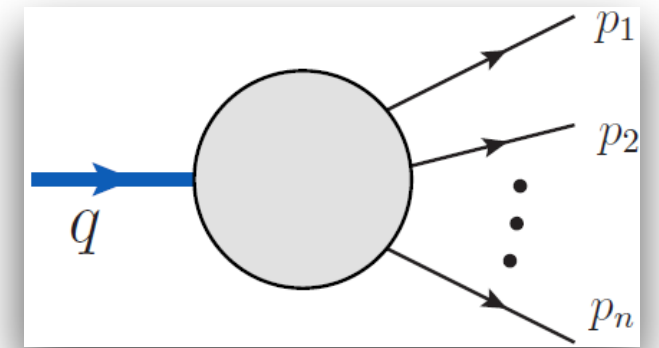




形状因子及其应用

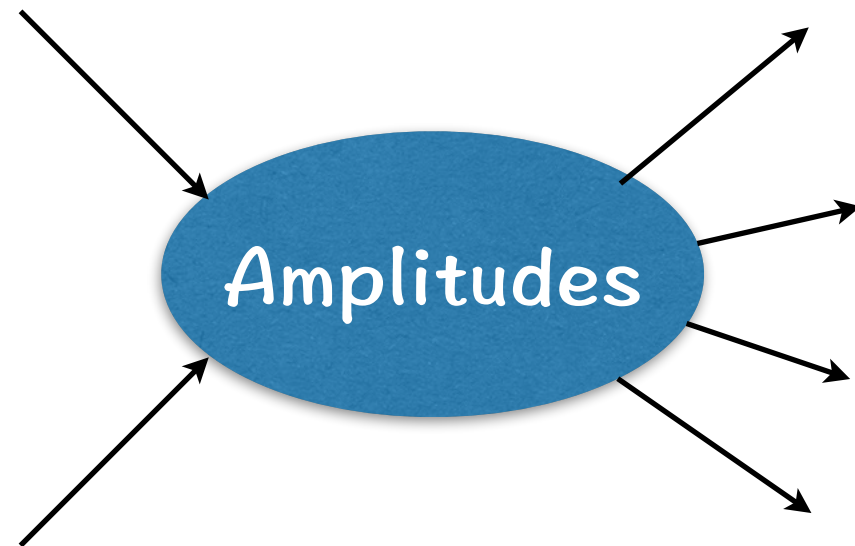
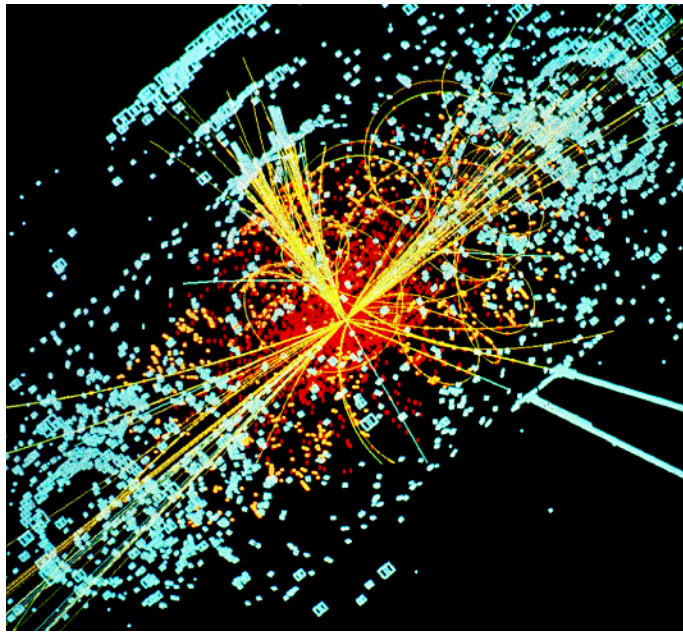
Gang Yang

ITP-CAS



第五届量子场论及其应用研讨会
2025 年 10 月 30 日至 11 月 2 日，北京

Scattering amplitudes

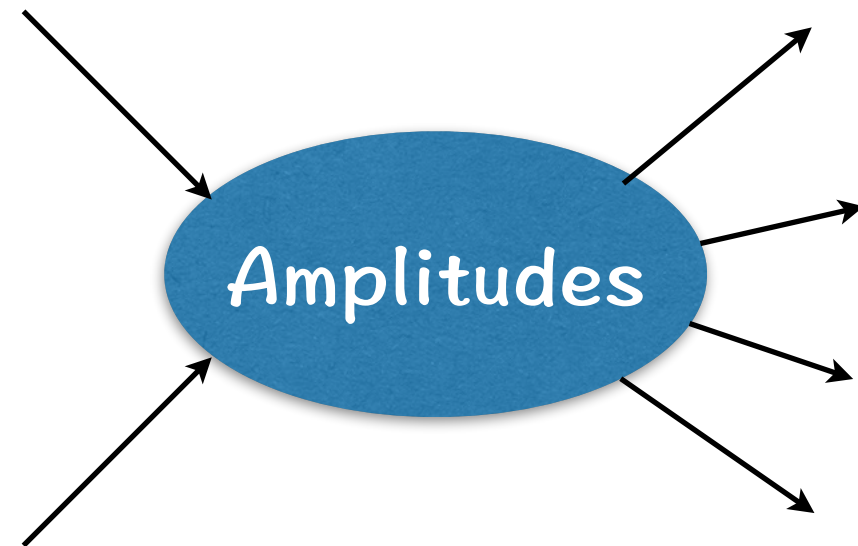
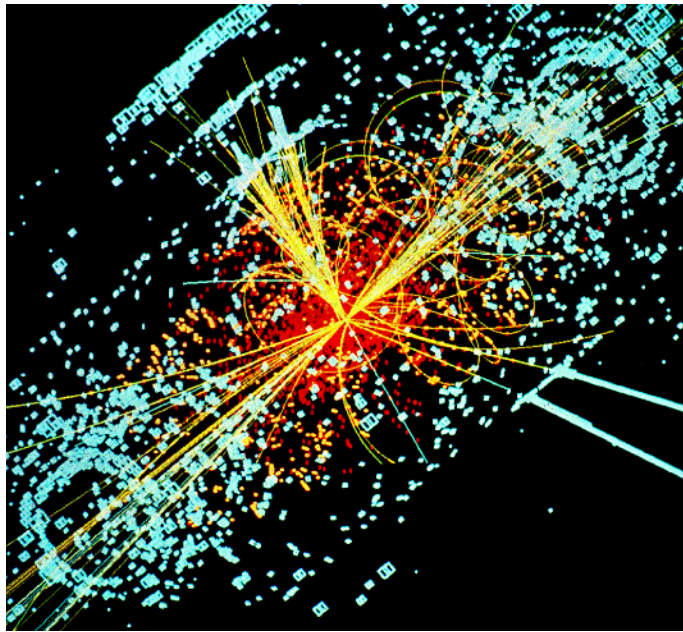


