红外发散与因子化：

1. 红外发散与KLN定理．　　虚图顶角修正．与实图单圈修正．

2. 算符乘积展开：

ⓐ 以4夸克有效算符为例．：　small-x expansion

ⓑ $e^+e^-\to q\bar{q}$ 中的 OPE　formal and phenuma

ⓒ DIS中的 OPE：　light cone expansion

3. 因子化：共线因子化

ⓐ DIS中的共线因子化：按区域展开：hard-collinear-soft-separation

ⓑ DIS单圈散射核的计算　　★　单圈散射核

4. Drell-Yan与共线因子化．　<u>选讲</u>

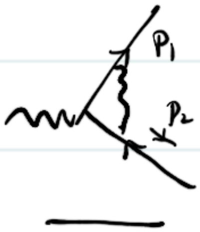
PDF：物理解释：从粒子数密度到规范不变．

$\underline{0}$ 发散:

- 紫外发散 $\underline{\int \frac{d^4k}{(k^2+m^2)^2}}$ $k\to\infty$. 对应 $x\to 0$
- 红外发散: $\underline{k^2\to 0}$
  - 软发散: 所有分量为0
  - 共线发散 $k^0 \sim |\vec{k}|$
- 快度发散: 不是软发散

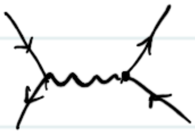
进行上课之前. 先做练习.

计算右图的顶点修正.



$q = p_1+p_2$

H. 红外发散：以 $e^+e^-\to q\bar{q}$ 为例.

树图  $e^+e^-\to\mu^+\mu^-$ $(-ie\gamma_\mu)$ $e^+e^-\to q\bar{q}$: $(-iee_q\gamma_\mu)$

单圈：

$\underline{\alpha_s\sim g_s^2}$

虚图 实图.

虚图： 直接计算

$$\underline{\Gamma_\mu} = \frac{\alpha_s C_F}{4\pi}\left(e^{-\gamma_E}\frac{4\pi}{q^2}\right)^{\varepsilon}\gamma_\mu\left(-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}+\frac{7\pi^2}{6}-8\right)(1+i\varepsilon\pi)$$ ☆ <u>提前完成</u>

$$\sigma_V = \frac{\alpha_s C_F}{2\pi}\sigma_B\left(e^{-\gamma_E}\frac{4\pi}{q^2}\right)^{\varepsilon}\left(-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}+\frac{7\pi^2}{6}-8\right)$$

$\sigma_B = \frac{4\pi\alpha^2}{3s}$. 树图层次上散射截面.

说明 ⓐ LSZ reduction formula, 需要考虑：

抽取 Z

ⓑ 自能修正. 用于提取$\sqrt{Z}$.

$$\sim\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2}\frac{i}{\not{k}}=0$$. 无标度积分

自能修正为0的原因：uv发散与IR发散抵消. $\left(\frac{A}{\varepsilon_{uv}}-\frac{A}{\varepsilon_{IR}}\right)=0$

ⓒ 实修正： $\left.-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}\right|_{\substack{uv\\IR}}+\cdots$

.实修正中 uv + 自能图中 uv | = 0 . 复合算符重整化. 或 矢量流守恒

+

自能图中 IR

‖

0

实修正中uv发散 可以认为是来自于 $\sqrt{Z}$ 的红外发散贡献！

正确做法：

$$\begin{tikzcd}
B & A \\
\mathrm{UV}. \arrow[dd, leftrightarrow, bend right, "\text{相差负号 ②}"'] & \underline{\mathrm{IR}} \arrow[d, "\text{等价}"'] \\
D & C \\
\mathrm{UV}. \arrow[r, leftrightarrow, bend right, "\text{① 相差负号}"'] & \underline{\mathrm{IR}}
\end{tikzcd}$$

$$\Bigg\} \rightarrow \underset{\text{正确做法}}{\underline{A+C}} \xrightarrow[\text{由①+②}]{C\to B} \underset{\underline{\text{实际做法}}}{\underline{A+B}}$$

红外发散(理论分析): 无质量 $p_1^2=p_2^2=0$

$$i\Gamma^\mu = i g_s^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \frac{1}{(k+p_1)^2} \frac{1}{(k-p_2)^2} \cdot \gamma^\alpha (\not p_1+\not k) \gamma^\mu (\not k-\not p_2) \gamma_\alpha$$

分子记为 N^μ. 则

$$i\Gamma^\mu = i g_s^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{N^\mu}{k^2 (k^2+2k\cdot p_1)(k^2-2k p_2)}$$

① 将 $\frac{1}{k^2}$ 写为:

$$\frac{1}{k^2} = \frac{1}{2|\vec k|}\left[\frac{1}{k_0-|\vec k|+i\varepsilon} - \frac{1}{k_0+|\vec k|-i\varepsilon}\right]$$

② 采用留数定理方法将 k_0 积分出来

$$\Gamma^\mu \propto \frac{g_s^2 C_F}{8(2\pi)^3} \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \int_0^\infty \frac{d|\vec k|}{|\vec k|} \frac{N^\mu}{[p_{10}-|\vec p_1|\cos\theta][p_{20}+|\vec p_2|\cos\theta]}$$

θ 为 $\vec k$ 与 $\vec p_1$ 的夹角.

- $\int \frac{d|\vec k|}{|\vec k|}$: 当 $|\vec k|\to 0$ 时, 所有动量分量为0. 软发散
- $\int \frac{d\cos\theta}{[p_{10}-|\vec p_1|\cos\theta]} \gtrsim \int \frac{d\cos\theta}{1-\cos\theta}$

  $p_1^2=0 \to p_{10}=|\vec p_1|$

  当 $\theta=0$ 即 $\cos\theta=1$ 时 $\vec k \parallel \vec p_1$ 存在共线发散

类似地 当 $\theta=\pi$ 时 $\cos\theta=-1$. $\vec k \parallel \vec p_2$ 存在共线发散

另一种分析方法：采用光锥坐标系：

$$k^\mu = (k^+, k^-, k_\perp) \qquad k^\pm = \frac{k^0 \pm k^z}{\sqrt{2}} \qquad \underline{\underline{k_\perp}} = k_T = (k^x, k^y)$$

$$p_1^2 = p_2^2 = 0$$

选择 $p_1^+ \neq 0$ $p_2^- \neq 0$ 即 $p_1^- = p_{1T} = p_{1\perp} = 0$ $p_2^+ = p_{2\perp} = p_{2T} = 0$

$$(k+p_1)^2 = 2(k^+ + p_1^+)k^- - k_\perp^2$$

$$(k-p_2)^2 = k^2 - 2k\cdot p_2 = 2(k^- - p_2^-)k^+ - k_\perp^2$$

$$k^2 = 2k^+ k^- - k_\perp^2$$

$$I = \int \frac{dk^+ dk^- d^2k_T}{(2\pi)^4} \frac{1}{2(k^+ + p_1^+)k^- - k_\perp^2} \frac{1}{2k^+(k^- - p_2^-) - k_\perp^2} \frac{1}{2k^+ k^- - k_\perp^2}$$

power counting (幂次估计)：

- soft $k^\mu = (k^+, k^-, k_\perp) \sim (\Lambda, \Lambda, \Lambda)$ $p_1^+ \sim p_2^- \sim Q$

$$(k+p_1)^2 = 2(k^+ + p_1^+)k^- - k_\perp^2 \sim Q\Lambda, \quad (k-p_2)^2 \sim Q\Lambda, \quad k^2 \sim \Lambda^2$$

则积分 I 的幂次为：

$$\frac{\Lambda\,\Lambda\,\Lambda^2}{\Lambda\,\Lambda\,\Lambda^2} \sim O(\Lambda^0)$$

- collinear. $k^\mu = (k^+, k^-, k_\perp) \sim (Q, \frac{\Lambda^2}{Q}, \Lambda)$ $k^2 \sim \Lambda^2$

$$(k+p_1)^2 = 2(k^+ + p_1^+)k^- - k_\perp^2 \sim \Lambda^2$$

$$(k-p_2)^2 = 2(k^- - p_2^-)k^+ + k_\perp^2 \sim Q^2 \sim \Lambda^0$$

则积分幂次为

$$\frac{\Lambda^0\,\Lambda^2\,\Lambda^2}{\Lambda^2\,\Lambda^0\,\Lambda^2} \sim O(\Lambda^0)$$

都是领头幂次

○ 围道积分： k^- 的极点为：

$$k^- = \frac{k_T^2 - i\varepsilon}{2(k^+ + p_1^+)} \qquad k^- = p_2^- + \frac{k_\perp^2 - i\varepsilon}{2k^+} \qquad k^- = \frac{k_\perp^2 - i\varepsilon}{2k^+}$$

当 $\underline{-p_1^+ < k^+ < 0}$ 时（对么？）极点位置分别位于实轴上下方

积分变为：
$$\int \frac{dk^+ d^2k_T}{(2\pi)^3} \frac{1}{2(k^+ + p_1^+)\left[2k^+\left(\frac{k_T^2 - i\varepsilon}{2(k^+ + p_1^+)} - p_2^-\right) - k_\perp^2\right]} \frac{1}{2k^+\left[\underline{\frac{k_T^2 - i\varepsilon}{2(k^+ + p_1^+)}}\right] - k_T^2 - i\varepsilon}$$

$$= \int \frac{dk^+ d^2k_T}{(2\pi)^3} \frac{2(k^+ + p_1^+)}{2k^+(k_T^2 - 2(k^+ + p_1^+)p_2^-) - 2(k^+ + p_1^+)k_T^2} \frac{1}{(2k^+ - 2k^+ - 2p_1^+)k_T^2}$$

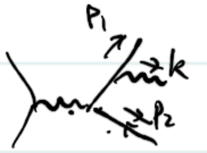
$$\sim \int \frac{dk^+}{k^+} \frac{d^2k_\perp}{k_\perp^2}$$

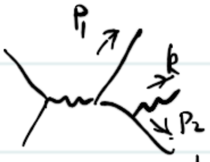
软 共线

① 当 $k_\perp = 0$ 时（k^+ 可以不为0） $k \parallel \vec{p}_1$（线） 共线发散

② 当 $k^+ = 0$ 时 加上所有分量为0. 软发散

1-2 实图：





$$S_{\mu\nu} = \gamma_\mu \frac{1}{m - \not{p}_1 \not{k}} \gamma_\nu + \gamma_\nu \frac{1}{m + \not{p}_2 + \not{k}} \gamma_\mu$$

加上旋量部分，忽略分子上k贡献，

$$\varepsilon^{*\lambda} \bar{u}(p_1) S_{\lambda\mu} v(p_2) = \left(-\frac{p_{1\lambda}}{p_1 \cdot k} + \frac{p_{2\lambda}}{p_2 \cdot k}\right) \underline{\bar{u}(p_1)\gamma_\mu v(p_2)} \varepsilon^*_\lambda$$

其中， $\bar{u}(p_1) \not{\varepsilon}^* \frac{\not{k} + \not{p} + m}{(k+p)^2 - m^2} \gamma_\mu v(p_2)$

$$= \bar{u}(p_1) \not{\varepsilon}^* \frac{\not{p}_1 + m}{2k \cdot p_1} \gamma_\mu \cdot v(p_2) = 2 \frac{\varepsilon^* \cdot p_1}{k \cdot p} \bar{u}(p_1) \gamma_\mu \cdot v(p_2)$$

$$\varepsilon^* \not{k} = 2\underset{0}{\underset{\downarrow}{\varepsilon^* \cdot k}} - \not{k} \cdot \not{\varepsilon}^*$$

软 直接压低
共线： $k \parallel p_1$ $\underline{\bar{u}(p_1)\not{k} \sim 0}$

则振幅模方为：

$$F_R = \left(\frac{p_1\lambda}{p_1 \cdot k} - \frac{p_2\lambda}{p_2 \cdot k}\right)^2 tr\left[(\not{p}_1 + m)\gamma_\mu (\not{p}_2 - m)\gamma_\nu\right] \times 另一半.$$

散射截面 $d\sigma_R \sim \sigma_B \cdot \frac{d^3k}{(2\pi)^3 2E} \sum_{\lambda=1,2} \left| \frac{p_1 \cdot \varepsilon^*}{p_1 \cdot k} - \frac{p_2 \cdot \varepsilon^*}{p_2 \cdot k} \right|^2$

将 $\sum_r \varepsilon^*_\mu \varepsilon_\nu \to -g_{\mu\nu}$，则积分部分为:

$$\sim \int \frac{d^3k}{(2\pi)^3} (-g_{\mu\nu}) \left(\frac{p_1^\mu}{k\cdot p_1} - \frac{p_2^\mu}{k\cdot p_2}\right)\cdot\left(\frac{p_1^\nu}{k\cdot p_1} - \frac{p_2^\nu}{k\cdot p_2}\right)$$

$$\sim \int \frac{d^3k}{(2\pi)^3} (+1) \quad \frac{2p_1\cdot p_2}{k\cdot p_1\, k\cdot p_2}$$

选择: $k^\mu = (k^0, \vec{k})$ $\quad p_1^\mu = E(1, \vec{v})$ $\quad p_2^\mu = E(1, \vec{v}')$ $\quad ((|\vec{v}| = |\vec{v}'| = 1$ 当 $m=0)$

$$p_1\cdot p_2 = 1 - \vec{v}\cdot\vec{v}', \quad k\cdot p_1 = k^0E(1 - \vec{v}\cdot\hat{k}) \quad k\cdot p_2 = k^0E(1-\vec{v}'\cdot\hat{k})$$

对k积分分解为大小与角度部分:

$$\int d\sigma_R \sim \sigma_B \frac{\alpha}{\pi} \int_0^{|\vec{q}|} \frac{dk^0}{k^0}\, \underline{I(v, v')}$$

$$\frac{d^3k}{(2\pi)^3 2E} \to \frac{d^4k}{(2\pi)^4} (2\pi)\delta c \quad)$$

$$I(v, v') = \int \frac{d\Omega_k}{4\pi} \frac{1-\vec{v}\cdot\vec{v}'}{[1-\hat{k}\cdot\vec{v}][1-\hat{k}\cdot\vec{v}']}$$

大小积分有发散 (红外)。角度部分仍可拆分为 $(1-\cos\theta)(1+\cos\theta)$

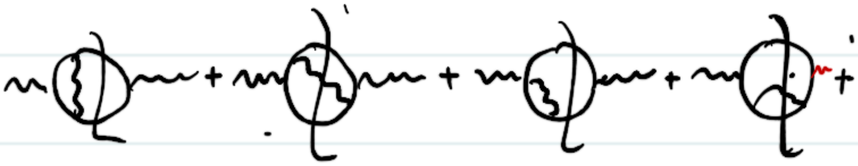
(*) 在DR下 完整计算结果为:

$$\sigma_R = \frac{\alpha_s C_F}{2\pi} \sigma_B \left(e^{-\gamma_E} \frac{4\pi}{q^2}\right)^\varepsilon \cdot \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \frac{7\pi^2}{6}\right)$$ (直接计算) ☆ 提前完成任务.

注意: ① 先计算出 DR下 σ_B

(*) $\sigma_B + \sigma_V + \sigma_R = \sigma_B \left(1 + \frac{3\alpha_s C_F}{4\pi}\right)$

(*) 可以用 KLN 定理描述: 下面所有图贡献相加后 无红外发散!



+ 交叉图

2、算符乘积展开 (operator product expansion)

2-1 四夸克有效算符

粒子物理电弱标准模型中 $b\to c\ell\tilde{\nu}$

$$iM = \bar{u}(p_c)\frac{ig_2}{2\sqrt{2}}\gamma_\mu(1-\gamma_5)V_{cb}\,u(p_b)$$

$$\times\ \bar{u}(p_{e^-})\frac{ig_2}{2\sqrt{2}}\gamma_\nu(1-\gamma_5)\,v(p_{\bar{\nu}_e})\ \times\ \frac{-ig^{\mu\nu}}{k^2-m_W^2}$$

其中 $k^\mu=(p_b-p_c)^\mu$

(*) $m_b: 4.8$ GeV

$m_c: 1.5$ GeV $\ll$ $m_W = 81.4$ GeV

$m_e\sim m_{\bar{\nu}_e}\sim 0$

$$k^2=(p_b-p_c)^2=p_b^2-2p_b\cdot p_c+p_c^2\ \underset{\text{b夸克静止系}}{\le}\ p_b^2=m_b^2\ll m_W^2$$

$$iM\approx\frac{-ig_2^2}{8m_W^2}V_{cb}\,\bar{u}(p_c)\gamma_\mu(1-\gamma_5)u(p_b)\ \ \bar{u}(p_e)\gamma^\mu(1-\gamma_5)v(p_{\bar{\nu}_e})$$

$$\frac{G_F}{\sqrt{2}}=\frac{g_2^2}{8m_W^2}$$

$\underline{iM}\sim\langle ce\tilde{\nu}|\exp[i\int d^4x\,H_{eff}]|b\rangle$

① 有效理论: $H_{eff}=\frac{G_F}{\sqrt{2}}V_{cb}\ \bar{c}\gamma^\mu(1-\gamma_5)b\ \ \bar{e}\gamma^\mu(1-\gamma_5)\tilde{\nu}$ △

c.b.e.ν 均为费米子场算符.

S矩阵元:

$$\langle ce\bar{\nu}_e|-i\int d^4x\,H_{eff}(x)|b\rangle=(2\pi)^4\delta^4(\ )\cdot\underline{iM}$$

如何从完整理论得到了该有效理论?

