Some new ideas of Feynman integrals computation

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Based on works in progress

Outline

I. Introduction

II. Small parameter expansion

III. Large mass expansion

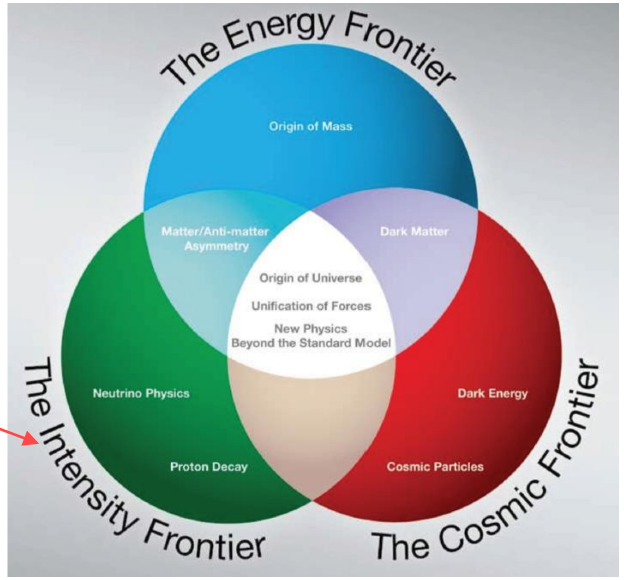
IV. Large index expansion

V. Summary and outlook

Precision: gateway to discovery

➢ New particles/physics have not been discovered yet at LHC

- Currently main strategy: search anomalous deviations from theory
- Interplay between exp. and th.



To make full use of data: theoretical errors should be much smaller than experimental errors, ideally:

$$\boxed{Error_{th} < \frac{1}{3} Error_{exp}}$$

Feynman integrals: a key obstacle in high-order computation

$$\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu}jk}(D,\vec{s})I_{\vec{\nu}'}(D,\vec{s}) = 0$$

1) Reduce loop integrals to basis (Master Integrals)

2) Calculate MIs

Integration-by-parts reduction: the bottleneck!

➢ The state-of-the-art IBP method: very challenging

- 4-loop DGLAP kernel cannot be obtained
- $H+2j$ production: exact two-loop contribution is missing — Chen, et al., JHEP2022
- $H+t\bar{t}$ production: exact two-loop contribution is missing — Catani, et al., PRL2023

➢ Improvements for IBPs

- Syzygy equations: trimming IBP system
  - Gluza, Kajda, Kosower, PRD2011
  - Böhm, Georgoudis, Larsen, Schulze, Zhang, PRD2018
  - NeatIBP: Wu, et al. CPC2024
- Block-triangular form: search simple IBP system
  - Liu, YQM, PRD2019
  - Guan, Liu, YQM, CPC2020
  - Blade: Guan, Liu YQM, Wu, 2405.14621

Improve efficiency by a hundredfold $\approx$ half order in α_s

Need to calculate two more orders in α_s! How?

Ways to bypass IBP

The new representation

- **Any L-loop amplitude, including Feynman Integral, takes the form**

$$\mathcal{M} \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D l_i}{\mathrm{i}\pi^{D/2}} \frac{P(l)}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_N^{\nu_N}},$$

- **Can be written in the new representation**

$$\mathcal{M} = \int [\mathrm{d}\mathbf{X}]\, \hat{\mathcal{M}}(\mathbf{X}) \qquad \hat{\mathcal{M}}(\mathbf{X}) = \mathcal{U}^{-\frac{(L+1)D}{2}} \sum_{\Delta,\vec{\nu}'} K_{\vec{\nu}'}^{\Delta}(\mathbf{X})\, I_{\vec{\nu}'}^{\Delta}(\mathbf{X})$$

