重整化：第15章 + 17.5 — 17.8

① 有限单圈

② 表观发散度

③ 正规化与单圈重整化：

(bare perturbation, renormalized perturbation theory)

④ QED 重整化 — g-2

⑤ Ward-Takahashi 等式 — △

⑥ QCD 单圈重整化

⑦ BRST △

1. 有限单圈的例子:

some example calculations.

3 tree level. 2 1-Loop

① $\Phi \to \phi\phi$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi^2$$

$$\mathcal{L}_I = -\mu\Phi\phi^2 \qquad \mathcal{H}_I = \mu\Phi\phi^2$$

$$\langle\phi(p_1)\phi(p_2)|S|\Phi(p_A)\rangle = \langle\phi(p_1)\phi(p_2)|-i\int d^4x\, \mu\Phi\phi^2|\Phi(p_A)\rangle$$

$$= -i\mu\int d^4x\, e^{-ip_A x}e^{ip_1 x}e^{ip_2\cdot x}\cdot 2$$

$$= -2i\mu\,(2\pi)^4\delta^4(p_A-p_1-p_2)$$

$$\Rightarrow i\mathcal{M} = -2i\mu$$

$$d\Gamma = \frac{1}{2M}\cdot(2\pi)^4\delta^4(p_A-p_1-p_2)\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\overline{|\mathcal{M}|}^2$$

$$\overline{|\mathcal{M}|}^2 = \underset{\uparrow}{\frac{1}{2}}\cdot|\mathcal{M}|^2 = 2\mu^2$$

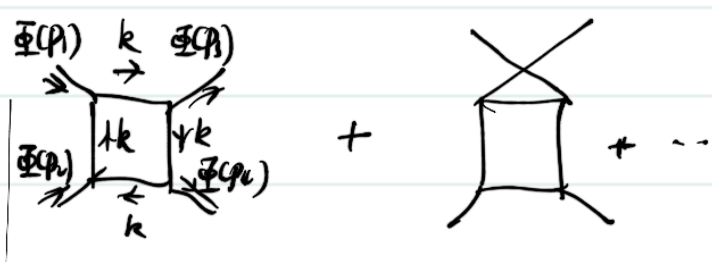
全同对称因子

② 1-loop: $\mathcal{L}$ 与之前一样.

$$\Phi(p_1)\Phi(p_2)\to\Phi(p_3)\Phi(p_4)$$

$$\langle\Phi(p_3)\Phi(p_4)|S|\Phi(p_1)\Phi(p_2)\rangle$$

展开 - - -



(a)

= 对应费曼图.

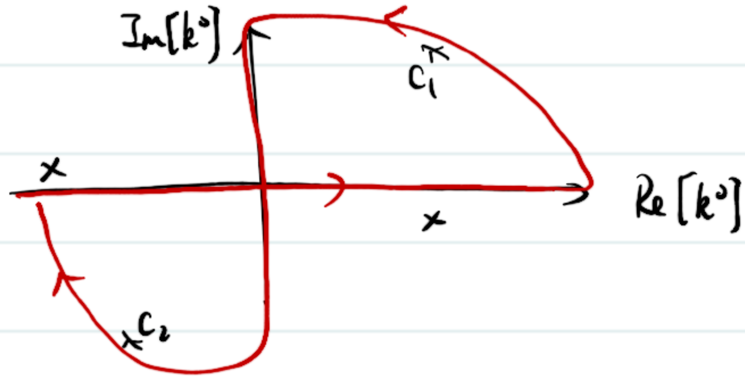
ⓐ 取 $p_1=p_2=p_3=p_4=0$ 极限.

$$i\mathcal{M}_a = \int\frac{d^4k}{(2\pi)^4}\cdot(-2i\mu)^4\cdot\frac{i}{(k^2-M_\Phi^2+i\varepsilon)^4}$$ (注意 symmetry factor)

$$= 16\mu^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m_\phi^2+i\varepsilon)^4}$$

针对 k^0 有两个根.

$$k^0 = \pm\sqrt{k^{i2}+m_\phi^2-i\varepsilon}$$



选择封闭回路得到积分为0

$$\int_{-\infty}^{+\infty} dk^0 + \int_{+i\infty}^{-i\infty} dk^0 + \int_{C_1} dk^0 + \int_{C_2} dk^0 = 0$$

在 C_1 和 C_2 上:

$$\int \frac{dk^0}{(k^{02}-x)^4} \sim \frac{\infty}{\infty^8} \to 0$$

所以 $\int_{-\infty}^{+\infty} dk^0 + \int_{+i\infty}^{-i\infty} dk^0 = 0$

定义 $k^0 = i k_E$ 则 $\int_{-\infty}^{+\infty} dk^0 = +\int_{-i\infty}^{+i\infty} dk^0$

$$= \int_{-\infty}^{+\infty} i\, dk_E = i\int_{-\infty}^{+\infty} dk_E \qquad \text{Wick rotation}$$

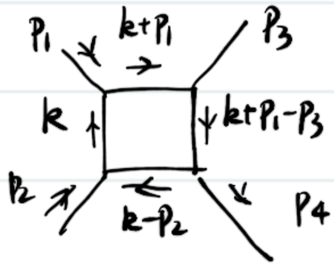
积分为: $iM_a = 16\mu^4 \int \frac{dk^0 d^3k}{(2\pi)^4} \frac{1}{(k^2-m_\phi^2+i\varepsilon)^4}$

$$\overset{k^0=ik_E}{=} 16\mu^4 \int_{-\infty}^{+\infty} \frac{i\, dk_E\, d^3k}{(2\pi)^4} \frac{1}{[-k_E^2-k^{i2}-m_\phi^2+i\varepsilon]^4}$$

$$= 16 i \mu^4 \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + m_\phi^2 - i\varepsilon)^4}$$

$$= 16 i \mu^4 \cdot \frac{1}{96\pi^2 m_\phi^4}$$

ⓑ 如果 $p_i \neq 0$.



a

$$i\mathcal{M}_a = \int \frac{d^4k}{(2\pi)^4} (-2i\mu)^4$$

$$\times \frac{i}{k^2 - m_\phi^2} \frac{i}{(k+p_1)^2 - m_\phi^2} \frac{i}{(k+p_1-p_3)^2 - m_\phi^2 + i\varepsilon} \frac{i}{(k-p_2)^2 - m_\phi^2 + i\varepsilon}$$

Feynman Parametrization:

$$\frac{1}{\alpha_1 \alpha_2} = \int_0^1 dx \frac{1}{[x\alpha_1 + (1-x)\alpha_2]^2}$$

$$\text{右边} = \int_0^1 dx \frac{1}{[(\alpha_1 - \alpha_2)x + \alpha_2]^2}$$

$$= \frac{1}{\alpha_1 - \alpha_2} \int d(\alpha_1 - \alpha_2) x \frac{1}{[(\alpha_1 - \alpha_2)x + \alpha_2]^2}$$

$$= \frac{1}{\alpha_1 - \alpha_2} (-1) \frac{1}{[(\alpha_1 - \alpha_2)x + \alpha_2]} \Bigg|_{x=0}^{x=1} = \frac{1}{\alpha_1 - \alpha_2} (-1) \left\{ \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right\} = \frac{1}{\alpha_1 \alpha_2} = \text{左边}$$

则 $$\frac{1}{k^2 - m^2} \frac{1}{(k+p_1)^2 - m^2} = \int \frac{dx}{[x(k^2 - m^2) + (1-x)((k-p_1)^2 - m^2)]^2}$$

$$i\mathcal{M}_a = 16\mu^4 \int \frac{d^4k}{(2\pi)^4} \int dx_1 dx_2 dx_3 \frac{3!}{\begin{aligned}[x_1((k+p_1)^2 - m_\phi^2) + x_2((k+p_1-p_2)^2 - m^2) + x_3((k-p_2)^2 - m^2) \\ + (1 - x_1 - x_2 - x_3)(k^2 - m^2)]^4\end{aligned}}$$

$$\text{分母} = (k + xp_1 + x_2(k + p_1 - p_2) - x_3 p_2)^2 - m^2 + xp_1^2 + x_2(p_1 - p_2)^2 + x_3 p_2^2$$
$$- [xp_1 + x_2(p_1 - p_2) - x_3 p_2]^2$$

$$= \hat{k}^2 - \Delta$$

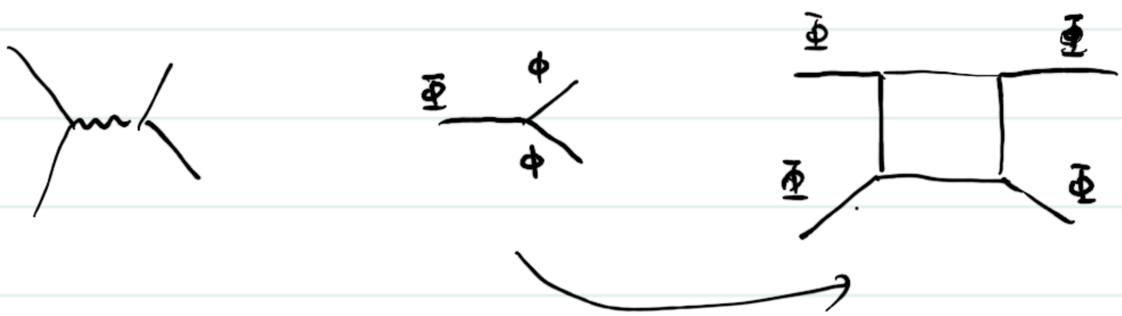
$$i\mathcal{M}_a = 16\mu^4 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \quad \frac{3!}{(k^2-\Delta)^4}$$

$$\overset{\text{平移}}{=} 16\mu^4 \int \frac{d^4k}{(2\pi)^4} \int [dx] \frac{3!}{(k^2-\Delta)^4}$$

$$= 16\mu^4 \int [dx] \frac{3!}{96\pi^2\Delta^2}$$

integration can be performed in principle.

c. ϕ^4 散射



$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m_\phi^2)^4} = \frac{1}{96\pi^2 m_\phi^4}$$

1. What kind of theory can produce UV divergence?

UV. ultraviolet

2. If divergences. what kind of modification?

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

$$\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4 \qquad H_I = \frac{\lambda}{4!}\phi^4$$

$$\phi(p_1)\,\phi(p_2) \to \phi(p_3)\,\phi(p_4)$$

$$\langle\phi(p_3)\phi(p_4)|\,S\,|\phi(p_1)\phi(p_2)\rangle \qquad S = T\exp\{-i\int H_I\, d^4x\}$$

$$= \langle\phi(p_3)\phi(p_4)|\left[-i\int d^4x\, \frac{\lambda}{4!}\,\phi^4(x)\right]|\phi(p_1)\phi(p_2)\rangle$$

$$= -i\int d^4x\, \frac{\lambda}{4!}\; e^{-ip_1x}\, e^{-ip_2x}\, e^{ip_3x}\, e^{ip_4x}\cdot \times 4!$$

$$= \underline{\underline{-i\lambda}}\, (2\pi)^4\, \delta^4(p_1+p_2-p_3-p_4) \equiv i\mathcal{M}\, (2\pi)^4\, \delta^4(\;)$$

$$\Rightarrow iM = -i\lambda$$

$$d\sigma = \frac{1}{2E_A\, 2E_B\, |v_A - v_B|}\, d\Phi_1\, d\Phi_2 \cdot (2\pi)^4 \delta^4(\quad) \cdot |iM|^2$$

d. 1-loop

$$\langle \phi(p_3)\phi(p_4) |\, \frac{(-i)^2}{2!} \int d^4x\, d^4y\, \frac{\lambda}{4!}\frac{\lambda}{4!}\, \phi^4(x)\, \phi^4(y)\, | \phi(p_1)\phi(p_2) \rangle$$

$$\Downarrow$$

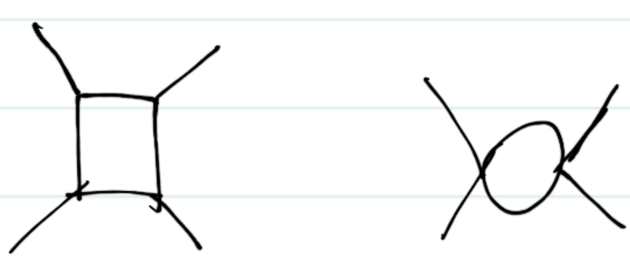
$$\downarrow$$

$$iM = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2 - m_\phi^2}\, \frac{i}{[k-(p_1+p_2)]^2 - m_\phi^2} \cdot (-i\lambda)^2$$

$$\xrightarrow{k \to \infty} \frac{\infty^4}{\infty^4} \sim \log \infty$$ 发散　　紫外发散 UV

$$\int \frac{dx}{x} \sim \frac{\infty}{\infty}$$

$\downarrow \log \infty$



2. Superficial degree of divergence

某种振幅：圈/自由动量：l $\qquad\qquad d^4k$

传播子：玻色 n_B $\qquad \dfrac{i}{k^2-m_\phi^2}$

费米：n_F $\qquad \dfrac{i}{\not k-m} \qquad \dfrac{i(\not k+m)}{k^2-m^2}$

当 $k\to\infty$：

$$D = \underline{4l-2n_B-n_F}$$

There are V vertices:

$S=\int \mathcal{L}\, d^4x$

$[\mathcal{L}]=4$

$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-\frac{1}{2}m^2\phi^2$

$[\phi]=1$

$\mathcal{L}=\bar\psi(i\not\partial-m)\psi$

$[\psi]=\frac{3}{2}$

$[\lambda]=d_\lambda$

$$\begin{cases} 4\cdot V = 2n_B+\underset{\substack{\uparrow\\ \text{外线}}}{N_B} + \frac{3}{2}\cdot(2n_F+N_F)+V d_\lambda \\ l = n_B+n_F+1-V \end{cases}$$

