# Theory and Phenomenology of Scattering Amplitudes

multi-loop multi-leg case …

2025 fifth QFT and application workshop
2025.11.1

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# Outline

Introduction

Analytic Feynman integral frontier

Case 1: 2loop 6point Feynman integrals

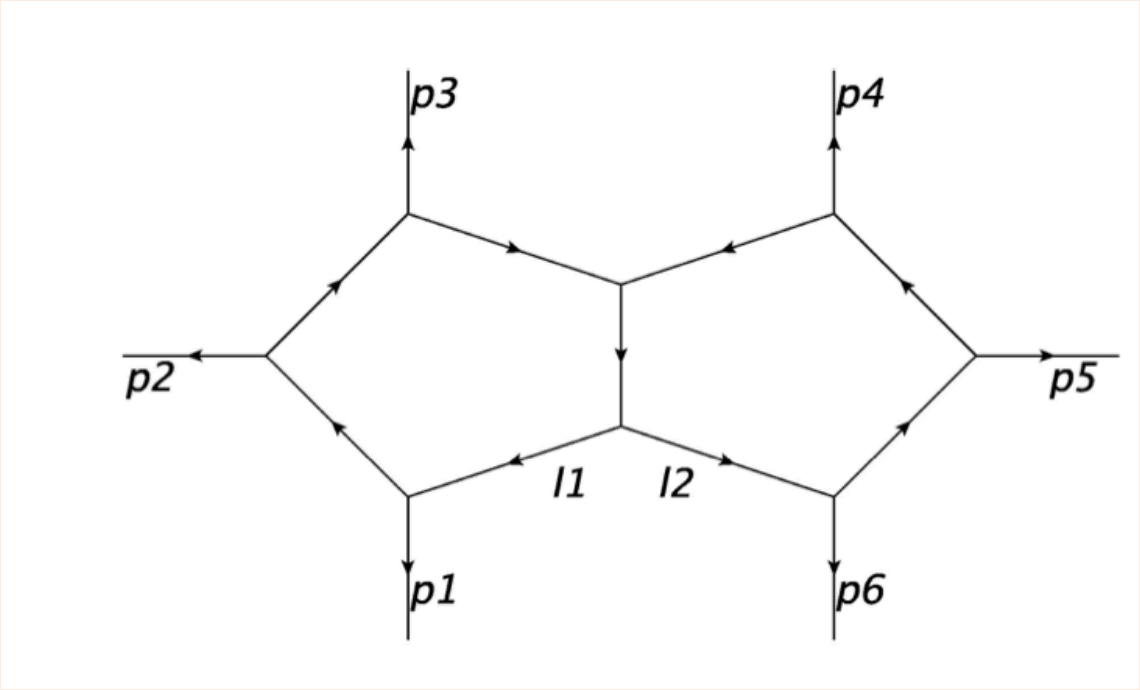
Case 2: 3loop 5point Feynman integrals

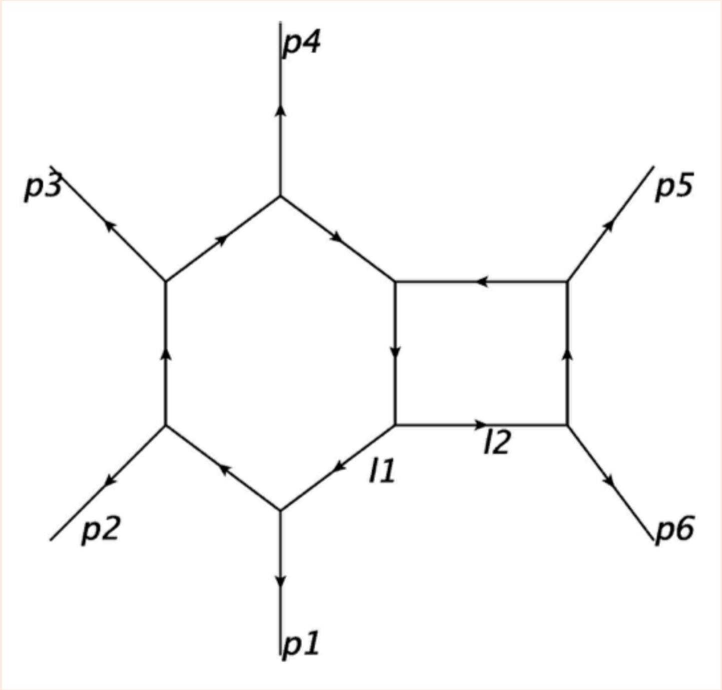
Phenomenology amplitudes frontier

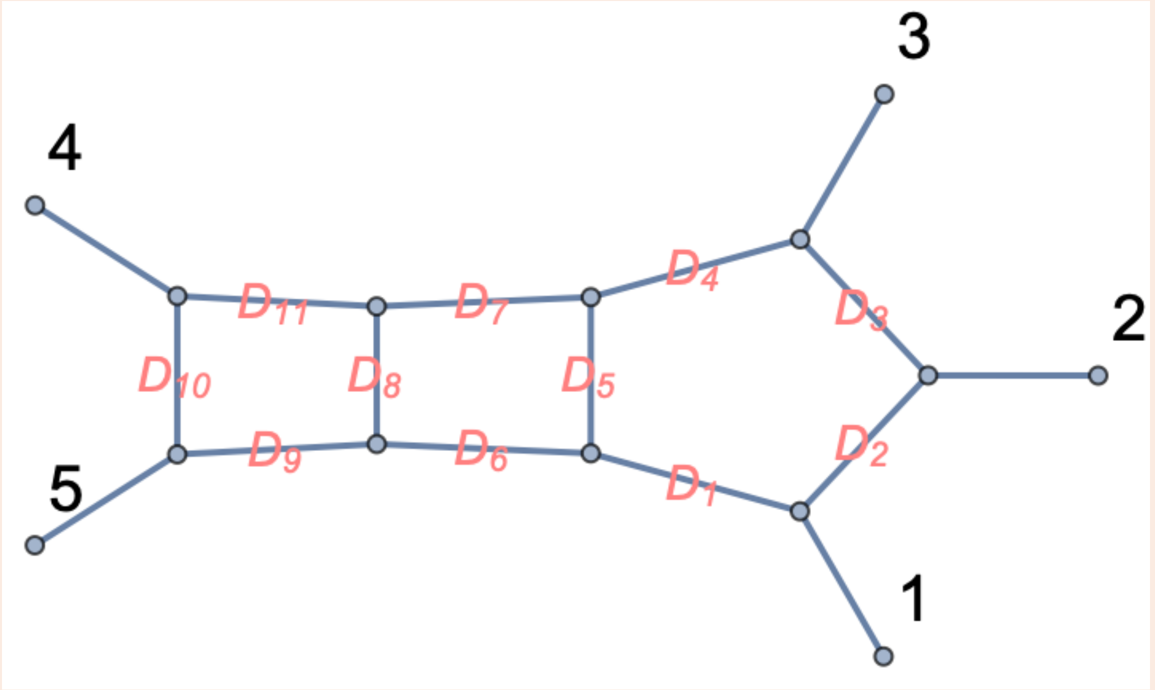
Theory exploration: QCD amplitude bootstrap

# Based on

## Feynman integral Integral evaluation




analytic computation of all 2loop 6point massless planar integrals is done

analytic computation of all 3loop 5point Feynman integral family

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021 (Editors' Suggestion)

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027

# Based on package development

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

"Performing integration-by-parts reductions using **NeatIBP 1.1** + Kira"
Wu, Boehm, Ma, Usovitsch, Xu, YZ, Comput.Phys.Commun. 316 (2025) 109798

NeatIBP **collaboration**

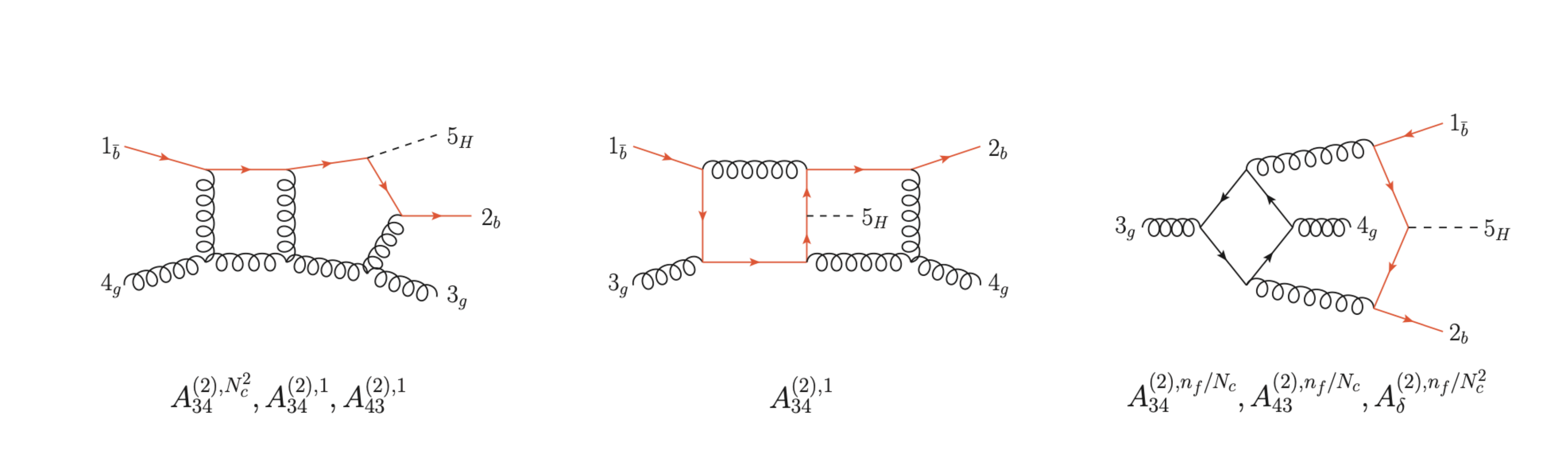
# Based on perturbative QCD computations

"Two-loop amplitudes for $O(\alpha_s^2)$ corrections to Wγγ production at the LHC"

Badger, Hartanto, Wu, YZ, Zoia, JHEP12(2024) 221

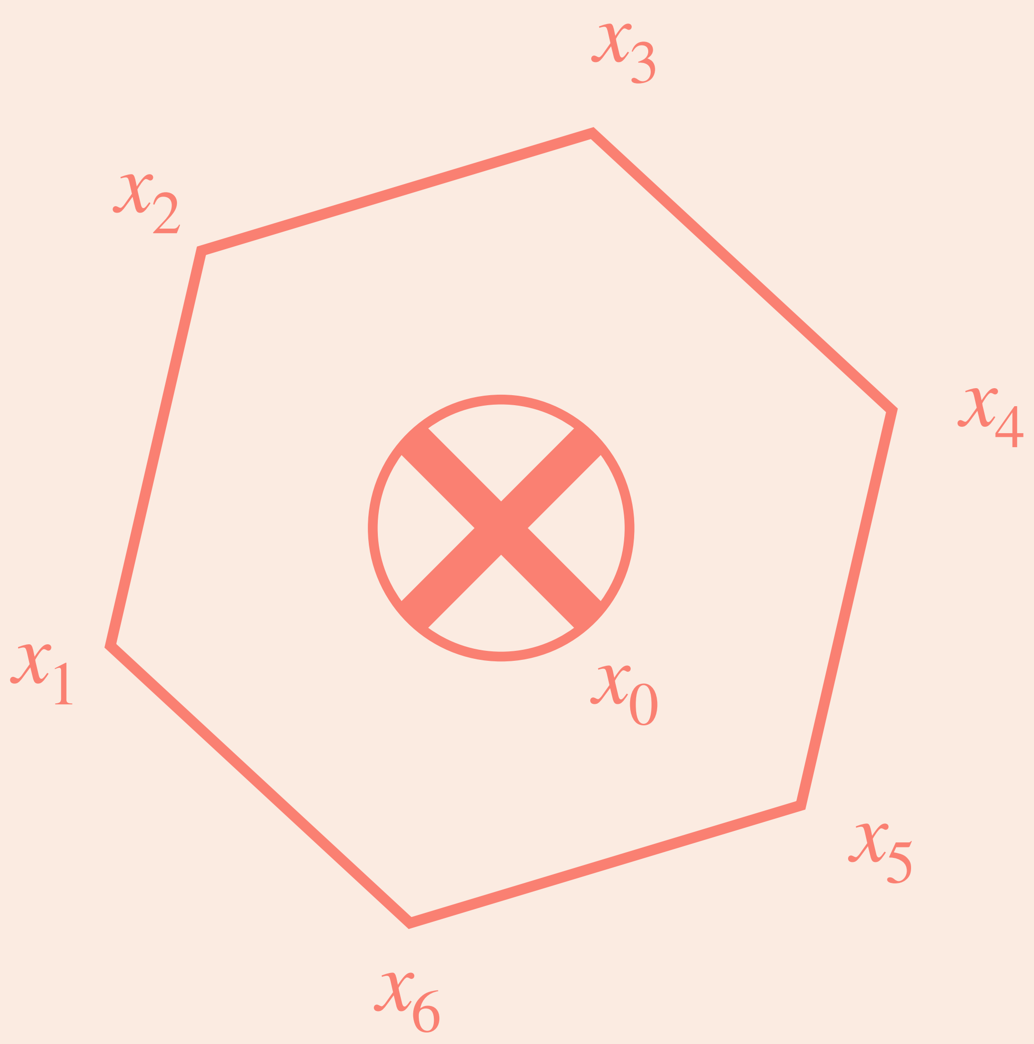
"Full-colour double-virtual amplitudes for associated production of a Higgs boson with a bottom-quark pair at the LHC"