- $\Delta = \frac{\mathrm{LD}}{2}$, K's are rational in $\boldsymbol{X}$
- Fixed-Branch Integrals (FBIs) defined as

$$I_{\vec{\nu}}^{\Delta}(\mathbf{X}) = \frac{(-1)^{\nu}\Gamma(\nu-\Delta)}{\prod_{\alpha=1}^{N}\Gamma(\nu_\alpha)} \int [\mathrm{d}\mathbf{y}] \frac{\prod_{\alpha=1}^{N} y_\alpha^{\nu_\alpha-1}}{\left(\frac{1}{2}\mathbf{y}^T\cdot R\cdot\mathbf{y} - \mathrm{i}0^+\right)^{\nu-\Delta}}$$

- The same as one-loop integrals, except for more delta functions $[\mathrm{d}\mathbf{y}] \equiv \prod_{\alpha=1}^{N} \mathrm{d}y_\alpha \prod_{b=1}^{B} \delta\left(1 - \sum_{i=1}^{n_b} y_{(b,i)}\right)$
- $\mathrm{U}^{-(\mathrm{L}+1)\mathrm{D}/2}$ can sometimes be absorbed into the definition of FBI

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Introduce a parameter for FI

- **For 2-loop,** 3 branch variables at most, **after** integrating delta functions: two-fold integration

$$F^D_{\vec{\nu}} = \sum_{\vec{\nu}'} \int_0^1 \mathrm{d}X \int_0^1 \mathrm{d}Y \, K^{\Delta}_{\vec{\nu},\vec{\nu}'}(X,Y) I^{\Delta}_{\vec{\nu}'}(X,Y)$$

  - Here, we rescale the integral region to [0,1], with the new integration variables named X and Y.
  - $\mathbf{U}^{-(\mathbf{L}+1)\mathbf{D}/2}$ is absorbed into the definition of FBI

- **Introduce a parameter into FI**

$$F^D_{\vec{\nu}}(\delta) = \sum_{\vec{\nu}'} \int_{a-a\delta}^{a+(1-a)\delta} \mathrm{d}X \int_{b-b\delta}^{b+(1-b)\delta} \mathrm{d}Y \, K^{\Delta}_{\vec{\nu},\vec{\nu}'}(X,Y) I^{\Delta}_{\vec{\nu}'}(X,Y)$$

  - The original FI can be achieved by taking the δ to 1
  - (a, b) could be any rational regular point of the integrand

Expand generalized FI at $\delta = 0$

$$F^D_{\vec{\nu}}(\delta) = \sum_{\vec{\nu}'} \int_{a-a\delta}^{a+(1-a)\delta} \mathrm{d}X \int_{b-b\delta}^{b+(1-b)\delta} \mathrm{d}Y \; K^{\Delta}_{\vec{\nu},\vec{\nu}'}(X,Y) I^{\Delta}_{\vec{\nu}'}(X,Y)$$

- **Expand F(δ) at δ=0 to n-th order with complexity $O(n^2)$**
  - We have FBIs' differential eqautions w.r.t. X and Y
  - K is rational on X and Y
  - Mainly Taylor expansion of X and Y, the integration is straightforward

Need $X^l Y^m$ whose $l + m \leq n$

- **The expansion takes the following form**

$$F^D_{\vec{\nu}}(\delta) = \delta^2 \sum_{\vec{\nu}'} b_{\vec{\nu}'} \left(\sum_{i=0}^{n} c_{\vec{\nu},\vec{\nu}',i}\, \delta^i + \mathcal{O}(\delta^{n+1}) \right)$$

  - b is boundary condition of the FBI, which is the fisrt order coefficient of the FBI
  - b could be irrational, but we don't really need b
  - c comes from DEs and K, and since (a, b) is rational, c is rational

Search relation between generalized FIs

$$F^D_{\vec{\nu}}(\delta) = \delta^2 \sum_{\vec{\nu}'} b_{\vec{\nu}'} \left(\sum_{i=0}^{n} c_{\vec{\nu},\vec{\nu}',i}\, \delta^i + \mathcal{O}(\delta^{n+1}) \right)$$