$l=1. \qquad n_B=2. \qquad V=2$

$$\begin{aligned} D &= 4l-2n_B-n_F \\ &= 4n_B+4n_F+4-4V-2n_B-n_F \\ &= 4n_B+4n_F+4-[2n_B+N_B+3n_F+\tfrac{3}{2}N_F+Vd_\lambda]-2n_B-n_F \end{aligned}$$

$$= \underline{4 - V\cdot d_\lambda - N_B - \tfrac{3}{2}N_F}$$

d_λ:

$\mathcal{L}_I = \frac{\lambda}{4!}\phi^4$ $\qquad [\lambda] = d_\lambda = 0$

$\Rightarrow D = 4 - N_B - \frac{3}{2}N_F$

$N_B=0$ $\qquad D=4$ 

$N_B=1$ 

$N_B=+3$

$N_B=2$ 

$N_B=4$ 

讨论 d_λ:

$d_\lambda>0$ 有限个图 super renormalizable

$d_\lambda=0$ 有限类 无限个图 renormalizable

$\underline{d_\lambda<0}$ 无限类 non-renormalizable

$\boxed{\text{Superficial}}$ d. of divergence.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

$[\phi] = 1$. $\quad [\lambda] = 0$

$D = 4 - N_B$

$n_B = 2 \qquad D = 2$

$n_B = 4 \qquad D = 0$

$n_B = 6 \qquad D = -2$

$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m_\phi^2} \cdots$$

QED

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}(i\not{\partial} - m)\psi + \bar{\psi}e\not{A}\psi$$

$[e] = 0$

$D = 4 - N_B - \frac{3}{2}N_F$

q, μ

$N_B = 4 \qquad N_F = 0$

$D = 0$

真正 $\underline{D = -4}$

$(q^\mu q^\nu - g^{\mu\nu}q^2)$

3. 正规化与重整化

ϕ^4  

a. 正规化:

① cutoff Λ

$$\int_{-\Lambda}^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_\phi^2+i\varepsilon}\sim\Lambda^2$$

$$\int_{-\Lambda}^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-m_\phi^2][(k-(p_1+p_2))^2-m^2]}\sim\log\Lambda$$

② Pauli-Villars

$$\frac{1}{k^2-m_\phi^2}\to\frac{1}{k^2-m_\phi^2}-\frac{1}{k^2-M^2}=(m^2-M^2)\frac{1}{(k^2-m^2)(k^2-M^2)}$$

③ dimensional regularization:

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-m^2][(k-(p_1+p_2))^2-m^2]}\to\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)((k-p)^2-m^2)}$$

当 $d=4$ 原来 $D=0$

当 $d<4$. 记 $d=4-2\varepsilon$ $D=\underline{-2\varepsilon}<0$ 不发散

$$\frac{\infty^d}{\infty^4}\sim\infty^{d-4}\sim\infty^{-2\varepsilon}\Rightarrow\frac{1}{d-4}=\frac{1}{\varepsilon}$$

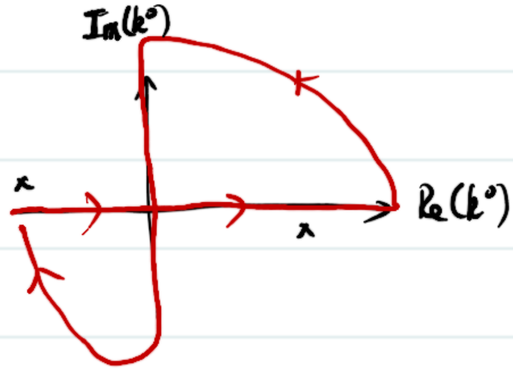
$\varepsilon\to0$. 发散 $\left(\frac{1}{\varepsilon}\right)$ 形式.

正规化例子 ϕ^4

$N_B=2$ $\qquad\qquad$ $N_B=4$

$N_B=2.$ $\qquad iM = -i\lambda\ \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ \frac{i}{k^2-m^2+i\varepsilon}$

Wick rotation



$k^0 = \pm\sqrt{k^{i^2}+m_\phi^2-i\varepsilon}$

$\to k^0 = ik_E^0$

$d^4k \to i\,d^4k_E$

$k^2-m^2 = k^{0^2}-k^{i^2}-m^2 = -k_E^2-k_i^2-m^2$

$= -(k_E^2+m^2)$

$$iM = \frac{1}{2}\lambda\int\frac{i\,d^4k_E}{(2\pi)^4}\ \frac{1}{-(k_E^2+m^2)}$$

spherical coordinate

$k_E^0 = |k_E|\cos\theta_1 \qquad k_E^1 = |k_E|\sin\theta_1\cos\theta_2 \qquad k_E^2 = |k_E|\sin\theta_1\sin\theta_2\cos\phi$

$k_E^3 = |k_E|\sin\theta_1\sin\theta_2\sin\phi$

$$d^4k_E = \frac{1}{2}k_E^2\,dk_E^2\cdot d\Omega_4$$

$$\int d\Omega_4 = 2\pi^2$$

$$iM = -\frac{1}{2}i\lambda\cdot 2\pi^2\int\frac{\frac{1}{2}k_E^2\,dk_E^2}{(2\pi)^4(k_E^2+m^2)}$$

$$\sim\int_0^\infty\frac{dk_E^2\,k_E^2}{k_E^2+m^2}$$

$$\xrightarrow{\text{cutoff}}\int_0^\Lambda\frac{dk_E^2\,k_E^2}{k_E^2+m^2} = \underline{\Lambda^2-m^2\cdot\log\frac{\Lambda^2+m^2}{m^2}}$$

b. 相加重整化. 与 bare perturbation theory

what is the fate of the divergences.

$$\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}(iM)\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\, iM\, \frac{i}{p^2-m^2}\, iM\, \frac{i}{p^2-m^2} + \cdots$$

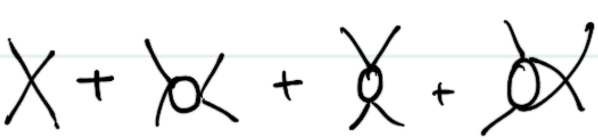
$$= \frac{i}{p^2-m^2+M} = \frac{i}{p^2-m^2-\Lambda^2+m^2\log\frac{\Lambda^2+m^2}{m^2}} = \frac{i}{p^2-\hat{m}^2}$$

$m^2+M \Rightarrow \hat{m}^2$.

Redefine mass $\quad m^2+\frac{\lambda}{32\pi^2}\left(\Lambda^2-m^2\log\frac{\Lambda^2+m^2}{m^2}\right)=\hat{m}^2$

Renormalization: redefine a parameter to absorb the UV divergence

dR.
$$iM = (-i\lambda)^2\,\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2}\,\frac{i}{(k-p)^2-m^2} \qquad s=p^2$$
$$= \frac{1}{2}\lambda^2\int_0^1 dx\int\frac{d^dk'}{(2\pi)^d}\,\frac{1}{[k'^2-\Delta]^2} \qquad \Delta = x^2 s+m^2-xs-i\varepsilon$$
$$= \frac{1}{2}\lambda^2\int_0^1 dx\,\frac{(-1)^2 i}{(4\pi)^{d/2}}\,\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$
$$= \frac{i}{32\pi^2}\lambda^2\frac{1}{\varepsilon} + 有限项$$



$$iM = -i\lambda + 3\times\frac{i}{32\pi^2}\lambda^2\frac{1}{\varepsilon}+\cdots \overset{s=0}{\equiv} -i\hat{\lambda}$$

重新定义 $s=0$, $iM = -i\hat{\lambda}$

$$\hat{\lambda} = \lambda + 3\times\frac{\lambda^2}{32\pi^2}\frac{1}{\varepsilon}$$

C. 抵消项、相乘重整化与 renormalized perturbation theory

Previously

$$D = 4 - \frac{3}{2}N_F - N_B - (V \cdot d_\lambda)$$

$d_\lambda = 0$： 有限类 无限个发散图.

marginal . renormalizable

$$D = 4 - \frac{3}{2}N_F - N_B$$

Example: a. $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$ $S = \int d^4x\, \mathcal{L}$

$[\mathcal{L}] = 4$

$[m] = 1$. $[\phi] = 1 \Rightarrow [\lambda] = 0$

b. 6维情况:

$[\mathcal{L}] = 6$

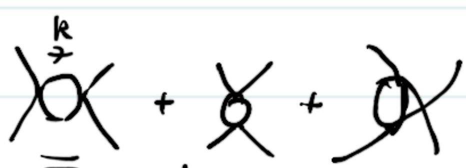
$[m] = 1$ $[\phi] = 2$. $\Rightarrow [\lambda] = -2$

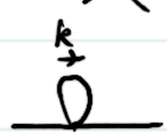
ϕ^4: in 4-dim: renormalizable

$$D = 4 - N_B$$

$N_B = 0$  bubble diagram ×

$N_B = 2$ 

$N_B = 4$ 

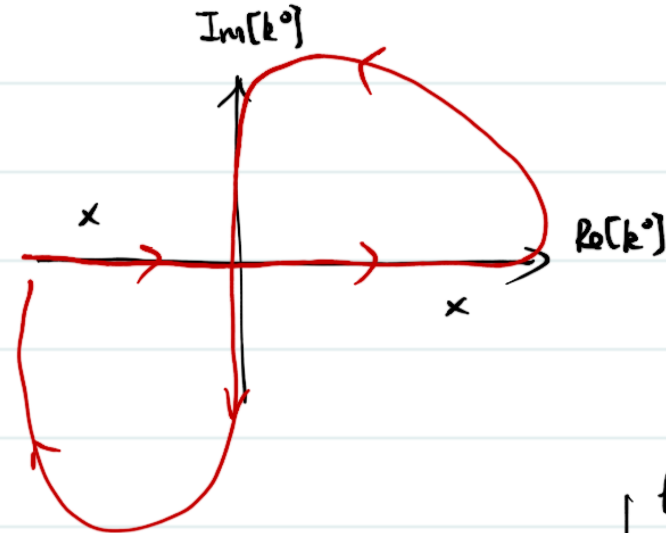
self-energy  $(-i\lambda)$

$$iM = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \cdot (-i\lambda)$$

$$\langle\phi(p)|\, S\, |\phi(p)\rangle = \langle\phi(p)|\, -i\int d^4x\, \frac{\lambda}{4!}\phi^4(x)\, |\phi(p)\rangle$$

$$\underset{k\to\infty}{=} \frac{\infty^4}{\infty^2} = \infty^2 \Rightarrow D=2$$

正规化



重整化:

$$k^0 = i k^4$$

$$d^4k = i\,dk^4 dk^1 dk^2 dk^3 = i\, d^4k_E$$

$$k^2 - m^2 = (k^0)^2 - (k^i)^2 - m^2$$

$$= -k^{4\,2} - (k^i)^2 - m^2 = -k_E^2 - m^2$$

$$= -(k^2 + m^2)$$

$$iM = \frac{1}{2}\lambda \int \frac{i\, d^4k_E}{(2\pi)^4} \frac{1}{-(k_E^2+m^2)}$$

$$= -\frac{1}{2} i\lambda \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2+m^2}$$

spherical coordinate.

$$k^4 = |k_E|\cos\omega \qquad k^1 = |k_E|\sin\omega\cos\theta \qquad k^2 = |k_E|\sin\omega\sin\theta\cos\phi$$

$$k^3 = |k_E|\sin\omega\sin\theta\sin\phi$$

$$d^4k_E = \frac{1}{2}k_E^2\, dk_E^2 \cdot d\Omega_4 \qquad \int d\Omega_4 = 2\pi^2$$

$$iM = -\frac{1}{2} i\lambda\, 2\pi^2 \int \frac{\frac{1}{2}k_E^2 dk_E^2}{(2\pi)^4} \frac{1}{(k_E^2+m^2)}$$

$$\propto \int_0^\infty \frac{dk_E^2\, k_E^2}{k_E^2+m^2}$$

$k_E \to \infty$ divergent integral

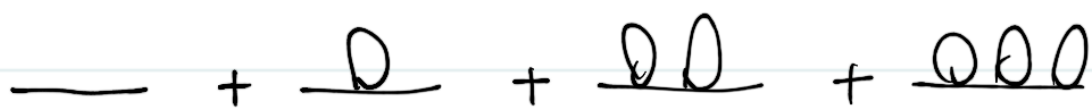
cutoff: $\displaystyle\int_0^{\Lambda^2} \frac{dk_E^2\, k_E^2}{k_E^2+m^2} = \int_0^{\Lambda^2} dk_E^2 \left[1-\frac{m^2}{k_E^2+m^2}\right]$

$$= \Lambda^2 - m^2 \log\frac{\Lambda^2+m^2}{m^2}$$

renormalization:

$k\to\infty \qquad x\to 0 \qquad$ 未至之境

考虑2pt: $\langle\Omega|T[\Phi(x)\,\Phi(y)]|\Omega\rangle$



$$\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}(iM)\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\,iM\,\frac{i}{p^2-m^2}\,iM\,\frac{i}{p^2-m^2} + \cdots$$

$$= \frac{i}{p^2-m^2+M}$$

$$p^2-m^2+M = p^2-m^2-\frac{\lambda}{32\pi^2}\left(\Lambda^2-m^2\log\frac{\Lambda^2+m^2}{m^2}\right)$$

Redefine: $\underline{m^2+\frac{\lambda}{32\pi^2}\left(\Lambda^2-m^2\log\frac{\Lambda^2+m^2}{m^2}\right)=\hat{m}^2}$

renormalization: redefine a parameter to remove the divergence (UV)

renormalizable: enough parameters to remove the div.