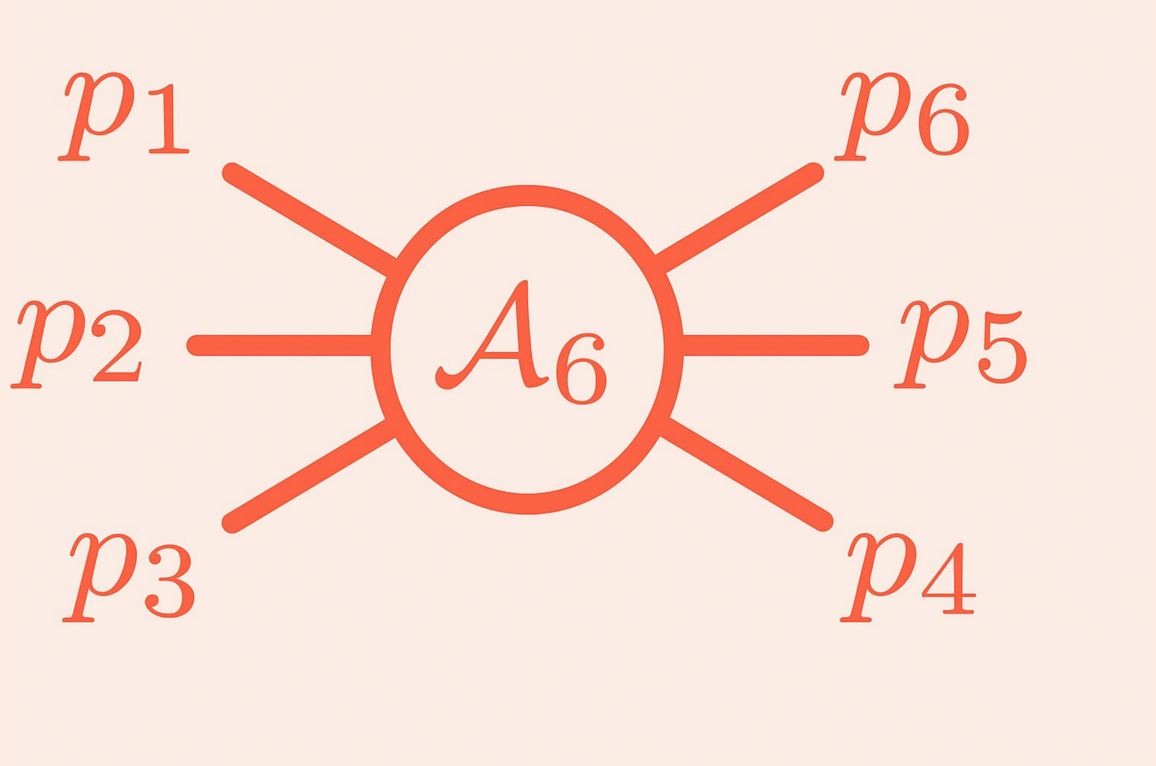
Badger, Hartanto, Poncelet, Wu, YZ, Zoia, JHEP03(2025) 066



# Based on new bootstrap ideas

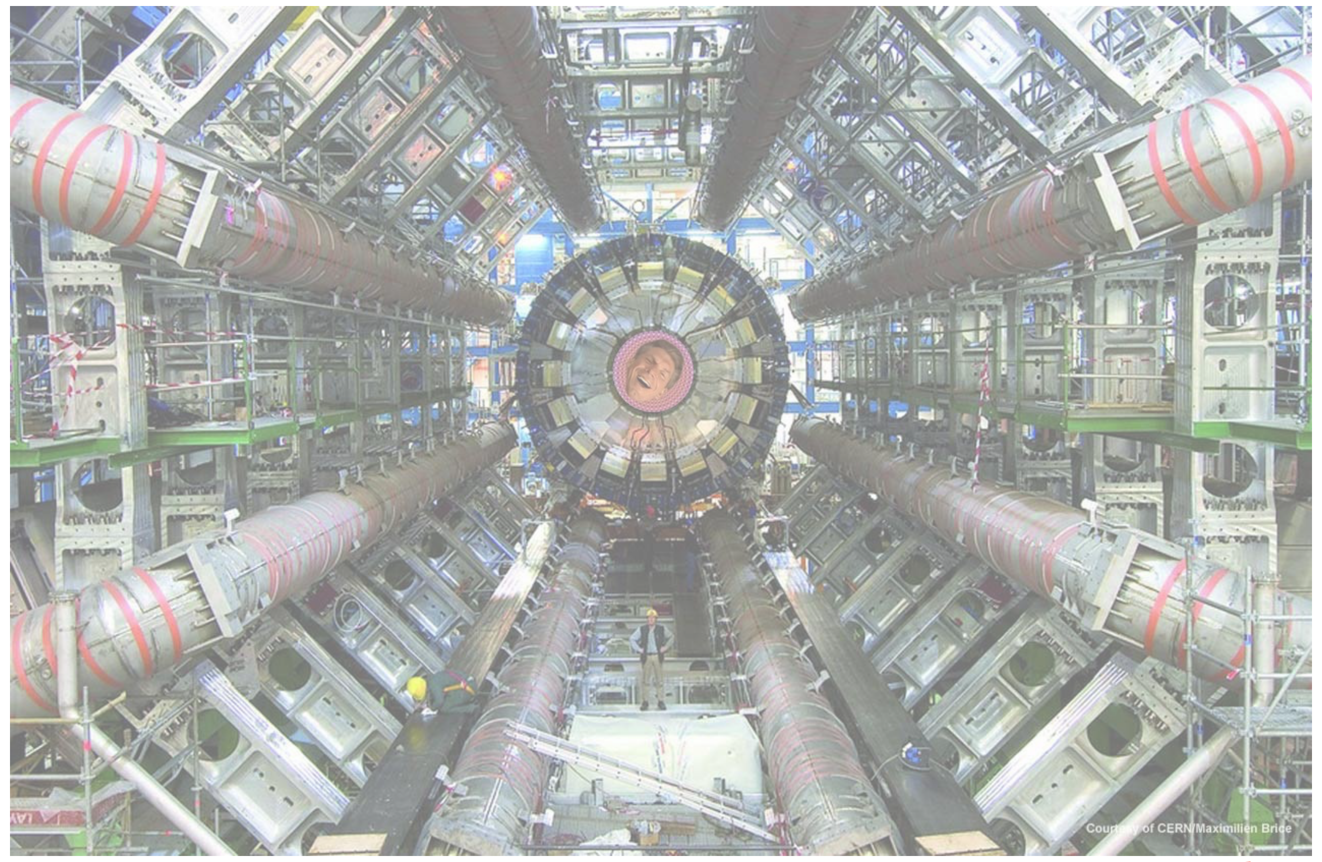
“Hexagonal Wilson loop with Lagrangian insertion at two loops in N=4 super Yang-Mills theory”,
Carrôlo, Chicherin, Henn, Yang, YZ, JHEP 07 (2025) 214




“Bootstrapping Six-Gluon QCD Amplitudes”
Carrôlo, Chicherin, Henn, Yang, YZ, arXiv:2510.20565

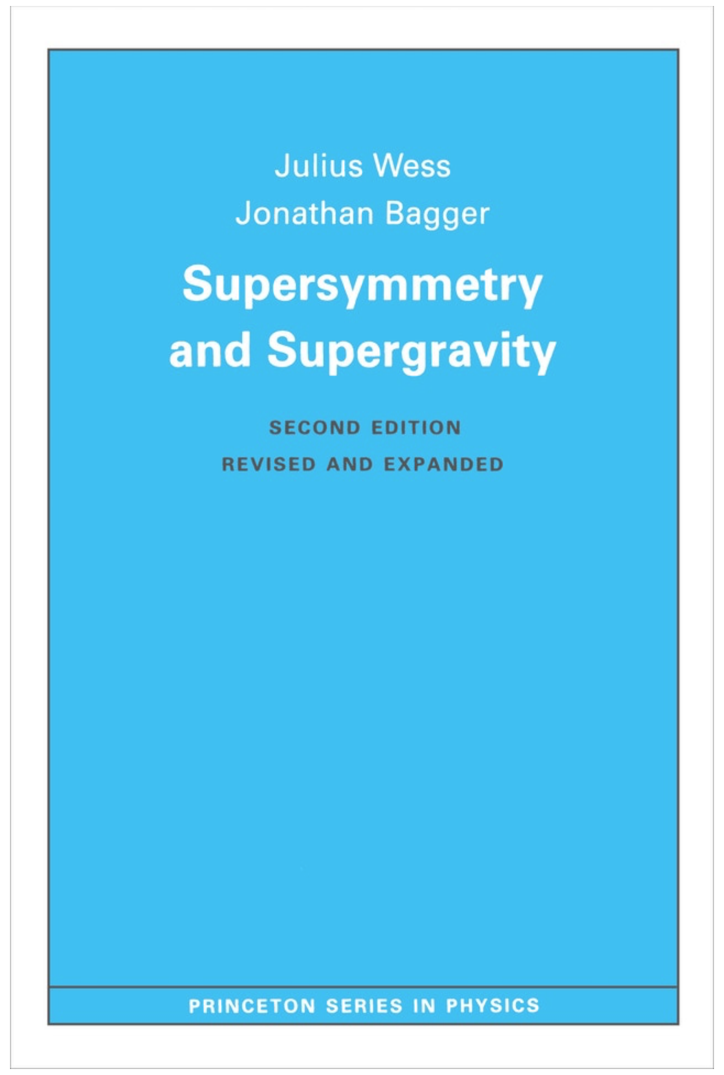
# Introduction



Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

Feynman integrals, loop amplitudes




Formal theory

N=8 supergravity UV finiteness

Gravitational wave template computations

for instance,
Driesse, Jakobsen , Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch
Nature 641 (2025) 8063, 603-607

# Why *analytic* multi-loop Feynman integrals/amplitudes?

- Analytic results can be very fast for numeric results
- Analytic methods can be sources for developing new numeric methods
- Theoretical aspects of quantum field theory

  for examples: 2loop N=4 SYM theory spacelike splitting amplitude

  Henn, Ma, Xu, Yan, *YZ*, Zhu, Phys.Rev.D 112 (2025) 7, 076003
- Quantum field theory computation of gravitational wave

  analytic continuation/ Fourier transform is sometimes needed
- Function space bootstrap for physical observable

  Carrôlo, Chicherin, Henn, Yang, YZ, JHEP 07 (2025) 214

# Analytic Feynman integral computation and the methodology

# Current status of analytic Feynman integral computation

| | 4-point | 5-point | 6-point | 7-point | 8-point |
|---|---|---|---|---|---|
| One-loop | known | known | known | known | known |
| Two-loop | most known | some results | ? | ? | ? |
| Three-loop | some results | ? | ? | ? | ? |
| Four-loop | some results | ? | ? | ? | ? |

with dimensional regulation

Frontier

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, PRL. 135, 031601

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027

Henn, Peraro, Xu, YZ, JHEP 03 (2022) 056

Abreu, Monni, Page, Usovitsch JHEP 06 (2025) 112

Liu, Matijasic, Miczajka, Xu, Xu, YZ,

Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion

# Canonical Differential Equation, new insights

$\epsilon$-factorized differential equation algorithm — $\epsilon$-collaboration 2506.09124