- **Use series expansions of generalized FIs to search relations between them**
  - Choose a set of generalized FIs for search $S = \{\vec{\nu} | \vec{\nu} \text{ chosen for search}\}$
  - Assume a polynomial relation for the integral, with degree d

$$\sum_{\vec{\nu} \in S} \left(\sum_{i=0}^{d} a_i \delta^i \right) F^D_{\vec{\nu}}(\delta) = 0$$

  - Considering a and c are rational and b should be independent irrational numbers, we can further write down independent relations

$$\sum_{\vec{\nu} \in S} \left(\sum_{i=0}^{d} a_i \delta^i \right) \left(\sum_{j=0}^{n} c_{\vec{\nu},\vec{\nu}',j} \delta^j + \mathcal{O}(\delta^{n+1}) \right) = 0, \quad \forall \vec{\nu}' \in \text{ Master FBIs}$$

Solve equations between generalized FIs

$$\sum_{\vec{\nu}\in S}\left(\sum_{i=0}^{d} a_i \delta^i\right)\left(\sum_{j=0}^{n} c_{\vec{\nu},\vec{\nu}',j}\delta^j + \mathcal{O}(\delta^{n+1})\right) = 0, \quad \forall \vec{\nu}' \in \text{ Master FBIs}$$

- **Solving the above equations, we may get the relations between (generalized) FIs**

  - To search relations between more complicated integrals, #S and d should increase, which in turn requires an increase in n
  - For $\mathbf{O(n^2)}$ complexity, n can easily go up to $\mathbf{10^2}$ or even $\mathbf{10^3}$
  - Since a and c are all rational, we can use *Finite Field* to speed up the calculation

An alternative approach to introducing a parameter

- **Introduce a parameter for FI by restricting the integration to a region near a specific U**

$$F_{\vec{\nu}}^{D} = \int [\mathrm{d}\mathbf{X}] K_{\vec{\nu},\vec{\nu}'}^{\Delta}(\mathbf{X}) I_{\vec{\nu}'}^{\Delta}(\mathbf{X}) = \int_0^{\frac{1}{3}} \mathrm{d}t \int [\mathrm{d}\mathbf{X}] K_{\vec{\nu},\vec{\nu}'}^{\Delta}(\mathbf{X}) I_{\vec{\nu}'}^{\Delta}(\mathbf{X}) \delta\left(\frac{1}{3} - \mathcal{U} - t\right)$$

$$F_{\vec{\nu}}^{D}(\lambda) = \int_0^{\lambda} \mathrm{d}t \int [\mathrm{d}\mathbf{X}] K_{\vec{\nu},\vec{\nu}'}^{\Delta}(\mathbf{X}) I_{\vec{\nu}'}^{\Delta}(\mathbf{X}) \delta\left(\frac{1}{3} - \mathcal{U} - t\right)$$

$\mathbf{X_1 X_2 + X_2 X_3 + X_3 X_1 \in [0, 1/3]}$

- Expand near λ ~ 0, so t ~ 0, U ~ 1/3, (X1, X2, X3) ~ (1/3, 1/3, 1/3), mainly Taylor region
- Integrate the δ-function in the mesure to get a two-fold integral
- Expand the integrand and the integration takes the standard form

$$\int_{-\infty}^{\infty} \mathrm{d}x \mathrm{d}y \, x^l y^m \delta(1 - x^2 - y^2)$$

- With series expansions of λ, the search process is the same

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η expansion

➢ **FBI with an auxiliary parameter η:**

$$\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta)=\frac{(-1)^{\nu}\Gamma(\nu-\Delta)}{\prod_{\alpha=1}^{N}\Gamma\left(\nu_{\alpha}\right)}\int[\mathrm{d}\mathbf{y}]\,\frac{\prod_{\alpha=1}^{N}y_{\alpha}^{\nu_{\alpha}-1}}{\left(\frac{1}{2}\mathbf{y}^{T}\cdot R\cdot\mathbf{y}+\eta\right)^{\nu-\Delta}}$$