$\mathcal{L}$: $m.\ \lambda\ \phi$ 3 parameters

耦合:

$D = d =$

dimensional regularization:

$$d^4 k_E = dk^1 dk^2 dk^3 dk^4 \rightarrow d^d k_E = \underbrace{dk^1 \cdots dk^{d-1}}_{\text{空间}} dk^d$$

定义 $k^0 = i k^D = i k^d$

Spherical coor.: $d^d k_E = k_E^{d-1} dk_E \cdot d\Omega_d$

$$\underline{d\Omega_d = \prod_{l=1}^{d} \sin^{d-1-l}\theta_l \, d\theta_l}$$

$$\underline{\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

$d=4$:

$$\int d\Omega_4 = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2$$

$$d\Omega_4 = \sin^2\theta_1 d\theta_1 \, \sin\theta_2 d\theta_2 \cdot d\theta_3$$

$$(\sin^2 w \, dw \quad \sin\theta \, d\theta \cdot d\phi)$$

$$\Gamma(2) = (2-1)! = 1$$

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2+\Delta)^n} = \frac{1}{(2\pi)^d} \int \frac{k_E^{d-1} dk_E \, d\Omega_d}{(k_E^2+\Delta)^n} = \frac{2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int \frac{k_E^{d-1} dk_E}{(k_E^2+\Delta)^n}$$

$$= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty \frac{(k_E^2)^{\frac{d-2}{2}} dk_E^2}{(k_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} (\Delta)^{\frac{d}{2}-n} \int_0^\infty \frac{(k_E^2)^{\frac{d-2}{2}} dk_E^2}{(k_E^2+1)^n}$$

$$k_E^2 = \Delta(\hat{k}_E^2)$$

$$B(p,q) = \int_0^\infty dt \frac{t^{p-1}}{(1+t)^{p+q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\underline{= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \frac{\Gamma(\frac{d}{2})\Gamma(n-\frac{d}{2})}{\Gamma(n)} \cdot (\Delta)^{\frac{d}{2}-n}}$$

$$\Rightarrow \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

p_1+p_2-k
$\equiv p-k$

$$iM = (-i\lambda)^2 \times \frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m^2} \frac{i}{(p-k)^2-m^2}$$

$$= \frac{1}{2}\lambda^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2-m^2} \frac{1}{(k-p)^2-m^2}$$

$$= \frac{1}{2}\lambda^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[x(k-p)^2 - xm^2 + (1-x)(k^2-m^2)\right]^2}$$

$$iM\Big|_s = \frac{1}{2}\lambda^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[(k-xp)^2 - \Delta\right]^2}$$

$$\Delta = x^2p^2 - xp^2 + m^2 - i\varepsilon$$

$$k^2 - 2xk\cdot p + xp^2 - m^2 = (k-xp)^2 - x^2p^2 + xp^2 - m^2 + i\varepsilon$$

$$iM\Big|_s = \frac{1}{2}\lambda^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-\Delta)^2}$$

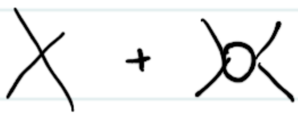
$$= \frac{1}{2}\lambda^2 \int_0^1 dx \frac{(-1)^2 i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

取 $d = 4-2\varepsilon$ $\quad \Gamma(2-\frac{d}{2}) = \Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon)$

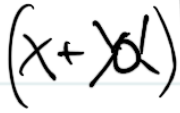
$$\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \left(\frac{1}{\Delta}\right)^{\varepsilon} = 1 - \varepsilon \ln\Delta$$

$$= \frac{i}{32\pi^2}\lambda^2 \frac{1}{\varepsilon} + \text{finite terms}$$

$$\doteq 3\,\frac{i}{32\pi^2}\lambda^2\frac{1}{\varepsilon} + \text{finite terms}$$

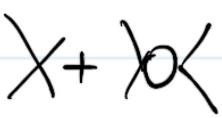


$$= -i\lambda + \frac{3i\lambda^2}{32\pi^2}\frac{1}{\varepsilon} + \underline{\text{finite terms}}$$

renormalization: 定义  $\Big|_{s=4m^2}$

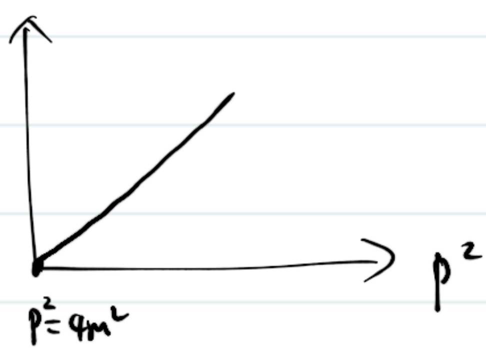
$$= -i\underline{\underline{\hat{\lambda}}}$$

$$= \left(-i\lambda + \frac{3i\lambda^2}{32\pi^2}\frac{1}{\varepsilon} + \text{finite}\right)\Big|_{p^2=4m^2}$$



$$= -i\lambda + \frac{3i\lambda^2}{32\pi^2}\frac{1}{\varepsilon} + \text{finite terms}\Big|_{p^2=4m^2} \Rightarrow -i\hat{\lambda}$$

$$+ \underbrace{\text{finite terms} - \text{finite term}\Big|_{p^2=4m^2}}_{\text{predictible}}$$



<u>相加性重整化:</u>

$$M + A\frac{1}{\varepsilon} \to \hat{M}$$

$$\lambda + B\frac{1}{\varepsilon} \to \hat{\lambda}$$

相乘性重整化: (抵消项)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

bare Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\,\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$ (加下标 指 bare)

$$\phi_0 = \sqrt{Z_\phi}\,\phi$$

$$m_0^2 = Z_m m^2$$

$$\lambda_0 = Z_\lambda \lambda$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \cdot Z_\phi - \frac{1}{2}m^2\phi^2 \cdot Z_m Z_\phi - \frac{\lambda}{4!}\phi^4 Z_\lambda Z_\phi^2$$

$$= \underline{\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4}$$

$$\underbrace{+ \frac{1}{2}(Z_\phi - 1)\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2(Z_m Z_\phi - 1) - \frac{\lambda}{4!}\phi^4(Z_\lambda Z_\phi^2 - 1)}_{\mathcal{L}_I}$$

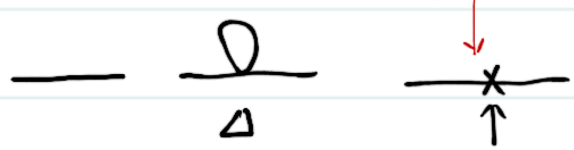
$\boxed{\text{bare pt}}$

$\boxed{\text{renormalized pt:}}$

counter term

$-i\lambda(Z_\lambda Z_\phi^2 - 1)$

① 2pt:



⇒ 有限值

$$i(Z_\phi - 1)p^2 - (Z_\phi Z_m - 1)m^2$$

$$\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\cdot iM \cdot \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left[i(Z_\phi-1)p^2 - i(Z_\phi Z_m - 1)m^2\right]\cdot\frac{i}{p^2-m^2}$$

$$= \frac{i}{p^2-m^2+M+\left[(Z_\phi-1)p^2-(Z_\phi Z_m-1)m^2\right]}$$ ⇒ 有限值

$$iM = -\frac{i\lambda}{32\pi^2}\left(\Lambda^2 - m^2\log\frac{\Lambda^2+m^2}{m^2}\right)$$ ⇒ DR $iM = \frac{-i\lambda}{32\pi^2}m^2\cdot\frac{1}{\varepsilon}$

$$Z_\phi - 1 = 0$$

$$-\frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2\log\frac{\Lambda^2+m^2}{m^2}\right) - (Z_m-1)m^2$$ 有限值

$$\Rightarrow Z_m = 1 - \frac{\lambda}{32\pi^2}\left(\frac{\Lambda^2}{m^2} - \log\frac{\Lambda^2+m^2}{m^2}\right) \quad \Rightarrow \quad Z_m = 1 - \frac{\lambda}{32\pi^2}\frac{1}{\varepsilon}$$

② 4pt:

$\to$ c.t.

$$-i\lambda + \underline{\underline{\frac{3i\lambda^2}{32\pi^2}\frac{1}{\varepsilon}}} + (-i\lambda)\cdot(Z_\lambda Z_\phi^2 - 1) \Longrightarrow$$ 有限值

$$Z_\lambda = 1 + \frac{\lambda}{32\pi^2}\frac{3}{\varepsilon}$$

4. QED 单圈重整化

a $\qquad \mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underline{\frac{1}{2\xi}(\partial\cdot A)^2} + \bar{\psi}e\not{A}\psi$

量纲：4维 $\qquad [\mathcal{L}] = 4$

$[\psi] = \frac{3}{2} \qquad [A_\mu] = 1 \quad \Rightarrow \quad [e] = 0$

$\rightarrow$ renormalizable

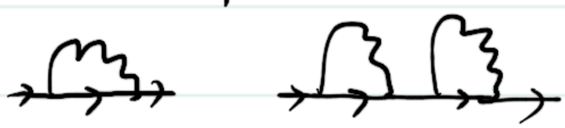
$D = 4 - \frac{3}{2}N_F - N_B$

1. $N_F = N_B = 0$

$D = 4.$

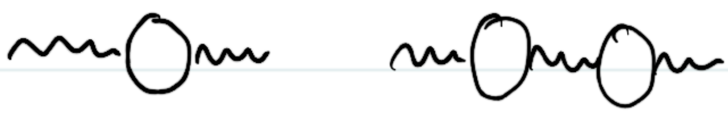
bubble diagram

2. $N_B = 0 \quad N_F = 2 \qquad D = 1$



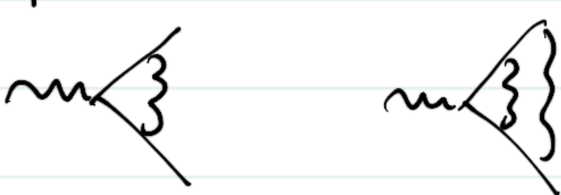
self energy of electron

3. $N_B = 2. \quad N_F = 0 \qquad D = 2$



self energy of photon

4. $N_B = 1 \quad N_F = 2 \qquad D = 0$



vertex diagram

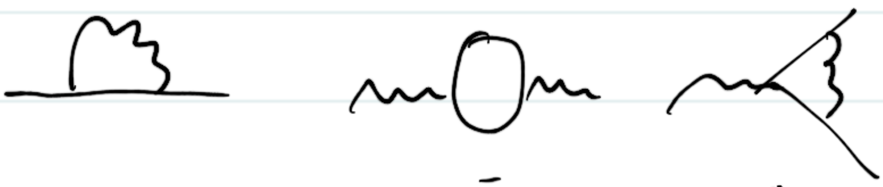
5. $N_F=0 \quad N_B=4 \qquad D=0$

gauge invariance

瑞丽散射

无

总结:



$=$ $\qquad$ Δ

6. DR 在d维下 $\qquad \mathcal{L}=\bar{\psi}(i\not{\partial}-m)\psi-\frac{1}{4}\underline{\underline{F_{\mu\nu}}}F^{\mu\nu}+\bar{\psi}e\not{A}\psi$

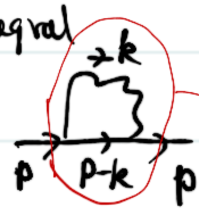
$d=4-2\varepsilon \qquad [\mathcal{L}]=d$

$$[A_\mu]=\frac{d-2}{2} \qquad\qquad 2\times\frac{d-2}{2}+2$$

$$[\psi]=\frac{d-1}{2}$$

$$[e]+[\psi]\times2+[A_\mu]=d \Rightarrow [e]=d-\frac{d-2}{2}-\frac{d-1}{2}\times2=\frac{4-d}{2}=\varepsilon$$

2. divergent integral



$$\equiv -i\Sigma_2(p)$$

$$=\int\frac{d^dk}{(2\pi)^d}\cdot ie\gamma_\mu\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}\cdot ie\gamma_\nu\times\frac{-ig^{\mu\nu}}{k^2}$$

$d=4.\quad k\to\infty \quad -i\Sigma_2\propto\frac{\infty^4\infty}{\infty^2\infty^2}\propto\infty$ 发散

$$-i\Sigma_2(p) = -e^2\int\frac{d^dk}{(2\pi)^d}\,\frac{\gamma_\mu(\not p-\not k+m)\gamma^\mu}{[(k-p)^2-m^2]\,k^2}$$

d维Dirac矩阵 (P300)

$$\gamma_\mu\gamma_\nu\gamma^\mu = (2-d)\gamma_\nu$$

$$\gamma_\mu\gamma^\mu = d$$

$$-i\Sigma_2(p) = -e^2\int\frac{d^dk}{(2\pi)^d}\,\frac{(2-d)(\not p-\not k)+dm}{[k^2][(k-p)^2-m^2]}$$

$$= -e^2\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\,\frac{(2-d)(\not p-\not k)+dm}{[x(k-p)^2-xm^2+(1-x)k^2]^2}$$

$$= -e^2\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\,\frac{(2-d)(\not p-\not k)+dm}{[(k-xp)^2-\Delta]^2}$$

$$\text{分母} = xk^2-2xk\cdot p+xp^2-xm^2+(1-x)k^2$$

$$= (k-xp)^2-x^2p^2+xp^2-xm^2 \equiv \hat{k}^2-\Delta$$

$$= -e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\,\frac{(2-d)(\not p-\not k-x\not p)+dm}{(k^2-\Delta)^2}$$

$$= -e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\,\frac{(2-d)(1-x)\not p+dm}{(k^2-\Delta)^2}$$

$$= -e^2\int_0^1 dx\,[(2-d)(1-x)\not p+dm]\,\underline{\frac{(-1)^2 i}{(4\pi)^{d/2}}\cdot\frac{\overset{\Gamma(\varepsilon)}{\Gamma(2-\frac{d}{2})}}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}}$$

$$\left(\frac{1}{\Delta}\right)^{\varepsilon} = 1-\varepsilon\ln\Delta$$

$d=4-2\varepsilon$

$$= -e^2\int_0^1 dx\left[\underbrace{(2-d)}_{-2+2\varepsilon}(1-x)\not p + dm\right]\frac{i}{(4\pi)^{d/2}}\left(\frac{1}{\varepsilon}-\underset{0.572}{\gamma_E}\right)\cdot(1-\varepsilon\ln\Delta)$$

只留领头项

$$= -ie^2\int_0^1 dx\left[-2(1-x)\not p+4m\right]$$

$$\times\frac{1}{(4\pi)^2}\left(\frac{1}{\varepsilon}\right)$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon}-\gamma_E+O(\varepsilon)$$

$$= -i\frac{e^2}{(4\pi)^2}\left[-\not p+4m\right]\cdot\frac{1}{\varepsilon}$$

包含发散

3. renormalized pt.