Better Integration-by-parts (IBP) reduction

**Blade,** Guan, Liu, Ma, Wu 2024
Comput.Phys.Commun. 310 (2025) 109538

**NeatIBP,** Wu, Boehm, Ma, Xu, *YZ* 2023
Comput.Phys.Commun. 295 (2024) 108999

Alphabet searching

**Effortless,** Matijasic, Miczajka to appear
https://github.com/antonela-matijasic/Effortless

**BaikovLetter**, Jiang, Liu, Xu, Yang,
Phys.Lett.B 864 (2025) 139443

**SOFIA** Correia, Giroux, Mizera 2503.16601

Solving differential equation

Novel representation of one-fold integration

Liu, Matijasic, Miczajka, Xu, Xu, YZ,
*Phys.Rev.D* 112 (2025) 1, 016021

# 2loop 6point Feynman integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett. 135 (2025) 3, 031601*

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP 08(2024) 027*

Abreu, Monni, Page, Usovitsch *JHEP 06 (2025) 112*

# 2loop Feynman integral: Scale frontier

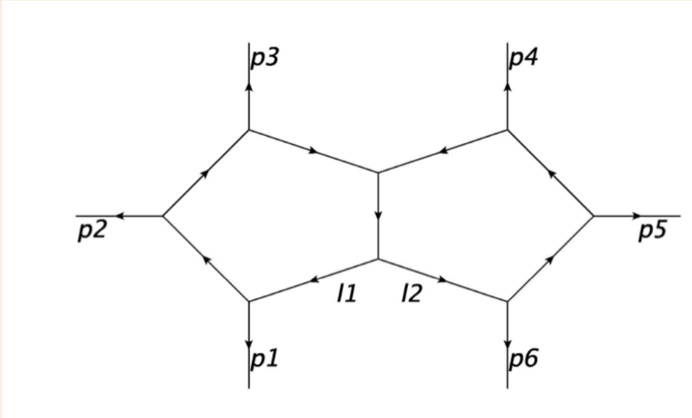
| | | |
|---|---|---|
| 2loop 5point massless | *Gehrmann, Henn, Lo Presti 2015*<br>*Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019* | 5 scales |
| 2loop 5point one-mass | *Papadopoulos, Tommasini, Wever 2019*<br>*Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020*<br>*Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2023*<br>*Jiang,Liu,Xu,Yang 2024*<br>*Badger, Becchetti,Giraudo, Zoia 2024* | 6 scales |
| 2loop 5point two-mass | *Cordero, Figueiredo, Kraus, Page and Reina 2023*<br>for leading-Color pp→ttH amplitudes with a light-quark loop | 7 scales |
| 2loop 6point massless | *Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2025*<br>*for NNLO 4 jets production, 2 jets+ 2photons* | 8 scales! |

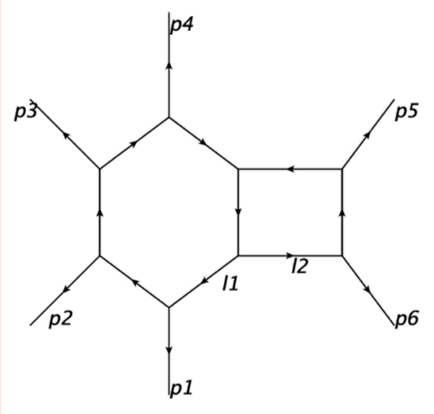
$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{345}$

# All planar 2loop 6point integrals

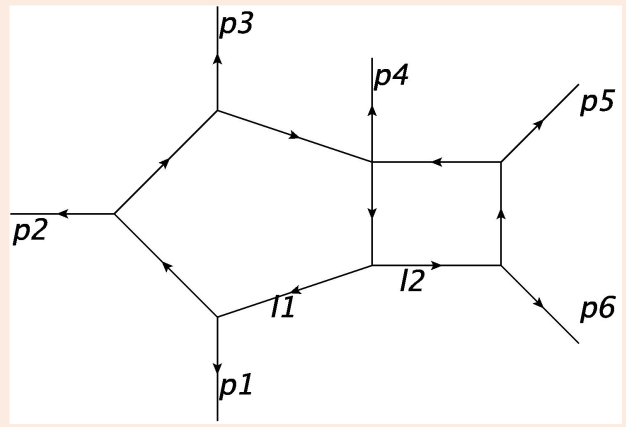
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett. 135 (2025) 3, 031601*
Abreu, Monni, Page, Usovitsch, JHEP 06 (2025) 112

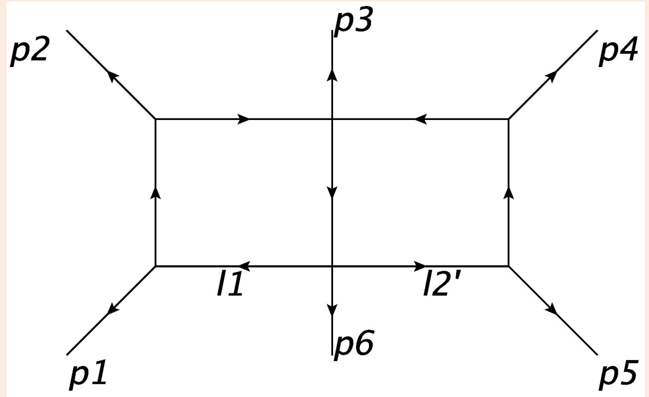
267
UT integrals

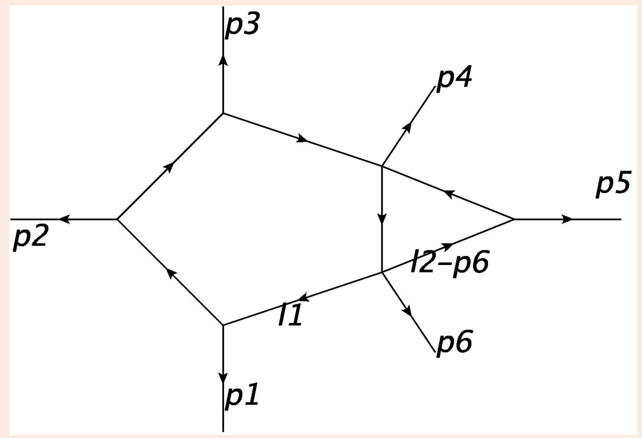


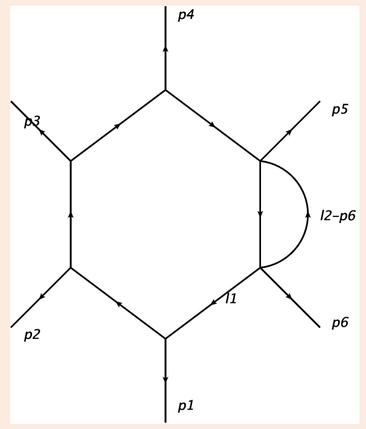




202
UT integrals



# Momentum Twistor

external momenta $d=4$

$$p_i = x_{i+1} - x_i$$

dual coordinates

$$\epsilon_{\dot\beta\dot\alpha} x^{\dot\alpha\gamma}\lambda_{A,\gamma} = \mu_{A,\dot\beta}$$
$$\epsilon_{\dot\beta\dot\alpha} x^{\dot\alpha\gamma}\lambda_{B,\gamma} = \mu_{B,\dot\beta}$$

$(Z_A, Z_B) \to x$ $\quad$ $(Z_i, Z_{i+1}) \to x_{i+1}$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot\beta} \end{pmatrix}, \quad i = 1, \ldots 6$$

momentum twistor

$\langle Z_A Z_B Z_C Z_D \rangle$ is dual conformally invariant

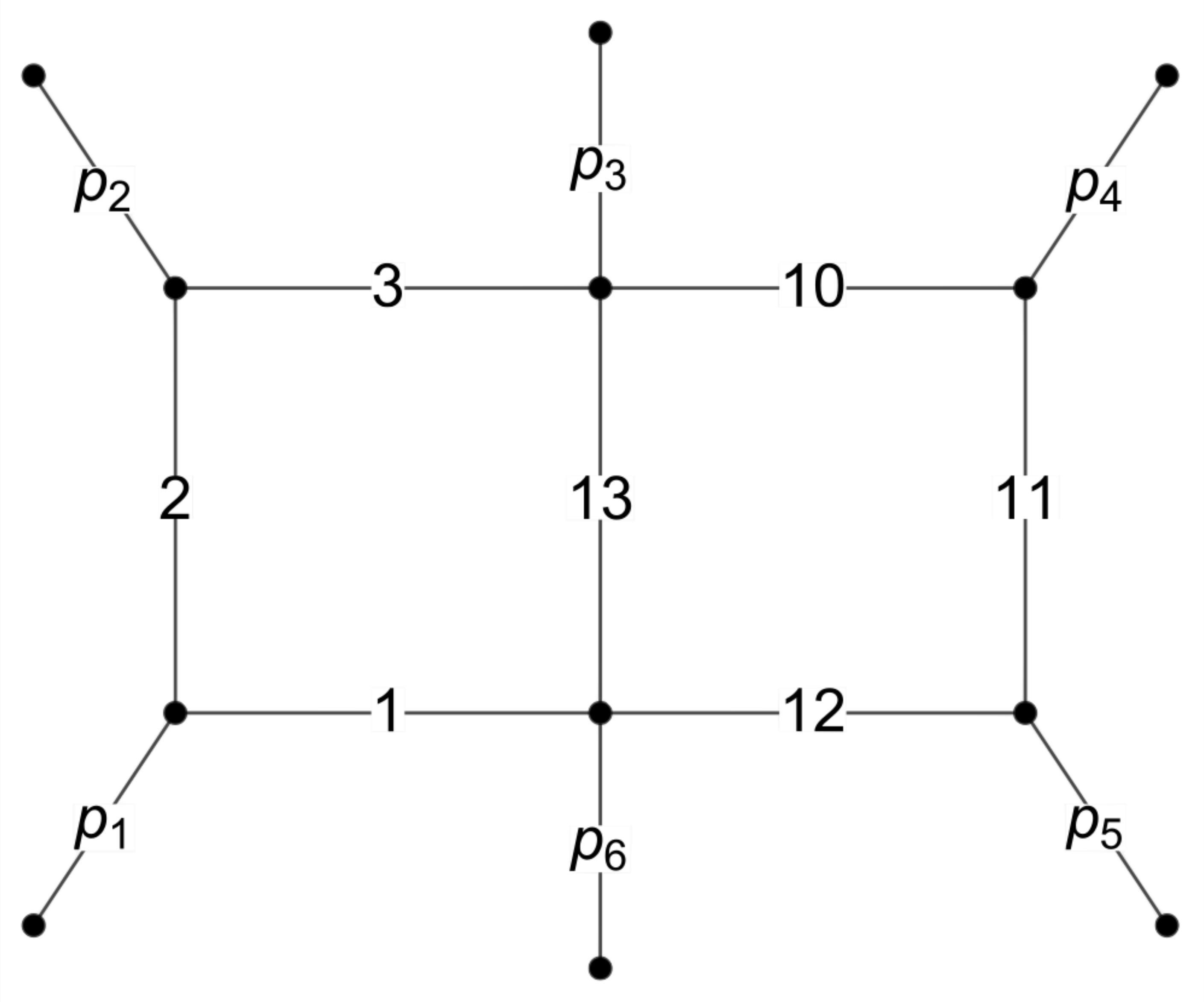
$$\tilde\lambda_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle\langle i-1,\, i\rangle},$$

