Renormalization in HQET.

- HQET Lagrangian
- field strength renormalization
- Heavy to light current
- Heavy to heavy current.

HQET Lagrangian

QCD: $\mathcal{L}_Q = \bar{Q}(i\not{D} - m)Q$

$$Q(x) = e^{-im_Q v\cdot x}\left[Q_v(x) + \mathcal{Q}_v(x)\right]$$

$$Q_v(x) = e^{im_Q v\cdot x}\frac{1+\not{v}}{2}Q(x)$$

$$\mathcal{Q}_v(x) = e^{im_Q v\cdot x}\frac{1-\not{v}}{2}Q(x)$$

① $e^{-im_Q v\cdot x}$: only describe quark

$$Q(x) \sim \int \frac{d^3p}{(2\pi)^3 2E_p}\left[b(p)\,u(p)\,e^{-ip\cdot x} + d(p)\,v(p)\,e^{ip\cdot x}\right]$$

$$p^\mu = \underbrace{m_Q v^\mu}_{\text{大分量}} + \underbrace{k^\mu}_{\text{小分量}}$$

$$Q(x) \sim e^{-im_Q v\cdot x}\int \frac{d^3k}{(2\pi)^3 2E_p}\left[b(k)\,u_v(k)\,e^{-ik\cdot x} + \cdots\right]$$

② $\frac{1+\not{v}}{2}$ $\frac{1-\not{v}}{2}$

动量空间: $\not{p}Q = m_Q Q$

领头阶 $m_Q\not{v}Q = m_Q Q \rightarrow \not{v}Q = Q$

可以用 $\frac{1+\not{v}}{2}$ 投影出大分量, 满足

$$\not{v}\frac{1+\not{v}}{2}Q = \frac{\not{v}+\not{v}\not{v}}{2}Q = \frac{\not{v}+1}{2}Q$$

- $\mathcal{L}_{HQET} = \bar{Q}_v(iv\cdot D)Q_v = \bar{Q}_v(iv\cdot\partial + g t^a v\cdot A^a)Q_v$

$$D_\mu = \partial_\mu - ig t^a A^a_\mu$$

$$\mathcal{L}_{QCD} = \bar{Q}(i\slashed{D} - m_Q)Q$$

$$= e^{im_Q v\cdot x}[\bar{Q}_v + \bar{\mathcal{Q}}_v][i\slashed{D} - m_Q]\, e^{-im_Q v\cdot x}[Q_v + \mathcal{Q}_v(x)]$$

$$\overset{LO}{=} e^{im_Q v\cdot x}\,\bar{Q}_v(i\slashed{D} - m_Q)\,e^{-im_Q v\cdot x}Q_v$$

$$= e^{im_Q v\cdot x}\,\bar{Q}_v\left((\underbrace{m_Q\slashed{v} - m_Q}_{=0})\,e^{-im_Q v\cdot x} + e^{-im_Q v\cdot x}\, i\slashed{D}\right)Q_v$$

$$= \bar{Q}_v(i\slashed{D})Q_v$$

$$= \bar{Q}_v\,\frac{1+\slashed{v}}{2}\,i\slashed{D}\,Q_v$$

$$= \bar{Q}_v\left(\frac{1+2iv\cdot D - i\slashed{D}\slashed{v}}{2}\right)Q_v = \bar{Q}_v(iv\cdot D)Q_v + \overset{\to 0}{\bar{Q}_v i\slashed{D}\frac{1-\slashed{v}}{2}Q_v}$$

$$= \bar{Q}_v(iv\cdot D)Q_v$$

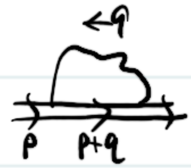
② quark propagator

$$\frac{i(\slashed{p}+m_Q)}{p^2-m_Q^2} \to \frac{i}{v\cdot k}$$

$$\frac{i(m_Q\slashed{v}+\slashed{k}+m_Q)}{(m_Qv+k)^2-m_Q^2} = \frac{im_Q(1+\slashed{v})}{m_Q^2+2m_Qv\cdot k+k^2-m_Q^2} = \frac{im_Q(1+\slashed{v})}{2m_Qv\cdot k} = \frac{1+\slashed{v}}{2}\,\frac{i}{v\cdot k}$$

③ 顶点 $\mathcal{L}_I = \bar{Q}_v\, g t^a v\cdot A^a Q_v \quad \to \quad igt^a v_\mu$

• field strength renormalization = self energy



$$iM = \int \frac{d^dq}{(2\pi)^d}\; i g_s t^a v_\mu \mu^\varepsilon \frac{i}{v\cdot(p+q)}\; i g_s t^a v^\mu \mu^\varepsilon \frac{-i}{q^2}$$

$$= -g_s^2 C_F \mu^{2\varepsilon} \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2\; v\cdot(p+q)}$$