(*) 算符分析：

$$S = T\exp\left[i\int d^4x\, \mathcal{L}_I\right] \qquad \mathcal{H}_I = -\mathcal{L}_I$$

$$= T\Big\{ \underline{\frac{ig_2}{2\sqrt{2}} V_{cb} \int d^4x\, \bar{c}\gamma_\mu(1-\gamma_5) b\, W^{+\mu}} \qquad i\int d^4x\, \mathcal{L}_I$$

$$\times \frac{ig_2}{2\sqrt{2}} \int d^4y\, \bar{e}\gamma_\nu(1-\gamma_5)\tilde{\nu}_e\, W^{-\nu}(y) \Big\} \qquad i\int d^4y\, \mathcal{L}_I$$

$$T\left\{ W^{+\mu}(x)\, W^{-\nu}(y) \right\} = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \underline{\frac{-ig^{\mu\nu}}{k^2-m_W^2+i\varepsilon}}$$

$$\overset{k \ll m_W}{=\!=} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-ig^{\mu\nu}}{-m_W^2+i\varepsilon}$$

$$= \frac{-ig^{\mu\nu}}{-m_W^2} \delta^4(x-y)$$

则 $$S \cong \frac{-ig_2^2}{8m_W^2} T\left\{ \int d^4x\, \bar{c}\gamma^\mu(1-\gamma_5) b \int d^4y\, \bar{e}\gamma^\nu(1-\gamma_5)\tilde{\nu}_e(y) \cdot \delta^4(x-y) \right\}$$

$$= \frac{-iG_F}{\sqrt{2}} T\left\{ \underline{\int d^4x\, \bar{c}\gamma^\mu(1-\gamma_5) b\, \bar{e}\gamma_\mu(1-\gamma_5)\tilde{\nu}_e} \right\}$$

有效理论的形成. 相当于

$$\underbrace{O_1(x)\, O_2(y)}_{y\sim x} = O_1(x)\, O_2(x+y-x) = \underbrace{O_1(x)\, O_2(x)}_{\hat{O}_1(x)} + (y-x)\, O_1(x)\, O_2'(x) + \cdots$$

$$O_1(x)\, O_2(y) = \hat{O}_1(x) + \cdots$$

把算符的乘积变成了新的定域算符，这就是算符乘积展开. 也是有效理论前提.

有漏洞么？

ⓐ $y-x$ 很大呢？ 幂次压低

ⓑ 复合算符重整化呢？

$$O_1(x)\, O_2(y) = \sum_i \underset{\uparrow}{C_i}\, \hat{O}_i(x)$$

弥补 UV 之间的差别，称为短程系数. 或 Wilson 系数

(四) 路径积分方法:

$$Z_W=\int[dW^+][dW^-]\ \exp[\ i\int d^4x\ \mathcal{L}_I]$$

$$\mathcal{L}_W=-\frac{1}{2}(\partial_\mu W^+_\nu-\partial_\nu W^+_\mu)(\partial^\mu W^{-\nu}-\partial^\nu W^{-\mu})+m_W^2\,W^+_\mu W^-_\nu$$

$$\underbrace{+\frac{g_2}{\sqrt{2}}(J^+_\mu W^{+\mu}+J^-_\mu W^{-\mu})}_{\text{相互作用}}$$

利用路径积分中方法,可将生成泛函写为:

$$Z_W=\int[dW^+][dW^-]\ \exp\left\{[\,i\int d^4x\,d^4y\ W^+_\mu(x)\ K^{\mu\nu}(x,y)\ W^-_\nu(y)]+\frac{ig_2}{2\sqrt{2}}\int d^4x\ [J^{+\mu}W^+_\mu+J^{-\mu}W^-_\mu\,]\right\}$$

$$K_{\mu\nu}(x,y)=\delta^4(x-y)\,[g_{\mu\nu}(\partial^2+m_W^2)-\partial_\mu\partial_\nu]$$

它的逆为 $\Delta_{\mu\nu}(x,y)=\int\frac{d^4k}{(2\pi)^4}\ e^{-ik\cdot(x-y)}\frac{-i}{k^2-m_W^2}\ [g_{\mu\nu}-\frac{k_\mu k_\nu}{m_W^2}]$

② 积分出 $W^\pm$, 则生成泛函为

$$Z_W\sim\ \exp[-\frac{ig_2^2}{8}\int d^4x\,d^4y\ J^-_\mu(x)\,\Delta^{\mu\nu}(x,y)\,J^+_\nu(y)\}$$

③ 对 $\Delta_{\mu\nu}(x,y)$ 取截断,即

$$\Delta_{\mu\nu}(x,y)=\frac{g_{\mu\nu}}{M_W^2}\ \delta^4(x-y)$$

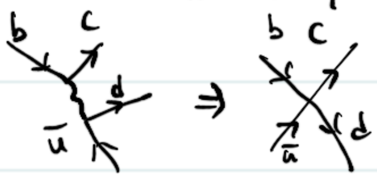
④ 代入后得

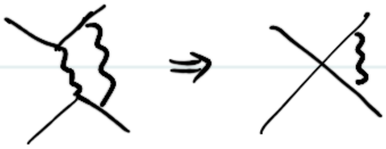
$$Z_W\approx\ \exp[-i\frac{G_F}{\sqrt{2}}\int d^4x\,J^+_\mu(x)\,J^{-\mu}(x)]$$

这与算符分析方法一致:

⑤ 一般地,需要考虑 QCD 修正:

$$H_{eff}=\frac{G_F}{\sqrt{2}}\left\{V_{ub}V^*_{ud}\,(C_1\underline{\underline{\hat{O}_1}}+C_2\underline{\underline{\hat{O}_2}})-V_{tb}V^*_{td}\sum_{i=3}^{10}C_iO_i\right\}$$

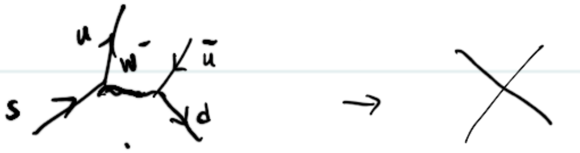




③ 对于半轻衰，复合算符重整化系数为1.

④ 对于非轻衰：则需计算演化，见 Peskin P607.

非轻：



$s \to u\bar{u}d$

树图：$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \underbrace{(\bar{d}\gamma^\mu(1-\gamma_5)u)(\bar{u}\gamma_\mu(1-\gamma_5)s)}_{O_1}$ $\underline{k^2 \ll m_W^2}$

加入QCD修正后：

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}}(\underline{C_1}O_1 + \underline{C_2O_2}) \cdot V_{ud}^* V_{us}$$

$$O_1 = (\bar{d}\gamma^\mu(1-\gamma_5)u)(\bar{u}\gamma_\mu(1-\gamma_5)s)$$

$$O_2 = (\bar{u}\gamma^\mu(1-\gamma_5)u)(\bar{d}\gamma_\mu(1-\gamma_5)s) = (\bar{d}^\alpha\gamma^\mu(1-\gamma_5)u^\beta)(\bar{u}^\beta\gamma_\mu(1-\gamma_5)s^\alpha)$$

颜色不一样

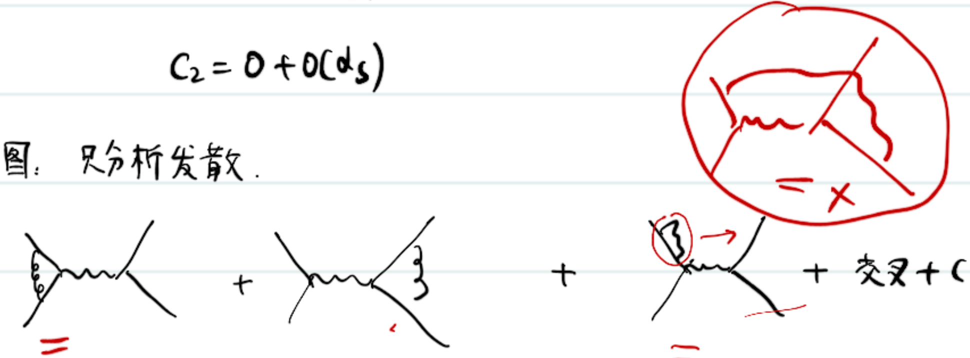
C_1 与 C_2 为短程系数

$$C_1 = 1 + O(\alpha_s)$$

$$C_2 = 0 + O(\alpha_s)$$

一单圈图：只分析发散.

完整：



+ 交叉 + ct

(*) 其实不用加ct，因为矢量流守恒，所有UV发散都抵消

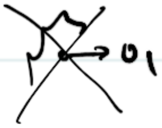
有效

+ 交叉 + ... + ... + 复合算符重整化.

自动抵消

(*) 

$$= - g_s^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\left[\frac{1}{2}\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s - \frac{1}{6}\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s\right]$$



$$= + g_s^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\frac{1}{4}\left[\frac{1}{2}\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s - \frac{1}{6}\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s\right]$$

四张图相加 $iM = -\frac{3}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\left[\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s - \frac{1}{3}\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s\right]$

(*) 对于 O_2 也是类似地，需交换 $\langle O_1\rangle$ 与 $\langle O_2\rangle$

可以类似于复合算符重整化：

$$O_1^0 = \underline{Z_{11}O_1 + Z_{12}O_2} \qquad O_2^0 = Z_{21}O_1 + Z_{22}O_2$$

$$\begin{pmatrix} O_1^0 \\ O_2^0 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} O_1 \\ O_2 \end{pmatrix} \qquad \begin{pmatrix} O_1 \\ O_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}^{-1}\begin{pmatrix} O_1^0 \\ O_2^0 \end{pmatrix}$$

$$\begin{tikzcd}
\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \arrow[d, phantom, "\parallel"] & \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}^{-1} \\
\begin{pmatrix} 1+\delta_{11} & \delta_{12} \\ \delta_{21} & 1+\delta_{22} \end{pmatrix}\begin{pmatrix} O_1 \\ O_2 \end{pmatrix} \arrow[d] & \\
\text{逆矩阵为：}\begin{pmatrix} 1-\delta_{11} & -\delta_{12} \\ -\delta_{21} & 1-\delta_{22} \end{pmatrix} \arrow[ruu] &
\end{tikzcd}$$

$\langle O_1\rangle$ 的矩阵元为：$\langle O_1^t\rangle + \langle O_1^{1\text{loop}}\rangle + \text{c.t.} \rightarrow$ 有限

$$-\frac{3}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\left[\langle O_2^t\rangle - \frac{1}{3}\langle O_1^t\rangle\right] - \delta_{11}\langle O_1^t\rangle - \delta_{12}\langle O_2^t\rangle = \text{有限}$$

$$\delta_{11} = \frac{1}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2} \qquad \delta_{12} = -\frac{3}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}$$

类似地 $\delta_{21} = \delta_{12} \qquad \delta_{22} = \delta_{11}$

$$\gamma = \frac{1}{2}\frac{\partial Z}{\partial \ln\mu} = \mu\frac{\partial}{\partial\mu}\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$$

$$= \frac{g_s^2}{(4\pi)^2}\begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix}$$

(4) 对角化算符：

$$O^{1/2} = \frac{1}{2}\left(\overset{O_1}{\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s} - \overset{O_2}{\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s}\right)$$

$$O^{3/2} = \frac{1}{2}\left(\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s + \bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s\right)$$

$$\gamma_{1/2} = -8\frac{g_s^2}{(4\pi)^2} \qquad \gamma_{3/2} = 4\frac{g_s^2}{(4\pi)^2}$$

$$\underline{\langle O_1\rangle}\Big|_{\mu=m_W} = \langle O^{1/2}\rangle_{\mu=m_W} + \langle O^{3/2}\rangle_{\mu=m_W}$$

$$= \left(\frac{\log m_W^2/\Lambda^2}{\log m_K^2/\Lambda^2}\right)^{4/b_0}\langle O^{1/2}\rangle\Big|_{\mu=m_K} \qquad b_0 = 11-\frac{2}{3}n_f$$

$$+ \left(\frac{\log m_W^2/\Lambda^2}{\log m_K^2/\Lambda^2}\right)^{-2/b_0}\langle O^{3/2}\rangle\Big|_{\mu=m_K}$$

$$= 2.1\,\langle O^{1/2}\rangle\Big|_{\mu=m_K} + 0.7\,\langle O^{3/2}\rangle\Big|_{\mu=m_K}$$

$$= 2.1\left(\frac{1}{2}\langle O_1\rangle + \frac{1}{2}\langle O_2\rangle\right)\Big|_{\mu=m_K} + 0.7\left(\frac{1}{2}\langle O_1\rangle - \frac{1}{2}\langle O_2\rangle\right)\Big|_{\mu=m_K}$$

$$= 1.4\,\langle \underline{\underline{O_1}}\rangle\Big|_{\mu=m_K} + \underline{\underline{0.7}}\,\langle O_2\rangle\Big|_{\mu=m_K}$$

$$H = \frac{G_F}{\sqrt{2}}\left(C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)\right)$$

$$\rightarrow \quad C_1(m_W) = 1 \qquad C_2(m_W) = 0$$

$$C_1(m_K) = 1.4 \qquad C_2(m_K) = 0.7.$$

↓ Wilson 系数的演化.

正负电子湮灭与算符乘积展开：

① 运动学

② 无相互作用 OPE

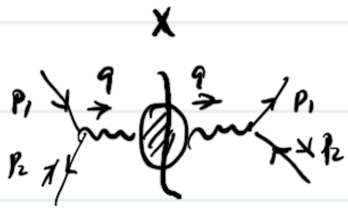
③ 领头项近似

④ OPE 的一般形式

⑤ OPE 适用范围与解析延拓

1. $e^+e^-\to X$ 的散射截面：

a. 图像：　$s=(p_1+p_2)^2=q^2$



光学定理：

$$\sigma(e^+e^-\to X)=\frac{1}{2s}\,\mathrm{Im}\,\mathcal{M}(e^+e^-\to e^+e^-)$$

$e^-(p_1)e^+(p_2)\to e^-(p_1)e^+(p_2)$ 向前散射

$$i\mathcal{M}(e^+e^-\to e^+e^-)=(-ie)^2\,\bar{u}(p_1)\gamma_\mu v(p_2)\cdot\frac{-i}{s}\cdot i\Pi_h^{\mu\nu}(q^2)\quad\frac{-i}{s}\quad\bar{v}(p_2)\gamma_\nu u(p_1)$$

$$\Pi_h^{\mu\nu}(q^2)=(q^2g^{\mu\nu}-q^\mu q^\nu)\,\Pi_h(q^2)$$

其中 $q^\mu q^\nu$ 项不产生贡献.（EOM 或矢量流守恒）

$$\frac{1}{4}\sum_{spin}\bar{u}(p_1)\gamma_\mu v(p_2)\ \bar{v}(p_2)\gamma^\mu u(p_1)=\frac{1}{4}\mathrm{tr}\left[\not{p}_1\gamma_\mu\not{p}_2\gamma^\mu\right]=\frac{1}{4}(-2)4(p_1\cdot p_2)=-s$$

所以　$\sigma(e^+e^-\to X)=-\frac{4\pi\alpha}{s}\,\mathrm{Im}\,\Pi_h(q^2)$

b. 算符分析：

不变振幅为：$i\mathcal{M}(e^+e^-\to X)=\bar{v}(p_2)(-ie\gamma_\mu)u(p_1)\cdot\frac{-i}{s}\cdot\underline{\langle X|\,(-ieJ^\mu_{em})|0\rangle}$

散射截面为：

$$\sigma=\frac{1}{2s}\ \frac{1}{4}\sum_{spin}\ \sum_X (2\pi)^4\,\delta^4(p_X-q)\ |i\mathcal{M}|^2$$

注意此处的求和也包含动量空间积分　$\frac{d^3p_1}{(2\pi)^3 2E_1}\cdots\frac{d^3p_X}{(2\pi)^3 2E_X}$

化简振幅模方：

$$\frac{1}{4}\sum_{spin}|iM|^2 = \frac{1}{4}\frac{e^4}{s^2}\, tr[\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu]\cdot\langle 0|J^\mu_{em}|X\rangle\langle X|J^\nu_{em}|0\rangle$$

$$= \frac{e^4}{s^2}\left(p_{2\mu}p_{1\nu} - g_{\mu\nu}\,p_1\cdot p_2 + p_{1\mu}p_{2\nu}\right)\langle 0|J^\mu_{em}|X\rangle\langle X|J^\nu_{em}|0\rangle$$

则 $\quad \sigma = \frac{e^4}{2s^3}\, \ell^{\mu\nu}\, W_{\mu\nu}$

其中 $\quad \ell^{\mu\nu} = p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{q^2}{2}g^{\mu\nu}$

$$\underline{W^{\mu\nu} = \sum_X (2\pi)^4\delta^4(p_X-q)\,\langle 0|\underline{J^\mu_{em}(x)}|X\rangle\langle X|J^\nu_{em}|0\rangle}$$

可以将强子矩阵元重新表述为 $\longrightarrow \langle 0|J^\mu_{em}(x)|X\rangle = \langle 0|J^\mu_{em}(0)|X\rangle \cdot e^{-ip_X\cdot x}$

$$W^{\mu\nu} = \int d^4x\, e^{iq\cdot x}\langle 0|J^\mu_{em}(x)\,J^\nu_{em}(0)|0\rangle$$

$I = \sum_X |X\rangle\langle X| \qquad$ 这里求和包含 $\frac{d^3p_1}{(2\pi)^3 2E_1}\cdots\frac{d^3p_X}{(2\pi)^3 2E_X}$

由于 $q^0>0$，则可以改写为 $\qquad J^\mu_{em}(x)J^\nu_{em}(0) - J^\nu_{em}(0)J^\mu_{em}(x)$

$$\underline{W^{\mu\nu} = \int d^4x\, e^{iq\cdot x}\langle 0|[J^\mu_{em}(x), J^\nu_{em}(0)]|0\rangle}, \qquad \underline{q^0\sim Q} \quad \underline{x^0}\sim\frac{1}{Q}$$

参数化： $\quad W^{\mu\nu} = (q^\mu q^\nu - q^2 g^{\mu\nu})\,\frac{1}{6\pi}\,W(s)$

则 $\quad \sigma = \frac{e^4}{2s^3}\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{q^2}{2}g^{\mu\nu}\right)\cdot\left(-q^2 g_{\mu\nu}\right)\frac{1}{6\pi}W(s)$

$$= \frac{(4\pi\alpha)^2}{2s^3}(-q^2)\Big(2\overset{q^2}{p_1\cdot p_2} - \frac{q^2}{2}\cdot 4\Big)\frac{1}{6\pi}W(s)$$

$$= \frac{4\pi\alpha^2}{3s}W(s)$$

$$\int d^4x\, e^{iq\cdot x}\langle 0|J^\nu_{em}(0)\,J^\mu_{em}(x)|0\rangle$$

$$= \int d^4x\, e^{iq\cdot x}\sum_X\langle 0|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(x)|0\rangle$$

$$= \int d^4x\, e^{iq\cdot x}\sum_X\langle 0|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(0)|0\rangle e^{ip_X\cdot x}$$

$$= \sum_X(2\pi)^4\underset{\times}{\delta^4(p_X+q)}$$

(4) 无相互作用分析：

假设忽略 QCD 相互作用，分析下面组合

$$T[J^{\mu}(x)J^{\nu}(0)] = -\mathrm{Tr}\left[\langle 0|T[\psi(0)\bar{\psi}(x)]|0\rangle \times \gamma^{\mu}_{x} \langle 0|T[\psi(x)\bar{\psi}(0)]|0\rangle \gamma^{\nu}\right]$$

$$+N\left[\bar{\psi}(x)\gamma^{\mu}\left[\langle 0|T[\psi(x)\bar{\psi}(0)]|0\rangle\right]\gamma^{\nu}\psi(0)\right]$$

$$+N\left[\bar{\psi}(0)\gamma_{\nu}\left[\langle 0|T[\psi(0)\bar{\psi}(x)]|0\rangle\right]\gamma^{\mu}\psi(x)\right]$$

$$+N\left[\bar{\psi}(x)\gamma^{\mu}\psi(x)\bar{\psi}(0)\gamma^{\nu}\psi(0)\right]$$