- Expand eta asymptotically in the limit toward infinity

$$\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta)=\frac{(-1)^{\nu}\Gamma(\nu-\Delta)}{\prod_{b=1}^{B}\Gamma\left(\sum_{i=1}^{n_b}\nu_{(b,i)}\right)}\eta^{\Delta-v}\left(1+\sum_{k=1}^{\infty}a_{\vec{\nu},k}^{\Delta}\eta^{-k}\right)$$

- Determine the coefficients by solving the differential equation

$$(2z_0\eta-C)\frac{\mathrm{d}}{\mathrm{d}\eta}\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta)=(2\Delta-\nu-B)\,z_0\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta)+\sum_{\alpha=1}^{N}z_{\alpha}\mathcal{I}_{\vec{\nu}-\vec{e}_{\alpha}}^{\Delta-1}(\eta)$$

η expansion

➢ **Coefficients can be computed directly**

$$a^{\Delta}_{\vec{\nu},k} = \frac{1}{z_0(B-2k-\nu)}\left[C(\Delta-\nu-k+1)a^{\Delta}_{\vec{\nu},k-1} - \sum_{\alpha=1}^{N}\left(\sum_{\substack{i \text{ in the same} \\ \text{branch with } \alpha}} \nu_{(b,i)} - 1\right) z_\alpha a^{\Delta}_{\vec{\nu}-\vec{e_\alpha},k}\right]$$

- **Final result**

$$a^{\Delta}_{\vec{\nu},k} \sim X^{(L+1)k} \qquad X = \{X_1, X_2, \ldots, X_B\}$$

- **Substitute η with $-i\mathcal{U}\eta$ so that η becomes a real number with the dimension of mass squared.**
- **Specifically, amplitude with an auxiliary parameter η becomes:**

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{P(l)}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}}$$

η expansion

➢ **Amplitude with the substituted η in the new representation:**

$$\mathcal{M}(\Delta, s, \eta) = \int [\mathrm{d}X]\, \mathcal{U}^{-\frac{(L+1)D}{2}} \sum_{\Delta,\vec{\nu}'} K_{\vec{\nu}'}^{\Delta}(X) \mathcal{I}_{\vec{\nu}}^{\Delta}(X, \eta)$$

$$\mathcal{I}_{\vec{\nu}}^{\Delta}(\eta) = \frac{(-1)^{\nu}\Gamma(\nu - \Delta)}{\prod_{b=1}^{B} \Gamma\left(\sum_{i=1}^{n_b} \nu_{(b,i)}\right)} (-\mathrm{i}\mathcal{U}\eta)^{\Delta-v} \left(1 + \sum_{k=1}^{\infty} a_{\vec{\nu},k}^{\Delta} (-\mathrm{i}\mathcal{U}\eta)^{-k}\right)$$

➢ **The computation of the η-expansion of the amplitude reduces to the evaluation of vacuum bubble diagrams.**

$$M(\Delta, \vec{s}, \eta) = \sum_{\lambda_0,\vec{\lambda}} M^{\lambda_0\ldots\lambda_r}(\Delta)\eta^{\lambda_0} s_1^{\lambda_1} \ldots s_r^{\lambda_r}$$

η expansion

➤ **Suppose we have a set of integrals:** $G = \{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$

- **Linear relations among them can be written as:**

$$\sum_{i=1}^{n} Q_i(\Delta, \vec{s}, \eta)\mathcal{M}_i(\Delta, \vec{s}, \eta) = 0$$

- Q_i **are homogeneous polynomials of** η **and kinematic variables s.**
- **Denote the mass dimension of** M_i **by** $Dim(M_i)$ **and the degree of** Q_i **by** d_i**.**

$$2d_1 + \mathrm{Dim}(\mathcal{M}_1) = \cdots = 2d_n + \mathrm{Dim}(\mathcal{M}_n)$$

$$Q_i(\Delta, \vec{s}, \eta) = \sum_{(\lambda_0, \vec{\lambda}) \in \Omega_{d_i}^{r+1}} Q_i^{\lambda_0 \ldots \lambda_r}(\Delta)\eta^{\lambda_0} s_1^{\lambda_1} \ldots s_r^{\lambda_r}$$

