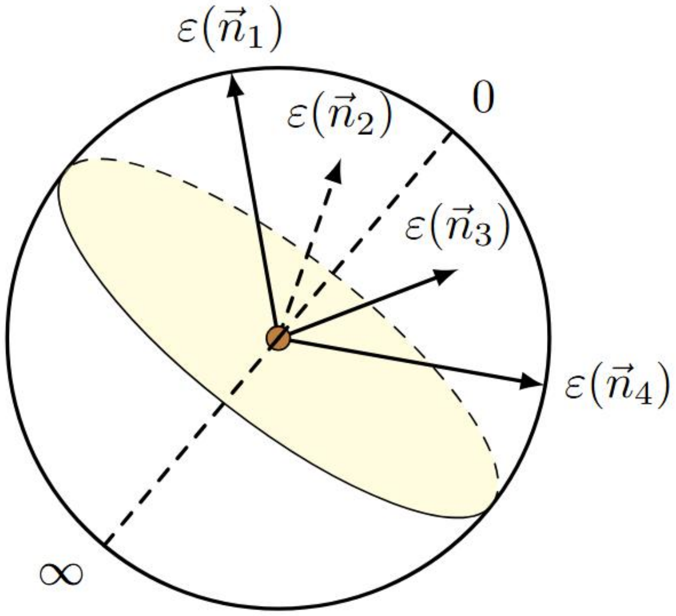
On-shell Phase Space of final states

energy fraction: $x_i = \frac{2q\cdot p_i}{q^2} \qquad i = 1,\cdots,n\,;$

angle parameter: $\zeta_{ij} = \frac{q^2(p_i\cdot p_j)}{2(q\cdot p_i)(q\cdot p_j)} \qquad i,j = 1,\cdots,n\,;$

$$\mathrm{E^nC}(\zeta_{ij}) = \frac{1}{\sigma_{\mathrm{tot}}}\int_0^1 \mathrm{d}x_1\cdots\mathrm{d}x_n\,(x_1\cdots x_n)^2\,\delta(1-Q_n)\left|\mathcal{F}_{n+1}^{(0)}\right|^2$$

$$\delta(1-Q_n) = \delta\left(1 - \sum_i x_i + \sum_{1\le i<j\le n}\zeta_{ij}\,x_i\,x_j\right)$$



Differential Equations: Mainstream Analytic Method

- Integration by Parts (IBP)   Analytic Tool for Feynman Integrals, Smirnov

e.g $$F(a)=\int\frac{\mathrm{d}^dk}{(k^2-m^2)^a}$$

Use this set of relations to solve the reduction problem

IBP Identity $$\int \mathrm{d}^dk\frac{\partial}{\partial k_\mu}\cdot k_\mu\frac{1}{(k^2-m^2)^a}=0$$

$$(d-2a)F(a)-2am^2F(a+1)=0$$

- (Canonical) Differential Equations   J. Henn 13'

D=$d-2\epsilon$

$$\mathrm{d}\vec{f}(\vec{x},\epsilon)=\epsilon\left[\mathrm{d}\tilde{A}(\vec{x})\right]\vec{f}(\vec{x},\epsilon)$$

$$\mathrm{d}\tilde{A}(\vec{x})=\sum_k a_k \mathrm{d}\log(\alpha_k)$$

Constant Matrix   Symbol letters

$$\xrightarrow[\text{integrals}]{\text{Chen's iterated}}$$

$$\vec{f}(\vec{x})=P\exp\left(\epsilon\int\mathrm{d}\tilde{A}\right)\vec{f}(\vec{x}_0)$$

EEC comparing to Feynman Integrals:

- Always Finite Integrals
- No dim-reg
- Different Types of Propagators

Differential Equations: Overview

3-Points

Verify Finite Integrals → Partial Fraction Decomposition to Simplify the integrand

⇩

Syzygy Equations for IBP → Combine Divergent Integrals into Finite Integrals

⇩

Lift Equations for Finite Integrals → Also need Additional Relations

⇩

Boundary IBP

⇩

Canonical Differetial Equations

4-Points

Difficulties of Higher Points

- Increasing number of integral variables
- Polynomial Propagators
- Complex Measurement because of the delta function

E3C Integrand and Simplification

- Integrand: Form Factor Square

$$\frac{(q^2)^2}{s_{12}s_{13}s_{24}s_{34}}+\frac{(q^2)^2}{s_{12}s_{13}s_{124}s_{134}}+\frac{4(q^2)^2}{s_{12}s_{34}s_{124}s_{134}}+\frac{4(q^2)^2}{s_{13}s_{24}s_{123}s_{124}}+\frac{2(q^2)^2}{s_{23}s_{24}s_{123}s_{124}}+\frac{(q^2)^2}{s_{24}s_{34}s_{124}s_{134}}.$$

where $s_{ij} = x_i x_j \zeta_{ij}$ and $p_4 = q - p_1 - p_2 - p_3$

- Using delta function $\delta(1-Q_n) = \delta\left(1 - \sum_i x_i + \sum_{1\le i<j\le n} \zeta_{ij}\, x_i\, x_j\right)$

All the propagators in E3C is linear!

- By Partial Fraction Decomposition (PFD)

$$\begin{aligned}
&\text{family 1} \quad && D_1 = -1 + x_2,\ D_2 = -1 + x_3,\ D_3 = -1 + x_1 + x_2 + x_3,\\
&\text{family 2} \quad && D_1 = -1 + x_2,\ D_2 = -1 + x_1 + x_2 + x_3,\ D_3 = -1 + x_1\zeta_{12} + x_3\zeta_{23},\\
&\text{family 3} \quad && D_1 = -1 + x_3,\ D_2 = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ D_3 = -1 + x_1\zeta_{12} + x_3\zeta_{23}.
\end{aligned}$$

E3C Finite Integrals

- Check the finiteness of the Integrals

An integral is finite if it is finite in every potentially divergent region ⇒ Check all possible regions!