In past over 30 years, significant progress has been made in the studies of scattering amplitudes.

[Parke, Taylor, 1986]

$$A_n^{\text{tree}}(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

Scattering amplitudes

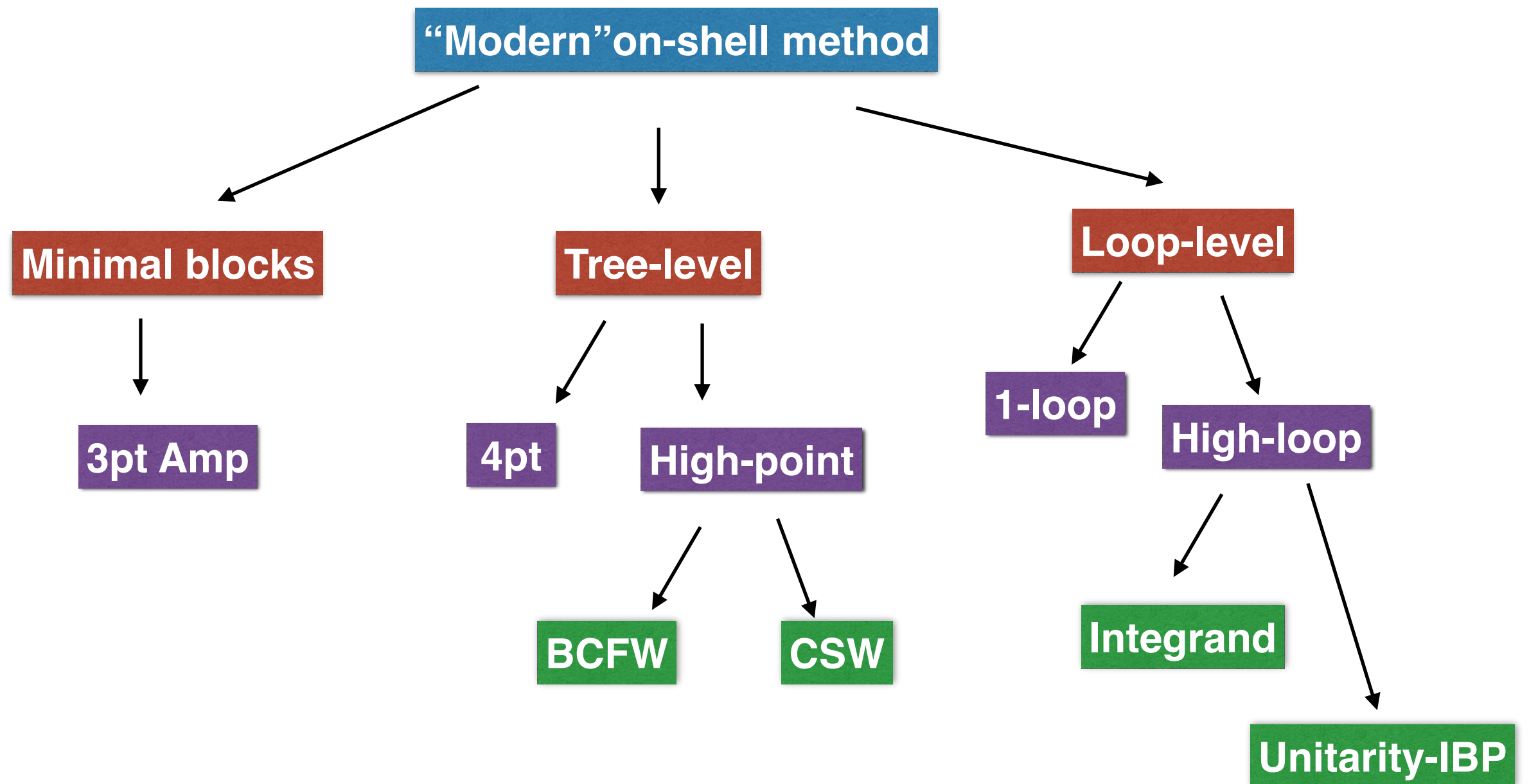


In past over 30 years, significant progress has been made in the studies of scattering amplitudes.