利用坐标空间传播子的表达式

$$\Delta(x) = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{i}{p^2-m^2+i\varepsilon}$$

$$= \frac{-1}{4\pi^2}\frac{1}{x^2-i\varepsilon} + \text{less singular term}$$

$$S_F(x) = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$$

$$= (i\not{\partial}+m)\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{i}{p^2-m^2+i\varepsilon}$$

$$= (i\not{\partial}+m)\left(\frac{-1}{4\pi^2}\frac{1}{x^2-i\varepsilon}\right)$$

$$= \cdots$$

$$x^{-6} \qquad x^{-3}$$

代入可得： $$T[J_{\mu}(x)J_{\nu}(0)] = \frac{x^2 g_{\mu\nu}-2x_{\mu}x_{\nu}}{\pi^4(x^2-i\varepsilon)^4} + \frac{ix^{\lambda}}{2\pi^2(x^2-i\varepsilon)^2}s_{\mu\lambda\nu\rho}O_V^{\rho}(x,0)$$

$$+\frac{x^{\lambda}}{2\pi^2(x^2-i\varepsilon)^2}\varepsilon_{\mu\lambda\nu\rho}O_A^{\rho}(x,0) + O_{\mu\nu}(x,0)$$

$$O_V^{\mu}(x,y) = \bar{\psi}(x)\gamma^{\mu}\psi(y) - \bar{\psi}(y)\gamma^{\mu}\psi(x)$$

$$O_A^{\mu}(x,y) = \bar{\psi}(x)\gamma^{\mu}\gamma_5\psi(y) - \bar{\psi}(y)\gamma^{\mu}\gamma_5\psi(x)$$

$$O_{\mu\nu}(x,y) = \bar{\psi}\gamma^{\mu}\psi(x)\;\bar{\psi}\gamma^{\nu}\psi(y)$$

$$s_{\mu\lambda\nu\rho} = g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\rho}$$

$\varepsilon_{0123} = +1$

利用编时乘积组合： $\Theta(x^0)$

$$T[J_\mu(x) J_\nu(0)] - T[J_\mu(x) J_\nu(0)]^{+} = \underset{\uparrow}{\varepsilon(x_0)} [J_\mu(x), J_\nu(0)]$$

$$\varepsilon(x_0) = \frac{x_0}{|x_0|}$$

且 $$\underline{\frac{1}{x^2 - i\varepsilon}} = \frac{P}{x^2} + i\pi\,\delta(x^2)$$

$$\frac{1}{(x^2 - i\varepsilon)^n} = \frac{P}{(x^2)^n} + i\pi \frac{(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(x^2)$$

可以得到：

$$\varepsilon(x_0)[\underline{J_\mu(x), J_\nu(0)}] = \frac{i}{3\pi^3} (2x_\mu x_\nu - x^2 g_{\mu\nu})\, \underline{\delta^{(3)}(x^2)} \quad ✓$$

$$+ \frac{1}{\pi} x^\lambda S_{\mu\lambda\nu\rho}\, \delta^{(1)}(x^2)\, O_V^\rho(x, 0) \quad ✓$$

$$- \frac{i}{\pi} x^\lambda \delta^{(1)}(x^2)\, \varepsilon_{\mu\lambda\nu\rho}\, O_A^\rho(x, 0) \quad ✓$$

$$+ O_{\mu\nu}(x, 0) - O_{\nu\mu}(0, x) \quad ✓$$

(*). 无相互作用. 领头阶近似 即 parton model., 只取 $\delta^{(3)}(x^2)$ 项：

$$W_{\mu\nu} = \Sigma e_q^2 \frac{i}{3\pi^2} \left(g_{\mu\nu} \frac{\partial}{\partial q} \cdot \frac{\partial}{\partial q} - 2\frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\nu}\right) I_3$$

$$\underline{I_n = \int d^4x\, e^{iq\cdot x}\, \varepsilon(x_0)\, \delta^{(n)}(x^2)}$$

可在动量空间化简得：

$$I_n = \frac{i\pi^2}{4^{n-1}(n-1)!} (q^2)^{n-1} \underline{\underline{\varepsilon(q_0)}}\, \theta(q^2)$$

代入矩阵元： $W_{\mu\nu} = \sum e_q^2 \frac{1}{6\pi} (q_\mu q_\nu - q^2 g_{\mu\nu}) \varepsilon(q_0) \theta(q^2)$

则 $\sigma = \frac{4\pi\alpha^2}{3s} N_c \sum e_q^2$.

$$\sigma(e^+e^- \to X) = \sum_q \sigma(e^+e^- \to q\bar{q})$$

(六) 一般性理论：包含相互作用之后；

$\underline{x很小}$. $\underline{\underline{J_\mu(x) J_\nu(0)}} \sim C^1_{\mu\nu}(x) \cdot 1 + C^{\bar{q}q}_{\mu\nu}(x)\, \bar{q}q(0) + C^{F^2}_{\mu\nu}(x) \cdot (F^a_{\alpha\beta})^2(0) + \cdots \underline{\bar{q}\overleftrightarrow{D}q}$

$\underline{\underline{(+6)}}$ $\qquad \underline{+4}$ $\qquad +4$

$\bar{q}\overleftrightarrow{D}q$

量纲分析： $C^1_{\mu\nu}(x) \sim x^{-6}$

$C^{q\bar{q}}_{\mu\nu} \sim m x^{-2}$

$C^{F^2}_{\mu\nu} \sim x^{-2}$

做傅立叶变换

$$-e^2 \int d^4x\, e^{iq\cdot x} J_\mu(x) J_\nu(0)$$

$$= -ie^2 (q^2 g_{\mu\nu} - q_\mu q_\nu) \left[C^1(q^2) \cdot 1 + C^{\bar{q}q}(q^2)\, m\bar{q}q + C^{F^2}(q^2) (F^a_{\alpha\beta})^2 + \cdots \right]$$

$C^1 \sim q^0 \qquad C^{\bar{q}q} \sim (q^2)^{-2} \qquad (C^{F^2}) \sim (q^2)^{-2}$

在领头阶

$$C^1(q^2) = -\left(3 \sum_f Q_f^2\right) \frac{\alpha}{3\pi} \log(-q^2 + i\varepsilon)$$

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s} \left\{ \text{Im}\, C^1(q^2) + \text{Im}\, C^{\bar{q}q}(q^2) \langle 0|\bar{q}q|0\rangle + \text{Im}\, C^{F^2}(q^2) \langle 0|(F^a_{\alpha\beta})^2|0\rangle + \cdots \right\}$$

领头阶： $\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s} \sum_q e_q^2$.

(*) 适用区域问题:

在OPE使用中需要保证小x为主类空. 但对于类时区域, 中间产生了大量强子, 不一定能保障适用性!

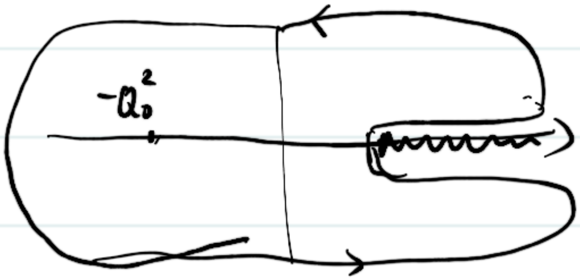
不过人们可以通过色散积分做延拓进行联系.

定义n阶矩

$$I_n = -4\pi\alpha \oint \frac{dq^2}{2\pi i} \frac{1}{(q^2+Q_0^2)^{n+1}} \Pi_h(q^2)$$

如果 contract the contour on the pole. we find

$$I_n = -4\pi\alpha \frac{1}{n!} \frac{d^n}{d(q^2)^n} \Pi_h(q^2)\Big|_{q^2=-Q_0^2}$$ ①



选择如上围道 可得

$$I_n = -4\pi\alpha \int \frac{dq^2}{2\pi i} \frac{1}{(q^2+Q_0^2)^{n+1}} \text{Disc}\, \Pi_h(q^2)$$

$$= -4\pi\alpha \int \frac{dq^2}{2\pi} \frac{1}{(q^2+Q_0^2)^{n+1}} \frac{1}{i} 2i\, \text{Im}\, \Pi_h(q^2)$$

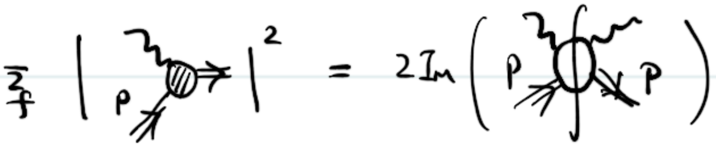
$$= \frac{1}{\pi} \int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma(s)$$

由公式①可得:

$$\int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma(s) = \frac{4\pi\alpha^2}{n(Q_0^2)^n} \sum_f Q_f^2 + O(\alpha_s(Q_0^2))) + O((Q_0^2)^{-2})$$

在$q^2=-Q_0^2$的OPE, 与散射截面的积分有关.

DIS.



运动学:

(*) DIS 为电子与核子深度非弹性散射:

$$e(k)+p(p)\to e(k')+X(p_X) \quad . \qquad q=k-k'$$

$$i\mathcal{M}(ep\to ef) = -ie\,\bar{u}(k')\gamma_\mu u(k)\,\frac{-i}{q^2}\,(ie)\langle X|J^\mu(0)|p\rangle$$

散射截面与下面物理量有关:

$$\sum_X (2\pi)^4\delta^4(q+p-p_X)\langle p|J^\mu_{em}(0)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle$$ 这里 $\sum$ 包括动量求和

(*) 引入矩阵元张量:

$$W^{\mu\nu}(p,q)=\frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\langle p|[J^\mu_{em}(x),J^\nu_{em}(0)]|p\rangle$$

利用态的完备性关系 $\sum_X |X\rangle\langle X$ 可得

$$W^{\mu\nu}(p,q)=\frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\sum_X\langle p|J^\mu_{em}(x)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle$$
$$-\frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\sum_X\langle p|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(x)|p\rangle$$

利用平移不变性: $\langle p|J^\mu_{em}(x)|X\rangle=\langle p|J^\mu_{em}(0)|X\rangle\cdot e^{i(p-p_X)\cdot x}$

$$\langle X|J^\mu_{em}(x)|p\rangle=\langle X|J^\mu_{em}(0)|X\rangle\, e^{i(p_X-p)\cdot x}$$

则 $$W^{\mu\nu}(p,q)=\frac{1}{4\pi}\sum_X(2\pi)^4\delta^4(p+q-p_X)\langle p|J^\mu_{em}(0)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle+\underbrace{\text{其它项}}_{\text{运动学不符合要求}}$$

(*) 利用 $W^{\mu\nu}$, 散射截面为:

$$\sigma(ep\to eX) = \frac{e^4}{2s}\int \frac{d^3k'}{(2\pi)^3 2E_{k'}}\ \frac{1}{2}\sum_{spin} \bar{u}(k)\gamma_\mu u(k')\ \bar{u}(k')\gamma_\nu u(k)$$

$$\times\ \frac{1}{(Q^2)^2}\ 4\pi W^{\mu\nu}$$

电子旋量部分: $\frac{1}{2}\sum_{spin} \bar{u}(k)\gamma_\mu u(k')\bar{u}(k')\gamma_\nu u(k) = 2(k_\mu k'_\nu - k\cdot k' g_{\mu\nu} + k_\nu k'_\mu)$

(*) 定义 $x = \frac{Q^2}{2p\cdot q} = \frac{2kk'(1-\cos\theta)}{2M(k-k')}$ 初态静止系

$$y = \frac{2p\cdot q}{2p\cdot k} = \frac{k-k'}{k}$$

$$\frac{\partial(x\ y)}{\partial(k'\cos\theta)} = \frac{2k'}{2M(k-k')} = \frac{2k'}{ys}$$

$$\int\frac{d^3k'}{(2\pi)^3 2k'} = \int\frac{2\pi\ dk'\ k'\ d\cos\theta}{(2\pi)^3\ 2} = \int dx\,dy\ \frac{ys}{(4\pi)^2}$$

(*) 代入截面公式:

$$\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2 y}{(Q^2)^2}\ (k_\mu k'_\nu - k\cdot k' g_{\mu\nu} + k_\nu k'_\mu)\ W^{\mu\nu}$$

(*) 利用 $W^{\mu\nu}$ 的协变性

$$W^{\mu\nu} = F_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{F_2}{pq}\left(p^\mu - \frac{p\cdot q\, q^\mu}{q^2}\right)\left(p^\nu - \frac{p\cdot q\, q^\nu}{q^2}\right)$$

也有人用 $F_L = F_2 - 2xF_1$ $F_T = F_1$ ··· 结构函数.

则
$$\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2 y}{Q^4}\left[\frac{2p\cdot k\, p\cdot k'}{p\cdot q}F_2 + 2k\cdot k' F_1\right]$$
$$= \frac{4\pi\alpha^2}{Q^4}\left(s(1-y)F_2 + xy^2 s F_1\right)$$

其中已使用 $2k\cdot k' = Q^2 = xys$.

(*) OPE in DIS

定义编时乘积算符　　　　$\bar{\psi}\gamma^{\mu}\psi \quad \bar{\psi}\gamma^{\nu}\psi$

$$t_{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\, T[\underline{J^{\mu}_{em}(x)\ J^{\nu}_{en}(0)}]$$

算符矩阵元为：　$T_{\mu\nu} = \langle p(p)|\, t_{\mu\nu}\, |p(p)\rangle$

$$T_{\mu\nu} = T_1\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right] + \frac{T_2}{p q}\left[p_\mu - \frac{p\cdot q}{q^2} q_\mu\right]\left[p_\nu - \frac{p q}{q^2} q_\nu\right]$$

可以证明：$\mathrm{Im}\, T_{1,2}(w+i\varepsilon, Q^2) = 2\pi\, f_{1,2}(w, Q^2)$

对于算符乘积：　$T[O_a(z)\,O_b(0)]$ 一般可展开为

$$T[O_a(z)O_b(0)] = \sum C_{abk}(z)\, O_k(0)$$

动量空间形式为：

$$\int d^4z\, e^{iq\cdot z}\, T[O_a(z)O_b(0)] = \sum C_{abk}(q)\, O_k(0)$$

从 T_1 与 T_2 可以看到，存在两种指标收缩情况，μ 与 ν 收缩，μ 与 μ_1，ν 与 ν_1 收缩

算符 O 中的指标要么是 μ, ν 要么与 q 收缩，现考虑全部指标与 q 收缩情况

可能的算符：$\underline{\bar{\psi}\gamma^{\mu_1}\psi}$，$\bar{\psi}D^{\mu_1}\psi$，$\bar{\psi}\gamma^{\mu_1}D^{\mu_2}\psi$　$\bar{\psi}D^{\mu_1}D^{\mu_2}\psi$ ……　量纲不同，对称指标数不同、

考虑一般情况：$C_{\mu_1\mu_2\cdots\mu_n}\, O^{\mu_1\cdots\mu_n}_{d,n}$　　其中 d 为量纲，n 为对称指标数

- $$\underbrace{\langle O^{\mu_1\cdots\mu_n}_{d,n}\rangle}_{d-2} = \underbrace{p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}}_{n}\cdot \underbrace{m_p^{d-n-2}}_{d-n-2}$$

- $$C_{\mu_1\mu_2\cdots\mu_n} \simeq q^{\mu_1}\underbrace{\overset{n}{\cdots}} q^{\mu_n} \times Q^{2-d-n}$$

- $$\boxed{C_{\mu_1\mu_2\cdots\mu_n}\langle O\rangle \text{ 的总体量纲为 } 0}$$

- $$C_{\mu_1\mu_2\cdots\mu_n}\langle O_{d,n}^{\mu_1\cdots\mu_n}\rangle \Rightarrow \frac{q_{\mu_1}q_{\mu_2}\cdots q_{\mu_n}}{Q^n}\cdot Q^{2-d} \times m_p^{d-n-2}\, p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}$$

$$\Rightarrow \frac{(p\cdot q)^n}{Q^n}\, Q^{2-d}\, m_p^{d-n-2}$$

$$\Rightarrow w^n \left(\frac{Q}{m_p}\right)^{2+n-d}$$

$$\underline{w=\frac{2p\cdot q}{Q^2}\sim 1} \qquad \Rightarrow w^n\left(\frac{Q}{m_p}\right)^{2-t}$$

$t = d-n$ = dimension − spin

算符的重要性可以按照 t 的增大而减小.

- 算符指标加 μ,ν 也是类似的··

例: $\bar\psi\gamma^\mu\psi$, $t=2$ $\quad$ $\bar\psi D^\mu\psi$: $t=3$. $\quad$ $\bar\psi\psi$: $t=3$. $\quad$ $\underline{最小\ t=2}$

说明: 这里对应 μ 指标取大分量 $\qquad$ $\underline{\bar\psi\gamma_\perp^\mu\psi,\ t=3}$

(*) 领头扭度算符 ($t=2$) 的一般形式为:

$$O_{q,V}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\bar q\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\overleftrightarrow{D}^{\mu_3}\cdots\overleftrightarrow{D}^{\mu_n}q\right\}$$

$$O_{q,A}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\bar q\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_n}\gamma_5 q\right\}$$

$$\bar A\overleftrightarrow{D}^\mu B = \bar A\overrightarrow{D}^\mu B - \bar A\overleftarrow{D}^\mu B$$

$$O_{g,V}^{\mu_1\cdots\mu_n} = -\frac{1}{2}\left(\frac{i}{2}\right)^{n-2} S\left\{G_A^{\mu_1\alpha}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_{n-1}}G_{A\alpha}{}^{\mu_n}\right\}$$

- 考虑到矢量流守恒, $q\leftrightarrow -q$ 的对称性.