Propagators = 0

- Example:

$$I(x_0, y_0) = \int_0^{x_0} \mathrm{d}x \int_0^{y_0} \mathrm{d}y \frac{1}{x+y} = (x_0 + y_0) \log(x_0 + y_0) - x_0 \log(x_0) - y_0 \log(y_0)$$

$$\Big\downarrow \ \boxed{x \to cx, y \to cy;\ c \to 0}$$

$$I(x_0, y_0) \to \int_0^{x_0} \mathrm{d}(cx) \int_0^{y_0} \mathrm{d}(cy) \frac{1}{cx+cy} \xrightarrow[\text{counting}]{\text{power}} 1 > 0$$

For infinite integrals, rewrite them into sum of finite integrals. (Find another basis)

E3C Divergent Region

- Consider the Delta Function

$$\delta(1 - x_1 - x_2 - x_3 + \zeta_{12}x_1x_2 + \zeta_{23}x_2x_3 + \zeta_{13}x_1x_3)$$

$$\begin{aligned}
&\text{region 1} \quad && \{x_1 \to 1 + (\zeta_{12} + \zeta_{13} - 2)cx_1,\ x_2 \to cx_2,\ x_3 \to cx_3\}|_{c\to 0}\,, \\
&\text{region 2} \quad && \{x_1 \to cx_1,\ x_2 \to 1 + (\zeta_{12} + \zeta_{23} - 2)cx_2,\ x_3 \to cx_3\}|_{c\to 0}\,, \\
&\text{region 3} \quad && \{x_1 \to cx_1,\ x_2 \to cx_2,\ x_3 \to 1 + (\zeta_{13} + \zeta_{23} - 2)cx_3\}|_{c\to 0}\,.
\end{aligned}$$

Potentially Divergent Region

- propagators

$$\mathcal{D}_1 = \frac{s_{134}}{q^2} = -1 + x_2,\ \mathcal{D}_2 = \frac{s_{124}}{q^2} = -1 + x_3,\ \mathcal{D}_3 = \frac{s_{123}}{q^2} = -1 + x_1 + x_2 + x_3,$$

$$\mathcal{D}_4 = \frac{s_{34}}{q^2 x_3} = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ \mathcal{D}_5 = \frac{s_{24}}{q^2 x_2} = -1 + x_1\zeta_{12} + x_3\zeta_{23}.$$

- Divergent Integrals

$$\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5\} \xrightarrow[\text{counting}]{\text{power}} \begin{cases} \text{region 1} & \{0, 0, 1, 0, 0\} \\ \text{region 2} & \{1, 0, 1, 0, 0\}. \\ \text{region 3} & \{0, 1, 1, 0, 0\} \end{cases}$$

Syzygy and Lift Equations

- Integration by parts (IBP) $\longrightarrow$ Syzygy Equations

$$\mathcal{O}_{\text{IBP}} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(a_i \cdot\ )$$

$$\sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0,$$

$$a_i,\ b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3]$$

Not increase the power of propagators!

- Derivative of a certain parameter $\longrightarrow$ Lift Differential Equations

$$\mathcal{O}_{\partial\zeta_{**}} = \frac{\partial}{\partial \zeta_{**}} + \mathcal{O}_{\text{IBP}} \equiv \boxed{\frac{\partial}{\partial \zeta_{**}} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i}$$

Find the differential equations of a certain kinematic

May bring higher power of divergent propagators

$$\frac{\partial}{\partial \zeta_{**}} D_j + \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0,$$

$$a_i,\ b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3]$$

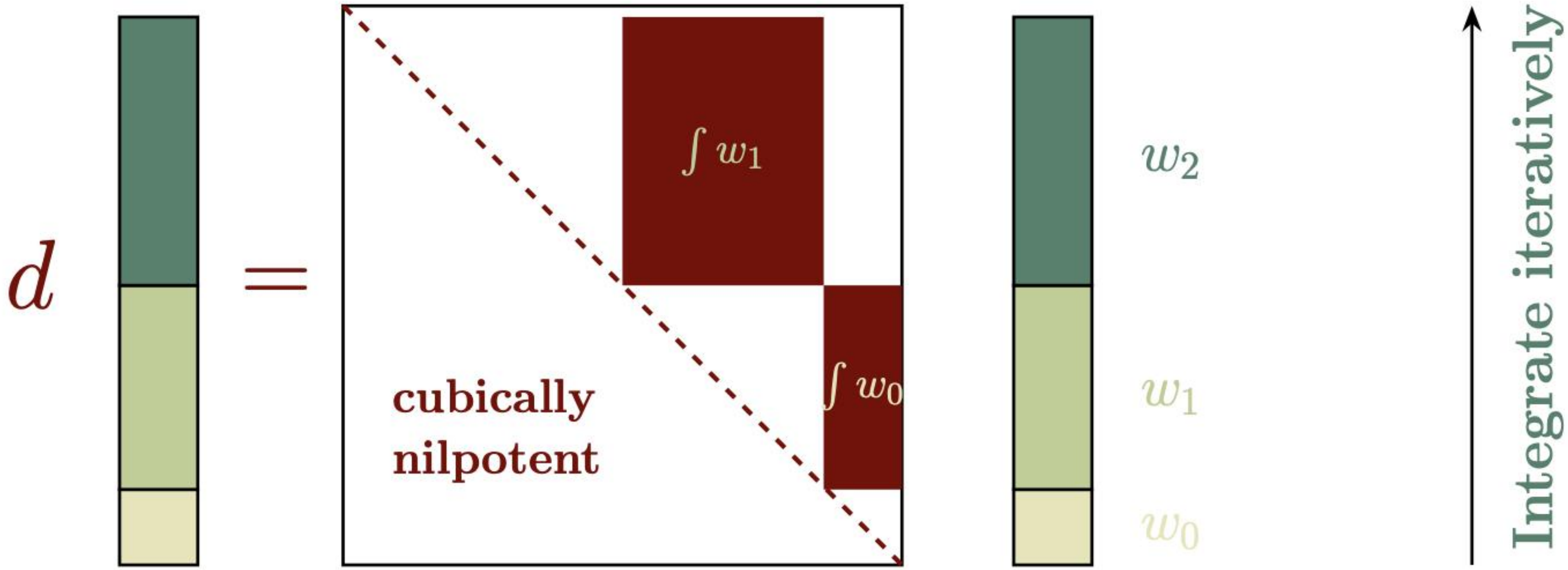
When the Lift equation is satisfied, the integral will be finite!