$$(\lambda_{i,\alpha}, \tilde\lambda_{i,\dot\alpha})$$

spinor helicity

# Uniformly transcendental (UT) basis determination

key step



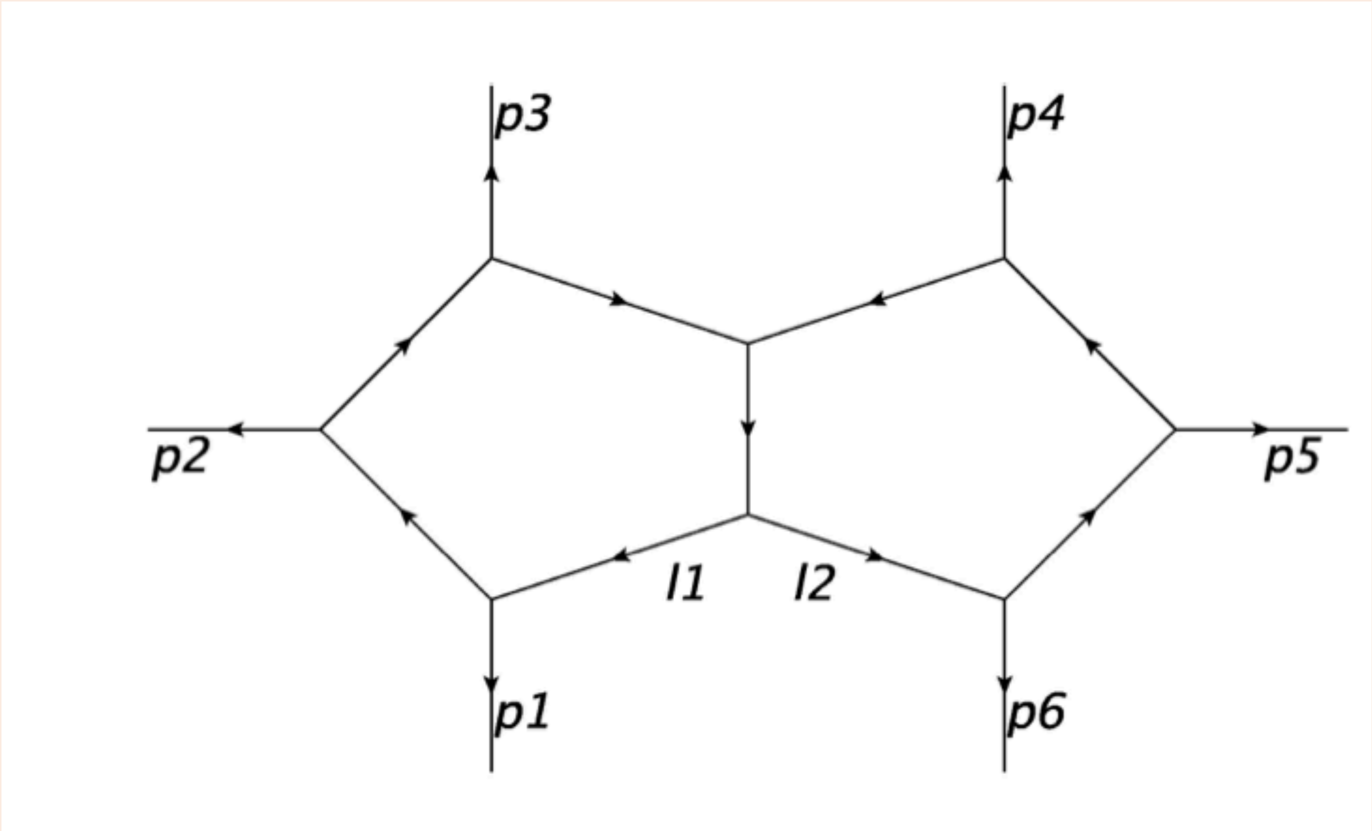

$$I_{\mathrm{db},i} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}}, \qquad i = 1, \ldots, 7$$

$$N_1 = -s_{12}s_{45}s_{156}\,,$$
$$N_2 = -s_{12}s_{45}(l_1+p_5+p_6)^2\,,$$
$$N_3 = \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5+p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix},$$
$$N_4 = \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2-p_6 & p_5 & p_4 & p_1+p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix},$$
$$N_5 = -\frac{1}{4}\frac{\epsilon_{1245}}{G(1,2,5,6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix},$$
$$N_6 = \frac{1}{8} G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2-p_6 & p_4 & p_5 \end{pmatrix} + \frac{D_2 D_{11}(s_{123}+s_{126})}{8}\,,$$
$$N_7 = -\frac{1}{2\epsilon}\frac{\Delta_6}{G(1,2,4,5)D_{13}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}.$$

Chiral numerator
*(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)*
/ Gram determinant
correspondence

$$\Delta_6 = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - \langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$$

# 2loop 6point top sector, UT integrals




Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \longrightarrow \text{evanescent, 6D weight-6 integral}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5\mu_{12}}{D_1 \dots D_9}$$

5 MIs (this sector)
267 MIs (whole family)

245 letters in total
except the 6D ones

“evanescent”: vanishing up to $\epsilon^0$

$N_1$, $N_2$, $N_3$ and $N_4$ are chiral numerators — Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010

# Complete canonical differential equation for 2l6p planar integrals

Use momentum twistor
Variables

$267 \times 267$ for double pentagon
$202 \times 202$ for hexagon box

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon)$$

$$A_i = \frac{\partial}{\partial x_i}\tilde{A}, \quad \tilde{A} = \sum_k \tilde{a}_k \log(W_k)$$

use alphabet to fit the canonical differential equation

# Even letter, Odd letter and the more complicated …

Even letter $F(s)$ a polynomial in Mandelstam variables or homogeneously linear in square roots

Conjecture: a Feynman integrals20 even letters are all from Landau singularity?

Odd letter $\frac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

"square roots": $\epsilon_{ijkl}, \quad \Delta_6, \quad , \sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

pseudo scalar

leading singularity hexagon

Källin function from massive triangle diagrams

More complicated letter $\frac{P(s)-\sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s)+\sqrt{Q_1(s)}\sqrt{Q_2(s)}}$

# Even letter, Odd letter and the more complicated …

245 letters

156 Even letters

$$s_{12}, \quad s_{123}$$
$$s_{12} - s_{123}$$
$$\cdots$$
$$-s_{12}s_{45} + s_{123}s_{345}$$
$$\cdots$$
$$\sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \quad \epsilon_{ijkl}$$

79 Odd letters

$$\frac{s_{12}+s_{34}-s_{56}-\sqrt{\lambda(s_{12},s_{34},s_{56})}}{s_{12}+s_{34}-s_{56}+\sqrt{\lambda(s_{12},s_{34},s_{56})}}$$
$$\cdots$$
$$\frac{s_{12}s_{23}-s_{23}s_{34}+s_{23}s_{56}+s_{34}s_{123}-s_{234}(s_{12}+s_{123})-\epsilon_{1234}}{s_{12}s_{23}-s_{23}s_{34}+s_{23}s_{56}+s_{34}s_{123}-s_{234}(s_{12}+s_{123})+\epsilon_{1234}},$$
$$\cdots \qquad \cdots$$
$$\frac{-s_{12}s_{45}s_{234}+s_{34}s_{61}s_{123}+s_{345}(-s_{23}s_{56}+s_{123}s_{234})-\Delta_6}{-s_{12}s_{45}s_{234}+s_{34}s_{61}s_{123}+s_{345}(-s_{23}s_{56}+s_{123}s_{234})+\Delta_6}$$

Then the canonical DE is derived analytically after ~50 times of numeric IBP running

10 More complicated letters

$$\frac{P-\epsilon_{1234}\sqrt{\lambda(s_{12},s_{34},s_{56})}}{P+\epsilon_{1234}\sqrt{\lambda(s_{12},s_{34},s_{56})}} \qquad \cdots$$

# Boundary Values

## Numeric boundary values

It is fine to use the package AMFlow to get ~100 digits as the boundary value for double-box, pentagon-triangle, hexagon-bubble diagrams

*Liu, Wang, Ma, 2018*
*Liu, Ma 2022*

## Analytic boundary values

It is still possible to get *fully analytic* boundary values

$$X_0: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \{-1,-1,-1,-1,-1,-1,-1,-1,-1\}$$

Solve the canonical DE on a curve starting with $X_0$ and require the finite solution
Some known integrals20 boundary values

} analytic boundary value

boundary value for a point in the physical region also obtained

# Boundary Values

## Analytic boundary values

Boundary values at the initial point, are combination of poly-logarithm of roots of unity

$$
\begin{aligned}
\epsilon^4 I_{\mathrm{db},1}(X_0) &= 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\,\mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,\\
\epsilon^4 I_{\mathrm{db},2}(X_0) &= 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\,\mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,\\
I_{\mathrm{db},3}(X_0) &= I_{\mathrm{db},4}(X_0) = I_{\mathrm{db},5}(X_0) = 0,\\
\epsilon^4 I_{\mathrm{db},6}(X_0) &= -\left(\frac{\pi^4}{540} + \frac{4}{3}\,\mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,\\
\epsilon^4 I_{\mathrm{db},7}(X_0) &= 0.
\end{aligned}
$$

from the ordinary differential equation

spurious pole asymptotic analysis

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

# Solution of canonical DE

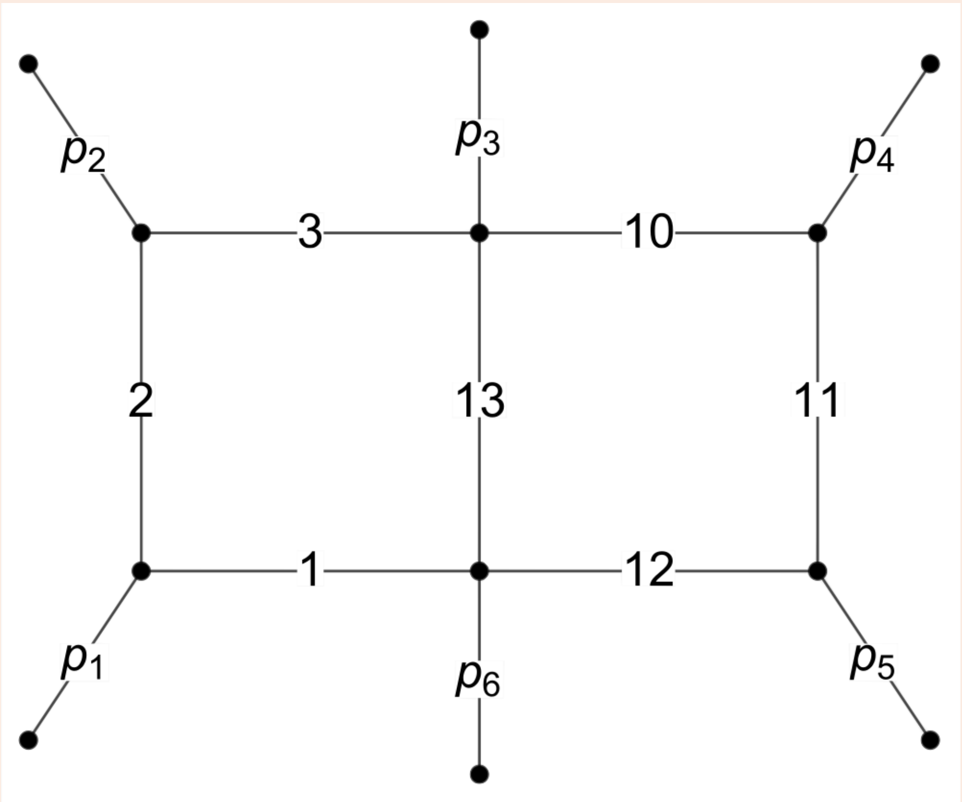
$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

## weight-1, weight-2

All in logarithm and classical poly-logarithm




$$\begin{aligned} I^{(2)}_{\mathrm{db},1} = \; & -\log(-v_1)\log(-v_2) - \log(-v_1)\log(-v_3) + \log(-v_1)\log(-v_4) - \log(-v_1)\log(-v_5) - \\ & \log(-v_1)\log(-v_6) + 4\log(-v_1)\log(-v_8) + \frac{1}{2}\log^2(-v_1) + \log(-v_2)\log(-v_3) - \\ & \log(-v_2)\log(-v_4) - \mathrm{Li}_2\left(1 - \frac{v_2 v_5}{v_7 v_8}\right) + \log(-v_2)\log(-v_6) + \log(-v_2)\log(-v_7) - \\ & 2\mathrm{Li}_2\left(1 - \frac{v_2}{v_8}\right) - \log(-v_2)\log(-v_8) - \log^2(-v_2) - \log(-v_3)\log(-v_4) + \log(-v_3)\log(-v_5) - \\ & \mathrm{Li}_2\left(1 - \frac{v_3 v_6}{v_8 v_9}\right) - 2\mathrm{Li}_2\left(1 - \frac{v_3}{v_8}\right) - \log(-v_3)\log(-v_8) + \log(-v_3)\log(-v_9) - \\ & \log^2(-v_3) - \log(-v_4)\log(-v_5) - \log(-v_4)\log(-v_6) + 4\log(-v_4)\log(-v_8) + \\ & \frac{1}{2}\log^2(-v_4) + \log(-v_5)\log(-v_6) + \log(-v_5)\log(-v_7) - 2\mathrm{Li}_2\left(1 - \frac{v_5}{v_8}\right) - \log(-v_5)\log(-v_8) - \\ & \log^2(-v_5) - 2\mathrm{Li}_2\left(1 - \frac{v_6}{v_8}\right) - \log(-v_6)\log(-v_8) + \log(-v_6)\log(-v_9) - \log^2(-v_6) - \\ & \log(-v_7)\log(-v_8) - \frac{1}{2}\log^2(-v_7) - \log(-v_8)\log(-v_9) + 3\log^2(-v_8) - \frac{1}{2}\log^2(-v_9) + \\ & \frac{\pi^2}{6} \end{aligned}$$