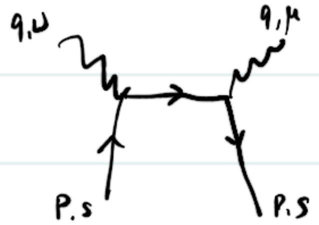
$$t_{\mu\nu} = \sum_{n=2,4}^{\infty}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{2^n q_{\mu_1}\cdots q_{\mu_n}}{(-q^2)^n}\sum_{j=q,g} 2\,C_{j,n}^{(1)}\,O_{j,V}^{\mu_1\cdots\mu_n}$$
$$+\sum_{n=2,4}^{\infty}\left(g_{\mu\mu_1}-\frac{q_\mu q_{\mu_1}}{q^2}\right)\left(g_{\nu\mu_2}-\frac{q_\nu q_{\mu_2}}{q^2}\right)\frac{2^n q_{\mu_3}\cdots q_{\mu_n}}{(-q^2)^{n-1}}\sum_{j=g,q} 2\,C_{j,n}^{(2)}\,O_{j,V}^{\mu_1\cdots\mu_n}$$

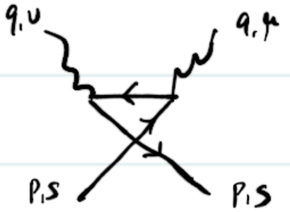
$\underline{C_{j,n}^{(1)}与\ C_{j,n}^{(2)}}$ 为相应短程系数, 它们的确定可以通过计算夸克矩阵元得到.

$\langle t_{\mu\nu}\rangle$: nonlocal 算符矩阵元

$\langle O_V\rangle$: local 算符矩阵元

- OPE for DIS tree-level (夸克矩阵元)





$$iM^{\mu\nu}=i\left\{\bar u(p,s)\gamma^\mu\frac{i(\not p+\not q)}{(p+q)^2}\gamma^\nu u(p,s)+\bar u(p,s)\gamma^\nu\frac{i(\not p-\not q)}{(p-q)^2}\gamma^\mu u(p,s)\right\}\qquad (i\text{为约定})$$

$$(p+q)^2=2p\cdot q+q^2=q^2\left[1+\frac{2p\cdot q}{q^2}\right]=q^2(1-\omega)$$

$$\bar u(p,s)\gamma^\mu(\not p+\not q)\gamma^\nu u(p,s)=\bar u(p,s)\left\{(p+q)^\mu\gamma^\nu+(p+q)^\nu\gamma^\mu-g^{\mu\nu}(\not p+\not q)+i\varepsilon^{\mu\nu\alpha\lambda}(p+q)_\alpha\gamma_\lambda\gamma_5\right\}u(p,s)$$

利用 $\not p\, u(p,s)=0$ $\quad\bar u(p,s)\gamma_\lambda u(p,s)=2p_\lambda$ $\quad\bar u(p,s)\gamma_\lambda\gamma_5 u(p,s)=2h\, p_\lambda$ (h: helicity)

可得. $\bar u(p,s)\gamma^\mu(\not p+\not q)\gamma^\mu u(p,s)=2(p+q)^\mu p^\nu+2(p+q)^\nu p^\mu-2g^{\mu\nu}p\cdot q$

代入并考虑 q 对称性:

$$iM^{\mu\nu}=-\frac{2}{q^2}\sum_{n=0}^{\infty}\omega^n\left[(p+q)^\mu p^\nu+(p+q)^\nu p^\mu-g^{\mu\nu}p\cdot q\right]+(\mu\leftrightarrow\nu,\ q\to-q,\ \omega\to-\omega)$$

$$=-\frac{4}{q^2}\sum_{n=0,2,4}^{\infty}\omega^n\,2p^\mu p^\nu-\frac{4}{q^2}\sum_{n=1,3,5}^{\infty}\omega^n(q^\mu p^\nu+q^\nu p^\mu-g^{\mu\nu}p\cdot q)$$

$$=-\frac{8}{q^2}\sum_{n=0,2,4}^{\infty}\frac{2^n(p\cdot q)^n}{(-q^2)^n}\left(p^\mu-\frac{p\cdot q\,q^\mu}{q^2}\right)\left(p^\nu-\frac{p\cdot q\,q^\nu}{q^2}\right)$$

$$-\frac{4}{q^2}\sum_{n=1,3,5}^{\infty}\frac{2^n(p\cdot q)^{n+1}}{(-q^2)^n}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)$$

将 p·q 拆分开可得:

$$iM^{\mu\nu}=-\frac{8}{q^2}\sum_{n=0,2,4}^{\infty}2^n\frac{q^{\mu_3}\cdots q^{\mu_{n+2}}}{(-q^2)^n}\left(g^{\mu\mu_1}-\frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2}-\frac{q^\nu q^{\mu_2}}{q^2}\right)\cdot\underline{p_{\mu_1}\cdots p_{\mu_{n+2}}}$$

$$-\frac{4}{q^2}\sum_{n=1,3,5}^{\infty}2^n\frac{q^{\mu_1}\cdots q^{\mu_{n+1}}}{(-q^2)^n}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)\underline{(p_{\mu_1}\cdots p_{\mu_{n+1}})}$$

* 夸克 local 算符矩阵元: $\frac{1}{2}\langle q(p)|\bar\psi\gamma^\mu\psi|q(p)\rangle=\frac{1}{2}\bar u\gamma^\mu u=\frac{1}{4}\mathrm{tr}[\not p\gamma^\mu]=.\,p^\mu$

$$\frac{1}{2}\,\frac{i}{2}\langle q(p)|\bar\psi\gamma^\mu\overleftrightarrow{D}^\nu\psi|q(p)\rangle=\frac{1}{2}p^\nu\bar u\gamma^\mu u=p^\mu p^\nu$$

(*) 利用矩阵元形式:

$$iM^{\mu\nu} = -\frac{8}{q^2}\sum_{n=0,2,4}^{\infty} \frac{2^n q^{\mu_3}\cdots q^{\mu_{n+2}}}{(-q^2)^n}\left(g^{\mu\mu_1}-\frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2}-\frac{q^\nu q^{\mu_2}}{q^2}\right)\langle p|O_{q,\nu\mu_1\cdots\mu_{n+2}}|p\rangle$$

$$-\frac{4}{q^2}\sum_{n=1,3,5}^{\infty} \frac{2^n q^{\mu_1}\cdots q^{\mu_{n+1}}}{(-q^2)^n}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)\langle p|O_{q,\nu\mu_1\cdots\mu_{n+1}}|p\rangle$$

平移 n 的求和可得:

$$iM^{\mu\nu} = 2\sum_{n=2,4,6}^{\infty} 2^n\frac{q^{\mu_3}\cdots q^{\mu_n}}{(-q^2)^{n-1}}\left(g^{\mu\mu_1}-\frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2}-\frac{q^\nu q^{\mu_2}}{q^2}\right) O_{q,\nu\mu_1\cdots\mu_n}$$

$$+2\sum_{n=2,4,6}^{\infty} 2^n\frac{q^{\mu_1}\cdots q^{\mu_n}}{(-q^2)^n}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right) O_{q,\nu,\mu_1\cdots\mu_n}$$

对比可得:

$$C^{(1,2)}_{q,\nu,n} = 1+O(\alpha_s)$$

$$C^{(1,2)}_{g,\nu,n} = 0+O(\alpha_s)$$

Twist:

(1) From OPE

(2) From EFT

(3) From conformal theory

(1) From OPE:

in 4-fermion

$$T[W_\mu^+(x)\, W_\nu^-(y)] = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i}{k^2-M_W^2} \curvearrowright \frac{i}{M_W^2-k^2} \times g_{\mu\nu}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i}{M_W^2}\left[1+\frac{k^2}{M_W^2}+\cdots\right] \times g_{\mu\nu}$$

$$= \frac{i}{M_W^2}\, \delta^4(x-y) + \frac{i}{M_W^2}(2\pi)^4\, \frac{\partial_x^2}{M_W^2}\, \delta^4(x-y) \times g_{\mu\nu}$$

$$= \frac{i}{M_W^2}\, g_{\mu\nu}\left[1+\frac{\partial^2}{m_W^2}+\cdots\right]\delta^4(x-y)$$

$$S \sim T\left[\frac{ig}{2\sqrt{2}}\int d^4x\, \bar\psi\gamma^\mu(1-\gamma_5)\psi W_\mu^+(x)\ \ \frac{ig}{2\sqrt{2}}\int d^4y\, \bar\psi\gamma^\nu(1-\gamma_5)\psi W_\nu^+(y)\right]$$

$$= T\left[\frac{-ig^2}{8M_W^2}\int d^4x\, d^4y\, \bar\psi\gamma^\mu(1-\gamma_5)\psi(x)\, \bar\psi\gamma_\mu(1-\gamma_5)\psi(y) \times \left\{1+\frac{\partial^2}{m_W^2}+\cdots\right\}\delta^4(x-y)\right]$$

$$= \frac{-ig^2}{8m_W^2}\left\{\int d^4x\, \bar\psi\gamma^\mu \underline{(1-\gamma_5)\psi(x) \times \bar\psi\gamma_\mu(1-\gamma_5)\psi(x)} + \frac{1}{m_W^2}\int d^4x\, \partial^2\left(\bar\psi\gamma^\mu(1-\gamma_5)\psi(x)\right)\bar\psi\gamma_\mu(1-\gamma_5)\psi\right\}$$

$$O(x) = \bar\psi\gamma^\mu(1-\gamma_5)\psi\, \bar\psi\gamma_\mu(1-\gamma_5)\psi$$

$$O'(x) = \partial^2(\bar\psi\gamma^\mu(1-\gamma_5)\psi)\, \bar\psi\gamma_\mu(1-\gamma_5)\psi$$

$O'(x)$ 相对于 $O(x)$ 量纲增加了 2, $\langle O'(x)\rangle / O(x) \sim k^2$.

考虑到系数 $\frac{1}{m_W^2}$, $\frac{\langle C'O'\rangle}{\langle CO\rangle} = \frac{k^2}{m_W^2} \ll 1$. 高量纲算符的贡献被压低

定域展开的重要性按照量纲即可判断.

(*) $\underline{\text{DIS呢?}}$ 按照量纲?

$\underline{\bar\psi\gamma^\mu\psi}$ $\underline{\bar\psi\gamma^\mu D^\nu\psi}$

$\langle\bar\psi\gamma^\mu\psi\rangle\sim p^\mu$ $\underline{\langle\bar\psi\gamma^\mu D^\nu\psi\rangle\sim p^\mu p^\nu}$

不要忘记 p^μ 的分量 (p^+) 可以很大, $\underline{\text{在Q量级}}$

所以即使考虑可能的系数压低, 高量纲算符也不压低! 那怎么办?

(*) 固定量纲:

例如 $\underline{\bar\psi\gamma^\mu D^\nu\psi}$ $\underset{4}{\underline{\bar\psi D^\mu\psi}}$

相应矩阵元 $\overset{-1}{\langle p(p)|}\,\overset{4}{\underline{\bar\psi\gamma^\mu D^\nu\psi}}\,\overset{-1}{|p(p)\rangle}\sim p^\mu p^\nu$

$\underset{-1}{\langle p(p)|}\,\underset{4}{\underline{\bar\psi D^\mu\psi}}\,\underset{-1}{|p(p)\rangle}\sim m p^\mu$

大 p^+: $\langle\bar\psi D^\mu\psi\rangle \ll \langle\bar\psi\gamma^\mu D^\nu\psi\rangle$

量纲相同, $\underline{\text{指标越多}}$ 贡献越大

(*) 固定指标:

例如: $\underset{4}{\underline{\bar\psi\gamma^\mu D^\nu\psi}}$ $\underset{5}{\underline{\bar\psi D^\mu D^\nu\psi}}$

$\langle\bar\psi\gamma^\mu D^\nu\psi\rangle\sim p^\mu p^\nu$

$\langle\bar\psi D^\mu D^\nu\psi\rangle\sim m p^\mu p^\nu$ 系数需要 $\frac{1}{Q}$

$\langle\bar\psi D^\mu D^\nu\psi\rangle \ll \langle\bar\psi\gamma^\mu D^\nu\psi\rangle$

指标相同, 量纲越小, 贡献越大

统一: 定义. $t = d-n$.

From EFT:

幂次估计，加些运动学描述接项.

p^+ 大分量

(*) 旋量场大小分量： $p^\mu \sim (p^+, p^-, p_\perp) \sim (Q, Q\lambda^2, \lambda Q)$ $\quad \not{p} = \underset{\sim}{\frac{n_- p}{2}}\not{n}_+ + \frac{n_+ p}{2}\not{n}_- + \not{p}_\perp$

$$T[\psi(x)\bar{\psi}(y)] = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i\not{p}}{p^2}$$

$$\psi(x) = \left(\frac{\not{n}_+\not{n}_-}{4} + \frac{\not{n}_-\not{n}_+}{4}\right)\psi(x) = \psi_c(x) + \eta_c(x)$$

$$T\Big[\underbrace{\frac{\not{n}_+\not{n}_-}{4}\psi(x)}_{\psi_c}\underbrace{\bar{\psi}(y)\frac{\not{n}_-\not{n}_+}{4}}_{\bar{\psi}_c}\Big] = \int\frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2}\underbrace{\frac{\not{n}_+\not{n}_-}{4}\not{p}\frac{\not{n}_-\not{n}_+}{4}}$$

$$\frac{\not{n}_+\not{n}_-}{4}\Big(\underset{只有这项非零}{\underline{\underline{\frac{n_- p}{2}}}}\not{n}_+ + \frac{n_+ p}{2}\not{n}_- + \not{p}_\perp\Big)\frac{\not{n}_-\not{n}_+}{4}$$

$$\sim \frac{Q\,Q\lambda^2(Q\lambda)^2}{Q^2\lambda^2}\cdot\times 1 \Rightarrow \lambda^2$$

$\Rightarrow \psi_c \sim \lambda$ $\quad$ 大分量

$$T\Big[\frac{\not{n}_-\not{n}_+}{4}\psi(x)\bar{\psi}(y)\frac{\not{n}_+\not{n}_-}{4}\Big] = \int\frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\frac{i}{p^2}\frac{\not{n}_-\not{n}_+}{4}\underset{\frac{n_+ p}{2}\not{n}_-}{\underset{=}{\not{p}}}\frac{\not{n}_-\not{n}_-}{4}$$

$$\sim \frac{Q\,Q\lambda^2(Q\lambda)^2}{Q^2\lambda^2}\cdot\lambda^2 \sim \lambda^4$$

$\Rightarrow \eta_c \sim \lambda^2$ $\quad$ 小分量

也有人写 γ^+

(*) $\bar{\psi}\frac{\not{n}_-}{2}\psi = (\bar{\psi}_c + \bar{\eta}_c)\frac{\not{n}_-}{2}(\psi_c + \eta_c) = \bar{\psi}_c\frac{\not{n}_-}{2}\psi_c$ $\quad$ 都是大分量， $\lambda^2 \to$ twist-2

$\bar{\psi}\times 1\,\psi = (\bar{\psi}_c + \bar{\eta}_c)(\psi_c + \eta_c) = \bar{\psi}_c\eta_c + \bar{\eta}_c\psi_c$ $\quad$ 一个小分量 $\quad \lambda^3 \to$ twist-3

$\bar{\psi}\frac{\not{n}_+}{2}\psi = \quad - \quad - \quad = \bar{\eta}_c\frac{\not{n}_+}{2}\eta_c$ $\quad$ 两个小分量 $\quad \lambda^4 \to$ twist-4

$\to$ $\bar{\psi}\gamma^\mu\psi$ 的领头项

胶子:

$$T[A_\mu(x)A_\nu(y)] = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{-i}{p^2}\left(g_{\mu\nu}-(1-\alpha)\frac{p_\mu p_\nu}{p^2}\right)$$

$$T[n_-\cdot A(x)\, n_-\cdot A(y)] = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{-i}{p^2}\left(0-(1-\alpha)\frac{n_-\cdot p\, n_-\cdot p}{p^2}\right)\sim\frac{\lambda^4}{\lambda^2}\cdot\frac{1}{\lambda^2}\sim\lambda^0$$

$$n_-\cdot A(x)\sim\lambda^0\sim 1$$ leading

$$T[A_{\perp\mu}(x)\,A_{\perp\nu}(y)] = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{-i}{p^2}\left(g_{\perp\mu\nu}+\cdots\right)\sim\frac{\lambda^4}{\lambda^2}\cdot 1\sim\lambda^2$$

$$A_\perp(x)\sim\lambda$$ next-to-leading

$$T[n_+\cdot A(x)\,n_+\cdot A(y)] = \cdots\quad\sim\frac{\lambda^4}{\lambda^2}\cdot\frac{\lambda^4}{\lambda^2}\sim\lambda^4$$

$$n_+\cdot A\sim\lambda^2$$ N^2L

幂次与动量一样！ $\underline{n_-\cdot D}$ 领头项

leading twist (t=2) 都是标量

(*) local: $\bar\psi_c\frac{\not n_-}{2}n_-\cdot D\,\psi_c = \underbrace{\frac{n_{-\mu}n_{-\nu}}{2}}_{\text{对称}}\,\overset{\text{对称}}{\underline{\underline{\bar\psi\gamma^\mu D^\nu\psi}}}$ leading twist moments

(*) nonlocal: $\bar\psi\frac{\not n_-}{2}\,\underline{P\exp\left[ig_s\int ds\, n_-\cdot A(sn_-)\right]}\psi$ --- leading twist PDF

(*) 纯胶子: $F^{\mu\nu}\ F_{\mu\nu}$ 最低量纲

$F^{++}=0$

$F^{+\perp}$. 最大贡献

$F^{+\perp}\cdot F^{+}_{\perp}$ leading twist.