New structures

New methods

“On-shell” method



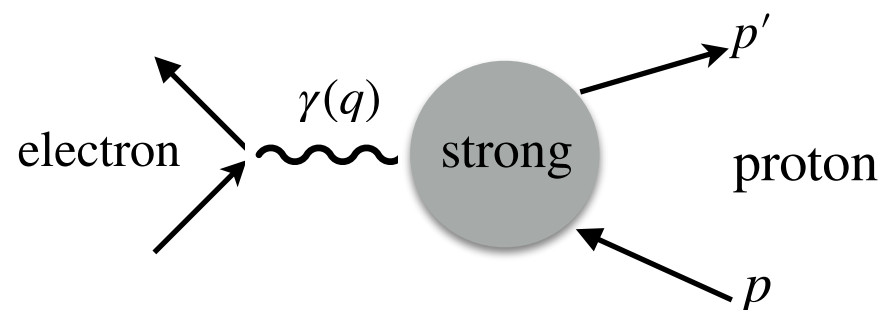
Amplitudes



Form Factors

Aspects of form factors

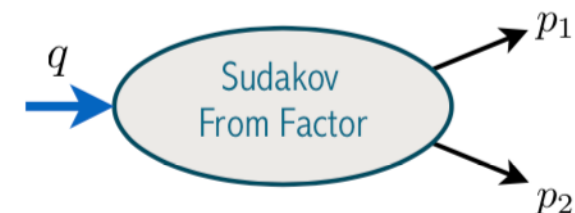
Nuclear form factor



$$F = \frac{4\pi}{q} \int_0^\infty \rho(r) \sin(qr) r dr$$

$$\bar{u}(p') \left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(p)$$

Sudakov form factor



$$\Gamma_\sigma(p, q; l) = \gamma_\sigma \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{e^2}{2\pi} \ln \left| \frac{l^2}{p^2} \right| \ln \left| \frac{l^2}{q^2} \right| \right)^n$$

$$= \gamma_\sigma \exp \left\{ -\frac{e^2}{2\pi} \ln \left| \frac{l^2}{p^2} \right| \ln \left| \frac{l^2}{q^2} \right| \right\}.$$

IR singularities

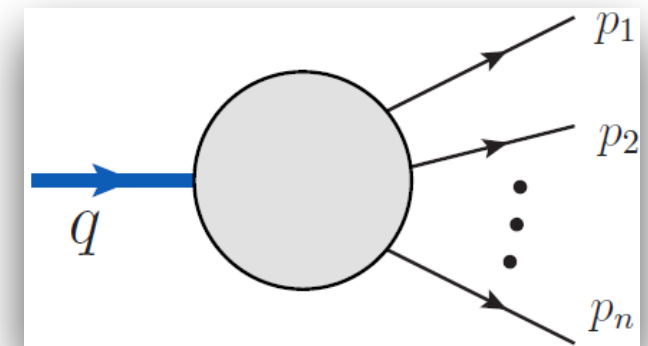
“Generalized” form factors

$$F_{n,\mathcal{O}}(1, \dots, n) = \int d^4x e^{-iq \cdot x} \langle p_1 \dots p_n | \mathcal{O}(x) | 0 \rangle$$

“Generalized” form factors

Hybrids of on-shell states and off-shell operators:

$$\begin{aligned} F_{n,\mathcal{O}}(1,\dots,n) &= \int d^4x e^{-iq\cdot x} \langle p_1 \dots p_n | \mathcal{O}(x) | 0 \rangle \\ &= \delta^{(4)}\left(\sum_{i=1}^n p_i - q\right) \langle p_1 \dots p_n | \mathcal{O}(0) | 0 \rangle \end{aligned}$$



$$q = \sum_i p_i, \quad q^2 \neq 0$$

form factors

$$\langle p_1 p_2 \dots p_n | 0 \rangle$$

Scattering amplitude



$$\langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle$$

Correlation functions

Applications of form factors

- Operator classification and spectrum
- EFT amplitudes
- IR divergences (Sudakov FF)
- Correlation functions (EEC, etc..)
- New hidden structures beyond amplitudes

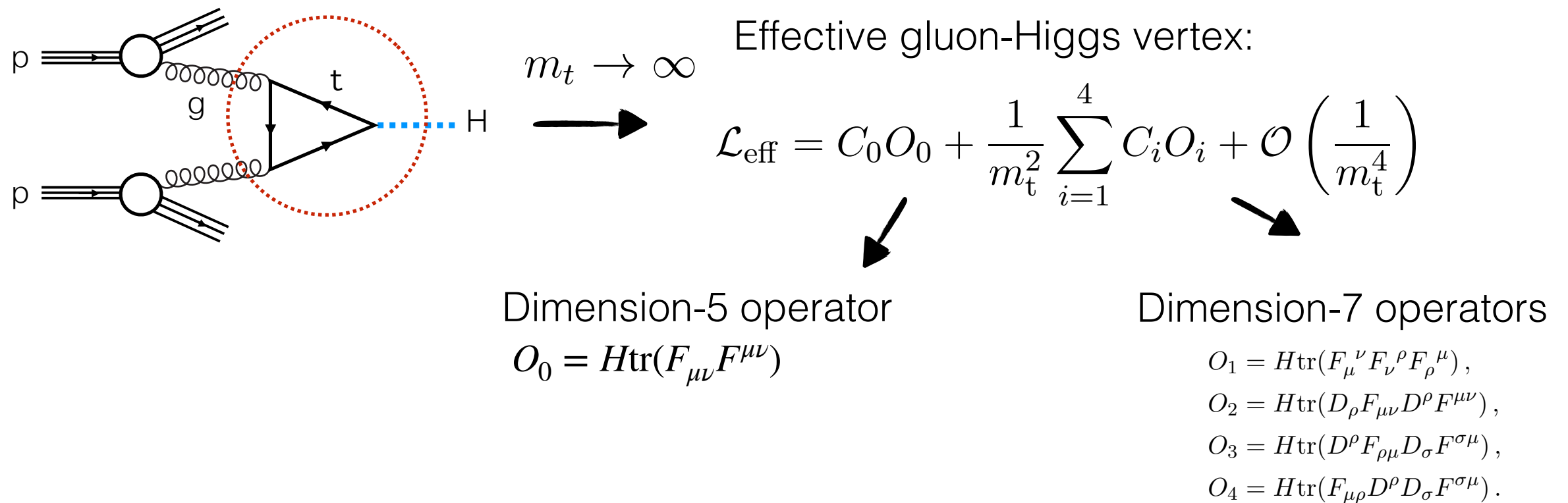
Loop form factor = (Universal IR div.) + (UV div.) + (Finite part)