## Solution of canonical DE

$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right) \qquad I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-3, weight-4

$$\begin{aligned}\vec{I}^{(4)} &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{I}^{(3)}(\vec{x}_0) + \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_1}\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_2}\vec{f}^{(2)}(t_2) \\ &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t)\right)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{f}^{(2)}(t)\right).\end{aligned}$$

one-fold integration

It takes minutes on a laptop to get 14 digits for all 2loop 6point integrals

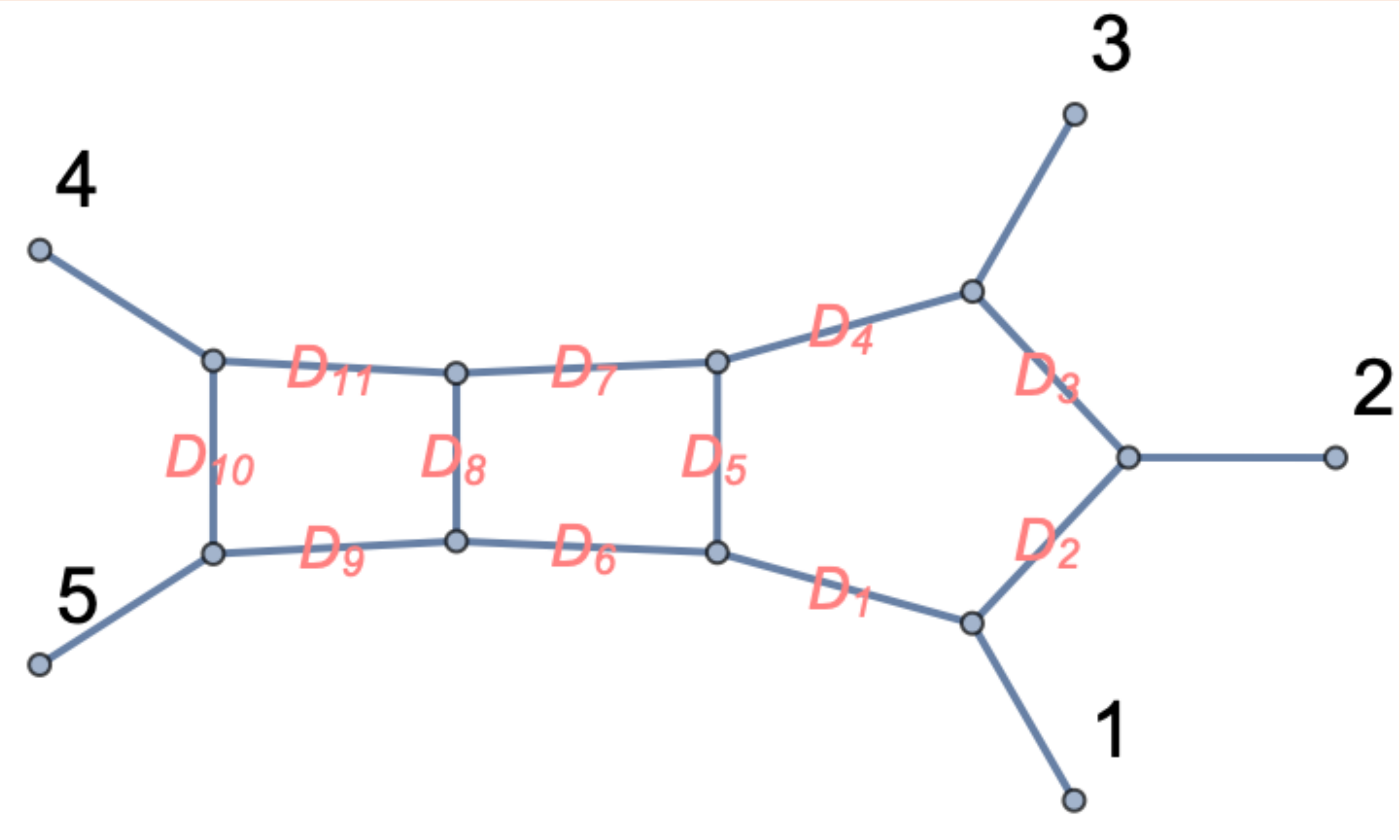
from our solution in both Euclidean and Physical regions refer to Yongqun Xu's talk

# 3loop 5point Feynman integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ,

Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion

# 3loop 5point planar family




5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021

✓ UT basis found!

Baikov analysis
Gram determinant

Canonical differential equation complicated ?

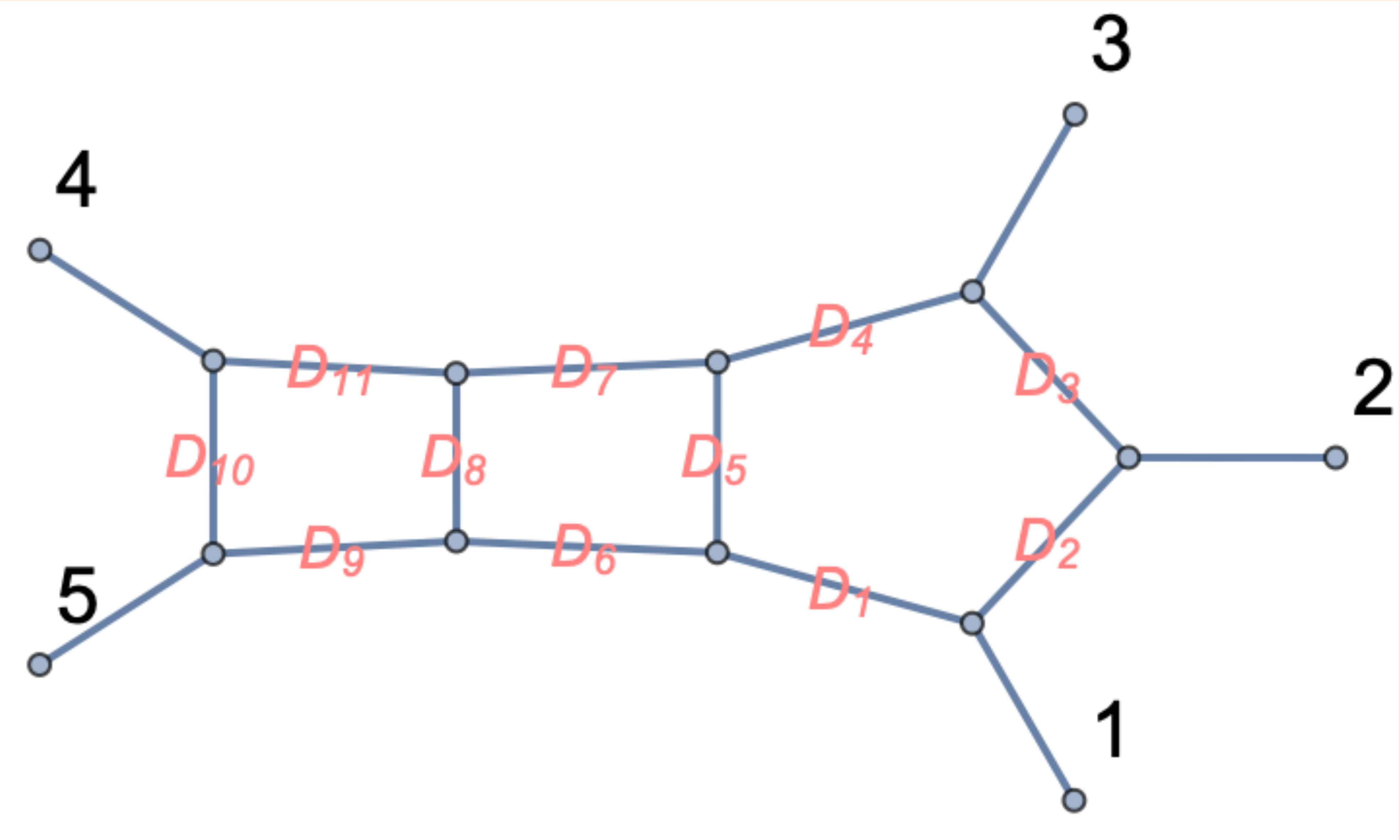
We use NeatIBP to derive the differential equation

~100 million IBPs → 85000 IBPs

hard to integrate to weight-6?

A novel one-fold representation

# First family of 3loop 5point calculated




5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

UT basis found!

Canonical differential equation
found with NeatIBP

31 letters …

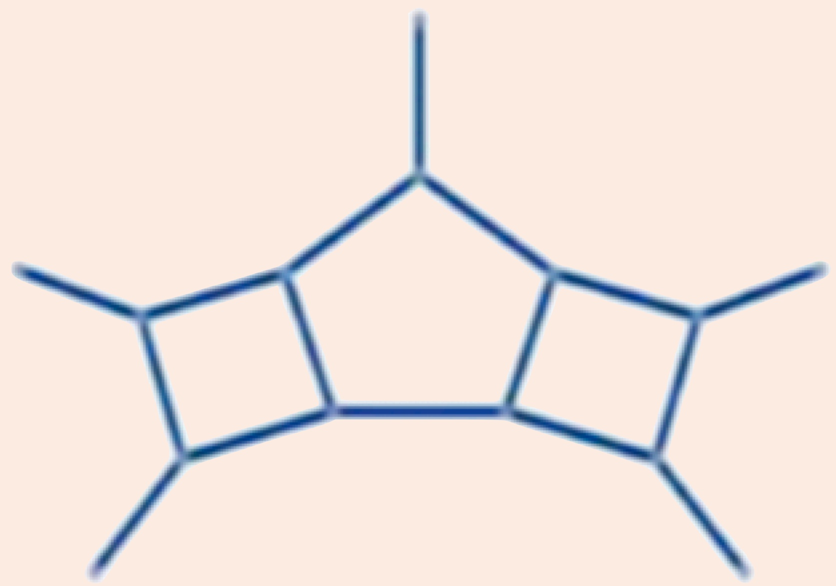
All boundary values up to weight-6 are obtained by spurious pole analysis

weight-1,2,3 classical polylogarithm, weight-4,5,6 one-fold integration
It takes 2 minutes on a laptop to get 10 digits from our analytic solution

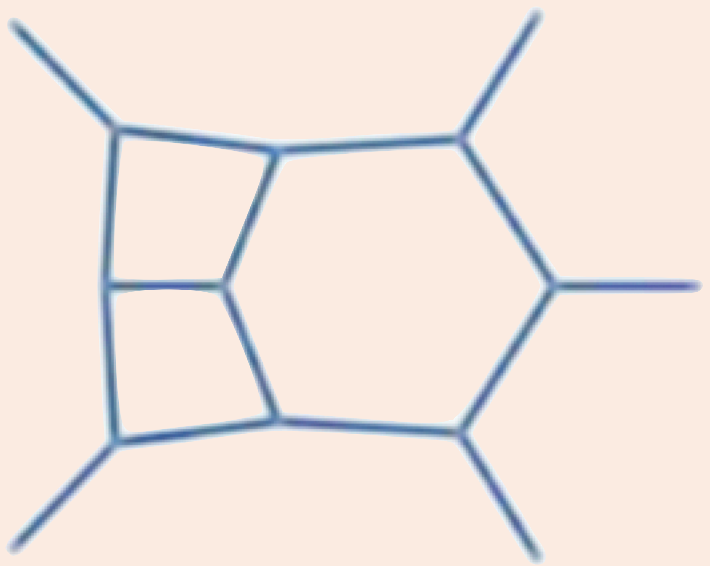
Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021 editors' suggestion

# Complete Result

All 3loop 5point integrals,
canonical DE/boundary values obtained



367 MIs



431 MIs

734 MIs

56 Letters (2loop 5point, 31 Letters)

small number
good news for bootstrap

| Weight | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 3loop 5point Symbols | 5 | 20 | 76 | 285 | 1000 | 2220 |