Boundary IBP and Canonical Differential Equations

- Differential Equations include a boundary term

$$\frac{\partial}{\partial \zeta_{**}} \mathrm{Int}[n_1, n_2, n_3, 1] - \mathcal{O}_{\partial \zeta_{**}} \mathrm{Int}[n_1, n_2, n_3, 1] = \sum_{i=1}^{3} (\mathrm{BT}_{x_i=0}).$$

Not only depends on lower point energy correlators but other terms from the other boundary of integral contour



E4C Difficulties

- More integration variables and parameters
- More complicated measurement

$$\delta(1-x_1-x_2-x_3-x_4+x_1x_2\zeta_{12}+x_1x_3\zeta_{13}+x_1x_4\zeta_{14}+x_2x_3\zeta_{23}+x_2x_4\zeta_{24}+x_3x_4\zeta_{34})$$

- Complex propagators (involving quadric propagators)

$$\frac{s_{45}}{x_4}=1-x_1\zeta_{14}-x_2\zeta_{24}-x_3\zeta_{34}\,,\quad \frac{s_{15}}{x_1}=1-x_2\zeta_{12}-x_3\zeta_{13}-x_4\zeta_{14}\,,\quad s_{2345}=1-x_1\,,$$
$$s_{1345}=1-x_2\,,\quad s_{1235}=1-x_4\,,\quad s_{1234}=1-x_1-x_2-x_3-x_4\,,$$
$$s_{345}=1-x_1-x_2+x_1x_2\zeta_{12}\,,\, s_{145}=1-x_2-x_3+x_2x_3\zeta_{23}\,,\, s_{125}=1-x_3-x_4+x_3x_4\zeta_{34}\,,$$
$$s_{123}=x_1x_2\zeta_{12}+x_1x_3\zeta_{13}+x_2x_3\zeta_{23}\,,\, s_{234}=x_2x_3\zeta_{23}+x_2x_4\zeta_{24}+x_3x_4\zeta_{34}\,,\, s_{1245}=1-x_3\,.$$

- More Divergent Integrals $\sum_n I_n^{\infty}=\sum_k I_k^{\text{finite}}$

At most 3 in each integral after partial fraction decomposition

Master Integrals of Three and Four-Point Any Angle

Three Points

Into 3 families

Propagators

family 1 $\quad D_1 = -s_{134},\ D_2 = -s_{124},\ D_3 = -s_{123},$

family 2 $\quad D_1 = -s_{134},\ D_2 = -s_{123},\ D_3 = \frac{s_{24}}{x_2},$

family 3 $\quad D_1 = -s_{124},\ D_2 = \frac{s_{34}}{x_3},\ D_3 = \frac{s_{24}}{x_2}.$

$$D_\delta = 1 - x_{123} - s_{123}$$

subfamily 1 $\quad D_1 = \frac{s_{24}}{x_2}|_{x_3\to 0},\ D_2 = \frac{s_{34}}{x_3}|_{x_3\to 0},\ D_\delta = (1 - x_{123} - s_{123})|_{x_3\to 0}\,,$

subfamily 2 $\quad D_1 = \frac{s_{14}}{x_1}|_{x_2\to 0},\ D_2 = \frac{s_{34}}{x_3}|_{x_2\to 0},\ D_\delta = (1 - x_{123} - s_{123})|_{x_2\to 0}\,,$

subfamily 3 $\quad D_1 = \frac{s_{24}}{x_2}|_{x_1\to 0},\ D_2 = \frac{s_{34}}{x_3}|_{x_1\to 0},\ D_\delta = (1 - x_{123} - s_{123})|_{x_1\to 0}\,.$

Master Integrals

Family 1

$$\left\{\frac{x_2}{D_2\,D_3\,D_4},\ \frac{D_1}{D_3\,D_4},\ \frac{D_2}{D_3\,D_4},\ \frac{1}{D_3\,D_4},\ \frac{1}{D_2\,D_4},\ \frac{D_2}{D_1\,D_4},\ \frac{1}{D_1\,D_4},\ \frac{1}{D_1\,D_2\,D_4},\ \frac{1}{D_{\mathrm{sub1},3}},\ \frac{1}{D_{\mathrm{sub2},3}},\ \frac{1}{D_{\mathrm{sub3},3}},\ 1\right\}$$

Family 2

$$\left\{\frac{D_2}{D_1\,D_3\,D_4},\ \frac{D_1}{D_2\,D_3\,D_4},\ \frac{1}{D_1\,D_3\,D_4},\ \frac{1}{D_2\,D_3\,D_4},\ \frac{D_3}{D_2\,D_4},\ \frac{D_1}{D_2\,D_4},\ \frac{1}{D_1\,D_4},\ \frac{1}{D_2\,D_4},\ \frac{1}{D_{\mathrm{sub1},3}},\ \frac{1}{D_{\mathrm{sub2},3}},\ \frac{1}{D_{\mathrm{sub2},2}\,D_{\mathrm{sub2},3}},\ \frac{1}{D_{\mathrm{sub3},3}},\ 1\right\}$$