Fierz transformation:

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$$(\Gamma_A)_{ij}\ (\Gamma_B)_{kl} = \sum_{CD} C_{ABCD} (\Gamma_C)_{il} (\Gamma_D)_{kj} \qquad \underline{(1)}$$

Γ: $I,\ \gamma_\mu,\ \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu],\ \gamma_\mu\gamma_5,\ \gamma_5$

可以证明：这些γ矩阵满足 $(\Gamma)^2 = \pm I$，且在求tr下相互正交

则 $tr[\Gamma_A \Gamma_B] = \underline{\pm 4}\,\delta_{AB}$

Fierz变换：在(1)式左右同乘上 $(\Gamma_C')_{li}\ (\Gamma_D')_{jk}$. 遍历不同$\gamma$即可得到.

下面以 $\Gamma_A = I$, $\Gamma_B = I$ 为例介绍：

$$\underline{(I)_{ij} (I)_{kl} = \sum_{CD} C_{CD} (\Gamma_C)_{il} (\Gamma_D)_{kj}}$$

① 左右同乘 $\underline{(I)_{li} (I)_{jk}}$:

左 $= I_{li} I_{ij} I_{jk} I_{kl} = tr[I] = 4$

右 $= \sum_{CD} C_{CD}\ tr[\Gamma_C]\ tr[\Gamma_D] \overset{C,D=I}{=} C_{II} \times 16$

$\rightarrow\ C_{II} = \frac{1}{4}$

② 左右同乘 $(\gamma_0)_{li}\ (\gamma_0)_{jk}$:

左 $= (\gamma_0)_{li} (I)_{ij} (\gamma_0)_{jk} (I)_{kl} = tr[\gamma_0 \gamma_0] = 4$

右 $= \sum_{CD} C_{CD}\ tr[\Gamma_C \gamma_0]\ tr[\Gamma_D \gamma_0] = C_{\gamma_0\gamma_0} \times 16$

$C_{\gamma_0\gamma_0} = \frac{1}{4}$

③ 左右同乘 $\gamma_i\ \gamma^i$. (注意.无求和)

左 $= (\gamma_i)_{li} (I)_{ij} (\gamma^i)_{jk} (I)_{kl} = tr[\gamma_i \gamma^i] = 4$

右 $= \sum_{CD} C_{CD}\ tr[\Gamma_C \gamma_i]\ tr[\Gamma_D \gamma^i] = C_{\gamma_i\gamma^i} \times 16$

$C_{\gamma_i\gamma^i} = \frac{1}{4}$

⊗左右同乘 γ_5 γ_5

左 = $(\gamma_5)_{li}(I)_{ij}(\gamma_5)_{jk}(I)_{kl} = tr[\gamma_5\gamma_5] = 4$

右 = $\sum_{CD} C_{CD}\, tr[\Gamma_C\gamma_5]\, tr[\Gamma_D\gamma_5] = C_{\gamma_5\gamma_5}\times 16$

$C_{\gamma_5\gamma_5} = \frac{1}{4}$

⊗左右同乘 $(\gamma_0\gamma_5)_{li}$ $(\gamma_0\gamma_5)_{jk}$

左 = $(\gamma_0\gamma_5)_{li}(I)_{ij}(\gamma_0\gamma_5)_{jk}(I)_{kl} = tr[\gamma_0\gamma_5\gamma_0\gamma_5] = -4$

右 = $\sum_{CD} C_{CD}\, tr[\Gamma_C\gamma_0\gamma_5]\, tr[\Gamma_D\gamma_0\gamma_5] = C_{\gamma_0\gamma_5,\ \gamma_0\gamma_5}\times 16$

$C_{\gamma_0\gamma_5\ \gamma_0\gamma_5} = -\frac{1}{4}$

⊗左右同乘 $(\gamma_i\gamma_5)_{li}$ $(\gamma_0\gamma_5)_{jk}$ (无求和)

左 = $(\gamma_i\gamma_5)_{li}(I)_{ij}(\gamma^i\gamma_5)_{jk}(I)_{kl} = tr[\gamma_i\gamma_5\gamma^i\gamma_5] = -4$

右 = $\sum_{CD} C_{CD}\, tr[\Gamma_C\gamma_i\gamma_5]\, tr[\Gamma_D\gamma^i\gamma_5] = C_{\gamma_i\gamma_5\ \gamma^i\gamma_5}\times 16$

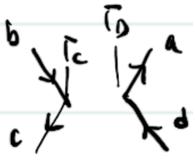
$C_{\gamma_i\gamma_5\gamma^i\gamma_5} = -\frac{1}{4}$

⊗左右乘 (σ_{ij}) - - -

所以 $$\left.\begin{aligned}(I)_{ij}(I)_{kl} = {} & \frac{1}{4} I_{il} I_{kj} + \underline{\underline{\frac{1}{4}(\gamma_\mu)_{il}(\gamma^\mu)}}_{kj} \\ & + \frac{1}{4}(\gamma_5)_{il}(\gamma_5)_{kj} - \frac{1}{4}(\gamma_\mu\gamma_5)_{il}(\gamma^\mu\gamma_5)_{kj} \\ & + \frac{1}{8}(\sigma_{\mu\nu})_{il}(\sigma^{\mu\nu})_{kj} \ \text{- - -} \ ?\end{aligned}\right\}$$

(*) 物理图像：

$$\begin{matrix} \bar{u}_a \Gamma_A u_b \\ \bar{u}_c \Gamma_B u_d \end{matrix} \xrightarrow{\text{Fierz}}$$



$$\begin{matrix} \bar{u}_a \Gamma_D u_d \\ \bar{u}_c \Gamma_C u_b \end{matrix}$$

(*) 另一个常用的 Fierz 变换：

$$(\gamma_\mu(1-\gamma_5))_{ij} \; (\gamma^\mu(1-\gamma_5))_{kl} = -(\gamma_\mu(1-\gamma_5))_{il} \; (\gamma^\mu(1-\gamma_5))_{kj}$$

(六) 物理应用过程：(simple model)

$\bar{B} \to l\tilde{\nu}$.

为简便，可以从4费米子理论出发：

$$H_{eff} = \frac{G_F}{\sqrt{2}} (\bar{l}\gamma_\mu(1-\gamma_5)\nu)(\bar{u}\gamma^\mu(1-\gamma_5)b$$

$$iM = \langle l\tilde{\nu}| -iH_{eff}(0)|\bar{B}\rangle$$

$$= -\frac{iG_F}{\sqrt{2}} \bar{u}_l\gamma_\mu(1-\gamma_5)v_\nu \times \langle 0|\bar{u}\gamma^\mu(1-\gamma_5)b|\bar{B}\rangle$$

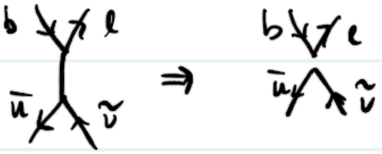
↓ 参数化

$if_B P^\mu$

轻子部分 + 强子部分 = 因子化

$$= -\frac{iG_F}{\sqrt{2}} \bar{u}_l\gamma_\mu(1-\gamma_5)u_\nu \times if_B P^\mu$$

(七) leptoquark：



$$O = [\bar{l}\gamma_\mu(1-\gamma_5)b][\bar{u}\gamma^\mu(1-\gamma_5)\nu]$$

不考虑颜色

一般有系数，这里不重要
↓

$$iM = -i\frac{G_F}{\sqrt{2}} \times c \times \langle l\tilde{\nu}|(\underset{=}{\bar{l}}\gamma_\mu(1-\gamma_5)\underset{\triangle}{b})(\underset{\triangle}{\bar{u}}\gamma^\mu(1-\gamma_5)\underset{=}{\nu})|\bar{B}\rangle$$

轻子与强子纠缠，如何分离？ - - - Fierz 变换

$$= -\frac{iG_F}{\sqrt{2}} \times c \times \langle l\tilde{\nu}|(\bar{l}\gamma_\mu(1-\gamma_5)\nu)(\bar{u}\gamma^\mu(1-\gamma_5)b|\bar{B}\rangle$$

剩下的类似.

换个路：

$$iM = -i\frac{G_F}{\sqrt{2}} \times c \times \langle l\tilde{\nu}| \left(\bar{l}\gamma_\mu(1-\gamma_5)\right)_i \underline{\underline{(I)}}_{ij}\, b_j \times \bar{u}_k \underline{\underline{(I)}}_{kl} \left(\gamma^\mu(1-\gamma_5)\nu\right)_l |\bar{B}\rangle$$

$$= -\frac{iG_F}{\sqrt{2}} \times c \times \langle l\tilde{\nu}| \left(\bar{l}\gamma_\mu(1-\gamma_5)\right)_i b_j\ \bar{u}_k \left(\gamma^\mu(1-\gamma_5)\nu\right)_l |\bar{B}\rangle$$

$$\times \left\{ \frac{1}{4}(I)_{il}(I)_{kj} + \frac{1}{4} - - - \right\}$$

$$= -\frac{iG_F}{\sqrt{2}} \times c \times \left\{ \frac{1}{4} \langle l\tilde{\nu}| \left(\bar{l}\gamma_\mu(1-\gamma_5)\right)_i (\gamma_5)_{il} \left(\gamma^\mu(1-\gamma_5)\nu\right)_l \times \left(-\bar{u}_k(\gamma_5)_{kj} b_j\right) |\bar{B}\rangle \right.$$
$$\left. + - - - \right\}$$

$$= -\frac{iG_F}{\sqrt{2}} \times c \times \left\{ \frac{1}{4}\bar{u}\gamma_\mu(1-\gamma_5)\overset{\substack{\text{轻子部分}\\ \downarrow}}{\gamma_5}\gamma^\mu(1-\gamma_5)\nu \times (-1) \langle 0| \overset{\substack{\text{强子部分}\\ \downarrow}}{\bar{u}}\gamma_5 b |\bar{B}\rangle + - - - \right.$$

(*) Factorization for DIS:

$$T^{\mu\nu}(q,p) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} \langle p| T[\, J^{\mu}(x)\, J^{\nu}(0)]|p\rangle$$

$$\underline{W^{\mu\nu}(q,p)} = 2\,\mathrm{Im}\, T^{\mu\nu}(q,p)$$

① leading order in perturbation theory:

$$T^{\mu\nu}(q,p) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\langle p|T[\,\underline{\bar{q}(x)\gamma^{\mu} q(x)}\ \ \underline{\bar{q}(0)\gamma^{\nu} q(0)}]|p\rangle + \overbrace{\bar{q}(x) \cdots\cdots q(0)}$$

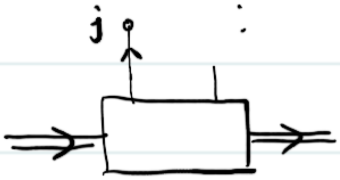
忽略 反夸克贡献

$$= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} \underline{\int \frac{d^4\ell}{(2\pi)^4} e^{-i\ell\cdot(x-0)}} \langle p|T[\bar{q}(x)\underset{\equiv (I)_{ij}}{\gamma^{\mu}} \underline{\underline{\frac{i\not{\ell}}{\ell^2}}} \underset{\equiv (I)_{kl}}{\gamma^{\nu}} q(0)]|p\rangle .$$

① 直接做 Fierz 变换

② 或分析: 算符部分 (动量空间): $\Gamma_{ji}(k,p) \equiv \int d^4x\, e^{-ik\cdot x}\langle p|T[\bar{q}_i(x)\, q_j(0)]|p\rangle$

对应:



$$\langle p|T[\bar{q}_i(x)\, q_j(0)]|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\ \Gamma_{ji}(k,p)$$

① i, j 为旋量指标、

先代入到 $W^{\mu\nu}$ 中:

$$W^{\mu\nu} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \left(\gamma^{\mu}(\not{k}+\not{q})\gamma^{\nu}\right)_{ij}\ \delta((k+q)^2)\ \ \Gamma_{ji}(k,p)$$

4维动量积分.

① $\Gamma_{ji}(k,p)$ 类似于波函数. 其中 k 的量纲为:

$$k_\perp \sim \Lambda, \quad k^- \sim \frac{\Lambda^2}{p^+}, \quad k^+ \sim p^+ \qquad \Lambda \sim \Lambda_{QCD}$$

no hard interaction inside. 引入 $\hat{k}^\mu = (k^+, 0, 0, 0)$ $\quad k^\mu = \hat{k}^\mu(1+O(\Lambda))$

② 如果仅保留领头项. 相当于对 $\langle p|\bar{q}_i(x)\, q_j(0)|p\rangle$ 做 $\vec{x}_\perp$ 和 x^+ 的展开.

③ 定义: $\langle p|\bar{q}_i(x)\, q_j(0)|p\rangle = \langle p|\bar{q}_i(x^- n)\, q_j(0)|p\rangle + O(x_\perp, x^+)$

$$\int \frac{dx^-}{2\pi} e^{-ik^+x^-} \langle p|T[\bar{q}_i(x^- n)\, q_j(0)]|p\rangle = \frac{1}{2}(\gamma^-)_{ji} f_{q/p}(z) \qquad z = k^+/p^+$$

$$\frac{1}{2}\int \frac{dx^-}{(2\pi)} e^{-ik^+x^-} \langle p|T[\bar{q}(x^- n)\gamma^+ q(0)]|p\rangle = f_{q/p}(z)$$

$$\hookrightarrow \langle p|T[\bar{q}_i(x^- n)\, q_j(0)]|p\rangle = \frac{1}{2}\int dk^+ e^{ik^+x^-} (\gamma^-)_{ji} f_{q/p}(z) \qquad z = k^+/p^+$$

重新代入至 $W^{\mu\nu}$

$$W^{\mu\nu} = \frac{1}{4}\int_0^{p^+} dk^+ \, \underline{f_{q/p}(z)}\, \delta((\hat{k}+q)^2)\, \mathrm{Tr}[\gamma^\mu \overset{q^-\gamma^+}{\underline{(\hat{k}+\not{q})}} \gamma^\nu \gamma^-]$$

到这里和做fierz变换一样.

$$(\hat{k}+q)^2 = 2(\hat{k}+q)^+ q^- = 0 \rightarrow \hat{k}^+ + \hat{q}^+ = 0 \rightarrow (\hat{k}+q)^\mu = (0, q^-, 0, 0)$$

可以展开为:

$$F_1(x, Q^2) = \frac{1}{2}\int dz\, \delta(x-z)\, f_{q/p}(z) = \frac{1}{2} f_{q/p}(x)$$

$$F_2(x, Q^2) = 2x F_1(x, Q^2) = x f_{q/p}(x)$$

部分子模型.

这里插入树图因子化证明.

(*) 直接用 Fierz 变换：

$$T^{\mu\nu}(p,q) = \frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x}\langle p|T[J^\mu(x)\, J^\nu(0)|p\rangle$$

$$= \frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x}\langle p|T[\bar{q}(x)\gamma^\mu \overbrace{q(x)\quad \bar{q}(0)}\gamma^\nu q(0)]|p\rangle + \cdots$$

$$= \frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x}\int\frac{d^4l}{(2\pi)^4}e^{-il\cdot(x-0)}\langle p|T[\underbrace{\bar{q}(x)_i\gamma^\mu}_{(I)_{ij}}\frac{i\not{l}}{l^2+i\varepsilon}\underbrace{\gamma^\nu q(0)_l}_{(I)_{kl}}]|p\rangle + \cdots$$

$$= \frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x}\int\frac{d^4l}{(2\pi)^4}e^{-il\cdot(x-0)}\times$$

$$\times\Big\{\frac{1}{4}\underline{\langle p|\bar{q}(x)_i(I)_{il}\, q(0)_l|p\rangle} * (I)_{kj}\underline{\left(\gamma^\mu\frac{i\not{l}}{l^2+i\varepsilon}\gamma^\nu\right)_{jk}}$$

$$+\frac{1}{4}\underbrace{\langle p|\bar{q}(x)_i(\gamma_\alpha)_{il}(q(0))_l|p\rangle}_{\text{强子矩阵元}}\times(\gamma_\alpha)_{kj}\underbrace{\left(\gamma^\mu\frac{i\not{l}}{l^2+i\varepsilon}\gamma^\nu\right)_{jk}}_{\text{微扰散射核}}$$

$$+\cdots\Big\}$$

(*) $p^\mu \sim (p^+, p^-, p_\perp)$ $p^+ \gg p_\perp \gg p^-$. 在领头阶, 矩阵元 $x^\mu \sim (0^+, x^-, 0_\perp)$

(*) 5种强子矩阵元. 它们的重要性按照 twist 展开

(*) leading twist $\bar{\psi}\gamma^+\psi$. $\bar{\psi}\gamma^+\gamma_5\psi$ $\bar{\psi}\gamma^+\gamma_\perp\psi$

非极化 (unpolarized) 极化 (helicity) 横向 (transversity)

(*) 需考虑 NLO QCD 修正.

- DIS Factorization at 1-loop

交换 collinear gluon

$\boxed{\langle P|\bar q\,\gamma^+ W_c(\infty,0)\,q(0)|P\rangle}$

等效于 expansion by region

$\boxed{\text{从最开始就积分出 } n_+k_1,\ k_{1\perp}}$

$$iM^{(1)} = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\langle p|T[J^\mu(x)\,J^\nu(0)]|P\rangle$$

$$= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\int d^4z_1 \langle p|T[\bar q\gamma^\mu \overbrace{q(x)\quad \bar q}\, \underline{i g_s \not{A}}\, \overbrace{q(z_1)\quad \bar q}\gamma^\nu q(0)]|P\rangle$$

$$= \frac{1}{4\pi}\int d^4x \int d^4z_1 \int\frac{d^4l_1}{(2\pi)^4}\int\frac{d^4l_2}{(2\pi)^4}\, e^{iq\cdot x}\, e^{-il_1(x-z_1)}\, e^{-il_2(z-0)}$$

$$\langle p|\bar q(x)\gamma^\mu \frac{i\not{l}_1}{l_1^2}\, i g_s t^a\gamma_\mu A^{a\mu}(z_1)\frac{i\not{l}_2}{l_2^2}\gamma^\nu q(0)|P\rangle$$ ---（矩阵元中只有一个大分量）

矩阵元中 $\langle p|\bar q_i(x^- n)\ t^a A^{a\mu}(z_1^- n)\ q(0)|P\rangle$ 只有 x^- 分量，则 x^+ 与 $x_\perp$ 分量可以积分出来

相应的 l_1 与 l_2 也可以积分出 l^- 与 $l_\perp$ 分量.