Phenomenology amplitudes frontier

Badger, Hartanto, Wu, *YZ*, Zoia, JHEP 12 (2025) 221

Badger, Hartanto, Poncelet, Wu, *YZ*, Zoia, JHEP 03 (2025) 066

# Frontier of multi-loop amplitudes for phenomenology

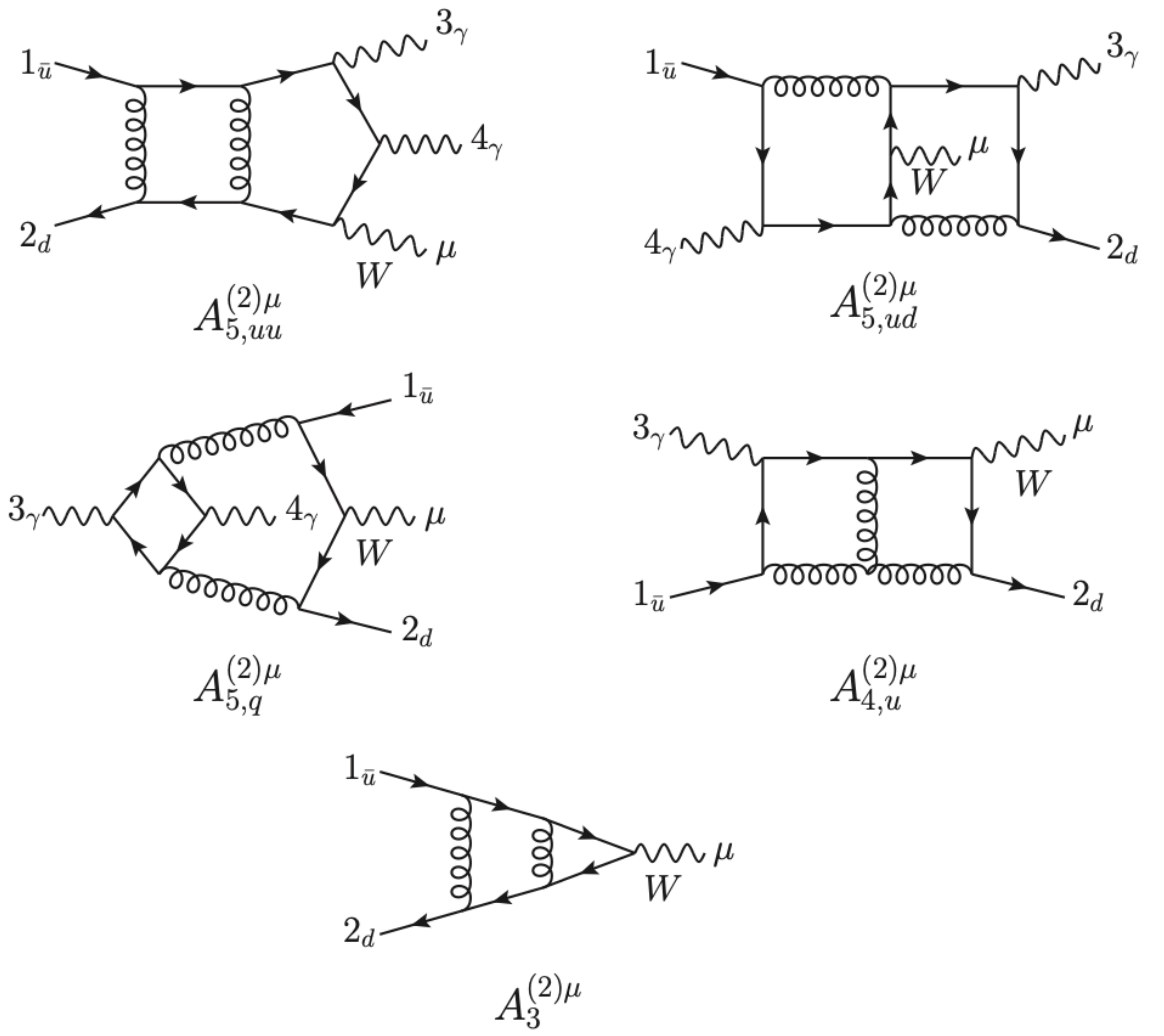
## Two-loop

| | | |
|---|---|---|
| Massless | $pp \to \gamma\gamma\gamma$ | Abreu, Page, Pascual, Sonikov 2020 |
| | | Chawdhry, Czakon, Mitov, Poncelet 2021 |
| | | Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov 2023 |
| | $pp \to \gamma\gamma j$ | Agarwal, Buccioni, von Manteuffel, Tancredi 2021 |
| | | Chawdhry, Czakon, Mitov, Poncelet 2021 |
| | | Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia 2021 |
| | | Buccioni, Chen, Feng, Gehrmann, Huss, Marcoli 2025 |
| | $pp \to \gamma jj$ | Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia 2023 |
| | $pp \to jjj$ | Abreu, Febres Cordero, Ita, Page, Sotnikov 2021 |
| | | De Laurentis, Ita, Klinkert, Sotnikov 2023 |
| | | Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi 2023 |
| | | De Laurentis, Ita, Sotnikov 2023 |
| Massive | $pp \to Wbb$ | Badger, Hartanto, Zoia 2021 Hartanto, Poncelet, Popescu, Zoia 2022 |
| | $pp \to Wjj$ | Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov 2022 De Laurentis, Ita, Page, Sotnikov 2025 |
| | $pp \to Hbb$ | Badger, Hartanto, Krys, Zoia 2021 Badger, HBH, Poncelet, Wu, *YZ*, Zoia 2024 |
| | $pp \to W\gamma j$ | Badger, Hartanto, Krys, Zoia 2022 |
| | $pp \to W/Zbb$ | Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini 2022 Mazzitelli, Sotnikov, Wiesemann 2024 |
| | $pp \to W\gamma\gamma$ | Badger, Hartanto, Wu, *YZ*, Zoia 2024 |
| | $pp \to ttH$ | Wang, Xie, Yang, Ye 2024 |
| | $pp \to tt\gamma$ | Wang, Yang 2025 |

adapt Matteo Becchetti's slide in HOCtools-II-2025

# NNLO QCD correction to W + 2 photon production

Badger, Hartanto, Wu, *YZ*, Zoia, JHEP 12 (2025) 221



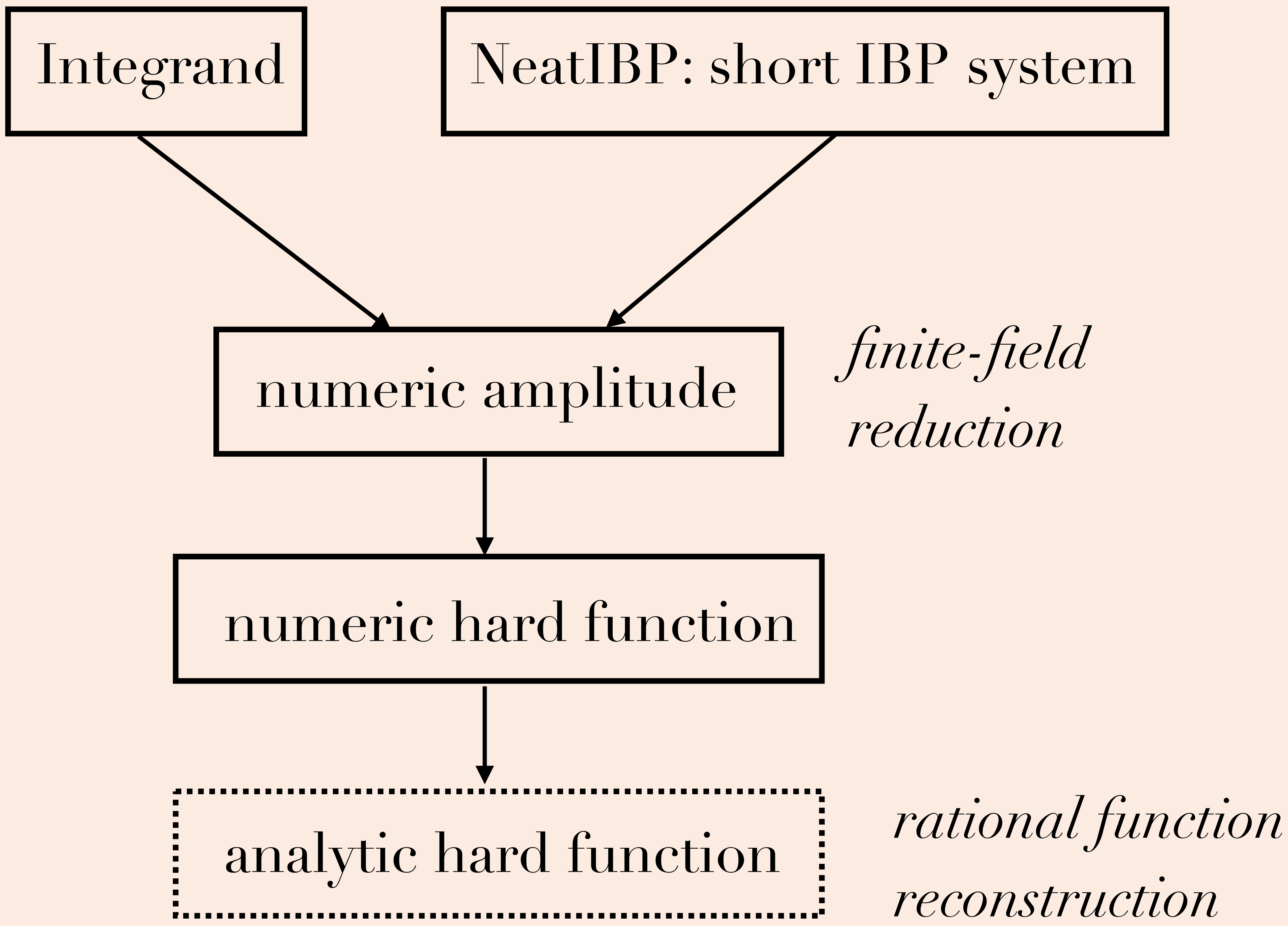

Most complicated part would be the two-loop five-point with one massive external leg (W boson)

6 scales: 5 Mandelstam + 1 mass

Laporta algorithm needs >10 million IBP reductions
Traditional IBP reduction is difficult

# NNLO QCD correction to W + 2 photon production

Badger, Hartanto, Wu, *YZ*, Zoia, JHEP 12 (2025) 221




Planar part:
NeatIBP+ Finiteflow 8 times faster,
3 times lower RAM usage,
than Finiteflow

non-Planar part:
NeatIBP+ Finiteflow works
Finiteflow itself does not provide the result

# Using NeatIBP

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

"Performing integration-by-parts reductions using NeatIBP 1.1 + Kira"
Wu, Boehm, Ma, Usovitsch, Xu, YZ, Comput.Phys.Commun. 316 (2025) 109798

Using algebraic geometry to find short IBP system, 2 or 3 orders of magnitudes shorter than that from Laporta algorithm.

| **Studies that used NeatIBP** | **Reference** |
|---|---|
| Differential Equations for Energy Correlators in Any Angle | arXiv:2506.02061 |
| One-loop amplitudes for $t\bar{t}j$ and $t\bar{t}\gamma$ productions at the LHC through $O(\epsilon^2)$ | arXiv:2505.10406 |
| Two-loop Feynman integrals for leading color $Wt\bar{t}$ production | JHEP 07 (2025) 001 |
| Two-loop QCD helicity amplitudes for $gg \to gt\bar{t}$ at leading color | JHEP 03 (2025) 070 |
| Full-color double-virtual amplitudes for $q\bar{q} \to b\bar{b}H$ | JHEP 03 (2025) 066 |
| Three-loop five-point pentagon-box-box Feynman diagram | arXiv:2411.18697 |
| Two-loop QCD corrections for $pp \to t\bar{t}j$ | arXiv:2411.10856 |
| Two-loop amplitudes for $W\gamma\gamma$ production at LHC | JHEP 12 (2025) 221 |
| NLO corrections to $J/\Psi c\bar{c}$ photoproduction | Phys.Rev.D 110 (2024) 9, 094047 |
| Two-loop five-point two-mass planar integrals | JHEP 10 (2024) 167 |
| Two-loop integrals for $t\bar{t}j$ production at hadron colliders in the leading color approximation | JHEP 07 (2024) 073 |

phenomenology application of NeatIBP

# NNLO QCD correction to W + 2 photon production

Badger, Hartanto, Wu, *YZ*, Zoia, JHEP 12 (2025) 221

Using NeatIBP 1.0 to generate short IBP systems

| family | deg. | # IBPs | # integrals | IBP disk size | running time |
|---|---|---|---|---|---|
| DPmz | 5 | 26673 | 27432 | 71.4 MB | 7h5m |
| DPzz | 5 | 61777 | 63880 | 375.8 MB | 21h54m* |
| HBmzz | 5 | 15428 | 15916 | 17.0 MB | 5h38m |
| HBzmz | 5 | 10953 | 11289 | 13.4 MB | 5h45m |
| HBzzz | 5 | 21126 | 21766 | 38.0 MB | 9h32m |
| PBmzz | 5 | 10224 | 10329 | 7.8 MB | 2h23m |
| PBzmz | 5 | 11610 | 11791 | 6.5 MB | 2h50m |
| PBzzz | 5 | 8592 | 8752 | 5.8 MB | 2h50m |
| HTmzzz | 4 | 3120 | 3176 | 1.5 MB | 1h12m |
| HTzmzz | 4 | 6594 | 6650 | 2.5 MB | 1h31m |
| HTzzzz | 4 | 4680 | 4631 | 4.0 MB | 2h31m |