利用公式:

$$\frac{1}{a^r b^s} = 2^s \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^\infty d\lambda \frac{\lambda^{s-1}}{[a+2b\lambda]^{r+s}}$$

$$iM = -g_s^2 C_F \mu^{2\varepsilon} \int \frac{d^dq}{(2\pi)^d} \int_0^\infty d\lambda \frac{2\Gamma(2)}{\Gamma(1)\Gamma(1)} \frac{1}{[q^2+2\lambda v\cdot(p+q)]^2}$$

需要注意: ① λ 有量纲, +1

② 在 $\lambda=0$ 时可能有红外发散, 可以被添加质量消除

③ 抵消项 $i v\cdot p\,[Z_h-1]$

所以不用得到 iM 的具体形式. 对其做 $v\cdot p$ 微分, 可简化

$$\frac{\partial(iM)}{\partial v\cdot p} = 4 g_s^2 \mu^{2\varepsilon} C_F \int_0^\infty d\lambda \int \frac{d^dq}{(2\pi)^d} \frac{2\lambda}{[q^2+2\lambda v\cdot(p+q)]^3}$$

$$= 8 g_s^2 \mu^{2\varepsilon} C_F \int_0^\infty d\lambda\, \lambda \frac{(-1)^3 i}{(4\pi)^{d/2}} \frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)} \cdot \frac{1}{[\lambda^2-2\lambda v\cdot p]^{3-\frac{d}{2}}}$$

$$= -4i \frac{g_s^2 C_F}{(4\pi)^2} (4\pi\mu^2)^\varepsilon \Gamma(1+\varepsilon) \underbrace{\int_0^\infty d\lambda\, \lambda \frac{1}{(\lambda^2-2\lambda v\cdot p)^{1+\varepsilon}}}_{\text{只看UV性质}}$$

只看UV性质 $\int_1^\infty d\lambda\, \lambda \frac{1}{\lambda^{2+2\varepsilon}} = \int_1^\infty d\lambda\, \lambda^{-1-2\varepsilon}$

$= -\frac{1}{2\varepsilon} \lambda^{-2\varepsilon}\Big|_1^\infty$

$= -\frac{1}{2\varepsilon}(0-1^{-2\varepsilon})$

$= \frac{1}{2\varepsilon} + O(\varepsilon^0)$

$$= -4i \frac{\alpha_s C_F}{4\pi} \frac{1}{2\varepsilon} + O(\varepsilon^0)$$

$$= -i \frac{\alpha_s C_F}{2\pi} \frac{1}{\varepsilon}$$

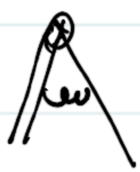
$$\rightarrow Z_h = 1 + \frac{\alpha_s C_F}{2\pi} \frac{1}{\varepsilon} \rightarrow \gamma_h = \frac{1}{2} \frac{d\ln Z_h}{d\ln\mu} = -\frac{\alpha_s C_F}{2\pi}$$

- Heavy-to-light current:

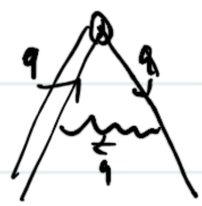
$$O_\Gamma = \bar{q}\,\Gamma\, Q_v$$

$$O_\Gamma = \frac{Q_\Gamma^{(0)}}{Z_0} = \frac{\sqrt{Z_q Z_h}}{Z_0}\left(\bar{q}\,\Gamma\, Q_v\right)$$

$$= \bar{q}\,\Gamma\, Q_v + \left(\frac{\sqrt{Z_q Z_h}}{Z_0} - 1\right)\bar{q}\,\Gamma\, Q_v$$

 $+$  $+ \left(\frac{\sqrt{Z_q Z_h}}{Z_0} - 1\right)$  无发散

计算顶点图:



$$iM = \int \frac{d^dq}{(2\pi)^d}\, i g t^a \mu^\varepsilon \gamma^\lambda \frac{i\slashed{q}}{q^2}\,\Gamma\, \frac{i}{v\cdot q}\, i g t^a \mu^\varepsilon v_\lambda \frac{-i}{q^2}$$

$$= -i g_s^2 \mu^{2\varepsilon} C_F \int \frac{d^dq}{(2\pi)^d} \frac{\slashed{v}\slashed{q}\,\Gamma}{(q^2)^2\, v\cdot q}$$

$$\frac{1}{(q^2)^2\, v\cdot q} = 2\frac{\Gamma(3)}{\Gamma(2)\Gamma(1)}\int_0^\infty d\lambda \frac{1}{\left[q^2 + 2\lambda v\cdot q\right]^3} = 4\int_0^\infty d\lambda \frac{1}{(q^2+2\lambda v\cdot q)^3} = 4\int_0^\infty d\lambda \frac{1}{\left[(q+\lambda v)^2 - \lambda^2\right]^3}$$