Family 3

$$\left\{\frac{D_3}{D_1\,D_2\,D_4},\ \frac{D_1}{D_2\,D_3\,D_4},\ \frac{1}{D_1\,D_2\,D_4},\ \frac{1}{D_2\,D_3\,D_4},\ \frac{1}{D_1\,D_4},\ \underbrace{\frac{1}{D_{\mathrm{sub1},3}},\ \frac{1}{D_{\mathrm{sub1},2}\,D_{\mathrm{sub1},3}},\ \frac{1}{D_{\mathrm{sub2},3}},\ \frac{1}{D_{\mathrm{sub3},3}}},\ 1\right\}$$

Also depends on subfamily integrals!
The subscripts describes which subfamily the propagtor belongs to.

Four Points

13 propagators, 26 families, 55 sectors

$$D_1 = \frac{s_{45}}{x_4},\quad D_2 = s_{123},\quad D_3 = \frac{s_{15}}{x_1},\quad D_4 = -s_{125},\quad D_5 = -s_{1234},$$

$$D_6 = -s_{1235},\quad D_7 = -s_{145},\quad D_8 = -s_{1245},\quad D_9 = -s_{345},\quad D_{10} = -s_{1345},$$

$$D_{11} = -s_{2345},\quad D_{12} = s_{234},\quad D_{13} = 1 - x_{1234} - s_{1234}$$

Propagators	Count	Numerators
D_2	2	$1, x_1$
D_4	3	$1, x_1, x_2$
D_{11}	1	1
D_7, D_{12}	6	$1, x_1, x_2, x_2^2, x_3^2, x_2x_4$
D_2, D_{10}	3	$1, x_1, x_3$
D_6, D_{10}	1	1
D_6, D_7	3	$1, x_2, x_3$
D_5, D_{12}	3	x_2, x_3, x_4
D_5, D_{11}	1	1
D_4, D_5	5	$1, x_1, x_1^2, x_2, x_3$
D_4, D_9	9	$1, x_1, x_1^2, x_2, x_3, x_1x_3, x_2x_3, x_4, x_1x_4$
D_4, D_7	4	$1, x_1, x_2, x_4$
D_3, D_5	1	1
D_2, D_{12}	9	$x_2, x_1x_2, x_1^2x_2, x_2^2, x_1x_2^2, x_3, x_1x_3, x_3^2, x_2x_4$
D_2, D_4	5	$x_1, x_2, x_3, x_1x_4, x_3x_4$
D_2, D_3	3	$1, x_1, x_2$
D_1, D_6	1	1
D_1, D_4	3	$1, x_1, x_2$
D_1, D_3	1	1
D_5, D_6, D_{12}	3	x_2, x_1x_2, x_3
D_3, D_5, D_6	2	$1, x_1$

Propagators	Count	Numerators
D_2, D_6, D_{12}	2	x_1x_2, x_2^2
D_2, D_6, D_{10}	2	x_1, x_2
D_2, D_6, D_7	4	x_1, x_1^2, x_2, x_1x_2
D_2, D_5, D_7	6	$x_1, x_1^2, x_1^3, x_1x_2, x_1x_3, x_2x_4$
D_2, D_4, D_5	3	x_1^2, x_2^2, x_1x_2
D_2, D_3, D_{10}	2	$1, x_1$
D_2, D_3, D_6	2	x_1, x_2
D_2, D_3, D_4	4	x_1, x_2, x_3, x_3^2
D_1, D_7, D_{12}	4	$1, x_1, x_2, x_4$
D_1, D_6, D_{10}	2	$1, x_1$
D_1, D_6, D_7	4	$1, x_1, x_1^2, x_2$
D_1, D_5, D_{12}	5	$x_2, x_1x_2, x_1^2x_2, x_3, x_4$
D_1, D_5, D_7	4	$1, x_1, x_1^2, x_2$
D_1, D_4, D_9	4	$1, x_1, x_2, x_4$
D_1, D_4, D_7	4	$1, x_1, x_2, x_4$
D_1, D_3, D_{10}	2	$1, x_1$
D_1, D_3, D_6	2	$1, x_1$
D_1, D_3, D_4	4	$1, x_1, x_1^2, x_2$
D_1, D_2, D_{12}	3	x_2, x_3, x_1x_4
D_1, D_2, D_6	2	x_1, x_2
D_1, D_2, D_4	3	x_1, x_2, x_3
D_1, D_2, D_3	2	$1, x_1$

Analytic Results of Master Integrals

Three Points

All Master Integrals are MPLs

Symbol Letters

Family 3 for instance

$$z_2,\ z_2-z_3,\ z_3, z_3-z_2,\ \bar{z}_3,\ \bar{z}_3-z_2,\ \bar{z}_3-z_3,$$
$$z_2^2+1,\ z_2z_3+1,\ z_2\bar{z}_3+1,\ z_3\bar{z}_3+1,$$
$$(z_3-z_2)(z_2z_3+1),\ (\bar{z}_3-z_2)(z_2\bar{z}_3+1)$$

Elliptic Integrals $\int \frac{N(x)\mathrm{d}x}{\sqrt{P_4(x)}}$ P_4 degree 4 polynomials

g=2 hyperelliptic $\int \frac{N(x)\mathrm{d}x}{\sqrt{P_6(x)}}$ P_6 degree 6 polynomials

Four Points

Including 3 elliptic curves and a hyperelliptic curve

Some MPLs cannot be integrated directly
Symbol Integration

Elliptic Curves

$$\frac{\delta(D_\delta)}{D_4D_5},\ \frac{\delta(D_\delta)}{D_5D_7},\ \frac{\delta(D_\delta)}{D_4D_9}$$