Laporta algorithm needs >10 million IBP reductions

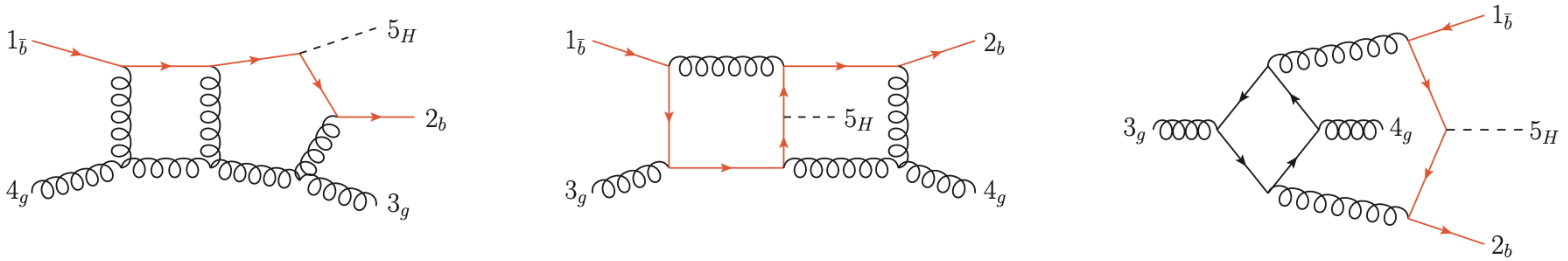
$\sim$10 million $\rightarrow$ $\sim$10 kilo

NeatIBP's IBP system has the size $\sim$10 kilo

2~3 orders of magnitude shorter

# Full color double virtual H + bbar production

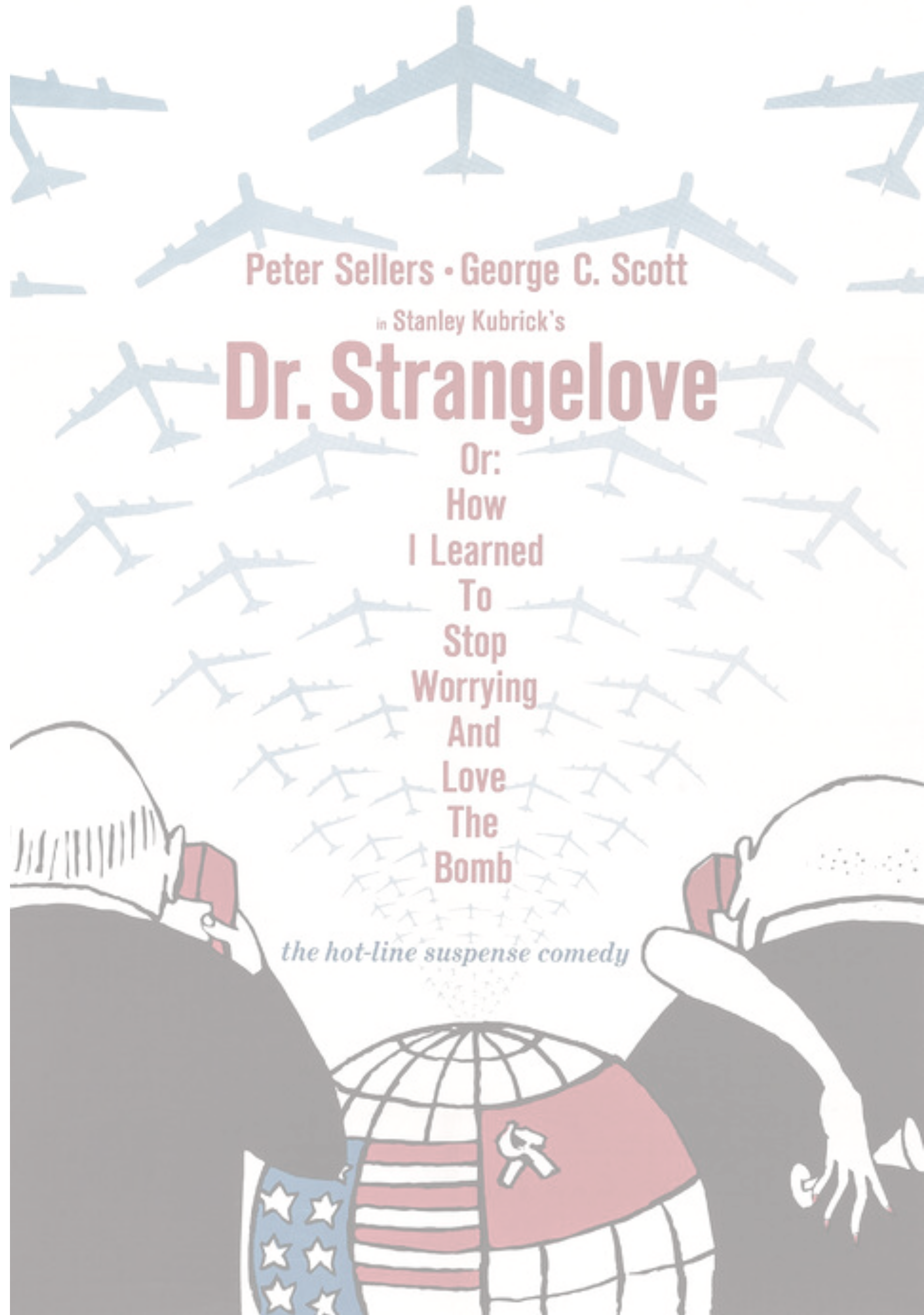
Badger, Hartanto, Poncelet, Wu, *YZ*, Zoia, JHEP 03 (2025) 066




$A_{34}^{(2),N_c^2}, A_{34}^{(2),1}, A_{43}^{(2),1}$ $\quad$ $A_{34}^{(2),1}$ $\quad$ $A_{34}^{(2),n_f/N_c}, A_{43}^{(2),n_f/N_c}, A_{\delta}^{(2),n_f/N_c^2}$

Reuse the same IBP system of NeatIBP for W + 2 photon production

*How I learned to stop worrying and calculate two-loop $pp \to gggg$*



# Bootstrap

Carrôlo, Chicherin, Henn, Yang, YZ

JHEP 07 (2025) 214
arXiv:2510.20565

# Bootstrap

Traditional bootstrap

Admissible letters

↓

Integrable symbols

↓

Ansatz

for N=4 sYM
planar amplitudes bootstrap

dual conformal: 9 letters for six points

Caron-Huot, Dixon, McLeod, von Hippel, 2016 (5loop)
Caron-Huot, Dixon, Dulat, von Hippel, McLeod, Papathanasiou, 2020 (6loop & 7loop)

larger number than

Our bootstrap

Feynman integrals

↓

symbols in integrals

↓

Ansatz

aims for QCD amplitudes

no IBP needed

Carrôlo, Chicherin, Henn, Yang, YZ

# Wilson loop and scattering amplitudes

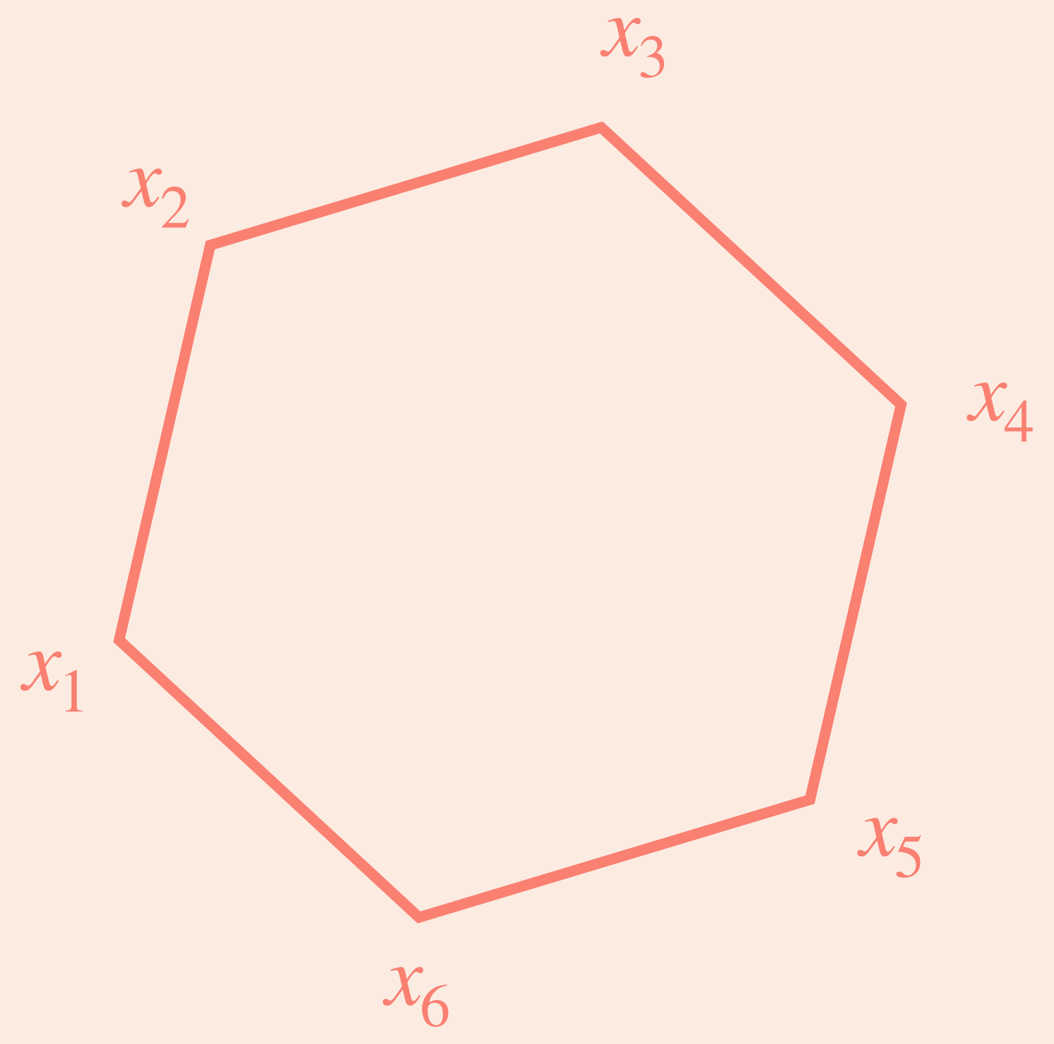
Alday and Maldacena, 2007

N=4 planar sYM,
Lightlike boundary Wilson loop expectation value is dual to MHV scattering amplitude

$$\log \frac{A_n^{\mathrm{MHV}}(p_1,\ldots,p_n)}{A_n^{\mathrm{MHV,\ tree}}(p_1,\ldots,p_n)} = \log\langle W(x_1,\ldots,x_n)\rangle \qquad p_i = x_{i+1}-x_i$$

infrared divergence

UV divergence from the gluon exchange near the corner




In the $x$-space, it has anomalous (dual) conformal invariance DCI

$$K^\mu \ \log\langle W(x_1,\ldots,x_n)\rangle_{\mathrm{finite}} = \frac{1}{2}\Gamma_{\mathrm{cusp}}\sum_{i=1}^{n} x^\mu_{i,i+1}\log\left(\frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}\right)$$

Bern-Dixon-Smirnov (BDS) Ansatz is a solution for this equation,
However the physical solution is BDS Ansatz plus a DCI remainder function