发散

$$\mathcal{L} = \bar\psi(i\not\partial-m)\psi - \frac14 F_{\mu\nu}F^{\mu\nu} + \bar\psi e\not A\psi$$

bare: $$\mathcal{L} = \bar\psi_0(i\not\partial-m_0)\psi_0 - \frac14(\partial_\mu A_{\nu 0}-\partial_\nu A_{\mu 0})(\partial^\mu A^\nu_0-\partial^\nu A^\mu_0)+\bar\psi_0 e\not A_0\psi_0$$

定义: $\psi_0 = \sqrt{Z_2}\,\psi$

$A^\mu_0 = \sqrt{Z_3}\,A^\mu$

$m_0 = Z_m m$

$e_0 = Z_e e$

$\xi_0 = Z_3\,\xi$

代入原来拉氏量

$$\mathcal{L} = Z_2\bar\psi i\not\partial\psi - Z_2 Z_m\bar\psi m\psi - \frac14 Z_3 F_{\mu\nu}F^{\mu\nu} + Z_2 Z_3^{1/2} Z_e\bar\psi e\not A\psi$$

$$= \bar{\psi} i\not{\partial} \psi - \bar{\psi} m \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} e \not{A} \psi$$

$$+ \Big(\underbrace{(Z_2 - 1) \bar{\psi} i\not{\partial} \psi - (Z_2 Z_m - 1) m \bar{\psi} \psi}_{\downarrow \ \text{2pt}} + \cdots \Big)$$

$i(Z_2 - 1)\not{p} - i(Z_2 Z_m - 1) m$. c.t. Feynman rule

$$\rightarrow Z_2 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon}\right)$$

$$Z_m = 1 - \frac{e^2}{(4\pi)^2} \frac{3}{\varepsilon}$$

QED: renormalization.

① $\mathcal{L} = \bar\psi_0 (i\not\partial - m_0)\psi_0 - e_0 \bar\psi_0 \not A_0 \psi_0 - \frac{1}{4} F^{0\mu\nu} F_{0\mu\nu}$ 其中 $F_0^{\mu\nu} = \partial^\mu A^{0\nu} - \partial^\nu A^{0\mu}$

$\psi_0 = Z_2^{1/2} \psi$

$A^{0\mu} = Z_3^{1/2} A^\mu$

质量 $\bar\psi_0 (-m_0) \psi_0 \to Z_2 m_0 \bar\psi\psi \equiv \underline{Z_2 Z_m m}\, \underline{\bar\psi\psi}$

相互作用 $\bar\psi_0 e_0 \not A_0 \psi_0 \to Z_2 Z_3^{1/2} e_0 \bar\psi \not A \psi \equiv Z_1 e \underline{\bar\psi \not A \psi}$

$\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{ct}$

$= \bar\psi (i\not\partial - m)\psi - e\bar\psi \not A \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ ... $\mathcal{L}_r$

$+ (Z_2 - 1)\bar\psi i\not\partial \psi - (Z_2 \underline{m_0} - m)\bar\psi\psi) - (Z_1 - 1)\bar\psi e \not A \psi - \frac{1}{4}(Z_3 - 1) F^{\mu\nu} F_{\mu\nu}$ ... $\mathcal{L}_{ct}$

$\underline{\delta_2 \bar\psi i\not\partial \psi - \delta_m \bar\psi\psi} - \underline{\delta_1 \bar\psi e \not A \psi} - \underline{\delta_3 \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$

抵消项

② $\langle p|S|p'\rangle = \langle p| e^{-i\int d^4x \mathcal{H}_I} |p'\rangle$

$= \langle p| i\int d^4x\, \mathcal{L}_I |p'\rangle$

$= \langle p| i\int d^4x\, \bar\psi(x) (i\underline{\delta_2} \not\partial - \delta_m) \psi(x) |p'\rangle$

$= i\int d^4x$

$\times\, \bar u(p) e^{ip\cdot x} (i\delta_2(-i\not p') - \delta_m)\, u(p) e^{-ip'x}$

$= (2\pi)^4 \delta^4(p - p')\, \bar u(p) (\underline{i\not p \delta_2 - i\delta_m}) u(p)$

$\Rightarrow \underline{i(\not p \delta_2 - \delta_m)}$

 $\Rightarrow \underline{-ie\gamma_\mu}$

ⓐ $\langle e^-(p')| S | \gamma(k) e^-(p)\rangle$

$$= \langle e^-(p')| \, i\int d^4x \, \mathcal{L}_I \, | \gamma(k) e^-(p)\rangle$$

$$= \langle e^-(p')| -ie\delta_1 \int d^4x \, \bar\psi(x) \not{A} \psi(x) \, | \gamma(k) e^-(p)\rangle$$

$$= -ie\delta_1 \int d^4x \;\; \bar u(p') e^{ip'x} \;\; \not\varepsilon(k) e^{-ik\cdot x} \, u(p) e^{-ip\cdot x}$$

$$= -ie\delta_1 \; \underline{(2\pi)^4 \delta^4(p+k-p')} \;\; \bar u(p') \gamma_\mu u(p) \;\; \varepsilon^\mu(k)$$

 $\Rightarrow \quad (-ie\delta_1\gamma_\mu) \quad \cdots \quad -ie\gamma_\mu$

ⓑ $\langle \gamma(k')| S | \gamma(k)\rangle = \langle \gamma(k')| \left(-\frac{1}{4}\delta_3\right) i\int d^4x \, F^{\mu\nu} F_{\mu\nu}(x) \, | \gamma(k)\rangle$

$$= -\frac{1}{4} i\delta_3 \int d^4x \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\langle \gamma(k')| (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) | \gamma(k)\rangle$$

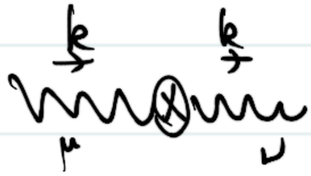
$$= -\frac{1}{4} i\delta_3 \int d^4x \, 2\times 2 \times (-1)$$

$$\times \langle \gamma(k')| \, A^\nu \, \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) \, | \gamma(k)\rangle$$

$$= i\delta_3 \int d^4x \;\; \varepsilon^{*\nu}(k') e^{ik'x} \;\; \partial^\mu\left[\partial_\mu \varepsilon_\nu(k) e^{-ik\cdot x} - \partial_\nu \varepsilon_\mu(k) e^{-ik\cdot x}\right]$$

$$= i\delta_3 \, (2\pi)^4 \delta^4(k-k') \;\; \varepsilon^*_\nu(k) \left[-k^2 g^{\mu\nu} - (-k_\mu k_\nu)\right] \varepsilon_\mu(k)$$

$$= (2\pi)^4 \delta^4(k-k') \;\; \varepsilon^*_\nu \left[(-i\delta_3)(k^2 g^{\mu\nu} - k^\mu k^\nu)\right] \varepsilon_\mu(k)$$



$$\Rightarrow \quad \underline{\underline{-i\delta_3 \, (k^2 g^{\mu\nu} - k^\mu k^\nu)}}$$

③ renormalized p.t.

$$\frac{i}{\not p - m} + \frac{i}{\not p - m}\left(-i\Sigma_2(p)\right)\frac{i}{\not p - m} + \frac{i}{\not p - m}\left(i(\not p\,\delta_2 - \delta_m)\right)\frac{i}{\not p - m} \Rightarrow \text{有限}$$

$$-i\Sigma(p) = -i\Sigma_2(p) + i(\not p\,\delta_2 - \delta_m)$$

重整化条件: $\Sigma(\not p = m) = 0$ $\quad \frac{d}{d\not p}\Sigma(\not p)\Big|_{\not p = m} = 0$.

$$-i\Sigma_2(p) = \int\frac{d^4k}{(2\pi)^4}(-ie\gamma_\mu)\frac{i(\not k+m)}{k^2-m^2}(-ie\gamma_\nu)\times\frac{-ig^{\mu\nu}}{(k-p)^2-\mu^2}$$

$$= -e^2\int\frac{d^dk}{(2\pi)^4}\gamma_\mu(\not k+m)\gamma^\mu\cdot\frac{1}{k^2-m^2}\frac{1}{(k-p)^2-\mu^2} \qquad \gamma_\mu\gamma_\nu\gamma^\mu = (2-d)\gamma_\nu$$

$$= -e^2\int\frac{d^dk}{(2\pi)^d}\left[(2-d)\not k + dm\right]\frac{1}{(k^2-m^2)((k-p)^2-\mu^2)} \qquad \gamma_\mu\gamma^\mu = d$$

$$= -e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\left[(2-d)\not k+dm\right]\frac{1}{\left[(1-x)(k^2-m^2)+x((k-p)^2-\mu^2)\right]^2}$$

$$= -e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\left[(2-d)\not k+dm\right]\frac{1}{\left[(k-xp)^2-\Delta\right]^2}$$

$$= -e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\left[(2-d)(\not k+x\not p)+dm\right]\times\frac{1}{(k^2-\Delta)^2}$$

$$D = k^2 - 2xk\cdot p - (1-x)m^2 - x\mu^2 + xp^2$$
$$= (k-xp)^2 - \underline{x^2p^2 - (1-x)m^2 - x\mu^2 + xp^2}$$
$$= (k-xp)^2 - \Delta$$

$$= -e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\left[(2-d)x\not p+dm\right]\times\frac{1}{(k^2-\Delta)^2}$$

$$= -e^2\int_0^1 dx\left[(2-d)(x\not p)+dm\right]\underline{\frac{(-1)^2 i}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}}$$

$$\underline{\underline{d=4-2\varepsilon}}\; -ie^2\int_0^1 dx\left[(-2+2\varepsilon)x\not p+(4-2\varepsilon)m\right]\cdot\frac{(4\pi)^{\varepsilon}}{(4\pi)^2}\left(\frac{1}{\varepsilon}-\gamma_E\right)\cdot\left(\frac{1}{\Delta}\right)^{\varepsilon}$$

$$\hookrightarrow (1-\varepsilon\ln\Delta)$$

$$\frac{\text{只保留}}{\text{发散}}\ -ie^2\int dx\left[-2x\not{p}+4m\right]\frac{1}{(4\pi)^2}\frac{1}{\varepsilon}$$

$$= -i\frac{e^2}{(4\pi)^2}\frac{1}{\varepsilon}\left[-\not{p}+4m\right]$$

Minimum subtraction (MS)

$$\frac{e^2}{4\pi}=\alpha$$

$$-i\frac{\alpha}{4\pi}\frac{1}{\varepsilon}\left[-\underline{\not{p}}+4m\right]+\underline{i(\not{p}\delta_2-\underline{\delta_m})}\ \text{--- 有限值}$$

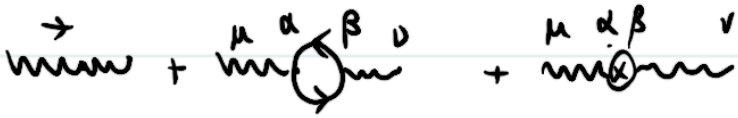
$$\Rightarrow\begin{cases}\delta_2=-\frac{\alpha}{4\pi}\frac{1}{\varepsilon}\\ m\delta_2-\delta_m=\frac{\alpha}{4\pi}\frac{1}{\varepsilon}\cdot 3m \quad (\text{令 } \not{p}=m)\\ \delta_m=\frac{\alpha}{4\pi}\frac{1}{\varepsilon}\cdot(-4m)\end{cases}$$

$$i(\not{p}\delta_2-\delta_m)\Rightarrow i\left[(\not{p}-m)\delta_2+(m\delta_2-\delta_m)\right]$$

$$\delta_2=Z_2-1\Rightarrow Z_2=\underline{1-\frac{\alpha}{4\pi}\frac{1}{\varepsilon}}$$

$$\delta_m=mZ_mZ_2-m\Rightarrow Z_m=1-\frac{\alpha}{4\pi}\frac{3}{\varepsilon}$$

光子自能项:

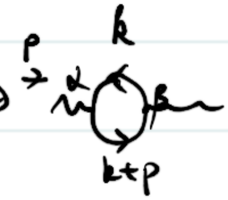


--- 有限

$$\frac{-ig_{\mu\nu}}{p^2} \quad \frac{-ig_{\mu\alpha}}{p^2}\left(i\Pi^{\alpha\beta}\right)\frac{-ig_{\nu\beta}}{p^2} + \frac{-ig_{\mu\alpha}}{p^2}\left(i\Pi_c\right)^{\alpha\beta}\cdot\frac{-ig_{\nu\beta}}{p^2}$$

$i\Pi^{\alpha\beta} + (i\Pi_c)^{\alpha\beta}$ --- 有限

$$i\Pi^{\alpha\beta} = (-1)\int\frac{d^4k}{(2\pi)^4}\cdot\mathrm{Tr}\left[-ie\gamma_\beta\frac{i(\not k+\not p+m)}{(k+p)^2-m^2}(-ie\gamma_\alpha)\frac{i(\not k+m)}{k^2-m^2}\right] \Leftarrow$$



$k\to\infty$. 原则为 k^2

但是 $\Pi^{\alpha\beta}\propto\left(p^2g^{\alpha\beta}-p^\alpha p^\beta\right)\Pi(p^2)$

$p_\alpha\Pi^{\alpha\beta} = p_\beta\Pi^{\alpha\beta} = 0$

$$\Pi(p^2) = -\frac{e^2}{12\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\cdots\right)$$

$\langle\gamma(p)|\,S\,|\gamma(p)\rangle$

$= \langle\gamma(p)|\; i\int d^4x\,\bar\psi e\not A\psi\; i\int d^4y\,\bar\psi e\not A\psi(y)|\gamma(p)\rangle$

$= \langle\gamma(p)|\, i\int d^4x\,d^4y\;\bar\psi e\not A$

$\times\int\frac{d^4q}{(2\pi)^4}\frac{i(\not q+m)}{q^2-m^2}e^{-iq\cdot(x-y)}\cdot\not A\psi(y)|\gamma(p)\rangle$

$= i\int d^4x\,d^4y\,d^4q\,(e^2)\,\langle\gamma(p)|$

$\times(-1)\mathrm{Tr}\left[\int\frac{d^4q'}{(2\pi)^4}e^{-iq'(y-x)}\frac{i(\not q'+m)}{q'^2-m^2}\right.$

$\left.\not A(x)\frac{i(\not q+m)}{q^2-m^2}e^{-iq\cdot(x-y)}\not A(y)\right]|\gamma(p)\rangle$

$$i\Pi^{\alpha\beta} + (i\Pi_c)^{\alpha\beta}\Big|_{\text{发散}} = 0$$

$$i(p^2 g^{\alpha\beta} - p^\alpha p^\beta)\pi(p^2) - i\delta_3(p^2 g^{\alpha\beta} - p^\alpha p^\beta)\Big|_{\text{发散}} = 0$$

$$\delta_3 = \pi(p^2) = -\frac{e^2}{12\pi^2}\left(\frac{1}{\varepsilon}\right) = Z_3 - 1$$

$$\Rightarrow Z_3 = 1 - \frac{e^2}{12\pi^2}\frac{1}{\varepsilon}$$

量纲: $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \underline{\underline{\bar{\psi} e A\!\!\!/ \psi}} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

d维 $[\mathcal{L}] = d$

$[\psi] = \frac{d-1}{2}$

$[A] = \frac{d-2}{2}$

$2\times[\psi] + [A] + [e] = d \Rightarrow d - 1 + \frac{d}{2} - 1 + [e] = d \Rightarrow \underline{[e] = 2 - \frac{d}{2}}$

如果 $d = 4 - 2\varepsilon$. $[e] = \varepsilon$

定义 $e = e_r \mu^\varepsilon$ $[\mu] = 1$

$$Z_3 = 1 - \frac{e^2}{12\pi^2}\frac{1}{\varepsilon} = 1 - \frac{e_r^2 \mu^{2\varepsilon}}{12\pi^2}\frac{1}{\varepsilon} = 1 - \frac{e_r^2}{12\pi^2}\frac{1}{\varepsilon}(1 + 2\varepsilon\ln\mu)$$

$$= 1 - \frac{e_r^2}{12\pi^2}\frac{1}{\varepsilon} - \underline{\frac{e_r^2}{12\pi^2} 2\ln\mu}$$

μ-dep Renormalization Group Equation.

3.

$$e(p)\,\gamma(q) \to e(p')$$

$$\bar{u}(p')\,\delta\Gamma^{\mu}\,u(p) = \int \frac{d^dk}{(2\pi)^d}\,\bar{u}(p')(-ie\gamma^{\nu})\frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\cdot(-ie\gamma^{\mu})\cdot\frac{i(\not{k}+m)}{k^2-m^2}\cdot(-ie\gamma_{\nu})\,u(p)$$
$$\times\frac{-i}{(k-p)^2}$$

$$\frac{1}{A^{\alpha}B^{\beta}C^{\gamma}} = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int dx\,dy\,dz\;\delta(1-x-y-z)\frac{x^{\alpha-1}y^{\beta-1}z^{\gamma-1}}{(xA+yB+zC)^{\alpha+\beta+\gamma}}$$

$$\frac{1}{[(k+q)^2-m^2][k^2-m^2][(k-p)^2]} = \Gamma(3)\int dx\,dy\,dz\;\delta(1-x-y-z)\frac{1}{[x[(k+q)^2-m^2]+y(k-p)^2+z(k^2-m^2)]^3}$$
$$= \int_0^1 dx\int_0^{1-x}dy\,\frac{2}{[\underbrace{x(k-p)^2+y((k+q)^2-m^2)+(1-x-y)(k^2-m^2)}_{D}]^3}$$

$$\begin{aligned}
D &= x(k-p)^2+y((k+q)^2-m^2)+(1-x-y)(k^2-m^2)\\
&= k^2-2x\,k\cdot p+2y\,k\cdot q+xp^2+yq^2-ym^2-(1-x-y)m^2\\
&= (k-xp+yq)^2-(xp-yq)^2+xp^2+yq^2-(1-x)m^2\\
&= (k-xp+yq)^2-\Delta_3
\end{aligned}$$

$$\Delta_3 = (1-x)^2m^2+(y^2+xy-y)q^2$$

分子： $\bar{u}(p')\not{q}\gamma^{\mu}\not{q}\,u(p) = -q^2\,\bar{u}(p')\gamma^{\mu}u(p)$

分子记为 $\bar{u}(p)N^{\mu}u(p) \approx \bar{u}(p')\gamma^{\mu}u(p) + \bar{u}(p')\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}u(p)$

$$\bar u(p') N^\mu u(p) = \bar u(p') \left\{ a p^\mu + b p'^\mu + c\gamma^\mu + d \frac{i\sigma^{\mu\nu}q_\nu}{2m} \right\} u(p)$$

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$

$$0 = q_\mu \bar u(p') N^\mu u(p) = \bar u(p')\left\{ a\, p\cdot q + b\, p'\cdot q + \underline{\underline{c \not q}} + 0 \right\} u(p)$$

如果 $a=b$. $a\, p\cdot q + a\, p'\cdot q = 0$

$$(p+p')\cdot(q) = (p+p')\cdot(p'-p) = p'^2 - p^2 = m^2 - m^2 = 0$$

$$\bar u(p') \not q\, u(p) = \bar u(p')(\not p' - \not p)\, u(p) = \bar u(p')(m-m)\, u(p) = 0$$

$$\bar u(p') N^\mu u(p) = \bar u(p')\left\{ a(p+p')^\mu + \underline{\underline{c\gamma^\mu}} + d\frac{i\sigma^{\mu\nu}q_\nu}{\underline{\underline{2m}}} \right\} u(p)$$

Gordon 恒等式

$$\bar u(p')(p+p')^\mu u(p) = 2m\,\bar u(p')\gamma^\mu u(p) - \bar u(p')\, i\sigma^{\mu\nu}q_\nu\, u(p)$$

分子: $$\bar u(p') N^\mu u(p) = \left\{ \frac{(2-d)^2}{d}(k')^2 - 2(x+y)(1-y)q^2 + 2m^2(4x - x^2 - 1) \right\} \bar u(p')\gamma^\mu u(p)$$
$$\underline{- 2m x(1-x)\, \bar u(p')\, i\sigma^{\mu\nu}q_\nu\, u(p)}$$

分母: $$\int_0^1 dx \int_0^{1-x} dy \frac{2}{[k'^2 - \Delta_3]^3}$$

振幅 $$= \int \frac{d^dk}{(2\pi)^d}\, \bar u(p') N^\mu u(p) \times \int_0^1 dx \int_0^{1-x} dy \frac{2}{[k'^2-\Delta_3]^3}$$

$$\infty^4 \qquad (\infty^2, \quad 1) \qquad \frac{1}{\infty^6}$$

只考虑发散: $(-ie)(-2ie^2)\int_0^1 dx \int_0^{1-x} dy \int \frac{d^d k'}{(2\pi)^d} \frac{1}{(k'^2-\Delta_3)^3} \cdot \frac{(2-d)^2}{d} k'^2 \quad \bar{u}(p')\gamma^\mu u(p)$

$$= (-ie)\bar{u}(p')\gamma^\mu u(p) \times (-2ie^2)\int_0^1 dx\int_0^{1-x}dy \int\frac{d^dk'}{(2\pi)^d}\frac{(2-d)^2}{d}\left[\frac{1}{(k'^2-\Delta_3)^2}+\frac{\Delta_3}{[(k')^2-\Delta_3]^3}\right]$$

只考虑发散

$$= \times(-2ie^2)\int_0^1 dx\int_0^{1-x}dy\ \frac{(2-d)^2}{d}\ \frac{(-1)^2 i}{(4\pi)^{d/2}}\ \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta_3}\right)^{2-\frac{d}{2}}$$

$$= \times(-2ie^2)\int_0^1 dx\ (1-x)\ 1\ \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma_E+\cdots\right).$$

$$= (-ie)\bar{u}(p')\gamma^\mu u(p) \times \frac{2e^2}{(4\pi)^2}\int_0^1 dx\,(1-x)\,\frac{1}{\epsilon}$$

$$= (-ie)\bar{u}(p')\gamma^\mu u(p)\ \frac{\alpha}{4\pi}\frac{1}{\epsilon}$$

 = 有限值

$$-ie\gamma^\mu \quad + \quad -ie\gamma^\mu\frac{\alpha}{4\pi}\frac{1}{\epsilon} \quad + \quad -ie\gamma^\mu(Z_1-1) = \text{有限值}$$

$$Z_1 = 1-\frac{\alpha}{4\pi}\frac{1}{\epsilon} = Z_2$$ Ward-Takahashi 恒等式.

② 第二种结构振幅为

$$(-ie)(-2ie^2)\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk'}{(2\pi)^d}\frac{1}{(k'^2-\Delta_3)^3}\cdot\left(-2mx(1-x)\right)\bar{u}(p')i\sigma^{\mu\nu}q_\nu u(p))$$

$$= (-ie)\bar{u}(p')i\sigma^{\mu\nu}q_\nu u(p)\cdot(-2ie^2)\int_0^1 dx\int_0^{1-x}dy\left(-2mx(1-x)\right)\frac{(-1)^3 i}{(4\pi)^2}\frac{\Gamma(3-2)}{\Gamma(3)}\left(\frac{1}{\Delta_3}\right)^{3-2}$$

$\Delta_3 = (1-x)^2m^2+(y^2+xy-y)q^2$

在 $q^2\to 0$ 极限 $\Delta_3 = (1-x)^2m^2$

$$\approx = (-ie)\,\bar{u}(p')\, i\sigma^{\mu\nu} q_\nu u(p) \times \frac{-e^2}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} \underline{\underline{dy}} \left(-2m(\underline{\underline{1-x}})\right) \cdot \frac{1}{(1-x)^2 m^2}$$

$$= (-ie)\,\bar{u}(p') \frac{i\sigma^{\mu\nu} q_\nu}{2m} u(p) \;\underline{\underline{\frac{\alpha}{2\pi} \times 1}}$$

$$\bar{u}(p')\Gamma^\mu u(p) = (-ie)\bar{u}(p') \left\{ F_1 \gamma^\mu + F_2 \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right\} u(p)$$

$$F_2(q^2=0) = \frac{\alpha}{2\pi}$$

$$a = \frac{g-2}{2} = 0 + \frac{\alpha}{2\pi} + \cdots \left(\frac{\alpha}{\pi}\right)^2 + \cdots \left(\frac{\alpha}{\pi}\right)^3 + \cdots \left(\frac{\alpha}{\pi}\right)^4 + \left(\frac{\alpha}{\pi}\right)^5$$

$\underline{72}$ $\quad \underline{891}$ $\quad \underline{12672}$

6. One loop renormalization in QCD.

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{GF} + \mathcal{L}_{FP}$$

⊗ bare perturbation: bare field → quantization → perturbative calculation → adjust parameters