Sudakov form factor

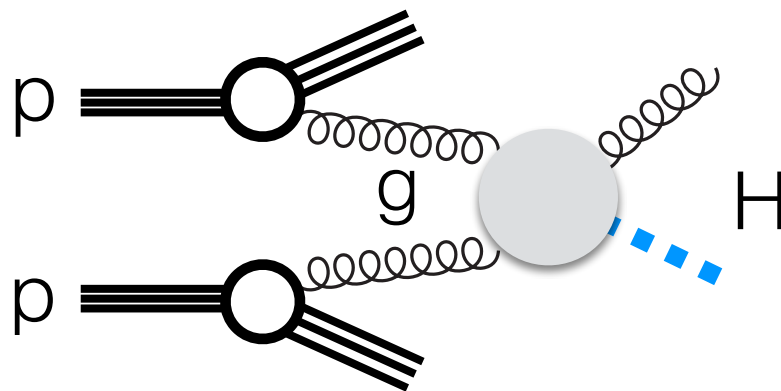
Renormalization

EFT amplitudes

“Higgs” EFT



Higgs plus jet production



$$A(q^H, 1^g, 2^g, \dots, n^g) = F_{\mathcal{O}=\text{tr}(F^2)}(1^g, 2^g, \dots, n^g)$$

Tree-level form factors

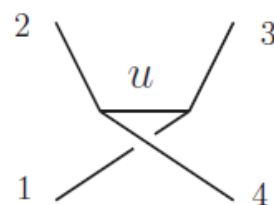
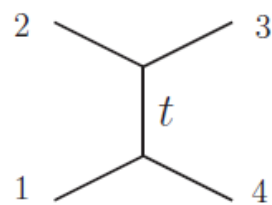
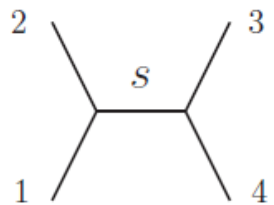
A warm-up example

Scattering amplitudes in massless scalar theory with ϕ^3 interaction:

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle \longrightarrow \text{(LSZ reduction)}$$

$$\left(\prod_{i=1}^4 \int d^4x_i e^{ip_i x_i} (\partial_{x_i}^2)\right) \langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = A_4(p_1, p_2, p_3, p_4)$$

$$\langle T\phi(x_1)\phi(x_2)\rangle = \Delta(x_1-x_2), \quad (\partial_{x_1}^2)\Delta(x_1-x_2) = \delta^4(x_1-x_2)$$



$$0 = \sum_i p_i, \quad p_i^2 = 0$$

$$A_4 = \delta^4(p_1+p_2+p_3+p_4) \left(\frac{1}{(p_1+p_2)^2} + \frac{1}{(p_2+p_3)^2} + \frac{1}{(p_1+p_3)^2} \right)$$

A warm-up example

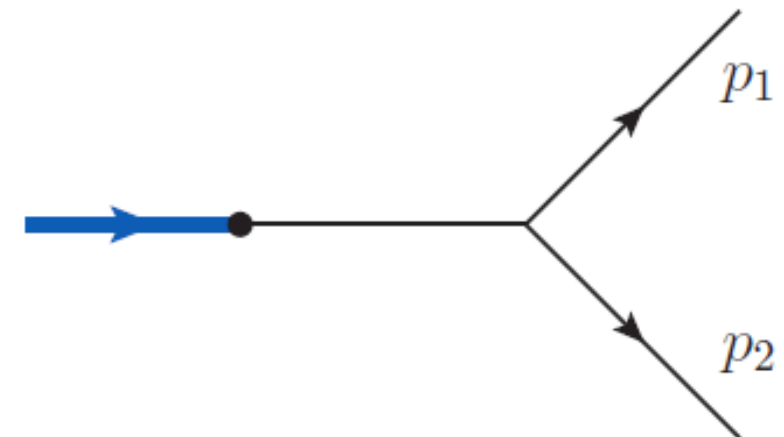
How about LSZ reduction for part of the fields?

$\langle 0|T\phi(x_1)\phi(x_2)\phi(y)|0\rangle \longrightarrow$ LSZ reduction for the first two fields, but only Fourier transformation for the third

$$\left(\int d^4y e^{-iqy} \prod_{i=1}^2 \int d^4x_i e^{ip_i x_i} (\partial_{x_i}^2) \right) \langle 0|T\phi(x_1)\phi(x_2)\phi(y)|0\rangle$$