Hyperelliptic Curves, g=2

$$\frac{\delta(D_\delta)}{D_2D_{12}}$$

Check by:

- Maximal Cut
- Leading singularities

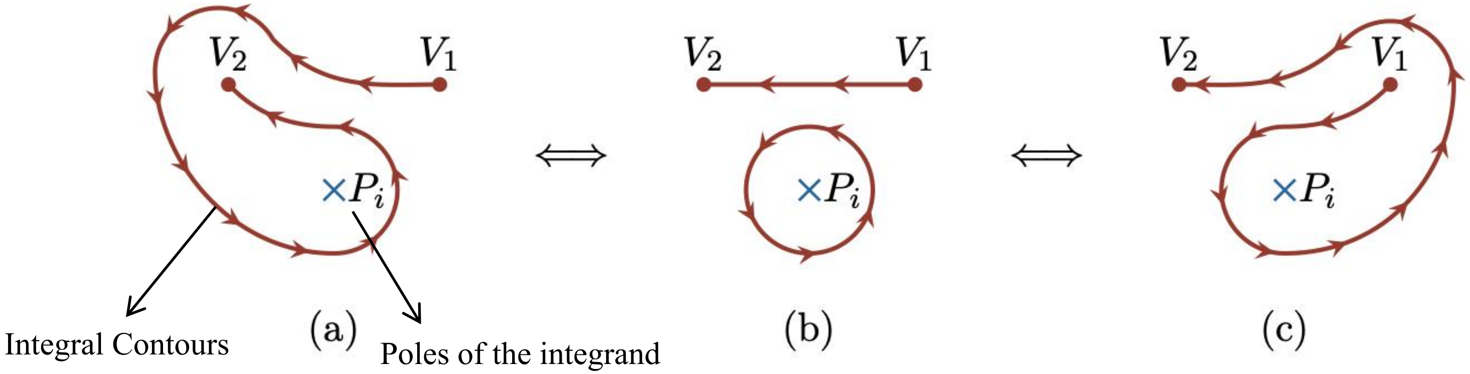
$\longrightarrow$ Algebraically

- Picard-Fuchs operators $\longrightarrow$ From DE

Symbol Integrations

Also useful in bootstrap method

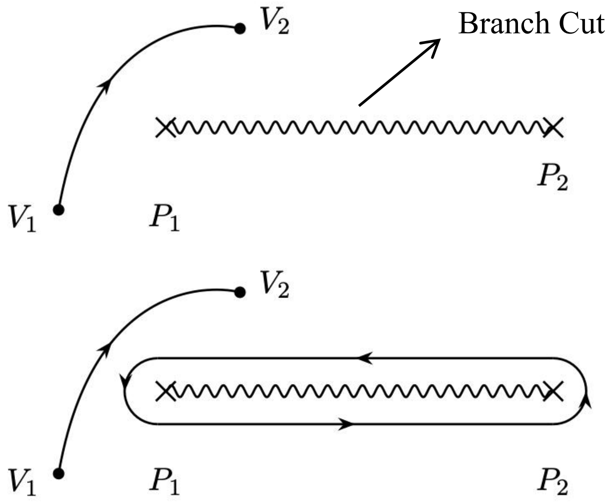
- Purpose: Calculate Symbols for integrals whose integrands have branch cuts!



When deforming the integral contour, it can wind around the pole and contribute to a residue contour, which indicates the discontinuity.

$$\mathcal{S}(\mathcal{I}) = S_1 \otimes S_2 \otimes \cdots \otimes S_n \quad \mathcal{S}(\mathrm{Disc}(\mathcal{I})) = S_2 \otimes \cdots \otimes S_n,$$

Symbol Discontinuity Relations



$$I = \int_a^b \frac{A_1(t) \otimes A_2(t) \ldots \otimes A_n(t)}{t-c} \mathrm{d}t,$$

$$\begin{aligned} \mathcal{S}(I) =& \frac{c-b}{c-a} \otimes A_1(c) \otimes A_2(c) \ldots \otimes A_n(c) \\ &+ \sum_e A(b) \otimes \left(\int_e^b \frac{1}{t-c} A_2(t) \otimes A_3(t) \ldots \otimes A_n(t) \right) \\ &+ \sum_e A(a) \otimes \left(\int_a^e \frac{1}{t-c} A_2(t) \otimes A_3(t) \ldots \otimes A_n(t) \right), \end{aligned}$$

A recursive relation to calculate symbols!

Z. Li, L. Ren and C. Zhang, to appear

Bootstrapping: Integrands in the Collinear Limit

- Three Point cE3C: Taking $n=3,\ \theta_{ij}\to 0$

Depends on the splitting functions!

$$\approx \left[\int \mathrm{dPS}_2 |\mathcal{M}^{\text{tree}}_{1\to 2}|^2\right] \times \left[\sum_{i,j,k} \int d\sigma_3^{\text{coll}} |\mathcal{M}^{\text{coll,tree}}_{1\to 3}|^2 \frac{E_i E_j E_k}{Q^3} \delta^3\left(x_{ij} - \frac{1-\cos\theta_{ij}}{2}\right)\right]$$

- N =4 sYM,

$$P_{1\to 3}(z_1,z_2,z_3) = N_c^2\left[\frac{s_{123}^2}{2s_{13}s_{23}}\left(\frac{1}{z_1 z_2} + \frac{1}{(1-z_1)(1-z_2)}\right) + \frac{s_{123}}{s_{12}z_3}\left(\frac{1}{z_1} + \frac{1}{1-z_1}\right) + \text{perms}\right]$$