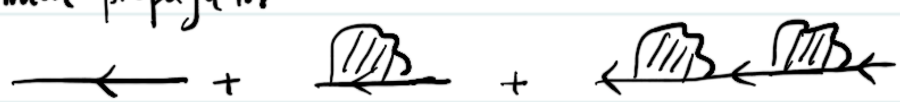
$\delta f(q^2) = f(q^2) - f(0)$

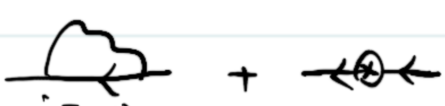
⊗ renormalized perturbation theory (use QED again)

$$\mathcal{L}_{QED} = \bar{\psi}^{(0)}(i\not{D}^{(0)} - m^{(0)})\psi^{(0)} - \frac{1}{4}F^{(0)\mu\nu}F^{(0)}_{\mu\nu} \qquad D_\mu = \partial_\mu + ieA_\mu$$

$$\left.\begin{aligned} \psi^{(0)} &= \sqrt{Z_2}\,\psi^{(r)} \\ A^{a(0)}_\mu &= \sqrt{Z_3}\,A^{a(r)}_\mu \\ e_0 Z_2 Z_3^{1/2} &= e Z_1 \\ m_0 &= Z_m m \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \delta_2 &= Z_2 - 1 \\ \delta_3 &= Z_3 - 1 \\ \delta_1 &= Z_1 - 1 = \frac{e_0}{e} Z_2 Z_3^{1/2} - 1 \\ \delta_m &= Z_2 m_0 - m = Z_2 Z_m m - m. \end{aligned}$$

- renormalization is equivalent to the determination of Z. δZ ...
- We need to use physical quantities to fix Z, they are called as "renormalization condition".
- fermion propagator





$\equiv -i\Sigma(\not{p})$

$-i\Sigma_2(\not{p})$

$$\left.\begin{aligned} &\Sigma(\not{p}=m) = 0 \\ &\frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m} = 0 \end{aligned}\right\} \rightarrow \quad \begin{aligned} m\delta_2 - \delta_m &= \Sigma_2(m) \\ \delta_2 &= \frac{d}{d\not{p}}\Sigma_2(\not{p}) \end{aligned}$$

QCD: $\mathcal{L} = -\frac{1}{4}(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu})^2 + \bar\psi_0 (i\not\partial - m_0)\psi_0 - \bar c^a_0 \partial^2 c^a_0$

$+ g_0 A^a_{0\mu} \bar\psi_0 \gamma^\mu t^a \psi_0 - g_0 f^{abc}(\partial_\mu A^a_{0\nu}) A^{b\mu}_0 A^{c\nu}_0$

$- \frac{1}{4} g_0^2 (f^{eab} A^a_{0\mu} A^b_{0\nu})(f^{ecd} A^{c\mu}_0 A^{d\nu}_0) - g_0 \bar c^a_0 f^{abc} \partial^\mu A^b_{0\mu} c^c_0.$

场的重整化: $A_{0\mu} = \sqrt{Z_3} A_\mu$

$\psi_0 = \sqrt{Z_2}\psi$

$c^a_0 = \sqrt{Z_2^c} c^a$

代入后: $\mathcal{L} = -\frac{1}{4} Z_3 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + Z_2 \bar\psi(i\not\partial - m_0)\psi - Z_2^c \bar c^a \partial^2 c^a$

$+ Z_2\sqrt{Z_3}\, g_0 A^a_\mu \bar\psi \gamma^\mu t^a \psi - g_0 Z_3^{3/2} f^{abc}(\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu}$

$- \frac{1}{4} g_0^2 Z_3^2 (f^{eab} A^a_{0\mu} A^b_{0\nu})(f^{ecd} A^{c\mu}_0 A^{d\nu}_0) - g_0 Z_2^c \sqrt{Z_3} f^{abc} \bar c^a \partial^\mu A^b_\mu c^c.$

$= \mathcal{L}_{ren}$

$- \frac{1}{4}(Z_3 - 1)(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \bar\psi(i\not\partial(Z_2 - 1) - (Z_2 m_0 - m))\psi - (Z_2^c - 1)\bar c^a \partial^2 c^a$

$+ (Z_2\sqrt{Z_3} g_0 - g) A^a_\mu \bar\psi \gamma^\mu t^a \psi - (g_0 Z_3^{3/2} - g) f^{abc}(\partial_\mu A^a_\nu)(A^{b\mu} A^{c\nu})$

$- \frac{1}{4}(g_0^2 Z_3^2 - g^2)(f^{eab} A^a_\mu A^b_\nu)(f^{ecd} A^{c\mu} A^{d\nu}) - (g_0 Z_2^c \sqrt{Z_3} - g) f^{abc} \bar c^a \partial^\mu A^b_\mu c^c.$

可以定义 $\delta_3 = Z_3 - 1$ $\qquad \delta_1 = \frac{g_0}{g} Z_2 \sqrt{Z_3} - 1$

$\delta_2 = Z_2 - 1$ $\qquad \delta_1^{3g} = \frac{g_0}{g} Z_3^{3/2} - 1$

$\delta_2^c = Z_2^c - 1$ $\qquad \delta_1^{4g} = \frac{g_0^2}{g^2} Z_3^2 - 1$

$\delta_m = Z_2 m_0 - m$ $\qquad \delta_1^c = \frac{g_0}{g} Z_2^c \sqrt{Z_3} - 1$

后面可证：$\delta_1-\delta_2=\delta_1^{3g}-\delta_3=\frac{1}{2}(\delta_1^{4g}-\delta_3)=\delta_1^{c}-\delta_2^{c}$

接下来以几类说明$U(1)$的重整化:

① 夸克自能： 17.5节

$$\frac{i}{\not p-m} + \frac{i}{\not p-m}(-i\Sigma)\frac{i}{\not p-m} + \frac{i}{\not p-m}(-i\Sigma)\frac{i}{\not p-m}(-i\Sigma)\frac{i}{\not p-m} + \cdots$$

$$= \frac{i}{\not p-m} + \frac{i}{\not p-m}\Sigma\frac{1}{\not p-m} + \frac{i}{\not p-m}\Sigma\frac{1}{\not p-m}\Sigma\frac{1}{\not p-m} + \cdots$$ 链近似

$$= \frac{i}{\not p-m-\Sigma} = \frac{i}{(\not p-m)\left(1-\frac{\Sigma}{\not p-m}\right)} = \frac{i}{\not p-m}\left(1+\frac{\Sigma}{\not p-m}+\cdots\right)$$

单圈或者 e^2 阶

$-i\Sigma$ $\qquad$ $-i\Sigma_2(\not p)$ $\qquad$ $i(\not p\delta_2-\delta m)$

重整化条件 $\quad -i\Sigma(\not p)\big|_{\not p=m}=0$

$$\frac{d}{d\not p}\Sigma(\not p)\Big|_{\not p=m}=0$$

Feynman gauge

Σ:

$$(T^a\cdot T^a)_{ij} = C_F\,\delta_{ij}$$

$$\gamma_\mu \not k \gamma^\mu = (2-d)$$

$$\gamma_\mu\gamma^\mu = d$$

$$-i\Sigma = \int\frac{d^4k}{(2\pi)^4}\, ig\gamma_\mu T^a \frac{i(\not k+\not p+m)}{(k+p)^2-m^2+i\varepsilon}\, ig\gamma^\mu T^a \times \frac{-i}{k^2}$$

$$= -g^2C_F\int\frac{d^4k}{(2\pi)^4}\,\gamma_\mu\frac{(\not k+\not p+m)}{(k+p)^2-m^2}\gamma^\mu\frac{1}{k^2}$$

$$= -g^2C_F\int\frac{d^4k}{(2\pi)^4}\frac{(dm+(2-d)(\not k+\not p))}{(k+p)^2-m^2}\frac{1}{k^2}$$

$$= -g^2C_F\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\,\frac{1}{[(k+xp)^2-\Delta]^2}\,(dm+(2-d)(\not k+\not p))$$

$$\frac{1}{(k+p)^2-m^2}\frac{1}{k^2}$$

$$=\int_0^1 dx\frac{1}{[x[(k+p)^2-m^2]+(1-x)k^2]^2}$$

$$=\int_0^1\frac{1}{\left((k+xp)^2-\underbrace{[(x^2-x)p^2+xm^2]}_{\Delta}\right)^2}$$

$$\Gamma(2-\tfrac{d}{2})=\frac{1}{\varepsilon}-\gamma_E$$

$d=4-2\varepsilon$

$$\downarrow k$$

$$=-g^2C_F\int_0^1dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta)^2}\left[dm+(2-d)(\not k-x\not p+\not p).\right]$$

$$=-g^2C_F\int_0^1dx\left[dm+(2-d)(1-x)\not p\right]\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta)^2}$$

$$\Downarrow$$

$$\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$=\underbrace{-ig^2C_F\int_0^1dx\left[dm+(2-d)(1-x)\not p\right]\frac{1}{(4\pi)^{d/2}}\Gamma(2-\tfrac{d}{2})\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}}_{\Sigma(p)}$$

结果包含发散需要被抵消项消除，即　　　　(δ_m定义，可能不一样)

$$-\Sigma(p)+\not p\overbrace{(Z_2-1)}^{\delta_2}-\delta_M=0\qquad\text{---- ① 为有限值}$$

?

令 $\not p=m$.　$-\Sigma(m)+m\delta_2-\delta_m$

$$\Sigma(m)=\frac{g^2C_F}{(4\pi)^2}\int_0^1dx\left[dm+(2-d)(1-x)m\right]\left[\frac{1}{\varepsilon}-\gamma+\ln4\pi+\ln(x^2m^2)\right]$$

$$m\delta_2-\delta m=+\frac{\alpha_sC_F}{4\pi}\times3\left(\frac{1}{\varepsilon}-\gamma+\ln4\pi\right)m$$

为了求Z_2，对①作微商

$$Z_2-1=\left.\frac{d\Sigma}{d\not p}\right|_{\not p=m}\qquad\downarrow\text{只保留发散项}$$

$$=\frac{g_s^2C_F}{(4\pi)^2}\int_0^1dx\,(2-d)(1-x)\left(\frac{1}{\varepsilon}-\gamma+\ln4\pi\right)$$

$$=-\frac{\alpha_sC_F}{4\pi}\left(\frac{1}{\varepsilon}-\gamma+\ln4\pi\right)$$

$$Z_2=1-\frac{\alpha_sC_F}{4\pi}\left(\frac{1}{\varepsilon}-\gamma+\ln(4\pi)\right)$$

如果采用 covariant gauge，则$-i\Sigma$为：

$$-i\Sigma=\int\frac{d^4k}{(2\pi)^4}\,ig\gamma_\mu T^a\frac{i(\not k+\not p+m)}{(k+p)^2-m^2}\,ig\gamma_\nu T^a\times\frac{-i}{k^2}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}(1-\xi)\right)$$

$k^\mu k^\nu$ 分子部分：$\not k(\not k+\not p+m)\not k=k^2\not k+\not k\not p\not k+mk^2$

费曼参数化: $\frac{1}{(k^2)^2[(k+p)^2-m^2]} = \int_0^1 dx \frac{2(1-x)}{[x((k+p)^2-m^2)+(1-x)k^2]^3}$

$$= \int_0^1 dx \frac{2(1-x)}{[(k+xp)^2-\Delta]^3} \qquad \Delta = xm^2+x^2p^2-xp^2$$

$$-i\Sigma(p) = -g^2 C_F \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \left\{ \frac{dm+(2-d)(\not k+\not p)-(1-\xi)(\not k+m)}{[(k+xp)^2-\Delta]^2} - (1-\xi)\frac{\not k\not p\not k\, 2(1-x)}{[(k+xp)^2-\Delta]^3} \right\}$$

$$= -g^2 C_F \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \left\{ \frac{dm+(2-d)(1-x)\not p-(1-\xi)(-x\not p+m)}{(k^2-\Delta)^2} \right.$$

$\not k\not p\not k = k_\mu k_\nu \gamma^\mu \not p \gamma^\nu$

$$\left. -(1-\xi)2(1-x)\frac{x^2\not p\not p\not p+\not k\not p\not k}{(k^2-\Delta)^3} \right\}$$

$= \frac{k^2}{d}\gamma^\mu\not p\gamma_\mu$

$$= -g^2 C_F \times \int_0^1 dx \left\{ [dm+(2-d)(1-x)\not p-(1-\xi)(m-x\not p)] \frac{i}{(4\pi)^{d/2}}\Gamma(2-\tfrac{d}{2})\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \right.$$

$= \frac{2-d}{d}k^2\not p$

$$\left. -(1-\xi)2(1-x)\frac{2-d}{d}\not p \frac{(-1)^2 i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(3-\frac{d}{2}-1)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{3-\frac{d}{2}-1} + \cdots \right\}$$

$$= -i\frac{g^2}{(4\pi)^2}C_F\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right)\times\left[4m-\not p-(1-\xi)(m-\tfrac{1}{2}\not p)+(1-\xi)\not p/2\right]$$

$$= -i\frac{g^2}{(4\pi)^2}C_F\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right)\left[3m-\not p+m\xi+\not p(1-\xi)\right]$$

$$= -i\frac{g^2}{(4\pi)^2}C_F\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right)\left[3m-\xi(\not p-m)\right] \equiv -i\Sigma$$

同样地 $\quad -\Sigma(m)+m\delta_2-\delta_m=0 \quad \to m\delta_2-\delta_m = \frac{\alpha_s}{4\pi}C_F\, 3m\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right)$

$$Z_2-1 = \frac{d\Sigma}{d\not p} \qquad \to Z_2 = 1-\frac{\alpha_s}{4\pi}\xi C_F\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right)$$

② 胶子自能修正：17.6节.