$$iM = -4 i g_s^2 \mu^{2\varepsilon} C_F \int_0^\infty d\lambda \int \frac{d^dq}{(2\pi)^d} \frac{\slashed{v}(\slashed{q} - \lambda\slashed{v})\Gamma}{(q^2-\lambda^2)^3}$$

$$= -4 i g_s^2 \mu^{2\varepsilon} C_F \int_0^\infty d\lambda\, (-\lambda)\, \frac{(-1)^3 i}{(4\pi)^{d/2}} \frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)} \frac{1}{(\lambda^2)^{3-\frac{d}{2}}}\,\Gamma$$

$$= \frac{4 g_s^2 C_F}{(4\pi)^2}(4\pi\mu^2)^\varepsilon \int_0^\infty \underline{d\lambda\,\lambda}\, \frac{\Gamma(1+\varepsilon)}{2} \frac{1}{(\lambda^2)^{1+\varepsilon}}\,\Gamma$$

$$\overset{\text{只留发散}}{=} \frac{\alpha_s C_F}{4\pi}\frac{1}{\varepsilon}\,\Gamma + O(\varepsilon^0)$$

考虑到 $Z_q = 1 - \frac{\alpha_s C_F}{4\pi}\frac{1}{\varepsilon}$

$Z_h = 1 + \frac{\alpha_s C_F}{2\pi}\frac{1}{\varepsilon}$

则 $Z_O = 1 + \frac{\alpha_s C_F}{4\pi}\frac{1}{\varepsilon} + \frac{1}{2}\left(-\frac{\alpha_s C_F}{4\pi}\frac{1}{\varepsilon}\right) + \frac{1}{2}\frac{\alpha_s C_F}{2\pi}\frac{1}{\varepsilon}$

$= 1 + \frac{3\alpha_s C_F}{8\pi}\frac{1}{\varepsilon}$

$\gamma_O = \frac{d\ln Z_O}{d\ln\mu} = -\frac{3\alpha_s C_F}{4\pi}$

- Heavy-to-Heavy: similar with heavy to light.

$$iM = \int \frac{d^dq}{(2\pi)^d}\, ig t^a \mu^\varepsilon v'_\mu \frac{i}{v'\cdot q}\, \Gamma\, \frac{i}{v\cdot q}\, ig t^a \mu^\varepsilon v^\mu \frac{-i}{q^2}$$

$$= -i g_s^2 C_F \mu^{2\varepsilon}\int \frac{d^dq}{(2\pi)^d}\frac{v\cdot v'}{v\cdot q\, v'\cdot q}\frac{1}{q^2}\,\Gamma \quad (w = v\cdot v')$$

$$\frac{1}{v\cdot q}\frac{1}{v'\cdot q} = \int_0^1 dx \frac{1}{[x v\cdot q + (1-x) v'\cdot q]^2}$$

$$\frac{1}{[q^2][v\cdot q]^2} = 4\frac{\Gamma(3)}{\Gamma(2)\Gamma(1)}\int_0^\infty d\lambda \frac{\lambda}{[q^2+2\lambda v\cdot q]^3}$$

$$iM = -8 i g_s^2 C_F \mu^{2\varepsilon} w \int_0^1 dx \int_0^\infty d\lambda\,\lambda \int\frac{d^dq}{(2\pi)^d}\frac{1}{[q^2+2\lambda q\cdot(xv+(1-x)v')]^3}$$

$$= -8 i g_s^2 C_F\mu^{2\varepsilon} w\int_0^1 dx\int_0^\infty d\lambda\,\lambda\, \frac{(-1)^3 i}{(4\pi)^{d/2}}\frac{\Gamma(3-\frac d2)}{\Gamma(3)}\frac{1}{(\lambda^2(xv+(1-x)v')^2)^{3-\frac d2}}$$

$$= -8\frac{\alpha_s C_F}{4\pi}(4\pi\mu^2)^\varepsilon\, w\, \frac{1}{2\varepsilon}\frac{\Gamma(1+\varepsilon)}{\Gamma(3)}\int_0^1 dx \frac{1}{[x^2v^2+(1-x)^2v'^2+2x(1-x)\,v\cdot v']^{1+\epsilon}}$$

- 对于现在情况，$v^2=v'^2=1$

$$iM = -2\frac{\alpha_sC_F}{4\pi}(4\pi\mu^2)^\varepsilon\Gamma(1+\epsilon)\frac{1}{\varepsilon}\, w\, r(w)$$

$$r(w)=\frac{1}{\sqrt{w^2-1}}\ln\left(w+\sqrt{w^2-1}\right)$$

于是 $Z_T = 1-\frac{\alpha_sC_F}{2\pi}\frac{1}{\varepsilon}(w\,r(w)-1)$

$$\gamma_T = \frac{\alpha_s C_F}{\pi}(w\,r(w)-1)$$

- 如果其中一个 $v^2=0$，此时对应光锥 Wilson line，这里存在新的发散，即为 cusp 发散.