注 change of momentum: $l_1 - q \to k$ $\qquad$ $l_1 - l_2 \to k_1$

$l_1 = q+k$ $\qquad$ $l_2 = l_1 - k_1$

$$iM^{(1)} = \frac{1}{4\pi}\int d^4x\, d^4z_1 \int\frac{d^4k}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\, e^{-ik\cdot x}\, e^{ik_1 z_1}\times$$

$$\langle p|\bar q(x^- n)\gamma^\mu \frac{i(\not{k}+\not{q})}{(k+q)^2}\, i g_s t^a\gamma_\mu A^{a\mu}(z_1 n)\frac{i(\not{k}+\not{q}-\not{k}_1)}{(k+q-k_1)^2}\times\gamma^\nu q(0)|P\rangle$$

$$= \frac{1}{4\pi}\int dx^- dz_1^- \int\frac{dk^+}{2\pi}\int\frac{dk_1^+}{2\pi}\, e^{-ik^+x^-}\, e^{ik_1^+z_1^-}\times$$

$$\times\langle p|\bar q(x^- n)\underset{\overset{\equiv}{(I)_{ij}}}{\gamma^\mu}\frac{i(\hat{\not{k}}+\not{q})}{(\hat k+q)^2}\, i g_s t^a\gamma_\alpha A^{a\alpha}(z_1 n)\frac{i(\hat{\not{k}}+\not{q}-\hat{\not{k}}_1)}{(\hat k+q-\hat k_1)^2}\underset{\overset{\equiv}{(I)_{kl}}}{\gamma^\nu} q(0)|P\rangle$$

Fierz 变换: $I_{ij} I_{kl} = \frac{1}{4}(\gamma_\beta)_{il}(\gamma^\beta)_{kj} + \frac{1}{4}\cdots$ —

$$iM^{(1)} = \frac{1}{4\pi}\int dx^- dz_1^- \int\frac{dk^+}{2\pi}\int\frac{dk_1^+}{2\pi}\, e^{-ik^+x^-}\, e^{ik_1^+z_1^-}$$

$$\times\frac{1}{4}\langle p|\bar q(x^- n)\gamma_\beta \times t^a A^{a\alpha}(z_1 n)\, q(0)|P(p)\rangle$$ 注意颜色指标

$$\times\, \mathrm{tr}\left[\gamma^\beta\gamma^\mu\frac{i(\hat{\not{k}}+\not{q})}{(\hat k+q)^2}\, i g_s\gamma_\alpha\frac{i(\hat{\not{k}}+\not{q}-\hat{\not{k}}_1)}{(\hat k+q-\hat k_1)^2}\gamma^\nu\right]$$

$$(\hat{k}+q)^2 = 2(\hat{k}^+ + q^+)q^- = 0$$

$$\hat{k}^\mu = (k^+,0,0,0) \qquad \hat{k}_1^\mu = (k_1^+,0,0,0)$$

取在壳

$$iM^{(1)} = \frac{1}{4\pi}\int dx^- dz_1^- \int\frac{dk^+}{(2\pi)}\int\frac{dk_1^+}{2\pi}\, e^{-ik^+x^-}\, e^{ik_1^+z_1^-} \qquad (2\pi)\delta((\hat{k}+q)^2)$$

$$\times \frac{1}{4}\langle p(p)|\bar{q}(x^-n)\,\gamma^+\, t^a A^{a+}(z_1^-n)\, q(0)\,|p(p)\rangle \qquad \otimes$$

$$\times\, \mathrm{tr}\Big[\gamma^-\gamma^\mu\; iq^-\gamma^+\; ig_s\underbrace{\gamma^-\,\frac{iq^-\gamma^+}{-2q^-\cdot\hat{k}_1^+ + i\varepsilon}}\;\gamma^\nu\Big] \qquad ●$$

$$(*) \;\rightarrow\; \frac{\gamma^-\gamma^+}{\underline{2}} \times \frac{i}{-\hat{k}_1^+ + i\varepsilon}$$

$\uparrow$

投影算符 可以拿走

$$(*) \qquad i\int\frac{dk_1^+}{2\pi}\,\frac{1}{-\hat{k}_1^+ + i\varepsilon}\, e^{ik_1^+z_1^-} = i\int\frac{dk_1^+}{2\pi}\,\frac{1}{\hat{k}_1^+ + i\varepsilon}\, e^{-ik_1^+z_1^-} = \underline{\theta(z_1^-)}$$

$$(*) \quad iM^{(1)} = \frac{1}{4\pi}\times\frac{1}{4} \int\frac{dk^+}{2\pi}\, e^{-ik^+x^-}\,(2\pi)\delta((\hat{k}+q)^2) \qquad \rightarrow$$

$$\times\int dx^- e^{-ik^+x^-}\int \underline{dz_1^-}\,\langle p(p)|\bar{q}(x^-n)\,\not{n}\times ig_s t^a\, n\cdot A^a(z_1^-n)\cdot q(0)\,|p(p)\rangle \quad \theta(z_1^-) \qquad \rightarrow$$

$$\times\, \mathrm{tr}[\gamma^-\gamma^\mu\; iq^-\gamma^+\;\gamma^\nu] \cdots \text{ tree-level Kernel}$$

中间行可写为：

$$\int dx^- e^{-ik^+x^-} \times \langle p(p)|\bar{q}(x^-n)\,\not{n}\, W_c(\infty,0)\; q(0)|p(p)\rangle \qquad \text{展开至一阶} \qquad ⓐ$$

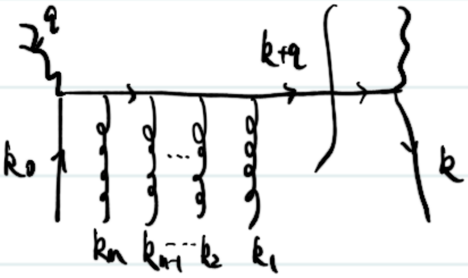
$$W_c = \mathrm{pexp}\Big[ig_s\int_0^\infty dz_1^-\; n\cdot A^a(z_1^-n)\,t^a\Big]$$

$\rightarrow$ 此为PDF定义：即 $f_{q/p}(z)$ - $z = \frac{k^+}{p^+}$

(*) 则： $iM^{(1)} = \frac{1}{4\pi}\times\frac{1}{4}\int\frac{dk^+}{2\pi}e^{-ik^+x^-}(2\pi)\delta((\hat{k}+q)^2)\ f_{q|p}(z)\cdot tr[\gamma^-\gamma^\mu iq^-\gamma^+\gamma^\nu]$

这就是单圈阶因子化. 只考虑了一个图.

(b) N gluon



$$W^{\mu\nu} = \frac{1}{4}\int\frac{dk^+}{2\pi}Tr[\gamma^\mu\underline{i\gamma\cdot(\hat{k}+q)}\gamma^\nu\gamma^-]\ \delta((\hat{k}+q)^2)$$

$$\times\frac{1}{2}\int\prod_{i=1}^{n}\frac{dk_i}{2\pi}\ \frac{ig_s}{-k_1^++i\varepsilon}\ \frac{ig_s}{-k_1^+-k_2^++i\varepsilon}\ \cdots\ \frac{ig_s}{-k_1^+-k_2^+\cdots-k_m^++i\varepsilon}$$

$$\times\int dx^-e^{-ix^-k^+}\prod_{i=1}^{n}dz_i^-\ \underline{e^{iz_ik_i^+}}\qquad ?$$

$$\times\langle p|\bar{q}(x^-n)\gamma^+A^+(z_1n)A^+(z_2n)\cdots A^+\quad,q(0)|p(p)\rangle$$

可以定义 $\tilde{k}_i^+ = k_1^++k_2^++\cdots+k_i^+$

$$W_c(\infty,x) = P\exp[ig_s\int_0^\infty dz^-A^+(z^-n+x)$$

$$= 1+\sum_i(ig_s)^i\int\prod_{j=1}^{i}dz_j^-A^+(z_1n+x)A^+(z_2n+x)\cdots A^+(z_jn+x)$$

$$\times\theta(z_1^--z_2^-)\theta(z_2^--z_3^-)\cdots\theta(z_{j-1}^--z_j^-)\theta(z_j^-)$$

利用 $\theta(\lambda) = i\int\frac{d\omega}{2\pi}e^{-i\omega\lambda}\frac{1}{\omega+i\varepsilon}$

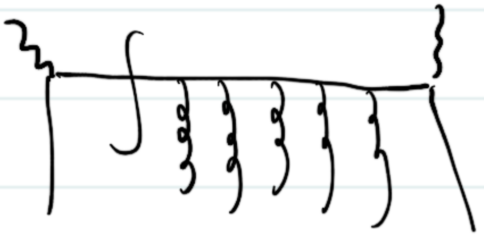
代入得：

$$W_c(\infty, x) = 1 + \sum_{n=1}^{\infty} (ig_s)^i \int \prod_{j=1}^{i} \frac{dk_j^+}{2\pi}$$
$$\times \frac{i}{-k_1^+ + i\varepsilon} \frac{i}{-k_2^+ + i\varepsilon} \cdots \frac{i}{-k_j^+ + i\varepsilon}$$
$$\times \int \prod_{j=1}^{i} dz_j^- \, e^{ik_j^+ z_j^+} A^+(z_1^- n + x) A^+(z_2^- n + x) \cdots A^+(z_j^- n + x)$$

代入 $W^{\mu\nu}$ 得

$$W^{\mu\nu} = \frac{1}{4} \int \frac{dk^+}{2\pi} \mathrm{Tr}[\gamma^\mu \gamma(\hat{k}+q)\gamma^\nu \gamma^-] \; \delta((\hat{k}+q)^2)$$
$$\times \frac{1}{2} \int dx^- e^{-ik^+x^-} \langle P(p)| \bar{q} \, \gamma^+ \overset{\hookleftarrow}{W_c(\infty, 0)} q(0) |P(p)\rangle$$

加入另一半：



其中的 PDF 改为:

$$\int dx^- e^{-ik^+x^-} \langle P(p)| \underline{\underline{\bar{q}(x^- n) W_c^+(\infty, x^- n)}} \gamma^+ W_c(\infty, 0) q(0) | P(p)\rangle$$

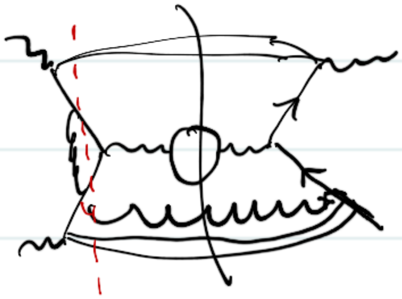
$$W_c(\infty, x) = P\exp\left[ig_s \int_0^\infty ds \, n\cdot A(x+sn)\right]$$

(*) 需要注意的是. 这里传播子需要用 $-i\varepsilon$.

$$\mathrm{disc} \frac{1}{p_1^2} \frac{1}{p_2^2} = \frac{1}{p_1^2+i\varepsilon} \frac{1}{p_2^2+i\varepsilon} - \frac{1}{p_1^2-i\varepsilon} \frac{1}{p_2^2-i\varepsilon}$$
$$= \frac{1}{p_1^2+i\varepsilon} \frac{1}{p_2^2+i\varepsilon} - \frac{1}{p_1^2+i\varepsilon} \frac{1}{p_2^2-i\varepsilon} + \frac{1}{p_1^2+i\varepsilon} \frac{1}{p_2^2-i\varepsilon} - \frac{1}{p_1^2-i\varepsilon} \frac{1}{p_2^3-i\varepsilon}$$
$$= \frac{1}{p_1^2+i\varepsilon} (-2\pi i \delta(p_2^2)) + (-2\pi i \delta(p_1^2)) \frac{1}{p_2^2-i\varepsilon}$$

(*) Keldysch formalism：（另一种方法）

cut diagram 与 discontinuity 不一定一一对应



见 ① Zh. Eksp. Teor. Fiz 47. 1515 (1964)

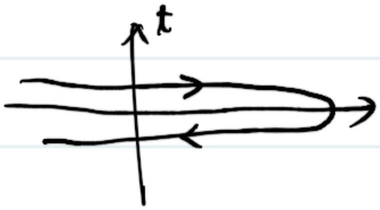
② 0710.0680. 附录 C.

此处也可以有 cut.

为了解决这一问题 采用 Keldysh formalism，

一般作用量为 $S = \int_{-\infty}^{+\infty} dt\, \mathcal{L}(\phi)$

修改为 $S = \int_C dt\, \mathcal{L}(\phi)$



C: $t: -\infty + i\delta \rightarrow +\infty + i\delta \rightarrow +\infty - i\delta \rightarrow -\infty - i\delta$

定义 $\phi_+(x) = \phi(x + i\delta)$ $\phi_-(x) = \phi(x - i\delta)$

则 $\int D\phi\, \phi_+(x_1) \cdots \phi_+(x_n)\, \phi_-(x_1) \cdots \phi_-(x_m)\, \exp[iS[\phi]]$

$= \langle 0| \bar{T}\{\phi(x_1)\phi(x_2) \cdots \phi(x_m)\}\, T\{\phi(x_1) \cdots \phi(x_n)\} |0\rangle$

T 为 path-order $\bar{T}$ 为反 path-order

类似的： $\sum_X (2\pi)^4 \delta^4(P_X - P)\, \left|\langle X| T[\phi(x_1) \cdots \phi(x_n)] |P\rangle\right|^2$

$= \int d^4x\, e^{-iP\cdot x} \langle P| \bar{T}[\phi(x_1+x) \cdots \phi(x_n+x)]\, T[\phi(x_1) \cdots \phi(x_n)] |P\rangle$

其中 $\langle 0|T[\phi_+(x_1)\phi_+(x_2)]|0\rangle = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x_1-x_2)}\frac{i}{p^2-m^2+i\varepsilon}$

$\langle 0|\bar{T}[\phi_-(x_1)\phi_-(x_2)]|0\rangle = \int\frac{d^4}{(2\pi)^4}e^{-ip\cdot(x_1-x_2)}\frac{i}{p-m^2-i\varepsilon}$

$\langle 0|\phi_-(x_1)\,\phi_+(x_2)|0\rangle = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x_1-x_2)}(2\pi)\,\delta(p^2-m^2)\,\theta(p^0)$

这与 cutting rule 是一致的

如果想得到其轮图的 Wilson line，建议从这里出发。★

I 光锥坐标系的约定: [如何约定 q^+ ?]

$n_+^\mu = (1,0,0,1)$ $\quad n_-^\mu = (1,0,0-1)$	$\longleftrightarrow$	$n_+^\mu = \frac{1}{\sqrt{2}}\underline{(1,0,0,1)}$ $\quad n_-^\mu = \frac{1}{\sqrt{2}}(1,0,0,-1)$
$n_+ \cdot n_- = 2$		$n_+ n_- = 1$
$n_+ \cdot p = p^0 - p^3 \quad n_- p = p^0 + p^3$		$n_+ p = \frac{1}{\sqrt{2}}(p^0 - p^3)$, $\quad n_- p = \frac{1}{\sqrt{2}}(p^0 + p^3)$
$p^\mu = \frac{n_+ \cdot p}{\underline{2}} n_-^\mu + \frac{n_- p}{2} n_+^\mu + p_\perp^\mu$		$p^\mu = \underline{n_+ p}\, n_-^\mu + n_- p\, n_+^\mu + p_\perp^\mu$
$p^2 = 2\frac{n_+ \cdot p}{2} n_-^\mu + \frac{n_- p}{2} n_{+\mu} - p_\perp^2 = \underline{n_+ p\, n_- p} - p_\perp^2$		$p^2 = \underline{2 n_+ p\, n_- p} - p_\perp^2$
$g^{\alpha\beta} = \frac{1}{2} n_-^\alpha n_+^\beta + \frac{1}{2} n_+^\alpha n_-^\beta + g_\perp^{\alpha\beta}$		$g^{\alpha\beta} = n_-^\alpha n_+^\beta + n_+^\alpha n_-^\beta + g_\perp^{\alpha\beta}$

→ 取 $p^\mu \sim n_+^\mu$ 此时 $n_- p$ 最大 $\Leftarrow$ $\boxed{\text{选择这个}}$ $\Uparrow$

$p^\mu \sim (n_- p,\ n_+ p,\ p_\perp) \sim (Q,\ Q\lambda^2,\ Q\lambda)$ $\qquad \lambda \sim 1/Q$

→ 取 $p^\mu \sim n_-^\mu$

Ⅱ DIS at tree-level

- in principle, we don't know how to compute the matrix element directly due to the hadron state
- one can replace the hadron state by suitable parton state.