$$= \left(\int d^4z \int d^4y e^{-iqy + i(p_1 + p_2)z} \right) \Delta(y - z)$$

$$= \delta^4(p_1 + p_2 - q) \frac{1}{(p_1 + p_2)^2}$$



$$q = p_1 + p_2, \quad p_i^2 = 0$$

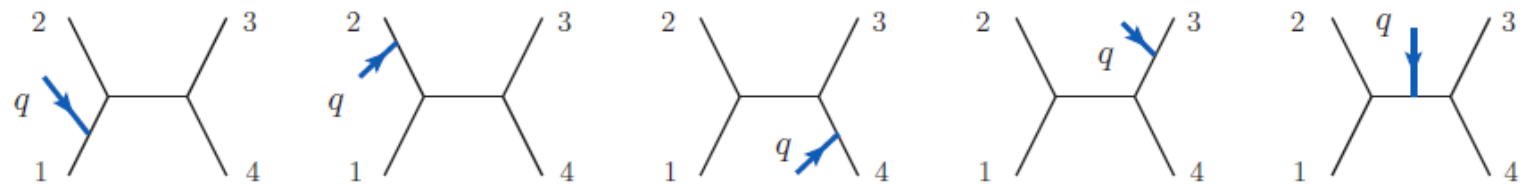
Composite operator

$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\text{tr}(\phi^2(y))|0\rangle \longrightarrow$ LSZ reduction for elementary fields,
Fourier transformation for the operator

$$\left(\int d^4y e^{-iqy} \prod_{i=1}^4 \int d^4x_i e^{ip_i x_i} (\partial_{x_i}^2) \right) \langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\text{tr}(\phi^2(y))|0\rangle$$

$$\longrightarrow F_4(p_1, p_2, p_3, p_4; q) = \delta^4 \left(\sum_{i=1}^4 p_i - q \right) \left(\frac{1}{s_{34}s_{234}} + \frac{1}{s_{34}s_{134}} + \dots \right)$$

$$q = \sum_{i=1}^4 p_i, \quad p^2 = 0, \quad q^2 \neq 0$$



+ (t, u channel like diagrams)

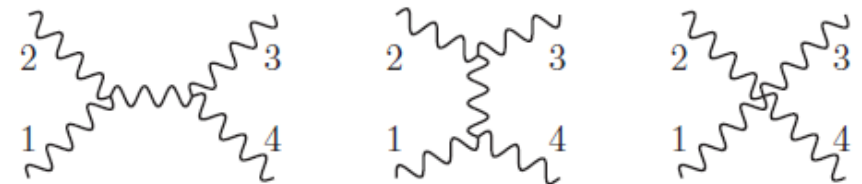
MHV form factor ?

MHV (color ordered) amplitudes (Parke-Taylor):

$$A_{\text{MHV}}(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = \delta^4\left(\sum_i p_i\right) \frac{\langle i j \rangle^4}{\langle 1 2 \rangle \cdots \langle n 1 \rangle}$$

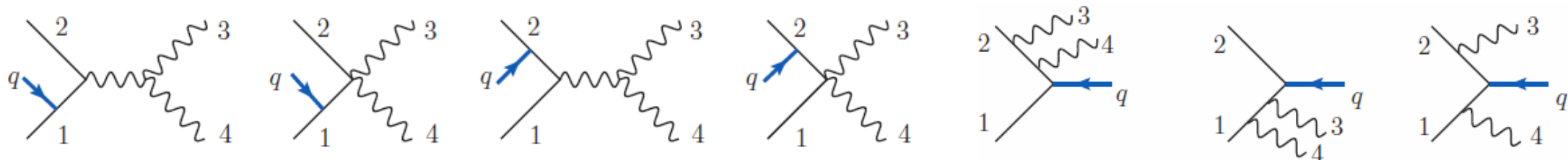
$$0 = \sum_i p_i, \quad p_i^2 = 0$$

(firstly found by computing Feynman diagrams)



Do we have MHV formula for (color ordered) form factor ?

A **four**-point example (in N=4): $F_4\left(\phi(p_1), \phi(p_2), g^+(p_3), g^+(p_4); \text{tr}(\phi^2)(q)\right)$



→
$$\delta^4\left(\sum_{i=1}^4 p_i - q\right) \frac{\langle 1 2 \rangle^2}{\langle 1 2 \rangle \cdots \langle 4 1 \rangle}$$