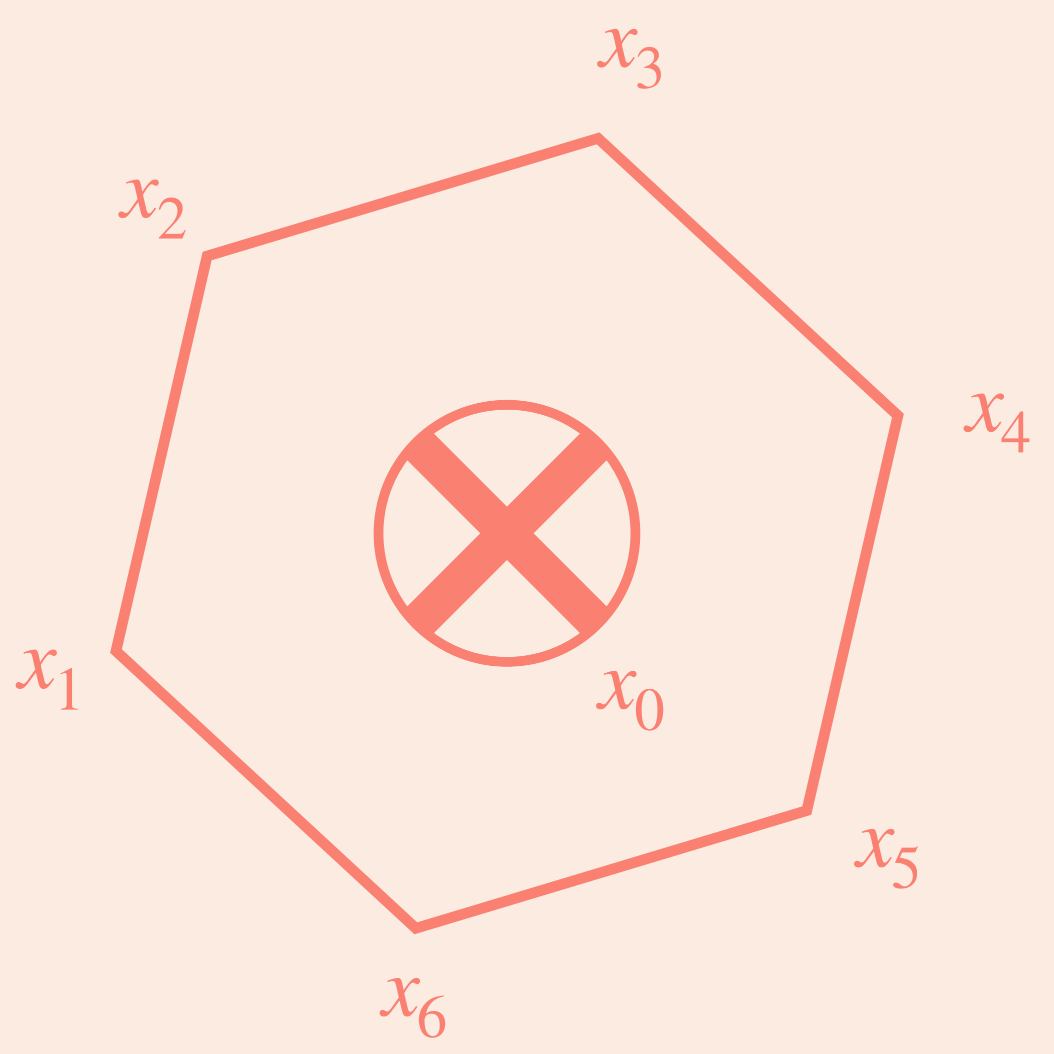
# Wilson loop + Lagrangian insertion and scattering amplitudes

$$F_n(x_1, \ldots, x_n; x_0) \equiv \pi^2 \frac{\langle W(x_1, \ldots, x_n) \mathcal{L}(x_0) \rangle}{\langle W(x_1, \ldots, x_n) \rangle}$$

Alday, Buchbinder and Tseytlin 2011
Alday, Heslop and Sikorowski 2013

N=4 sYM on-shell Lagrangian

This quantity is finite




Surprisingly, it is dual to the maximally transcendental part of all-plus helicity amplitude in pure Yang-Mills theory!

L-loop $F_n$ is dual to (L+1)-loop all-plus helicity amplitude in pure Yang-Mills theory

Chicherin and Henn, 2202.05596

## Bootstrap

Set up an ansatz and fit the coefficients

ansatz from all functions (all integrable symbols)

usually too many functions …

ansatz from functions in Feynman integrals!

leading singularity (rational function)

$$F_n = \sum_{i=1}^{22} \sum_{j} c_{ij} R_i f_j$$

transcendental function
from Feynman integrals

# Wilson loop + Lagrangian leading singularity

Chicherin and Henn, 2022
Brown, Henn, Mazzucchelli, Trnka 2025

$$B_{ijklm} := \frac{\langle AB(mij)\cap(jkl)\rangle^2}{\langle ABjm\rangle\langle ABij\rangle\langle ABjk\rangle\langle ABlj\rangle\langle ABmi\rangle\langle ABkl\rangle},$$

$$(abc)\cap(def) := (ab)\langle cdef\rangle - (ac)\langle bdef\rangle + (bc)\langle adef\rangle$$

Kermit functions

$$B_{ijkl} := \frac{\langle ijkl\rangle^2}{\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABli\rangle}.$$

Take the gauge, $\langle ABij\rangle \to \langle ij\rangle$ rational function in momentum twistor variables

There are only 20 (22) linearly independent leading singularities

## Function space: planar 2loop 6point massless integrals

weight-3
integrable symbols
$528 > 266$

A counting of functions, with the dihedral symmetry

| Transcendental weight | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| # All symbols | 9 | 62 | 319 | 945 |
| # Two-loop six-point symbols | 9 | 62 | 266 | 639 |
| # Two-loop five-point one-mass symbols | 9 | 59 | 263 | 594 |
| # One-loop squared symbols | 9 | 59 | 221 | 428 |
| # Genuine two-loop six-point symbols | 0 | 0 | 3 | 45 |

Surprisingly,
the number of genuinely two-loop six-point functions are very small …
It is a good news for bootstrap

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2501.01847

# First Application:
# Hexagonal Wilson loop with Lagrangian insertion at two loops in N=4 sYM

Carrôlo, Checherin, Henn, Yang, YZ, JHEP 07 (2025) 214

$$F_n(x_1,\ldots,x_n;x_0) := \pi^2 \frac{\langle 0|W_n[x_1,\ldots,x_n]L(x_0)|0\rangle}{\langle 0|W_n[x_1,\ldots,x_n]|0\rangle}$$

Using group representation to construct an Ansatz with dihedral symmetry →

| weight | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **unknowns** in dihedral ansatz | 5 | 22 | 139 | 644 | 1892 |
| genuine **unknowns** | 4 | 20 | 125 | 585 | 1718 |
| **constraints:** | | | | | |
| soft | 3 | 20 | 116 | 515 | 1439 |
| collinear | 3 | 20 | 121 | 551 | 1539 |
| spurious $s_{24}=0$ | 1 | 12 | 76 | 360 | 1044 |
| spurious $s_{25}=0$ | 1 | 6 | 36 | 165 | 483 |
| scaling dimension | 0 | 4 | 20 | 125 | 585 |
| triple collinear | 1 | 5 | 31 | 134 | 353 |
| total **constraints** | 4 | 20 | 125 | 585 | 1718 |
| unfixed **unknowns** | 0 | 0 | 0 | 0 | 0 |

Bootstrap

$F_2(x_1\ldots x_6;x_0)$ is fixed in the symbol level!

# Two-loop six-point QCD bootstrap

Carrôlo, Chicherin, Henn, Yang, YZ

arXiv:2510.20565

no supersymmetry

$- - + + + +$

two-loop six-point
pure-Yang-mills bootstrap done!

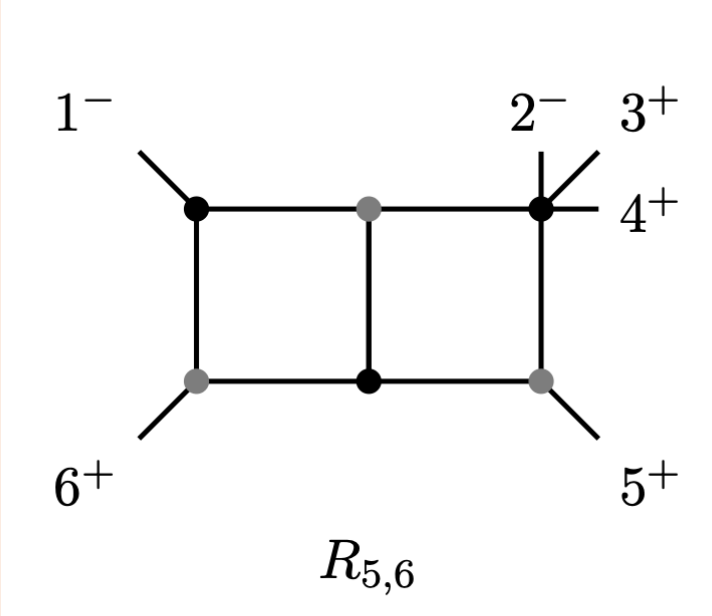
$$\mathcal{H}^{(2)} = \lim_{\epsilon\to 0}\left[\mathcal{A}^{(2)} - I^{(1)}\mathcal{A}^{(1)} + \frac{1}{2}\left(I^{(1)}\right)^2 \mathcal{A}^{(0)}\right],$$

Parke-Taylor

$$\mathcal{H}^{(2)}_{\mathrm{YM}} = R_1 G_1 + \sum_{2<i<j\leq 6} R_{i,j} G_{i,j}.$$

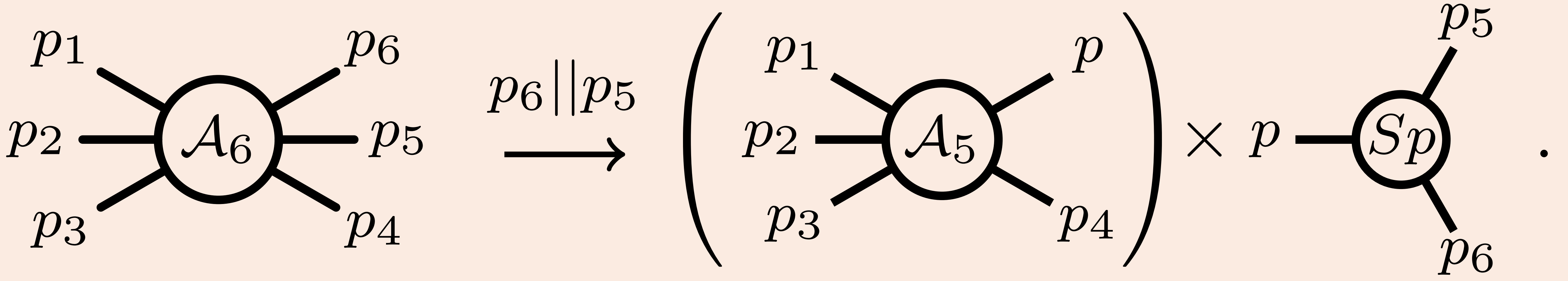
Ansatz: rational factor × transcendental functions

$$R_{i,j}/R_1 = -1 + 12u_{i,j} - 30u_{i,j}^2 + 20u_{i,j}^3\,, \quad u_{i,j} = \frac{\langle 1i\rangle\langle 2j\rangle}{\langle 12\rangle\langle ij\rangle}$$



# Two-loop six-point QCD bootstrap

| Condition on $\mathcal{H}^{(2)}_{\mathrm{YM}}$ | No. of constraints |
|---|---|
| dimensionless symbols $G$ | 996 |
| no spurious/high-order poles | |
| $\langle 36\rangle = 0$ | 333 |
| $\langle 35\rangle = 0$ | 614 |
| $\langle 34\rangle = 0$ | 629 |
| $\langle 45\rangle = 0$ | 343 |
| collinear limit $p_5 \parallel p_6$ | |
| $--++++$ | 1785 |
| $+--+++$ | 1307 |
| $++--++$ | 1785 |
| $+++--+$ | 1646 |
| $-++++-$ | 1646 |
| triple collinear limit $p_4 \parallel p_5 \parallel p_6$ | |
| $--++++$ | 1836 |
| $++--++$ | 724 |
| Total | 2412 |




2loop 6point pure-Yang-Mills $--++++$ helicity amplitude weight-4 part completely fixed by those conditions!

# Summary and Outlook

Analytic computation of all 2loop 6point planar massless integrals,
all 3loop 5point integrals are done;

NeatIBP, a powerful package for cutting-edge IBP reduction for precision physics

a lot of future applications

The dawn of QCD bootstrap

First two-loop six-point QCD amplitude bootstrap

Multi-loop multi-leg Feynman integrals are no longer that difficult!

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福寿双全，万事顺遂