$= i\Pi(q^2)(q^2 g^{\mu\nu}-q^\mu q^\nu)\delta^{ab}$

树图传播子为

$$iD^{ab}_{\mu\nu}(k) = \frac{i}{k^2+i\epsilon}\delta^{ab}\left(-g_{\mu\nu}+(1-\alpha)\frac{k_\mu k_\nu}{k^2+i\epsilon}\right)$$

先给出抵消项：(与之前方法稍有不同)

counterterm from: $\mathcal{L} = -\frac{1}{4} F^{a,0}_{\mu\nu}F^{a,0,\mu\nu} = -\frac{1}{4} Z_3 F^a_{\mu\nu}F^{a,\mu\nu}$

two-point c.t. $\mathcal{L}_c^{2pt} = -\frac{1}{4}(Z_3-1)(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a,\nu}-\partial^\nu A^{a,\mu})$

$$\langle 0|T[A^{a\mu}(x)\, i\int d^4z\, \mathcal{L}_c^{2pt}(z)\, A^{b,\nu}(y)]|0\rangle$$ (此为坐标空间)

$$= -\frac{i}{4}(Z_3-1)\int d^4z\, \langle 0|T[A^{a,\mu}(x)(\partial_\alpha A^c_\beta - \partial_\beta A^c_\alpha)(\partial^\alpha A^{c,\beta}-\partial^\beta A^{c,\alpha})A^{b,\nu}(y)]|0\rangle$$

$$= -\frac{i}{4}(Z_3-1)\times 2\int d^4z \int\frac{d^4k_1}{(2\pi)^4}\frac{ik_{1\alpha}d_{\mu\beta}-ik_{1\beta}d_{\mu\alpha}}{k_1^2+i\epsilon}\times e^{-ik_1(x-z)}\,\delta^{ac}\times(-i)$$

$$\times\int\frac{d^4k_2}{(2\pi)^4}e^{-ik_2(z-y)}\frac{-ik_2^\alpha d^{\nu\beta}-(-ik_2^\beta)d^{\nu\alpha}}{k_2^2+i\epsilon}\,\delta^{bc}\times(-i)$$

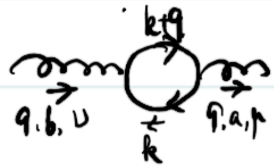
$$= -\frac{i}{2}(Z_3-1)\int\frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\frac{-i}{k^2+i\epsilon}\left[k^2 d_{\mu\beta}d^{\nu\beta}-k_\beta d_{\mu\alpha}k^\alpha d^{\nu\beta}\right]\times 2\times\frac{-i}{k^2+i\epsilon}\delta^{ab}$$

$$= \underline{-i(Z_3-1)}\int\frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\underline{\frac{-id_{\mu\alpha}}{k^2+i\epsilon}}\left[k^2 g^{\alpha\beta}-k^\alpha k^\beta\right]\frac{-id_{\nu\beta}}{k^2+i\epsilon}\,\delta^{ab}$$

$-i\delta_3(k^2 g^{\mu\nu}-k^\mu k^\nu)\delta^{ab}$

Now let us calculate the one loop diagrams, and mark the intermediate factor as $i\Pi^{\mu\nu,ab}$

夸克圈



$$i\Pi_2^{\mu\nu,ab} = (-1)\, tr[t^a t^b] \int \frac{d^4k}{(2\pi)^4}\, tr\left[ig\gamma_\mu \frac{i(\not k+\not q+m)}{(k+q)^2-m^2} ig\gamma_\nu \frac{i(\not k+m)}{k^2-m^2}\right]$$

$$tr[t^a t^b] = C(N)\delta^{ab} = T_F\delta^{ab}$$

$$= (-1) T_F \delta^{ab} 4g^2 \int\frac{d^4k}{(2\pi)^4} \frac{(k+q)^\mu k^\nu - 4 g^{\mu\nu}(k+q)\cdot k + k^\mu (k+q)^\nu}{[k^2-m^2][(k+q)^2-m^2]}$$

$$= -4g^2 T_F\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{N^{\mu\nu}}{[(k+xq)^2-\Delta]^2}$$

$$= -4g^2 T_F\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{(k-xq+q)^\mu (k-xq)^\nu - 4g^{\mu\nu}(k-xq+q)\cdot(k-xq) + (k-xq)^\mu(k-xq+q)^\nu}{(k^2-\Delta)^2}$$

其中使用了费曼参数化:

$$x(k+q)^2 - xm^2 + (1-x)(k^2-m^2) = (k+xq)^2 - m^2 + xq^2 - x^2q^2 = (k+xq)^2 - \underbrace{[m^2+x^2q^2-xq^2]}_{\Delta}$$

$$i\Pi^{\mu\nu}_{ab} := -4g^2 T_F\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{1}{(k^2-\Delta)^2}$$

$$\times\left\{2k^\mu k^\nu - g^{\mu\nu}k^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}x(1-x)q^2 + m^2 g^{\mu\nu}\right\}$$

$$= -4g^2 T_F\delta^{ab}\int_0^1 dx\left\{\left[\frac{2}{d}-1\right]g^{\mu\nu}\frac{-i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-\frac{d}{2}-1)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}-1}\right.$$

$$\left.+\left[-2x(1-x)q^\mu q^\nu + g^{\mu\nu}x(1-x)q^2 + m^2 g^{\mu\nu}\right]\times\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\right\}$$

$$\Gamma(2-\tfrac{d}{2}) = \frac{1}{\varepsilon}-\gamma$$

$$\Gamma(2-\tfrac{d}{2}-1) = \Gamma(-1+\varepsilon) = \frac{1}{-1+\varepsilon}\Gamma(\varepsilon) = \frac{1}{-1+\varepsilon}\left(\frac{1}{\varepsilon}-\gamma+\cdots\right)$$

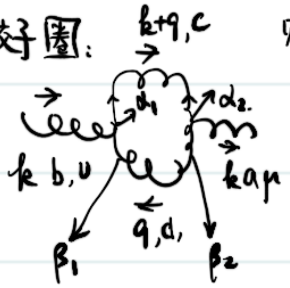
$$= -4ig^2 T_F\delta^{ab}\times\frac{1}{(4\pi)^2}\int_0^1 dx\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right)\times$$

$$\times\left[g^{\mu\nu}(x(1-x)q^2+m^2-\Delta) - 2x(1-x)q^\mu q^\nu\right]$$

$$= -\frac{8i\alpha_s}{4\pi}T_F\delta^{ab}\int_0^1 dx\, x(1-x)\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right)\left[g^{\mu\nu}q^2 - q^\mu q^\nu\right]$$

$$= -i\frac{\alpha_s}{4\pi}T_F\delta^{ab}\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi\right)\left(g^{\mu\nu}q^2-q^\mu q^\nu\right)$$

三胶子圈：　见 P_{409} 页



$$i\Pi^{\mu\nu,ab} = \frac{1}{2} g^2 f^{acd} f^{bcd} \int \frac{d^dq}{(2\pi)^d} \frac{-i d^{\alpha_1\alpha_2}}{(k+q)^2+i\varepsilon} \times \frac{-i d^{\beta_1\beta_2}}{q^2+i\varepsilon}$$

$k \leftrightarrow q$

$$\times \left\{ (-q-q-k)_{\beta_2} g_{\mu\alpha_2} + (k+q+k)_\mu g_{\alpha_2\beta_2} + (-k+q)_{\alpha_2} g_{\mu\beta_2} \right\}$$

$$\times \left\{ (q+k+q)_{\beta_1} g_{\nu\alpha_1} + (-k-q-k)_\nu g_{\alpha_1\beta_1} + (k-q)_{\alpha_1} g_{\nu\beta_1} \right\}$$

$\Delta = x(1-x)k^2$

$$= \frac{g^2}{2} f^{acd} f^{bcd} \int \frac{d^dq}{(2\pi)^d} \frac{1}{(k+q)^2 q^2} \left[A_{\mu\nu} + (1-\alpha) B_{\mu\nu} + (1-\alpha)^2 C_{\mu\nu} \right]$$

$f^{acd} f^{bcd} = \delta^{ab} C_G$

结果　$$\Pi^{ab}_{\mu\nu}(k) = \frac{g^2}{2(4\pi)^{2-\varepsilon}} \delta^{ab} C_G (-k^2)^{-\varepsilon} \frac{\Gamma(\varepsilon) B(2-\varepsilon, 2-\varepsilon)}{1-\varepsilon}$$

$$\times \Big\{ (19-12\varepsilon) k^2 g_{\mu\nu} - 2(11-7\varepsilon) k_\mu k_\nu$$

$$+ (k^2 g_{\mu\nu} - k_\mu k_\nu)(3-2\varepsilon)(1-\alpha)\left[2(1-4\varepsilon) + (1-\alpha)\varepsilon\right] \Big\}$$

③

为 0 （已可以见 Peskin 书的讨论）

④

$$\Pi^{ab}_{\mu\nu} = \frac{g^2}{2(4\pi)^{2-\varepsilon}} \delta_{ab} C_G (-k^2)^{-\varepsilon} \frac{\Gamma(\varepsilon) B(2-\varepsilon, 2-\varepsilon)}{1-\varepsilon} \left[k^2 g_{\mu\nu} + 2(1-\varepsilon) k_\mu k_\nu \right]$$

胶子+鬼圈为　$$\Pi^{ab}_{\mu\nu} = \frac{g^2}{(4\pi)^{2-\varepsilon}} \delta^{ab} C_G (-k^2)^{-\varepsilon} (k^2 g_{\mu\nu} - k_\mu k_\nu) \frac{\Gamma(\varepsilon) B(2-\varepsilon, 2-\varepsilon)}{1-\varepsilon}$$

$$\times \left[2(5-3\varepsilon) + (1-\alpha)(1-4\varepsilon)(3-2\varepsilon) + (1-\alpha)^2 \frac{\varepsilon}{2}(3-2\varepsilon) \right]$$

总结果为：　$$\Pi^{ab}_{\mu\nu}(k) = \delta^{ab} (k_\mu k_\nu - k^2 g_{\mu\nu}) \Pi(k^2)$$

$$\Pi(k^2) = -\frac{g^2}{(4\pi)^2} C_G \left[\left(\frac{13}{6} - \frac{\alpha}{2} \right) \left(\frac{1}{\varepsilon} - \gamma - \ln \frac{-k^2}{4\pi\mu^2} + \frac{31}{9} - (1-\alpha) + \frac{(1-\alpha)^2}{4} \right] \right.$$

$$+ \frac{g^2}{(4\pi)^2} T_R N_f \frac{4}{3} \left[\frac{1}{\varepsilon} - \gamma - \ln \frac{-k^2}{4\pi\mu^2} - 6 \int_0^1 dx\, x(1-x) \ln\left[x(1-x) - \frac{m^2}{k^2}\right] \right]$$

$$+ Z_3 - 1$$

于是　$$Z_3 = 1 - \frac{g^2}{(4\pi)^2} \left[\frac{4}{3} T_R N_f - \frac{1}{2} C_G \left(\frac{13}{3} - \alpha \right) \right] \frac{1}{\varepsilon} + \cdots$$

③ 夸克胶子顶点修正： 计算细节见 Peskin P529

$$\delta_1 = -\frac{g^2}{(4\pi)^2}\left(C_2(r)+C_2(G)\right)\left(\frac{1}{\varepsilon}-\gamma+\ln 4\pi+\cdots\right)$$

β function: $\beta(g) = g\mu\frac{d}{d\mu}\left(-\delta_1+\delta_2+\frac{1}{2}\delta_3\right)$

$$= -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G)-\frac{4}{3}n_f C(r)\right]$$

$C(r) = T_F = \frac{1}{2} \qquad C_2(G) = N.$

其它项结果：

⊗ 鬼粒子传播子：

$$\tilde{\Pi}^{ab}(k)=\delta^{ab}k^2\left[-\frac{g^2}{(4\pi)^2}C_G\frac{3-\alpha}{4}\frac{1}{\varepsilon}+\tilde{Z}_3-1\right]$$

$$\tilde{Z}_3=1+\frac{g^2}{(4\pi)^2}C_G\frac{3-\alpha}{4}\frac{1}{\varepsilon}$$

⊗ 三胶子顶角：

$$\Lambda^{abc}_{\mu\nu\lambda}=-igf^{abc}V_{\mu\nu\lambda}(k_1k_2k_3)$$
$$\times\left[\frac{g^2}{(4\pi)^2}\left(C_G\left(-\frac{17}{12}+\frac{3\alpha}{4}\right)+\frac{4}{3}T_RN_f\right)\frac{1}{\varepsilon}+Z_1-1\right]$$

$$Z_1=1-\frac{g^2}{(4\pi)^2}\left[C_G\left(-\frac{17}{12}+\frac{3\alpha}{4}\right)+\frac{4}{3}T_RN_f\right]\frac{1}{\varepsilon}$$

⊗ 鬼粒子-胶子顶角：

$$\Lambda^{abc}_{\mu}(k,p,p')=-igf^{abc}p_\mu\left[\frac{g^2}{(4\pi)^2}C_G\frac{\alpha}{2}\frac{1}{\varepsilon}+\tilde{Z}_1-1\right]$$

$$\tilde{Z}_1=1-\frac{g^2}{(4\pi)^2}C_G\frac{\alpha}{2}\frac{1}{\varepsilon}$$

⊗ 夸克-胶子耦合顶角：

$$\Lambda^{aij}_{F\mu}=g\gamma_\mu T^a_{ij}\left[\frac{g^2}{(4\pi)^2}\left(\frac{3+\alpha}{4}C_G+\alpha C_F\right)\frac{1}{\varepsilon}+Z_{1F}-1\right]$$

$$Z_{1F}=1-\frac{g^2}{(4\pi)^2}\left[\frac{3+\alpha}{4}C_G+\alpha C_F\right]\frac{1}{\varepsilon}$$

⊗ 四胶子顶角：

$$\Lambda^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2k_3k_4)=-g^2W^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}$$
$$\times\left[\frac{g^2}{(4\pi)^2}\left(\left(-\frac{2}{3}+\alpha\right)C_G+\frac{4}{3}T_RN_f\right)\frac{1}{\varepsilon}+Z_4-1\right]$$

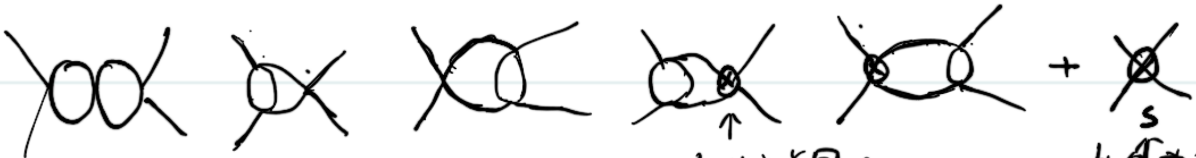
$$Z_4=1-\frac{g^2}{(4\pi)^2}\left[\left(-\frac{2}{3}+\alpha\right)C_G+\frac{4}{3}T_RN_f\right]\frac{1}{\varepsilon}$$

本部分最后练习!(也是从初等量子场论通往高等量子场论的转折点)

请在维数正规化下严格计算顶点修正,和实圆修正!