- QCD, quark jet for instance

J. M. Campbell and E. W. N. Glover 97'
S. Catani and M. Grazzini 98'

$$P_{\bar{q}q'q} = C_F T_F \frac{s_{123}}{2s_{12}}\left[-\frac{\left[z_1\left(s_{12}+2s_{23}\right) - z_2\left(s_{12}+2s_{13}\right)\right]^2}{\left(z_1+z_2\right)^2 s_{12}s_{123}} + \frac{4z_3 + \left(z_1-z_2\right)^2}{z_1+z_2} + (1-2\epsilon)\left(z_1+z_2-\frac{s_{12}}{s_{123}}\right)\right]$$

Bootstrapping: Integrands

- Collinear Phase Space

$$d\sigma_3^{\text{coll}} = \mathrm{d}s_{12}\mathrm{d}s_{13}\mathrm{d}s_{23}\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3\delta(1-\omega_1-\omega_2-\omega_3)\frac{4\Theta(-\Delta_3^{\text{coll}})(-\Delta_3^{\text{coll}})^{-\frac{1}{2}-\epsilon}}{(4\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)}$$

with
$$\Delta_3^{\text{coll}} = (\omega_3 s_{12} - \omega_1 s_{23} - \omega_2 s_{13})^2 - 4\omega_1\omega_2 s_{13} s_{23}$$

- Measurement
$$\frac{\omega_1\omega_2\omega_3}{8} \times \delta\left(x_{12} - \frac{1}{4}\theta_{12}^2\right)\delta\left(x_{13} - \frac{1}{4}\theta_{13}^2\right)\delta\left(x_{23} - \frac{1}{4}\theta_{23}^2\right)$$

- Combine and integrate over δ, for instance one term in $\mathcal{N}=4$

$$\int_0^1 \mathrm{d}\omega_1 \int_0^1 \mathrm{d}\omega_2 \int_0^1 \mathrm{d}\omega_3 \delta(1-\omega_1-\omega_2-\omega_3)\frac{s_{12}}{s_{123}(\omega_1+\omega_2+\omega_3)^2(\omega_1+\omega_2)}$$

A projective integral

Bootstrapping: Algorithms Overview

Writing Down the Integrands

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Using Feynman Parametrization to Combine All the Propagators

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Spherical Contours for Highest Weight

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Bootstrapping Lower Weight Functions From Spurious Poles and Data

Feynman Parametrization

- Purpose: One-denominator quadric singularity to perform Spherical Contour
- Three denominators

$$\frac{1}{D_1^{c_1} D_2^{c_2} D_3^{c_3}} = \frac{\Gamma(c_1+c_2+c_3)}{\Gamma(c_1)\Gamma(c_2)\Gamma(c_3)} \int_0^1 \mathrm{d}a_1 \int_0^1 \mathrm{d}a_2 \int_0^1 \mathrm{d}a_3 \delta(1-a_1-a_2-a_3) \frac{a_1^{c_1-1} a_2^{c_2-1} a_3^{c_3-1}}{(a_1 D_1 + a_2 D_2 + a_3 D_3)^{c_1+c_2+c_3}}$$

- Generally (used in four points bootstrapping or higher)

$$\frac{1}{D_1^{c_1} D_2^{c_2} \cdots D_k^{c_k}} = \frac{\Gamma(c_1+c_2+\cdots+c_k)}{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_k)} \int_0^1 \mathrm{d}a_1 \int_0^1 \mathrm{d}a_2 \cdots \int_0^1 \mathrm{d}a_k \delta(1-\sum_{i=1}^{k} a_k) \frac{a_1^{c_1-1} a_2^{c_2-1} \cdots a_k^{c_k-1}}{(a_1 D_1 + a_2 D_2 + \cdots + a_k D_k)^{\sum_i c_i}}$$

- Note: Functions inside of δ can change, like Cheng-Wu thoerem.

Rewrite into Projective Space

- Integrate over two δ functions $\Longleftrightarrow$ Putting one of ω and a_i to 1.

This equivalent to set any one of variable to 1 in three points

$$\int_0^\infty \mathrm{d}a_2 \int_0^\infty \mathrm{d}a_3 \int_0^\infty \mathrm{d}\omega_1 \int_0^\infty \mathrm{d}\omega_2 \frac{s_{12}a_2}{(s_{123}|_{\omega_3=1} + a_2(\omega_1+\omega_2+1) + a_3(\omega_1+\omega_2))^4}$$

- Match the homogenity, into projective measure

$$\int_\Delta \frac{\langle X\mathrm{d}X^4\rangle s_{12}x_3}{(s_{123}(X) + x_4(x_5+x_2+x_3) + x_4(x_1+x_2))^4} = \int_\Delta \frac{\langle X\mathrm{d}X^4\rangle N\left[X^3\right]}{(XQX)^4}$$

Where Q is a 4x4 matrix

with the volume element $\langle X\mathrm{d}X^4\rangle = \frac{1}{5}\epsilon_{I_1\cdots I_5} X^{I_1}\mathrm{d}X^{I_2}\wedge\cdots\wedge\cdots\mathrm{d}X^{I_5}$

- Requirement: Quadric Singularities! $\Longrightarrow$

Feyman Parameters of quadric term should be set to 1!