$$q = \sum_i p_i, \quad p_i^2 = 0, \quad q^2 \neq 0$$

MHV tree-level form factors

$$F_n(1^+, \dots, i_\phi, \dots, j_\phi, \dots, n^+; \text{tr}(\phi^2)(q))$$

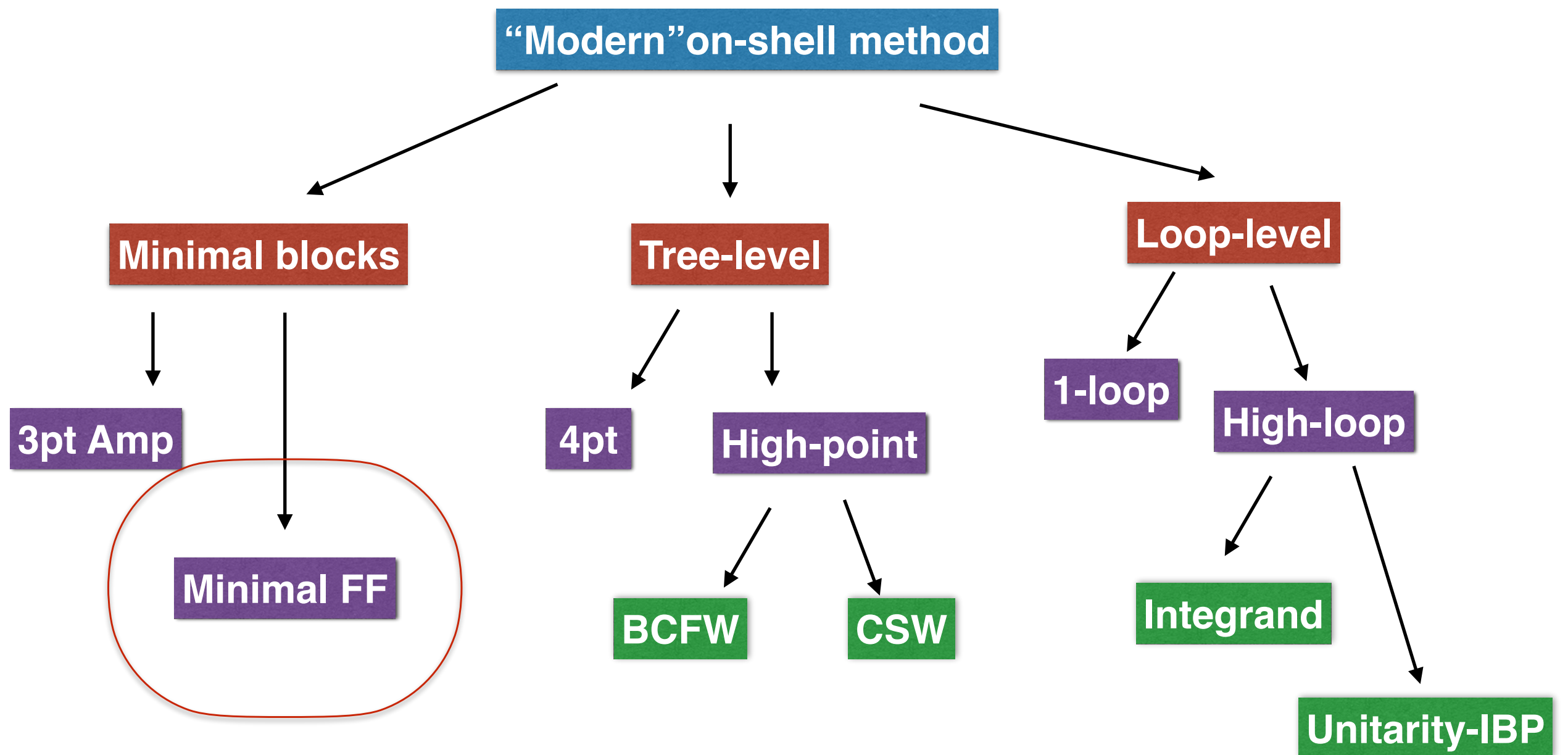
$$= \delta^n \left(\sum_{i=1}^n p_i - q \right) \frac{\langle i \ j \rangle^2}{\langle 1 \ 2 \rangle \cdots \langle n \ 1 \rangle}$$

$$q = \sum_i p_i, \quad p_i^2 = 0, \quad q^2 \neq 0$$

MHV like structure implies the **underlying simplicity** of form factor !

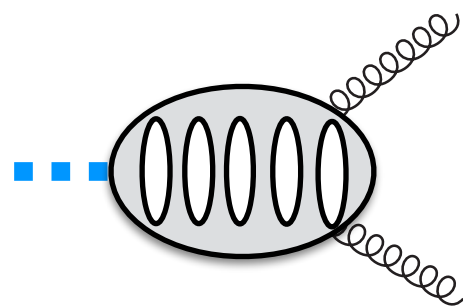
MHV rules, BCFW recursion relation, unitarity method can be applied efficiently.

“On-shell” method



Loop-level form factors

$$\mathcal{F}_n = \int d^4x e^{-iq \cdot x} \langle p_1, \dots, p_n | \text{tr}(F^2)(x) | 0 \rangle$$

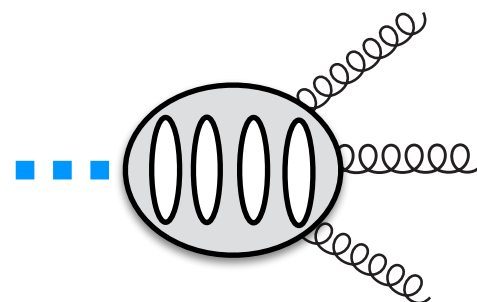


Full-color integrand up to 5 loops

Boels, Kniehl, Tarasov, GY 2012
GY, 2016

Integrated results at 4 loops

Boels, Huber, GY 2017
Huber, von Manteuffel, Panzer, Schabinger, GY 2020

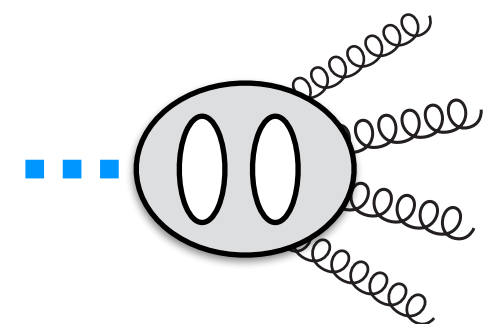


Full-color integrand up to 4 loops

Lin, GY, Zhang, 2021

Integrated results at 3 loops

Lin, GY, Zhang, 2021
Guan, Lin, Liu, Ma, GY 2023



Integrated results at 2 loops

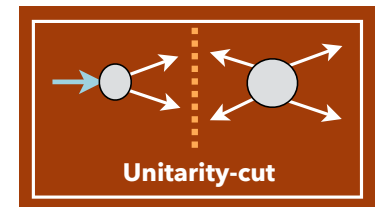
Guo, Wang, GY, 2022
Guo, Wang, GY, Yin 2024

See also: Dixon, Gurdogan, Liu, McLeod, Wilhelm 2021, 2022; Dixon, Xin 2024

Computational methods

- On-shell unitarity method Bern, Dixon, Durban, Kosower 1994;
Britto, Cachazo, Feng, 2003