$m=0$. $p_1^2=p_2^2=0$. $(p_1+p_2)^2=q^2$

最后练习2. 两圈图计算 Peskin 338.



此处为单圈ct 此处为双圈s道ct.

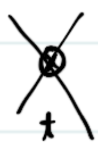
+ (s→t, s→u)交换

单圈s道:

$p \rightarrow$  $= (-i\lambda)^2 V(p^2) = (-i\lambda)^2 \left(-\frac{i}{2}\right) \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dx \frac{1}{[m^2 - x(1-x)p^2]^{2-\frac{d}{2}}}$

 $= -i\delta_\lambda = (-i\lambda)^2 \left[-iV(4m^2) - 2iV(q^2=0)\right]$

定义 

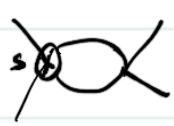
$= (-i\lambda)^2 \times (-iV(4m^2))$ 

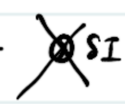
$=$ 

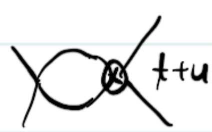
$= (-i\lambda)^2 \times (-iV(q^2=0))$

把s channel分为三类:

Group I:  + 

\+ 

\+ 

Group II:  + 

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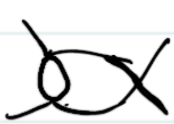
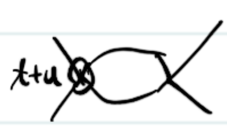
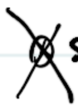
Group III:  

\+ 

Group I:

 $= (-i\lambda)^3 [iV(p^2)]^2$

 +  $= 2(-i\lambda)(-i\delta\lambda)(iV(p^2)) = 2(-i\lambda)^3\, iV(p^2)\,(-iV(4m^2))$

 SI $\equiv -i\delta_\lambda^{SI}$

$$iM\Big|_{SI} = (-i\lambda)^3 (iV(p^2))^2 + 2(-i\lambda)^3\, iV(p^2)\,(-iV(4m^2)) + (-i\delta_\lambda^{SI})$$

$$= (-i\lambda)^3 \left[(iV(p^2) - iV(4m^2))^2 - (iV(4m^2))^2 \right] + (-i\delta_\lambda^{SI})$$

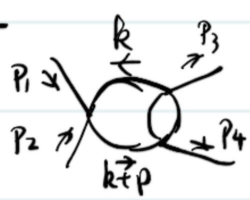
重整化条件:

$$iM\Big|_{SI,\ s=4m^2,\ t=u=0} = 0$$

$$-i\delta_\lambda^{SI} = (-i\lambda)^3 \left(+(iV(4m^2))^2\right)$$

其中 $V(p^2) - V(m^2) = \frac{1}{32\pi^2}\int_0^1 dx \log\left[\frac{m^2 - x(1-x)p^2}{m^2 - x(1-x)4m^2}\right]$

Group II:



$$(-i\lambda)^3 \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m^2} \frac{i}{(k+p)^2-m^2}\, iV((k+p_3)^2) \equiv iM_{SII}$$

将V的表达式代入，并对剩余的两个传播子做 Feynman 参数化可得

$$iM_{s_{\text{II}}}=-\frac{\lambda^3}{2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\int_0^1 dy\int\frac{d^dk}{(2\pi)^d}\frac{1}{[k^2+2yk\cdot p+yp^2-m^2]^2}\frac{1}{[m^2-x(1-x)(k+p_3)^2]^{2-\frac{d}{2}}}$$

接下来进一步做 Feynman 参数化：

$$\frac{1}{A^\alpha B^\beta}=\int_0^1 dw\,\frac{w^{\alpha-1}(1-w)^{\beta-1}}{[wA+(1-w)B]^{\alpha+\beta}}\,\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

则振幅变为：

$$iM_{s_{\text{II}}}=-\frac{\lambda^3}{2}\frac{\Gamma(4-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\int_0^1 dy\int_0^1 dw\int\frac{d^dk}{(2\pi)^d}$$

$$\times\frac{w^{1-\frac{d}{2}}(1-w)}{\left[w\left(m^2-x(1-x)(k+p_3)^2\right]+(1-w)\left(m^2-k^2-2yk\cdot p-yp^2\right)\right)^{4-\frac{d}{2}}}$$

现在化简分母：

$$-wx(1-x)(k+p_3)^2+(1-w)(-k^2-2yk\cdot p-yp^2)+m^2$$

$$=[-wx(1-x)-(1-w)]k^2-2wx(1-x)k\cdot p_3-2y(1-w)k\cdot p-wx(1-x)p_3^2-y(1-w)p^2+m^2$$

$$=[-wx(1-x)-(1-w)]\left[k+\frac{wx(1-x)p_3+y(1-w)p}{w(1-x)x+(1-w)}\right]^2$$

$$+\frac{[wx(1-x)p_3+y(1-w)p]^2}{wx(1-x)+(1-w)}-wx(1-x)p_3^2-y(1-w)p^2+m^2$$

$$\equiv-(\underbrace{(1-w)+wx(1-x)}_{\geq 0})\ell^2-P^2+m^2-i\varepsilon=(-1-i\varepsilon)((1-w)+wx(1-x))\left(\ell^2-\frac{m^2-P^2}{(1-w)+w(1-x)}+i\varepsilon\right)$$

其中 $P^2 = wx(1-x)m^2 + y(1-w)p^2 - \dfrac{w^2x^2(1-x)^2m^2 + 2wx(1-x)y(1-w)p\,p_3 + y^2(1-w)^2p^2}{wx(1-x) + (1-w)}$

$$iM_{sII} = -\frac{\lambda^3}{2}\frac{\Gamma(4-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\int_0^1 dy\int_0^1 dw\int\frac{d^dk}{(2\pi)^d}\, w^{1-\frac{d}{2}}(1-w)$$

$$\times\frac{1}{\left[-[(1-w)+wx(1-x)]\,l^2 - P^2 + m^2 - i\varepsilon\right]^{4-\frac{d}{2}}}$$

$$= -\frac{\lambda^3}{2}\frac{\Gamma(4-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\int_0^1 dy\int_0^1 dw\, w^{1-\frac{d}{2}}(1-w)\,(-1-i\varepsilon)^{-4+\frac{d}{2}}\left((1-w)+wx(1-x)\right)^{-4+\frac{d}{2}}$$

这两个$(-1)^{4-\frac{d}{2}}$相消，可用欧氏空间转动得到.

$$\times\frac{(-1)^{4-\frac{d}{2}}\, i}{(4\pi)^{d/2}}\frac{\Gamma(4-\frac{d}{2}-\frac{d}{2})}{\Gamma(4-\frac{d}{2})}\frac{1}{\left(\frac{m^2-P^2-i\varepsilon}{(1-w)+wx(1-x)}\right)^{4-\frac{d}{2}-\frac{d}{2}}}$$

$$= -\frac{i}{2}\lambda^3\frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1 dx\int_0^1 dy\int_0^1 dw\, w^{1-\frac{d}{2}}(1-w)\frac{1}{(m^2-P^2-i\varepsilon)^{4-d}}\frac{1}{[(1-w)+wx(1-x)]^{d/2}}$$

接下来需要将 $w=0$ 的极点贡献分离出来，记整体被积函数为 $f(w)$

$$iM_{sII} = -\frac{i}{2}\lambda^3\frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1 dw\, w^{1-\frac{d}{2}} f(w)$$

$$= -\frac{i}{2}\lambda^3\frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1 dw\, w^{1-\frac{d}{2}}\left(f(w)-f(0)+f(0)\right)$$

该操作类似于 plus function 定义，用以消除端点发散.

第项贡献中积分是有限的

$$-\frac{i}{2}\lambda^3\frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1 dw\, w^{1-\frac{d}{2}}\int_0^1 dx\int_0^1 dy\left(\frac{1-w}{[1-w+wx(1-x)]^{d/2}}\frac{1}{(m^2-P(w))^{4-d}} - \frac{1}{(m^2-P(w=0))^{4-d}}\right)$$

$$= -\frac{i}{2}\lambda^3\frac{\Gamma(4-d)}{(4\pi)^d}\times\left(1+\varepsilon\times\cdots\right)$$

将 $d=4-2\varepsilon$ 展开

如果$m=0$，积分为 $1+\frac{1}{2}\varepsilon\left[6+\pi^2-4\log(-P^2)\right]$

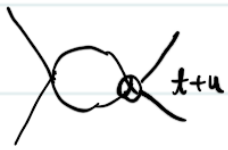
第二项贡献

$$-\frac{i}{2}\lambda^3 \frac{\Gamma(4-d)}{(4\pi)^d} \int_0^1 dx \int_0^1 dy \underline{\int_0^1 dw\, w^{1-\frac{d}{2}}} \cdot \frac{1}{[m^2-y(1-y)p^2]^{4-d}}$$

$$= -\frac{i}{2}\lambda^3 \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \frac{1}{\varepsilon} \int_0^1 dy \frac{1}{(m^2-y(1-y)p^2)^{2\varepsilon}}$$

$$= -\frac{i}{2}\lambda^3 \frac{1}{\varepsilon} \frac{1}{(4\pi)^4} \int_0^1 dy \left[\frac{1}{2\varepsilon} - \gamma + \ln 4\pi - \ln[m^2-y(1-y)p^2]\right]$$

在m=0时 $\frac{1}{2\varepsilon} - \gamma + \ln 4\pi + 2 - \ln(-p^2)$



$$iM_{s_{II}} = (-i\lambda)^3 (-2iV(0))\, iV(p^2)$$

$$= \frac{i\lambda^3}{2(4\pi)^d} \int_0^1 dy \frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{[m^2-y(1-y)p^2]^{2-\frac{d}{2}}}$$

$$= \frac{i\lambda^3}{2(4\pi)^4} \int_0^1 dy \left(\frac{1}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2\right)\left(\frac{1}{\varepsilon} - \gamma + \ln 4\pi - \ln(m^2-y(1-y)p^2)\right)$$

可以看到两部分相加后 non local divergence 即 $\frac{1}{\varepsilon}\ln(m^2-y(1-y)p^2)$ 相互抵消.

剩余的 local divergence 可以被下一阶抵消项所吸收.

现考虑无质量情况，$m=0$

单圈  $\equiv (-i\lambda)^2 iV(p^2) = (-i\lambda)^2\left(-\frac{i}{2}\right)\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{1}{[-x(1-x)p^2]^{2-\frac{d}{2}}}$

本节采用MS方案，即 单圈抵消项为

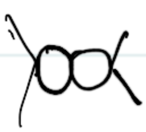
 $= -i\delta_\lambda = (-i\lambda)^2\left(+\frac{i}{2}\right)\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\times 3 \equiv (-i\delta_\lambda)^2\times 3\times(-i\hat{V}(0))$

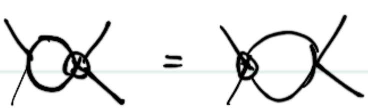
定义为

 $= -i\delta_\lambda\,(-i\hat{V}(0))$ 此处 $\hat{V}(0)$ 表示仅减除发散

 $t+u = -i\delta_\lambda\times 2\,(-i\hat{V}(0))$

考虑 2-loop，Group I：

 $= (-i\lambda)^3(iV(p^2))^2$

 $= (-i\lambda)^3\, iV(p^2)\cdot(-i\hat{V}(0))$

$$(-i\lambda)^3\left[(iV(p))^2 - 2iV(p^2)\,i\hat{V}(0)\right] = (-i\lambda)^3\left[+\left(iV(p^2)-i\hat{V}(0)\right)^2 - \left(i\hat{V}(0)\right)^2\right]$$

$$iV(p^2) - i\hat{V}(0) = \frac{1}{32\pi^2}\int_0^1 dx\,\ln\frac{-x(1-x)p^2}{\mu^2}$$

其余各项均和有质量类似

$$P(W) = y(1-w)p^2 - \frac{2wx(1-x)p_3 \cdot p + y^2(1-w)^2 p^2}{wx(1-x) + (1-w)}$$

$$p - p_3 = p_3 - p_4 = \frac{1}{2} p^2$$