- **There is only one degree of freedom in** $\{d_i\}$**, which can be chosen as** $d_{max} = \max\{d_i\}$ **.**

η expansion

- Vanishing the coefficients of these polynomials of η and $\vec{s}$ provides the condition to solve for Q
- Thus, the reduction relations among these integrals can be obtained.
  - Consider two sets of integrals, G1 and G2. Assuming that G1 can be reduced to G2, we provide an algorithm to find out relations to realize this reduction:

  - 1.Let $G = \{G_1, G_2\}$and $d_{max} = 0$.
  - 2.Solve the linear equations generated by vanishing the coefficients of these polynomials, to obtain all possible relations
  - 3.If the obtained relations are enough to express G_1 in terms of G_2 ,stop; otherwise, increase d_{max} by 1 and go to step 2

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Non-commutativity of IBP Equations

➢ IBP Equations for Integral Reduction

$$0 = \int [dl] \frac{\partial}{\partial l_i^\mu} \left(\frac{q_j^\mu}{\prod_a D_a^{\nu_a}} \right) \Rightarrow \left(\sum_{a,b} F_{ba}^{(m)} 1_b^- 1_a^+ + \sum_a F_a^{(m)} 1_a^+ + F_0^{(m)} \right) J(\nu) = 0$$

$$\begin{cases} 1_a^+ J(\nu) = \nu_a J(\nu + e_a) \\ 1_b^- J(\nu) = J(\nu - e_b) \end{cases}$$

- Linear Relations from Full Derivatives

➢ Difficulties: Non-commutativity $[1_b^-, 1_a^+] = \delta_{ab}$

- Laporta algorithms: Large Linear System.
- Symbolic Rules (Smirnov): complexity from non-commutative nature of IBP

S. Laporta: hep-ph/0102033

A.V. Smirnov, V.A. Smirnov: hep-ph/0606247

A.V. Smirnov, V.A. Smirnov: hep-lat/0509187

Large Index Limit & its Expansion

➢ Use large index (ν_a) limit to resolve the structure

- Subtitute $\nu_a \to n + \nu_a$, take limit $n \to \infty$; $\mathrm{j}(\nu) \coloneqq J(n+\nu)$
- Consider leading behavior in 1/n, then make perturbation

➢ Asymptotic behavior at large n limit

- Single out terms involving $1_a^{\pm}$
- Asymptotic IBP made up of commutative difference operator $\Delta_a^{\pm}$
- Linear Difference Equation at leading order

$$\Delta_a^{\pm} j(\nu) := j(\nu \pm e_a)$$

$$[\Delta_a^{\pm}, \Delta_b^{\pm}] = 0, \quad \Delta_a^{+}\Delta_a^{-} = 1$$

$$1_a^{+} = n\Delta_a^{+}(1 + O(1/n))$$

$$(F_{ba}^{(m)}\Delta_b^{-}\Delta_a^{+} + F_a^{(m)}\Delta_a^{+})j(\nu) = O(1/n)$$

Unperturbed Solution: Characteristic Equation

➢ IBP in leading order of 1/n

$$(F_{ba}^{(m)}\Delta_b^-\Delta_a^+ + F_a^{(m)}\Delta_a^+)j(\nu) = O(1/n)$$

- Opartors $\Delta_a^\pm$, $f^m \equiv \sum F_{ba}^{(m)}\Delta_b^-\Delta_a^+ + \sum F_a^{(m)}\Delta_a^+$ commute with each other
- Feynman Integrals (FIs) are lying in the zero eigenspaces of $f^{(m)}$

$$j(\nu) = \sum_k C_k j^{(k)}(\nu)$$

⟹ FIs can be expressed as linear combination of $\Delta_a^\pm$'s eigenfunctions.