① UV is not changed   ② IR will be changed

$$T^{\mu\nu}(p,q) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} \langle p|T[J^\mu(x)\, J^\nu(0)]|p\rangle \qquad \underline{p \to \text{quark}}$$

$$W^{\mu\nu}(p,q) = \text{Disc}\ \Pi^{\mu\nu}(p,q)$$

ⓐ tree-level: 完整理论.

$$T^{\mu\nu(0)} = \frac{1}{4\pi}\int d^4x\, \underline{e^{iq\cdot x}} \int \frac{d^4l}{(2\pi)^4}\, \underline{e^{-il\cdot(x-0)}}$$

$$\times\ \bar{u}(p)\, \underline{e^{ip\cdot x}}\, \gamma^\mu \frac{i\not{l}}{l^2+i\varepsilon} \gamma^\nu\, u(p)$$

$$\underline{(2\pi)^4\delta^4(l-p-q)} \xleftarrow{d^4x}$$

$$= \frac{1}{4\pi}\, \underline{\bar{u}(p)}\, \gamma^\mu \frac{i(\not{p}+\not{q})}{(p+q)^2+i\varepsilon} \gamma^\nu\, \underline{u(p)}$$

$$\underset{\text{unpolarized}}{=}\ \underline{\frac{1}{4\pi}\ \frac{1}{2}\, \text{tr}[\not{p}\,\gamma^\mu\ (\not{p}+\not{q})\,\gamma^\nu]}\ \ \underline{\frac{i}{(p+q)^2+i\varepsilon}}$$

$$W^{\mu\nu} = \text{Disc}\ \Pi^{\mu\nu}(p,q)$$

$$= \frac{1}{4\pi}\cdot i\,(-2\pi i)\,\delta((p+q)^2)\cdot \frac{1}{2}\, \underline{\text{tr}[\not{p}\,\gamma^\mu(\not{p}+\not{q})\gamma^\nu]}$$

需要强调的是 $\not{p}$ 只含 $\not{n}_+$. $\not{p}+\not{q}$ 只含 $\not{n}_-$

$$W^{\mu\nu} = \delta((p+q)^2) \times [p^\mu (p+q)^\nu - p\cdot q\, g^{\mu\nu} + p^\nu (p+q)^\mu].$$

$$= \delta((p+q)^2) \times [2p^\mu p^\nu + q^\mu p^\nu + p^\mu q^\nu - g^{\mu\nu} p\cdot q]$$

$$\delta(2x) = \underline{\tfrac{1}{2}}\,\delta(x)$$

$$(p+q)^2 = \underline{2p\cdot q + q^2} = \underline{2p\cdot q - Q^2} = Q^2\left[\frac{2p\cdot q}{Q^2} - 1\right] = \frac{Q^2}{x}(1-x)$$

$$x = \frac{Q^2}{2p\cdot q}\ \triangle \quad \text{Bjorken scaling} \qquad \delta\left(\frac{Q^2}{x}(1-x)\right)$$

$$W^{\mu\nu} = \underline{\frac{x}{Q^2}\,\delta(1-x)}\ \left[\underline{2p^\mu p^\nu + q^\mu p^\nu + p^\mu q^\nu - g^{\mu\nu}\, p\cdot q}\right]$$

参数化：$$\underline{\underline{W^{\mu\nu}}} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1 + \frac{\left(p^\mu - q^\mu \frac{p\cdot q}{q^2}\right)\left(p^\nu - q^\nu \frac{p\cdot q}{q^2}\right)}{p\cdot q}\, F_2$$

$$F_1 = \frac{x}{Q^2}\, p\cdot q\ \delta(1-x) = \frac{1}{2}\,\delta(1-x)$$

$$F_2 = \frac{x}{Q^2}\,\delta(1-x)\cdot 2\cdot p\cdot q = x\,\delta(1-x)$$ （这里看不出 $\frac{1}{x}$ 的来源）

ⓑ PDF 的微扰论展开：

$$f(x) = \int \frac{d(n_+\xi)}{2\pi}\, e^{-ix\, n_+\xi\, n_-p} \langle p|\, \bar{q}\, W_C^\dagger(\xi)\, \frac{\not n_-}{2}\, W_C\, q(0)\, |p\rangle$$

树图为：

$$f(x) = \int \frac{d(n_+\xi)}{2\pi}\, e^{-ix\, n_+\xi\, n_-p}\ \bar{u}(p)\, e^{i\, n_-p\, n_+\xi}\ \frac{\not n_-}{2}\ u(p)$$

$$= \delta(x\, n_-p - n_-p)\cdot \bar{u}(p)\,\frac{\not n_-}{2}\, u(p)$$

$$= \frac{1}{2}\,\delta(x n_-p - n_-p)\ \mathrm{tr}\left[\not p\, \frac{\not n_-}{2}\right]$$

$$= \delta(x-1)$$

ⓒ 因子化形式：

$$F_1 = \int \frac{d\xi}{\xi}\, f(\xi)\, C_1(\xi, x)$$ similar for F_2

$$\rightarrow\ \frac{1}{2}\,\delta(1-x) = \int \frac{d\xi}{\xi}\, \delta(\xi-1)\, C_1(\xi, x) \ \rightarrow\ C_1 = \frac{1}{2}\,\delta(\xi - x)$$

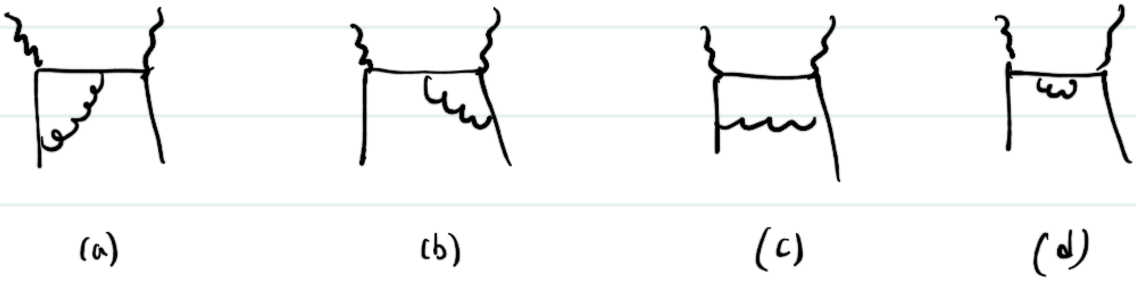
— 如果 $f(\xi)$ 不是部分子层次，而是非微扰：则

$$F_1 = \int \frac{d\xi}{\xi}\, f(\xi)\ \frac{1}{2}\,\delta(\xi - x) = \frac{1}{2} f(x)$$

$$F_2 = x\cdot f(x)$$

} <u>这就是部分子模型</u>.

III Factorization for DIS in momentum space with Method of region



ⓐ 以图a为例：

$$iM_a^{\mu\nu} = \frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x} \langle q(p)|\, \bar\psi \gamma^\mu \overbrace{\psi(x) \int d^4z_1\, \bar\psi}^{l_1}\, ig \overbrace{A\!\!\!/\, \overbrace{\psi(z_1)\, \bar\psi}^{l_2} \gamma^\nu \overbrace{\psi(0) \int d^4z_2\, \bar\psi}^{l_3}\, ig A\!\!\!/}^{k}\, \psi(z_2)\, |q(p)]\rangle$$

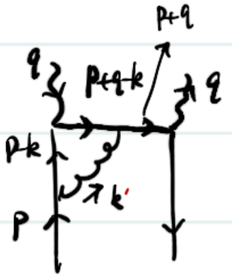
$$= \frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x} \int d^4z_1 \int d^4z_2 \int\frac{d^4l_1}{(2\pi)^4}\int\frac{d^4l_2}{(2\pi)^4}\int\frac{d^4l_3}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}$$

$$\times\ \bar u(p)\, e^{ip\cdot x}\gamma^\mu\, e^{-il_1(x-z_1)}\, \frac{i l\!\!\!/_1}{l_1^2+i\varepsilon}\, ig t^a \gamma_\alpha\, e^{-il_2(z_1-0)}\, \frac{i l\!\!\!/_2}{l_2^2}\, \gamma^\nu$$

$$\times\ e^{-il_3(0-z_2)}\, \frac{i l\!\!\!/_3}{l_3^2}\, ig t^a \gamma_\beta\, u(p)\, e^{-ip\cdot z_2} \times \frac{-i g^{\alpha\beta}}{k^2+i\varepsilon}$$

$$= \frac{1}{4\pi}\int\frac{d^4k}{(2\pi)^4}\, \underline{\bar u(p)}\, \triangle\, \gamma^\mu \frac{i(p\!\!\!/+q\!\!\!/)}{(p+q)^2+i\varepsilon}\, ig_s t^a \underline{\gamma_\alpha\, \frac{i(p\!\!\!/+q\!\!\!/-k\!\!\!/)}{(p+q-k)^2+i\varepsilon}}\, \gamma^\nu\, \triangle\, \frac{i(p\!\!\!/-k\!\!\!/)}{(p-k)^2+i\varepsilon}\, ig_s t^a \gamma_\beta\, \underline{u}(p)$$

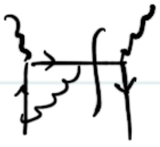
$$\times\ \frac{-i g^{\alpha\beta}}{k^2+i\varepsilon}$$



一般地：
$$\text{disc} = \frac{1}{p_1^2+i\varepsilon}\frac{1}{p_2^2+i\varepsilon} - \frac{1}{p_1^2-i\varepsilon}\frac{1}{p_2^2-i\varepsilon} = \frac{1}{p_1^2+i\varepsilon}\underline{\frac{1}{p_2^2+i\varepsilon} - \frac{1}{p_1^2+i\varepsilon}}\frac{1}{p_2^2-i\varepsilon} + \frac{1}{p_1^2+i\varepsilon}\frac{1}{p_2^2-i\varepsilon} - \frac{1}{p_1^2-i\varepsilon}\frac{1}{p_1^2-i\varepsilon}$$

$$= \underline{\frac{1}{p_1^2+i\varepsilon}(-2\pi i)\,\delta(p_2^2)} + (-2\pi i)\,\delta(p_1^2)\,\frac{1}{p_2^2-i\varepsilon}$$

现只考虑第一项：对应于



$$\rightarrow \frac{1}{-i\varepsilon}$$

$$W^{\mu\nu} = \frac{1}{4\pi}\int\frac{d^4k}{(2\pi)^4}\,\bar{u}(p)\,\gamma^\mu\,\underline{(\not p+\not q)}\,ig_s t^a\underbrace{\gamma_\alpha\,\frac{i(\not p+\not q-\not k)}{(p+q-k)^2+i\varepsilon}\,\gamma^\nu\,\frac{i(\not p-\not k)}{(p-k)^2+i\varepsilon}\,ig_s\gamma_\beta}\,t^a\,u(p)$$

$$\times\ \frac{-ig^{\alpha\beta}}{k^2+i\varepsilon}\cdot\underline{(2\pi)\,\delta((p+q)^2)}$$

$\uparrow$ 常数

$$\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\beta\gamma^\alpha = -2\gamma_\beta\gamma_\nu\gamma_\mu$$

$$= \frac{1}{4\pi}\int\frac{d^4k}{(2\pi)^4}\,\bar{u}(p)\,\gamma^\mu\,(\not p+\not q)\,ig_s t^a\cdot(-2)\,\frac{i(\not p-\not k)}{(p-k)^2+i\varepsilon}\,\gamma^\nu\,\frac{i(\not p+\not q-\not k)}{(p+q-k)^2}\,ig_s t^a\,u(p)$$

$$\times\ \frac{-i}{k^2}\,(2\pi)\,\delta((p+q)^2)$$

ⓑ 幂次估计 $\quad p^\mu\sim(Q,\lambda^2Q,\lambda Q)$

$\quad (p+q)^\mu\sim(Q\lambda^2,Q,\lambda Q)$

其实应该判断 F_1 与 F_2 修正的幂次. 这里直接判断 $W^{\mu\nu}$

hard: $\quad k^\mu\sim(Q,Q,Q)$

$$\begin{array}{cccccccccc} W^{\mu\nu}\sim & Q^4. & Q & Q & \frac{Q}{Q^2} & \frac{Q}{Q^2} & \frac{1}{Q^2} & \frac{1}{Q^2} & \sim Q^0\underline{\underline{\lambda}}^0 \\ & d^4k & \underline{u\bar{u}} & (\not p+\not q) & \frac{\not p-\not k}{(p-k)^2} & \frac{\not p+\not q-\not k}{(p+q-k)^2} & \frac{1}{k^2} & \delta((p+q)^2) & \end{array}$$

collinear: $\quad k^\mu\sim(Q,\lambda^2Q,\lambda Q)$

$$\begin{array}{cccccccccc} W^{\mu\nu}\sim & Q^4\lambda^4 & Q & Q & \frac{Q}{Q^2\lambda^2} & \frac{Q}{Q^2} & \frac{1}{Q^2\lambda^2} & \frac{1}{Q^2} & \sim Q^0\underline{\underline{\lambda}}^0 \\ & d^4k & u\bar{u} & \underset{\not n_-}{\not p+\not q} & \frac{\not p-\not k}{(p-k)^2} & \frac{\not p+\not q-\not k}{(p+q-k)^2} & \frac{1}{k^2} & \delta((p+q)^2) & \end{array}$$

soft: $\quad k^\mu\sim(Q\lambda,Q\lambda,Q\lambda)$

$$W^{\mu\nu}\sim\ Q^4\lambda^4\quad Q\quad Q\quad \frac{Q}{Q^2\lambda}\quad \frac{Q}{Q^2\lambda}\quad \frac{1}{Q^2\lambda^2}\quad \frac{1}{Q^2}\quad \sim Q^0\lambda^0$$

ⓒ collinear regim

$$W^{\mu\nu} = \frac{1}{4\pi}\int \frac{d^4k}{(2\pi)^4}\, \bar{u}(p)\,\gamma^\mu\, \overset{\underline{\underline{\not{n}_-}}}{(\not{p}+\not{q})}\; i g_s t^a \gamma_\alpha\, \frac{i(\overset{\underline{\not{n}_-}}{\not{p}+\not{q}}-\overset{\not{n}_+}{\not{k}})}{(p+q-k)^2}\,\gamma^\nu\, \frac{i(\overset{\not{n}_+}{\not{p}-\not{k}})}{(p-k)^2+i\varepsilon}\; i g_s\, \overset{\underline{\underline{\not{n}_+}}}{\gamma_\beta}\, t^a\, u(p)$$

$$\times\ \frac{-i g^{\alpha\beta}}{k^2}\cdot(2\pi)\,\delta((p+q)^2)$$

k 为 collinear. $k^\mu \sim (Q,\ \lambda^2 Q,\ Q\lambda)$

leading power: $\not{p}-\not{k}\sim\not{n}_+$ 不然之前的幂次估计就要改了.

$$\not{n}_+\,\underline{\gamma_\beta}\,u(p) \to \not{n}_+\,\underline{\not{n}_-\,n_{+\beta}}\,u(p)$$

$$\gamma^\mu(\not{p}+\not{q})\, i g_s t^a \gamma_\alpha\, \frac{i(\not{p}+\not{q}-\not{k})}{(p+q-k)^2}\,\gamma^\nu$$

$$= \gamma^\mu(\not{p}+\not{q})\, i g_s t^a (\not{n}_+ n_{-\alpha})\, \frac{i(\not{n}_-\, n_+\cdot(p+q))}{-2n_+\cdot(p+q)\, n_-k + i\varepsilon}\,\gamma^\nu$$

$$= \gamma^\mu(\not{p}+\not{q})\, i g_s t^a\, \not{n}_+ n_{-\alpha}\, \frac{i\not{n}_-}{-2n_-k+i\varepsilon}\,\gamma^\nu \to \qquad \underline{\frac{\not{n}_+\not{n}_-}{2}\to 1}$$

$$= \underline{\gamma^\mu(\not{p}+\not{q})\gamma^\nu}\quad \underline{i g_s t^a n_{-\alpha}\, \frac{i}{-n_-k+i\varepsilon}}$$

$$(I)_{ij}(I)_{kl} = \frac{1}{4}(I)_{il}(I)_{kj} + \frac{1}{4}(\gamma_\beta)_{il}(\gamma^\beta)_{kj} \leftarrow$$

Fierz 变换

$$\to W^{\mu\nu} = \frac{1}{4\pi}\int\frac{d^4k}{(2\pi)^4}\, \underline{\bar{u}(p)} \overset{\downarrow}{=} \gamma^\mu\, \underline{(\not{p}+\not{q})\gamma^\nu} \overset{\downarrow}{=} \underline{\underline{i g_s t^a n_{-\alpha}\, \frac{i}{-n_-k+i\varepsilon}}}\, \frac{i(\not{p}-\not{k})}{(p-k)^2+i\varepsilon}\, i g_s \gamma^\alpha t^a u(p)$$

$$\times\ \frac{-i}{k^2}\,(2\pi)\,\delta((p+q)^2)$$

$$= \frac{1}{4\pi}\times\frac{1}{4}\,\mathrm{tr}[\gamma_\beta\gamma^\mu(\not{p}+\overset{\not{n}_-}{\underline{\not{q}}})\gamma^\nu]\,(2\pi)\,\delta((p+q)^2) \quad ---$$

$$\times\int\frac{d^4k}{(2\pi)^4}\,\bar{u}(p)\,\gamma_\beta\, i g_s t^a n_{-\alpha}\, \underline{\frac{i}{-n_-k+i\varepsilon}}\, \frac{i(\not{p}-\not{k})}{(p-k)^2+i\varepsilon}\, i g_s \gamma^\alpha t^a u(p) \times \frac{-i}{k^2}$$

<u>领头项:</u> $\gamma_\beta = \not{n}_+ n_{-\beta} = \frac{n_{-\beta}}{n_-p}\, n_-p\,\not{n}_+ = \frac{n_{-\beta}}{n_-p}\,\not{p}$

$$W^{\mu\nu} = \frac{1}{4\pi}\times\frac{1}{2}\,\mathrm{tr}[\not{p}\gamma^\mu(\not{p}+\not{q})\gamma^\nu]\,(2\pi)\,\delta((p+q)^2) \quad ----\ 树图\ \underline{W^{\mu\nu(0)}}$$

$$\times\ \underbrace{\frac{1}{n_-p}\int\frac{d^4k}{(2\pi)^4}\,\bar{u}(p)\,\frac{\not{n}_-}{2}\, i g_s t^a n_{-\alpha}\, \frac{+i}{-n_-k+i\varepsilon}\, \frac{i(\not{p}-\not{k})}{(p-k)^2+i\varepsilon}\, i g_s \gamma^\alpha t^a u(p)\times\frac{-i}{k^2}}_{\uparrow\ \underline{M}}$$

$$f^{(1)} = \int \frac{d(n_+\xi)}{2\pi}\, e^{-ix(n_+\xi)n_-p} \langle p| \left[\bar q\, W_c^\dagger\, \frac{\not n_-}{2}\, W_c\, q(0)\right] |p\rangle$$

$$= \int \frac{d(n_+\xi)}{2\pi}\, e^{-ix(n_+\xi)n_-p}$$

$$\times \langle p|\, \bar q\, \frac{\not n_-}{2}\, \overset{W_c^{(1)}}{\underline{ig_s \int_0^\infty ds\, n_-A(sn_-)}} \cdot q(0) \overset{H^{(1)}}{\int d^4x\, \bar q\, ig_s \not A\, q(x)}\, |p\rangle$$

$$= \int \frac{d(n_+\xi)}{2\pi}\, e^{-ix\, n_+\xi\, n_-p}$$

$$\times\, \bar u(p)\, e^{i n_+\xi\, n_-p}\, \frac{\not n_-}{2}\, ig_s t^a \int_0^\infty ds\, n_{-\alpha}\, \cancel{A^{a\alpha}(sn_-)} \quad \int d^4x \times \int \frac{d^4l}{(2\pi)^4}\, e^{-il\cdot(0-x)}$$