Highest Weight: Spherical Contours

- Algorithms of Spherical Contour

- Using
$$\mathcal{S}[I] = a_1 \otimes a_2 \otimes \cdots \otimes a_k$$
$$\mathcal{S}[\mathrm{Disc}_{a_1}[I]] = a_2 \otimes \cdots \otimes a_k$$

A. Nima, E.Y. Yuan 17'

$$I = \int_\Delta \frac{\langle X \mathrm{d}X^{n-1}\rangle \sqrt{-\mathrm{det}Q}}{(XQX)^{\frac{n}{2}}}$$

$$Q = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$$\mathcal{S}e_n = C_n \sum_\rho \overline{\rho_1\rho_2} \otimes \overline{\rho_3\rho_4} \otimes \cdots \otimes \overline{\rho_{n-1}\rho_n}$$

- Consider n is even
- Q is symmetric matrix
- Q is not degenerate
- Δ is a canonical simplex defined by $x_i = 0$

- First Entries

$$\overline{ij}_{\text{first entry}} = \begin{cases} r(Q^{-1}_{\{i,j\},\{i,j\}}), & Q_{ii} \neq 0, Q_{jj} \neq 0, \\ \left(\frac{Q_{ij}^2}{Q_{jj}}\right)^{-\mathrm{sign}(Q_{ij})}, & Q_{ii} = 0, Q_{jj} \neq 0, \\ \left(\frac{Q_{ij}^2}{Q_{ii}}\right)^{-\mathrm{sign}(Q_{ij})}, & Q_{ii} \neq 0, Q_{jj} = 0, \\ Q_{ij}^{-2\,\mathrm{sign}(Q_{ij})}, & Q_{ii} = Q_{jj} = 0. \end{cases}$$

- Discontinuities

$$\frac{\sqrt{-1}}{2(n-2)} \int_{\Delta^{(ij)}} \frac{\sqrt{-\det Q^{(ij)}}\, \langle X_{(ij)} \mathrm{d}^{n-3} X_{(ij)}\rangle}{(X_{(ij)} Q^{(ij)} X_{(ij)})^{\frac{n-2}{2}}},$$

where $Q^{(ij)} = Q_{\widehat{\{i,j\}},\widehat{\{i,j\}}} - Q_{\widehat{\{i,j\}},\{i,j\}} (Q_{\{i,j\},\{i,j\}})^{-1} Q_{\{i,j\},\widehat{\{i,j\}}}.$

- Recursively read “First Entries” and discontinuities

Highest Weight: Spherical Contours Explanation

- Why Spherical Contour Works?

$$I=\int_{\Delta}\frac{\langle X\mathrm{d}X^{n-1}\rangle\sqrt{-\mathrm{det}Q}}{(XQX)^{\frac{n}{2}}} \xrightarrow{\text{choose i,j}} \text{ij First Entries} \xrightarrow[\text{Discontinuities}]{\text{Calculating}} I'=\int_{\Delta}\frac{\langle X\mathrm{d}X^{n-3}\rangle\sqrt{-\mathrm{det}Q'}}{(X'Q'X')^{\frac{n-2}{2}}}$$

How to?

- Perform Change of Variables $\begin{pmatrix} x_i \\ x_j \end{pmatrix} = R\begin{pmatrix} w_i \\ w_j \end{pmatrix} - Q^{-1}_{\{i,j\},\{i,j\}}Q_{\{i,j\},\widehat{\{i,j\}}}X_{(ij)}$ $\quad R^{\mathrm{T}}Q_{\{i,j\},\{i,j\}}R=\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

- After change of Variables $I=\sqrt{-\mathrm{det}Q}\int_{\Delta_{ij}}\langle X\mathrm{d}X^{n-3}\rangle\int \mathrm{d}w_i\mathrm{d}w_j\frac{1}{(w_iw_j+X_{(ij)}Q_{(ij)}X_{(ij)})^{\frac{n}{2}}}$

- Discontinuity Contours: Spherical Contours

$$\mathrm{Disc}[I]=\sqrt{-\mathrm{det}Q}\int_{\Delta_{ij}}\langle X\mathrm{d}X^{n-3}\rangle\int_{\boxed{S^2}}\mathrm{d}w_i\mathrm{d}w_j\frac{1}{(w_iw_j+X_{(ij)}Q_{(ij)}X_{(ij)})^{\frac{n}{2}}}$$

$$\boxed{w_i=re^{i\theta},\ w_j=re^{-i\theta},\quad r\in[0,+\infty],\ \theta\in[0,2\pi].}$$

This is related to that the discontinuities are similar to residue contours.

A Toy Example

- Consider a pure log function

$$\log(z) = \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \frac{z-1}{(x_2(x_1+1)+(x_1+z))^2} = \int_0^\infty \frac{\langle X \mathrm{d}^2 X\rangle\,(LX)}{(XQX)^2},$$

where $\quad L = [0:0:z-1], \quad Q = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2z \end{pmatrix}.$

$\big\downarrow$ $x_1 = y_+ - x_3, \; x_2 = y_- - x_3$ Perform a transformation

$(y_+ y_- + (z-1)x_3^2)$ in the denominator, which means poles!

$\big\downarrow$ Discontinuities are calculated by a S^2 contour

$y_\pm = r\, e^{\pm i\phi}$, $r \in [0, +\infty]$, $\phi \in [0, 2\pi]$.

$$-\int_0^{+\infty} \mathrm{d}r\,(-2ir) \int_0^{2\pi} \mathrm{d}\phi \frac{(z-1)x_3^2}{(r^2+(z-1)x_3^2)^2} = 2\pi i. \longrightarrow$$ Discontinuities of a pure log!