$$\Delta_a^+ j^{(k)} = A_a^{(k)} j^{(k)}$$

➢ Eigenvalue problem

- Eigenvalues of operators $\Delta_a^\pm$ satisfy characteristic equations:

$$\boxed{\begin{aligned} F_{ba}^{(m)}B_bA_a + F_a^{(m)}A_a &= 0, \quad \forall m \\ A_aB_a - 1 &= 0, \quad \forall a \end{aligned}}$$

$$\Delta_a^\pm J^{(k)} = (A_a^{(k)})^{\pm 1} J^{(k)}$$

- General solutions can be generated by action of $\Delta_a^\pm$:

$$j(\nu) = \sum_k C_k \prod_a (A_a^{(k)})^{\nu_a}$$

Perturbed Solution: Expansion by recursion

➢ IBP in all orders of n

- Keep all orders in 1/n:

$$\sum_a \left(\sum_b F_{ba}^{(m)} \Delta_b^- + F_a^{(m)} \right) \Delta_a^+ j(\nu) + \frac{1}{n} \left(\sum_{a,b} F_{ba}^{(m)} \Delta_b^- \Delta_a^+ \nu_a + \sum_a F_a^{(m)} \Delta_a^+ \nu_a + F_0^{(m)} \right) j(\nu) = 0$$

- Substitute the perturbation ansatz for each eigenfunction.

$$j(\nu) = \prod_a (A_a)^{\nu_a} \sum_{k=0} \frac{h_k(\nu)}{n^k} \qquad h_k(\nu) = \sum_{\vec{\alpha}} c_{\vec{\alpha}} \prod_a \nu_a^{\alpha_a}$$

- Matching coefficients in full order IBP. Starting from $h_0 = 1$, h_k at each order can be reconstructed recursively.

$$\sum_a \left(\sum_b \tilde{F}_{ba}^{(m)} \Delta_b^- + \tilde{F}_a^{(m)} \right) \Delta_a^+ h_{k+1}(\nu) + \left(\sum_{a,b} \tilde{F}_{ba}^{(m)} \Delta_b^- \Delta_a^+ \nu_a + \sum_a \tilde{F}_a^{(m)} \Delta_a^+ \nu_a + F_0^{(m)} \right) h_k(\nu) = 0$$

with $\tilde{F}_{ba}^{(m)} = F_{ba}^{(m)} B_b A_a, \quad \tilde{F}_a^{(m)} = F_a^{(m)} A_a$

Example

➢ Ex. Bubble Integral

- Propagators defined by $D_1 = l^2 + 1, \quad D_2 = (l+p)^2, \quad p^2 = s$
- Two Eigenvalues (for s=9): $(A_1^{(1)}, A_2^{(1)}) = (1/2, 1/4), \quad (A_1^{(2)}, A_2^{(2)}) = (4/5, 4/25)$
- Perturbation around 1st Eigenfunctions (1/2,1/4):

$$
\begin{aligned}
h_1(\nu_1,\nu_2) =& c_{00}^{(1)} - 3\nu_1 + \frac{5\nu_2}{2} - \nu_1^2 + 2\nu_1\nu_2 - \nu_2^2 \\
h_2(\nu_1,\nu_2) =& \frac{\nu_1^4}{2} - 2\nu_1^3\nu_2 + 3\nu_1^2\nu_2^2 - 2\nu_1\nu_2^3 + \frac{\nu_2^4}{2} \\
& - \frac{7\nu_2^3}{2} - \frac{27\nu_1^2\nu_2}{2} + 12\nu_1\nu_2^2 + 5\nu_1^3 \\
& + \nu_2^2\left(\frac{167}{8} - c_{00}^{(1)}\right) + \nu_1\nu_2\left(-\frac{91}{2} + 2c_{00}^{(1)}\right) + \nu_1^2(25 - c_{00}^{(1)}) \\
& + \nu_1\left(\frac{209}{2} - 3c_{00}^{(1)}\right) + \nu_2\left(-\frac{797}{8} + \frac{5c_{00}^{(1)}}{2}\right) + c_{00}^{(2)}
\end{aligned}
$$

Notice: Some coefficient cannot be determined by IBP, but they don't affect the reconstruction of linear relations for reduction.