Simple tree blocks -> Higher loop results



- Color-kinematics duality Bern, Carrasco, Johansson 2008

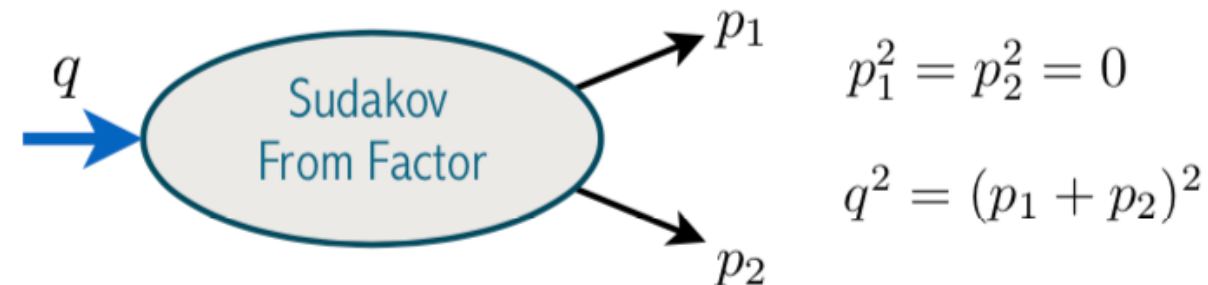
Large number of diagrams -> Very few “master” diagrams

- Master-integral bootstrap Guo, Wang, GY 2021

Construct final results directly using physical constraints

Sudakov form factor

Sudakov form factor



Logarithm behavior is well-understood:

For dim-reg representation, see:
Magnea and Sterman 1990;
Sterman and Tejeda-Yeomans 2002
Bern, Dixon, Smirnov 2005

$$\log F_2(1,2) \simeq - \sum_{l=1}^{\infty} g^{2l} \left(\frac{\gamma_{\text{cusp}}^{(l)}}{\epsilon^2} + \frac{\mathcal{G}_{\text{coll}}^{(l)}}{\epsilon} \right) (-q^2)^{-l\epsilon} + \mathcal{O}(\epsilon^0)$$

Leading IR singularity -> Cusp anomalous dimension

Color structure

Up to three loops, only quadratic Casimir appears:

<i>L-loop</i>	$L=1$	$L=2$	$L=3$	$L=4$
<i>Color Factor</i>	C_A	C_A^2	C_A^3	C_A^4, d_{44}

For $SU(N)$:

$$C_A = N$$

$$d_{44} = \frac{N^2(N^2 + 36)}{24}$$

At four-loop, there is a new quartic Casimir which contains non-planar part

Casimir scaling conjecture

In "On the Structure of Infrared Singularities of Gauge- Theory Amplitudes",
[JHEP 0906, 081 \(2009\)](#)

Thomas Becher and Matthias Neubert conjectured that:

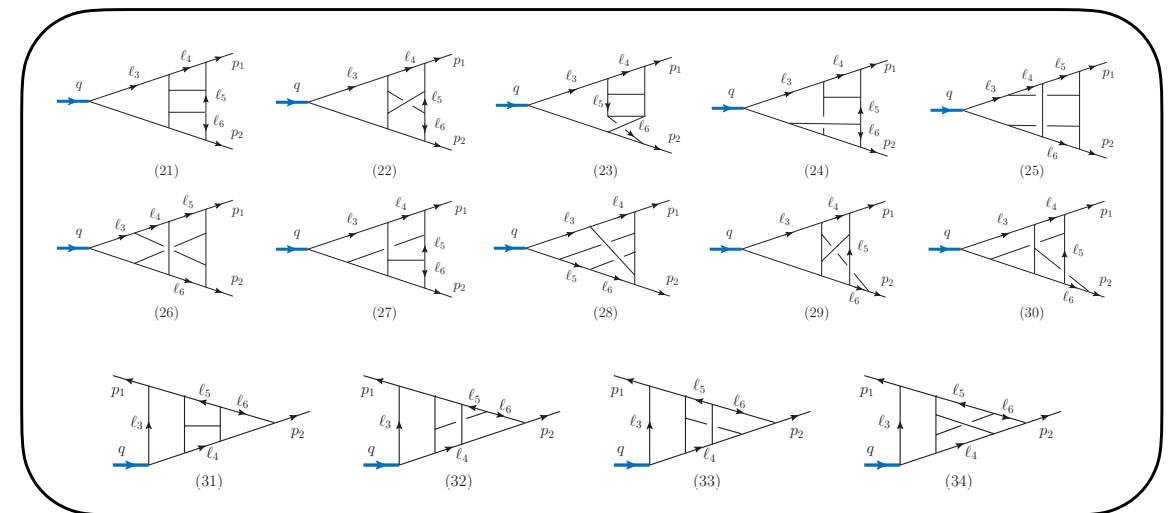
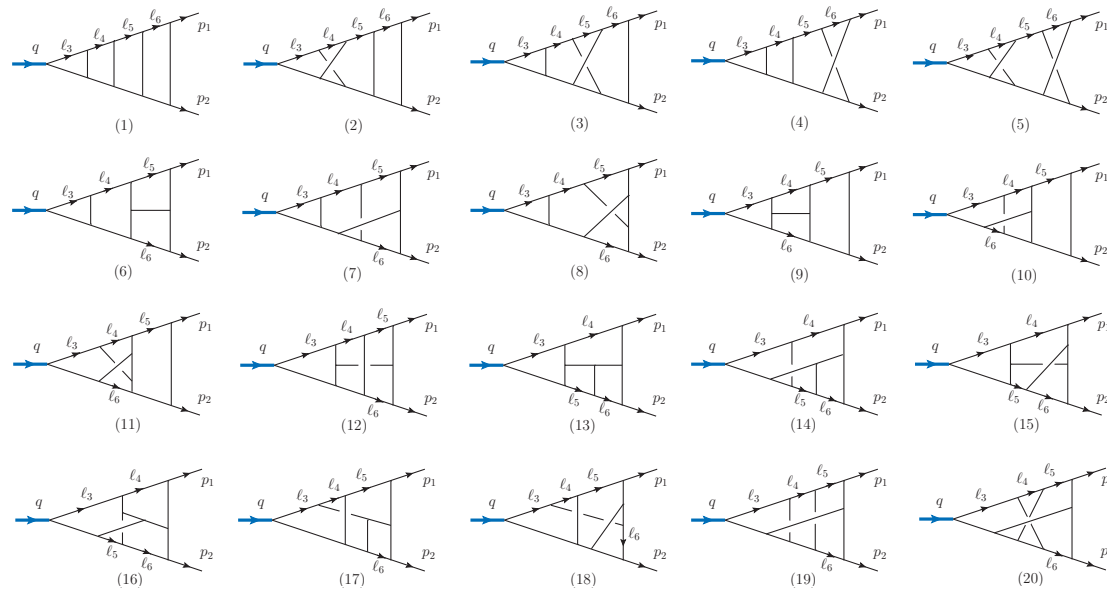
*“Our formula predicts Casimir scaling of the cusp anomalous dimension to all orders in perturbation theory, and we explicitly check that the constraints **exclude the appearance of higher Casimir invariants at four loops.**”*