Highest Weight: Spherical Contours with Numerators

- More information: Rational coefficient of symbols. Mixed weights.
- Symbol Entries will not change.
- Discontinuities can be calculated by the same spherical contour!

$$I=\int_{\Delta}\frac{\langle X\mathrm{d}X^{n-1}\rangle N\left[X^{k}\right]}{(XQX)^{\frac{n+k}{2}}}\xrightarrow{\text{choose i,j}}\text{ij First Entries}\xrightarrow[\text{Discontinuities}]{\text{Calculating}}\cdots\cdots\longrightarrow\overset{\text{Coeffecient}}{\boxed{\text{Last integral Rational}}}$$

- Consider ij variables $I=\int_{\Delta}\langle X\mathrm{d}X^{n-3}\rangle N\int \mathrm{d}x_i\mathrm{d}x_j\frac{x_i^{p_i}x_j^{p_j}}{(XQX)^{\frac{n+k}{2}}}$
- After performing change of variables and taking spherical contours

$$\mathrm{Disc}[I]=\sqrt{-\mathrm{det}Q}\int_{\Delta_{ij}}\langle X\mathrm{d}X^{n-3}\rangle N\boxed{\int_{S^2}\mathrm{d}w_i\mathrm{d}w_j\frac{x_i(w_i,w_j)^{p_i}x_j(w_i,w_j)^{p_j}}{(w_iw_j+X_{(ij)}Q_{(ij)}X_{(ij)})^{\frac{n}{2}}}}$$

This part can be integrated in a general form! Still a recursive relation.

Lower weight Functions: Symbol Bootstrapping

- Energy Correlations: Finite at the poles of highest terms as well as at infinity.
- Example: From a term of $\mathcal{N}=4$

$$F_1 = \int_\Delta \frac{\langle \omega d\omega^3 \rangle \omega_2 \omega_3}{[\omega_1\omega_2 + |z|^2\omega_2\omega_3 + |1-z|^2\omega_1\omega_3 + \omega_1 + \omega_2 + \omega_3]^4}$$

- Highest Weight

$\big\downarrow$ By Spherical Contour

Three different orders only two contribute

Combine to Block-Wigner Dilogarithm

$$\frac{|1-z|^2\left(-1-|z|^2+2|z|^4+2|1-z|^2-|z|^2|1-z|^2-|1-z|^4\right)}{(z-\bar{z})^5}\left(z\bar{z}\otimes\frac{1-z}{1-\bar{z}}+(1-z)\otimes\bar{z}-z\otimes(1-\bar{z})\right)$$

- Assumptions: $\displaystyle \frac{1}{(z-\bar{z})^4}\sum_{i,j=0}^{n}\left[a_{ij}z^i\bar{z}^j + b_{ij}z^i\bar{z}^j\log(z\bar{z}) + c_{ij}z^i\bar{z}^j\log(1-z)(1-\bar{z})\right]$
  - The weight two function when $z-\bar{z}\to\epsilon$ is order ϵ , so we assume the pole is $(z-\bar{z})^{-4}$.
  - The heighest power of z and $\bar{z}$ in the weight two numerators are power 3.
  - By this assumption we can recover all the solutions in N=4 sYM.

QCD results, H. Chen, M.-X. Luo, I. Moult, T.-Z. Yang, X. Zhang, and H. X. Zhu 19'

Problems in QCD Bootstrap

- The spurious pole condition cannot fully determine the coefficient!

About 90 remaining variables in each channel!

More Data!

- S_3 Symmetries from permutation of angles
- Squeeze Limit

H. Chen, M. Luo, T. Yang, X. Zhang and H.X. Zhu 19'

$$G(z)\Big|_{z\to 0} \simeq \frac{1}{|z|^2}\,, \qquad G(z)\Big|_{z\to 1} \simeq \frac{1}{|1-z|^2}\,, \qquad G(z)\Big|_{z\to \infty} \simeq \frac{1}{|z|^2}\,.$$

About 30 remaining variables in each channel!

- Light-Ray OPE data: Expansion at $z \to 0$. H. Chen, 23'

  e.g Expansion to z^5 is enough for quark channel.

- Any other physical information to use?

New Phenomena in Four-Points

- Feynman Parametrization will give more than quadrics: Cubics

For instance
$$I = \int \frac{\langle X\mathrm{d}X^3\rangle x_2\zeta_{23}}{s_{123}s_{234}x_{1234}} = \int \mathrm{dt} \int \frac{\langle X\mathrm{d}X^4\rangle x_2\zeta_{23}}{(s_{123}+\boxed{ts_{234}}+x_5x_{1234})^3}$$

Performing Spherical Contour with parameter t
$$\mathcal{S}[I] = \int \mathrm{dt} \frac{\mathcal{S}[t;z,\bar{z},w,\bar{w}]R(t)}{\boxed{a_3t^3+a_2t^2+a_1t+a_0}},$$

$$a_3 = w\bar{w}(1-z)^2(1-\bar{z})^2,\ a_2 = -(1-z)(1-\bar{z})(w\bar{w}(1+z)(1+\bar{z})-(w+\bar{w})(w\bar{w}+z\bar{z})),$$
$$a_1 = -(1-w)(1-\bar{w})(z\bar{z}(1+w)(1+\bar{w})-(z+\bar{z})(w\bar{w}+z\bar{z})),\ a_0 = (1-w)^2(1-\bar{w})^2z\bar{z},$$

$\mathcal{S}[t;z,\bar{z},w,\bar{w}]$ can still be calculated by Spherical Contour

  - Symbol Integrations: Do the quadrics with spherical contour, integration over the remaining variables.
  - Generalized Projections: Symbol Calculations from projective geometries

J.Y. Gong, E.Y. Yuan 25' to appear

- More kinematic variables:

  - We assume that all the spurious poles conditions are enough for bootstrapping N=4 sYM cE4C all the lower weight functions.

Summary and Outlook

- Differential Equations Method of Any Angle Scattering ENC in N=4 sYM

  E3C in Any Angle: Canonical Differential Equations

  E4C in Any Angle: Master Integrals

  Integral Family of EEC: Elliptic, HyperElliptic, or even Calabi-Yau (CY)?

- Bootstrap Method of Energy Correlators in the Collinear Limit in N=4 sYM and QCD

  Spherical Contour Method for highest weight. More direct bootstrap method?

  Lower weight bootstrap information by Squeeze Limit and Light-Ray OPE

  Does it work in E4C or even QCD?

  Does these analytic information related to factorization or other properties?

Thanks