Summary: Large Index Expansion

- **Large index (ν_a) limit:**
  - resolve the structure, separate commutative & non-comm. behavior
- **Leading order (unperturbed part):**
  - determined by Characteristic Equation of commutative difference operators.
- **Higher order (perturbation):**
  - recursive expansion coefficient calculation
- **Reduction Relation:**
  - using expansion to reconstruct relation suitable for integral reduction.

Surprising Connection to Syzygy-IBP

➢ Characteristic Equation & Syzygy

- Q: How to better understand the information encoded in characteristic equations? (and to facilitate its solution?)

Baikov Rep:

$$\int [dz] \frac{1}{\prod_a z_a^{\nu_a}} B(z)^{w(d)}$$

  - Consider Syzygy-IBP in Baikov Representation. A syzygy vector (g_a, g_0) defined by:

$$\sum_a g_a(z) \frac{\partial B(z)}{\partial z_a} = g_0(z) B(z)$$

  - Syzygy-constrained IBP:

$$0 = \int [dz] \left(\frac{\partial g_a(z)}{\partial z_a} + w(d) g_0(z) \right) B(z)^{w(d)}$$

  - Syzygy-IBP In operator form:

$$\left(1_a^+ g_a(1_b^-) + w(d) g_0(1_b^-)\right) J(\nu) = 0$$

  - Taking 1_a^+ -related part, syzygy g_a can give an alternate derivation of Char. Eqns.;

$$g_a^{(m)}(z) = \sum_b F_{ba}^{(m)} z_b + F_a^{(m)}$$

  - Moreover, linear syzygies ($\deg g_a \leq 1$) gives exactly the same Char. Eqns. as in previous derivation.

Now Char. Eqns. can be written as:

$$f^{(m)}(A, B) = \sum_a g_a^{(m)}(B) A_a = 0$$

Elimination of Char.Eqns & Module Intersection

➢ Intersected Syzygy Module

J. Böhm, A. Georgoudis, K.J. Larsen, H. Schönemann, Y. Zhang: hep-th/1805.01873

- Further constraint put on syzygy-IBP, make combination of $\{g^{(m)}\}_{m=1}^{K}$ so that no increment of power in denominators.

$$h_a(z) = \sum_m p_m g_a^{(m)}(z), \quad h_a \propto z_a$$

- In other word: $$h \in M_{int} = M_{syz} \cap M_{pd}, \quad M_{syz} = \left\langle g^{(m)} \right\rangle_{m=1}^{K}, \quad M_{pd} = \langle z_a e_a \rangle_{a=1}^{N}$$

➢ Module Intersection -> Elimination

- For the solution of Char. Eqns, module intersection gives a set of A-eliminated equations.

$$\tilde{h} = \sum_m p_m f^{(m)} = \sum_{a,m} p_m(B) g_a^{(m)}(B) A_a = \sum_a h_a(B) A_a = \sum_a \frac{h_a(B)}{B_a}$$

- As Groebner basis in lex. order of A-eliminated equations usually gives explicit form of the solutions $h_N(B_N) = 0, B_i = h_i(B_N)$.

Elimination of Char.Eqns & Module Intersection

➢ Inverse Problem: Elimination -> Module Intersection

- Inverse Problem: Can we constructed construct a module-intersected syzygy $h_a{}'(z)$ from each A-eliminated Equation $h(B)$ from $\{f^{(m)}(A,B)\}_{m=1}^{K}$? Is $h \mapsto h'$ a map (unique assignment exists)?

- Answer: h' exists, but not unique: they can differ by $h''(z)$, s.t $\sum_a h''_a/z_a = 0.$

➢ Summary

- Characteristic equations share the same structure and information as linear syzygy, both from 1_a^+-related part of the IBP.
- Module intersection offers a referential method to solve the characteristic equation (elimination of A's).

Summary and outlook

- IBP is the main obstacle of precise computation
- The new representation provides various ways of asymptotic expansion
- Large index expansion
- Optimistic to overcome multi-leg FIs computation beyond one-loop, and to meet the requirement of high-precision LHC data

Thank you !

Stay tuned!