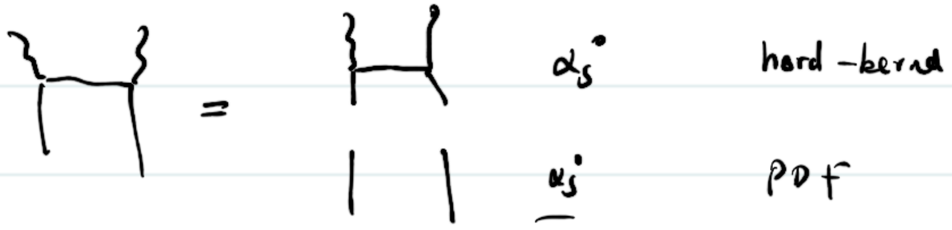
$$\times\, \frac{i\not l}{l^2+i\varepsilon}\, ig_s t^b \gamma_\beta\, \cancel{A^{b\beta}(x)}\, u(p)\, e^{-ip\cdot x}$$

$$\times \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(sn_- - x)}\, \frac{-ig^{\alpha\beta}}{k^2+i\varepsilon}\, \delta^{ab}$$

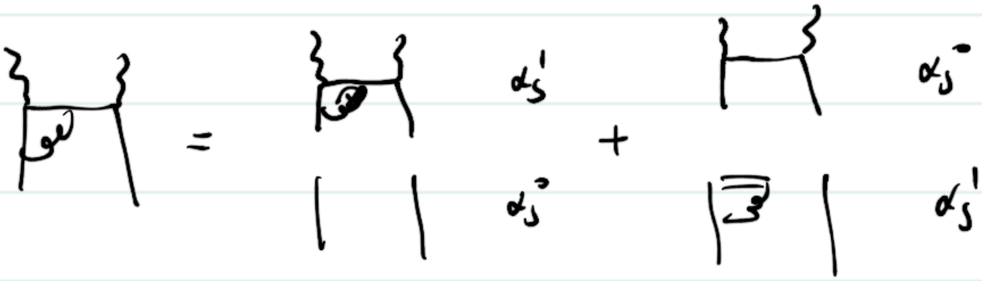
$$W^{\mu\nu(1)}\Big|_{\text{collinear}} = \underline{\hat W^{(0)} \otimes f^{(1)}}$$

$$W^{\mu\nu(1)}\Big|_{\text{hard}} = \hat W^{(1)} \otimes f^{(0)} \rightarrow F_1^{(1)}\Big|_{\text{hard}} = C_1^{(1)} \otimes f^{(0)}$$

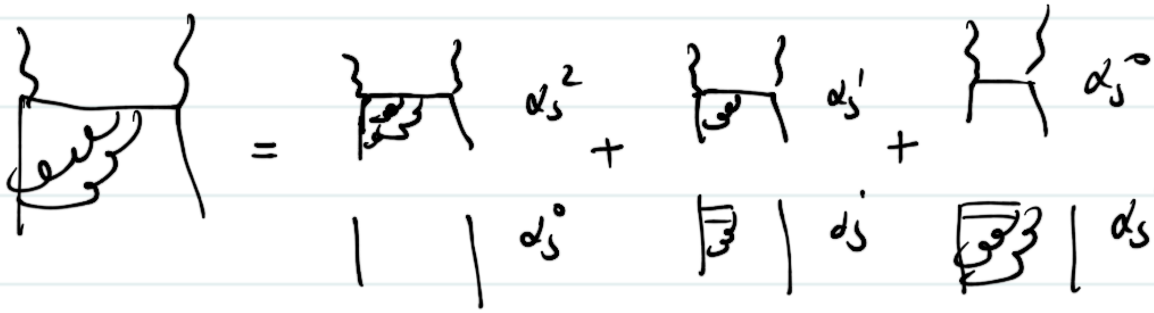
α_s^0



α_s^1



α_s^2



$\boxed{\not p+\not q}$

$\boxed{\text{做 Fierz 变换么?}}$

ⓒ

(4) eikonal 近似：

$$\overset{\not n_-}{(\not p+\not q)}\; i g_s t^a \overset{\not n_+}{\gamma_a}\; \frac{i(\not p+\overset{\not n_-}{\not q}-\not k)}{(p+q-k)^2}$$

$$= \frac{n_+ q}{2}\not n_- \times i g_s t^a \left(\frac{\not n_+}{2} n_{-\alpha} + \frac{\not n_-}{2} n_{+\alpha}\right) \frac{i\frac{n_+\cdot(p+q)}{2}\not n_- + \cdots}{-n_+(p+q)(n_-k)+i\varepsilon}$$

$$= \frac{n_+ q}{2}\not n_-\; i g_s t^a \left(\frac{\not n_+}{2} n_{-\alpha}\right) \frac{\not n_-}{2}\, \frac{i}{-n_-k+i\varepsilon} \quad - \cdots \qquad \frac{\not n_-\not n_+}{4}\to 1$$

$$\underline{\text{Disc}}\,(iM) = \frac{1}{4\pi}\cdot(2\pi)\,\delta((p+q)^2)\int\frac{d^4k}{(2\pi)^4}\,\bar u(p)_0\,\gamma^\mu\,\overset{(\not p+\not q)}{n_+q}\; i g_s t^a\, n_{-\alpha}\,\underline{\frac{\not n_-}{2}}\,\frac{i}{\underline{-n_-k}+i\varepsilon}\,\gamma^{\mu'}{}_0\,\frac{i(\overset{\boxed{\not n_+}}{\not p-\not k})}{(p-k)^2}$$

$$\times\, i g_s t^a\, \underline{\gamma_\beta}\, \underline{\underline{u}} \times \frac{-i g^{\alpha\beta}}{k^2+i\varepsilon}$$

(5) 计算 PDF 单圈.

(6) 得到结论

$$(iM)_0 \quad = (iH)_0 \times f_0$$

$$(iM)_1 \quad = (iH)_0 \times f_1 + (iH)_1 \times f_0$$

Method of region for quasi-PDF:

I Set notation:

$$e^{-ix\xi p}$$

ⓐ. PDF:

$$f_q(x) = \int \frac{d(n_+\xi)}{2\pi} \underline{e^{-ix(n_+\xi)(n_-p)}} \langle p(p)|[\bar{q}\, W_c^{\dagger}(\xi) \frac{\not{n}}{2} W_c(0) q(0)]|p(p)\rangle$$

$$W_c(\xi) = P\exp[ig_s \int_0^{\infty} ds\ n_- A_c(\xi + sn_-)]$$

ⓑ quasi-PDF:

$$\tilde{f}_q(x) = \int \frac{d(n_z\cdot\xi)}{2\pi} e^{-ix(n_z\xi)(\tilde{n}_z p)} \langle p(p)|\bar{q}\, W(\xi) \frac{\not{n}_z}{2} W q(0)]|p(p)\rangle$$

$$W(\xi) = P\exp[ig_s \int_0^{\infty} ds\ n_z\cdot A(\xi + sn_z)]$$

$$n_z^{\mu} = (0,0,0,-1) \qquad \tilde{n}_z^{\mu} = (0,0,0,1)$$

$$\tilde{f}_q(x) = \int \frac{dz}{2\pi} e^{ixzp^z} \langle p(p)|\ \bar{q}\, W(z) \frac{\gamma^z}{2} W q(0)]\, |p(p)\rangle$$

$$W(z) = P\exp[-ig_s \int_0^{\infty} ds\ A^z(z + sn_z)]$$

$$\xi\cdot p = \xi^0 p^0 - \xi^i p^i = \xi^0 p^0 - \xi^z p^z \rightarrow -z\cdot p^z$$

Ⅱ tree-level :

$|P(p)\rangle \to |q(p)\rangle$

ⓐ $$f_q(x) = \int \frac{d(n+\xi)}{2\pi} e^{-ix(n+\xi)(n-p)} \overbrace{\bar{u}(p) e^{i(n+\xi)(n-p)} \frac{\not{n}_-}{2} u(p)}$$

$$= \int \frac{d(n+\xi)}{2\pi} e^{-ix(n+\xi)(n-p)} e^{i(n+\xi)(n-p)} \frac{1}{2} \mathrm{tr}\left[\not{p} \frac{\not{n}_-}{2}\right]$$

$$= \delta(x n\text{-}p - n\text{-}p) \cdot \frac{1}{4} \cdot x 4\, n\text{-}p \;=\; \delta(x-1)$$

$$\sum_s u(p,s)\bar{u}(p,s) = \not{p}$$

$$\to \sum_{s_1 s_2, p_0} u(p,s_1)\bar{u}(p,s_2) = \delta_s \not{p}$$

ⓑ $$\tilde{f}_q(x) = \int \frac{dz}{2\pi} e^{ixzp^z} \bar{u}(p) e^{-izp^z} \frac{\gamma^z}{2} u(p)$$

$$\overset{unp}{=} \int \frac{dz}{2\pi} e^{ixzp^z - izp^z} \frac{1}{2} \mathrm{tr}\left(\not{p} \frac{\gamma^z}{2}\right)$$

$$= \delta(xp^z - p^z)\, p^z = \delta(x-1)$$

ⓒ $$\tilde{f}_q(x) = \int_0^1 dy\; C(x,y) \cdot f(y)$$

代入:

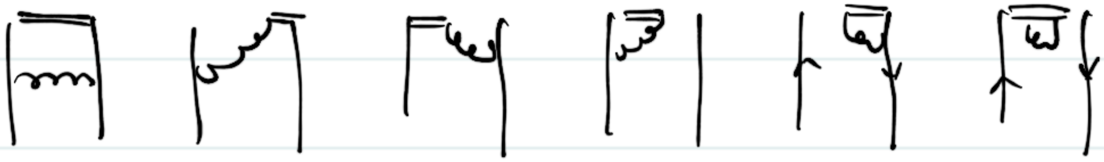
$$\delta(x-1) = \int_0^1 dy\; C(x,y)\, \delta(y-1) \;\to\; C^{(0)}(x,y) = \delta(x-y)$$

$$\Bigg\} \otimes$$

$$\tilde{f}_q(x) = \int \frac{dy}{y} C\left(\frac{x}{y}\right) f(y) \;\to\; C^{(0)}\left(\frac{x}{y}\right) = \delta\left(\frac{x}{y} - 1\right)$$

III. method of regim at $\underline{\alpha_s^1}$

perturbation theory:



(a) (b) (c) (d) (e) (f)

ⓐ 推导 (a) 图:

$$\tilde{f}^{(a)}(x) = \int \frac{dz}{2\pi} e^{ixzp^z} \langle q(p) | \left(\bar{q}(z) \frac{\gamma^z}{2} q(0) | q(p) \right)$$

$$= \int \frac{dz}{2\pi} e^{ixzp^z} \langle q(p) | \underline{ig_s \int d^4\eta_1\, \bar{q}(\eta_1) t^a A_\mu^a \gamma^\mu q(\eta_1)}\ \underline{\bar{q}(z) \frac{\gamma^z}{2} q(0)}$$

$$ig_s \int d^4\eta_2\, \bar{q}(\eta_2) t^b A_\nu^b \gamma^\nu q(\eta_2) \cdot \quad | q(p) \rangle)$$

$$= \int \frac{dz}{2\pi} e^{ixzp^z} \int d^4\eta_1 d^4\eta_2 \int \frac{d^4 l_1}{(2\pi)^4} \int \frac{d^4 l_2}{(2\pi)^4} \int \frac{d^4 l_3}{(2\pi)^4}$$

$$\times \bar{u}(p) \underline{e^{ip\cdot\eta_1}}\, ig_s t^a \gamma_\mu\ e^{-il_1(\eta_1 - z)} \frac{i\not{l}_1}{l_1^2} \frac{\gamma^z}{2}$$

$$\times e^{-il_2(0-\eta_2)} \frac{i\not{l}_2}{l_2^2} ig_s t^b \gamma_\nu \underline{e^{-ip\cdot\eta_2} u(p_2)}$$

$$\times \frac{-ig^{\mu\nu}}{l_3^2} \cdot \delta^{ab} e^{-il_3\cdot(\eta_1 - \eta_2)}$$

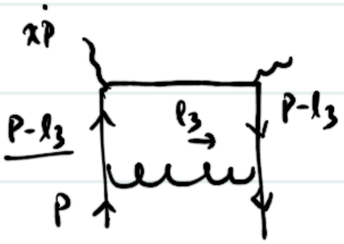
$$= \int \frac{dz}{2\pi} e^{ixzp^z} \int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4} \frac{d^4 l_3}{(2\pi)^4} \cdot (2\pi)^4 \delta^4(p - l_1 - l_3)\, (2\pi)^4 \delta^4(l_2 - p + l_3)$$

$$\times e^{-il_1^z z}$$

$$\times \bar{u}(p)\, ig_s t^a \gamma_\mu \frac{i\not{l}_1}{l_1^2} \frac{\gamma^z}{2} \frac{i\not{l}_2}{l_2^2} ig_s t^b \gamma_\nu u(p) \times \frac{-ig^{\mu\nu}\delta^{ab}}{l_3^2}$$

$$= \int \underline{\frac{dz}{2\pi} e^{ixzp^z}} \int \frac{d^4 l_3}{(2\pi)^4} \underline{e^{-i(p^z - l_3^z)z}} \hookrightarrow \delta(xp^z - p^z + l_3^z)$$

$$\times \bar{u}(p) \cdot ig_s t^a \gamma_\mu \frac{i(\not{p} - \not{l}_3)}{(p - l_3)^2} \frac{\gamma^z}{2} \frac{i(\not{p} - \not{l}_3)}{(p - l_3)^2} ig_s t^a \gamma_\nu u(p) \frac{-ig^{\mu\nu}}{l_3^2} \delta^{ab}$$



ⓑ. 幂次估计:

hard: $l_3 \sim (Q, Q, Q)$

collinear: $l_3 \sim (Q, \lambda^2 Q\ \lambda Q)$

soft $l_3 \sim (\lambda Q.\ \lambda Q\ \ \lambda Q)$

$$\tilde{f} = \underline{\int \frac{d^4 l_3}{(2\pi)^4} \delta(xP^2 - P^2 + l_3^2)}\ \ \underline{\bar{u}(P)}\ ig_s \underbrace{\gamma_\mu}_{(\not{n}_+ n_{-\mu} + \not{n}_- n_{+\mu} + \gamma_{\perp\mu})} t^a \frac{i(\not{P} - \not{l}_3)}{\underline{(P-l_3)^2}} \frac{\gamma^+}{2} \frac{i(\not{P} - \not{l}_3)}{(P-l_3)^2} ig_s \gamma^\mu t^a u(P) \times \frac{-i}{l_3^2}.$$

hard: $Q^4 \cdot \frac{1}{Q}\ Q\ \frac{Q}{Q^2}\ \frac{Q}{Q^2} \cdot \frac{1}{Q^2} \sim Q^0 \lambda^0$

collinear: $Q^4 \lambda^4 \cdot \frac{1}{Q} \cdot Q.\ \frac{\lambda Q}{Q^2 \lambda^2}\ \frac{\lambda Q}{\lambda^2 Q^2} \cdot \frac{1}{Q^2 \lambda^2} \sim Q^0 \lambda^0$

$$\underline{\overset{Q}{\int} dl_3^2 \cdot \overset{\frac{1}{Q}}{\delta}(xP^2 - P^2 + l_3^2) = 1}$$

soft: $Q^3 \lambda^3 \times Q \times \frac{\lambda Q}{\lambda Q^2} \times \frac{\lambda Q}{\lambda Q^2} \times \frac{1}{Q^2 \lambda^2} \sim Q^0 \lambda^1$

ⓒ: collinear:

$$\tilde{f} = \int \frac{d^4 l_3}{(2\pi)^4} \, \delta(xP^z - P^z + l_3^z) \cdot \bar{u}(p)\, i g_s t^a \gamma_\mu \frac{i(\not{p}-\not{l}_3)}{(P-l_3)^2} \cdot \frac{\gamma^z}{2} \frac{i(\not{p}-\not{l}_3)}{(P-l_3)^2} i g_s t^a \gamma^\mu u(p)$$

$$\times \frac{-i}{l_3^2}$$

$$\delta(xP^z - P^z + l_3^z) = \delta\left[\frac{xP^0 - P^0 + l_3^0 + (xP^z - P^z + l_3^z)}{2} + \frac{-(xP^0 - P^0 + l_3^0) + (xP^z - P^z + l_3^z)}{2} \right]$$

$$= \delta\left(\frac{1}{\sqrt{2}}\, n_- \cdot (xP - P + l_3) - \frac{1}{\sqrt{2}}\, \cancel{n_+ \cdot (xP - P + l_3)}\right)$$

$$= \sqrt{2}\, \delta(n_- \cdot (xP - P + l_3))$$

$$\frac{\gamma^z}{2} = \frac{1}{2} \cdot \left(\frac{1}{\sqrt{2}}(\not{n}_- - \not{n}_+)\right) = \frac{1}{\sqrt{2}}\left(\frac{\not{n}_-}{2} - \cancel{\frac{\not{n}_+}{2}}\right)$$

$$\tilde{f}\Big|_c = \int \frac{d^4 l_3}{(2\pi)^4} \sqrt{2}\, \delta(n_- \cdot (xP - P + l_3))\, \bar{u}(p)\, i g_s t^a \gamma_\mu \frac{i(\not{p}-\not{l}_3)}{(P-l_3)^2} \frac{1}{\sqrt{2}} \frac{\not{n}_-}{2} \frac{i(\not{p}-\not{l}_3)}{(P-l_3)^2} i g_s t^a \gamma^\mu u$$

$$\times \frac{-i}{l_3^2}$$

$$= \int \frac{d^4 l_3}{(2\pi)^4} \delta(n_- (xP - P + l_3))\, \bar{u}(p)\, i g_s t^a \gamma_\mu \frac{i(\not{p}-\not{l}_3)}{(P-l_3)^2} \frac{\not{n}_-}{2} \frac{i(\not{p}_3-\not{l}_3)}{(P-l_3)^2} i g_s t^a \gamma^\mu u(p) \times \frac{-i}{l_3^2}$$

$$= f(x)$$

PDF:

$$f = \int \frac{d(n+\xi)}{(2\pi)} e^{-ix(n+\xi)(n-p)} \langle q(p) | \bar{q}(\xi) \frac{\gamma_-}{2} q(0) | q(p) \rangle$$