An explicit four-loop computation is needed.

Four-loop compact integrand

[Boels, Kniehl, Tarasov, GY 2012]

Four-loop form factor integrand was obtained by:
color-kinematics duality and **unitarity**:

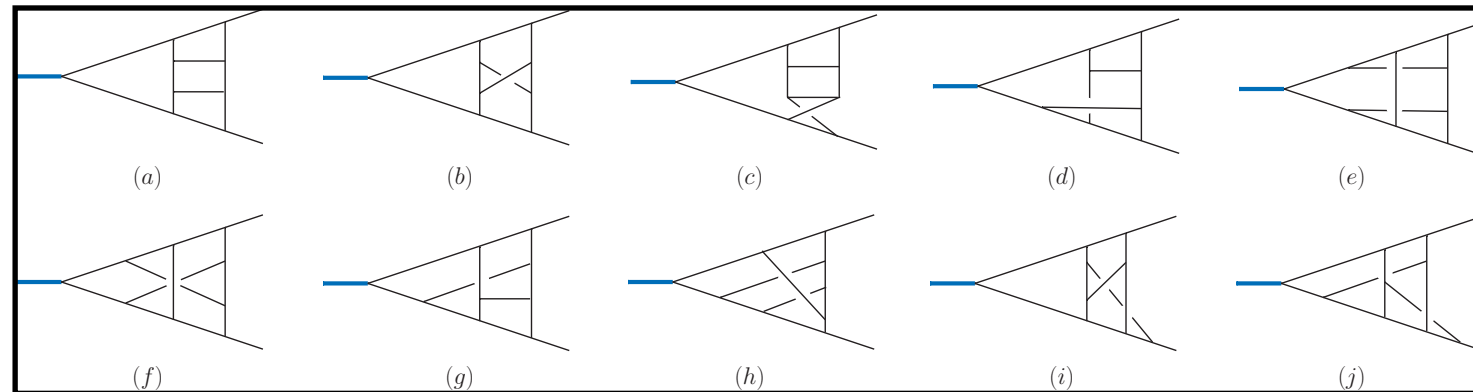


non-planar

compact form and with only quadratic loop momenta in the numerator.

$$\begin{aligned}
 N_{21} = & -(\ell_3 \cdot p_1)^2 - (\ell_3 \cdot p_2)^2 - 6(\ell_3 \cdot p_1)(\ell_3 \cdot p_2) \\
 & + (p_1 \cdot p_2)[2(\ell_3 \cdot \ell_3) + 4(\ell_3 \cdot p_1) + p_1 \cdot p_2] \\
 & + (\alpha_1 + 1)[(\ell_3 \cdot p_{12} - p_1 \cdot p_2)^2 \\
 & - \frac{2}{7}(\ell_3 \cdot (\ell_3 - p_{12}) + p_1 \cdot p_2)(p_1 \cdot p_2)]
 \end{aligned}$$

Four-loop Sudakov form factor



- Numerical integration: [Boels, Huber, GY 2017](#)

$$\gamma_{\text{cusp, NP}}^{(4)} = -3072 \times (1.60 \pm 0.19) \frac{1}{N_c^2}$$

Finding **Uniform Transcendental (UT) basis** is the key

- Analytic integration: [Huber, von Manteuffel, Panzer, Schabinger, GY 2020](#)
(See also: [Henn, Korchemsky, Mistlberger 2020](#))

$$\gamma_{\text{cusp, NP}}^{(4)} = -3072 \times \left(\frac{3}{8} \zeta_3^2 + \frac{31}{140} \zeta_2^3 \right) \frac{1}{N_c^2} = -3072 \times 1.52 \frac{1}{N_c^2}$$

Casimir scaling conjecture is incorrect.

Full-color five-loop N=4 integrand is also known. [GY, 2016](#) (N4LL resummation)

High-dimensional operators and two-loop renormalization

- 1804.04653, 1904.07260, 1910.09384, with Qingjun Jin (靳庆军)
- 2011.02494 with Qingjun Jin, Ke Ren (任可) ;
- 2202.08285, 2208.08976; 2301.01786, 2312.08445, 2510.19696 with Qingjun Jin, Ke Ren; Rui Yu (余睿)

High dimensional YM operators

Gauge invariant operators:

$$\mathcal{O} \sim c(a_1, \dots, a_n) (D_{\mu_{11}} \dots D_{\mu_{1m_1}} F_{\nu_1 \rho_1})^{a_1} \dots (D_{\mu_{n1}} \dots D_{\mu_{nm_n}} F_{\nu_n \rho_n})^{a_n}.$$

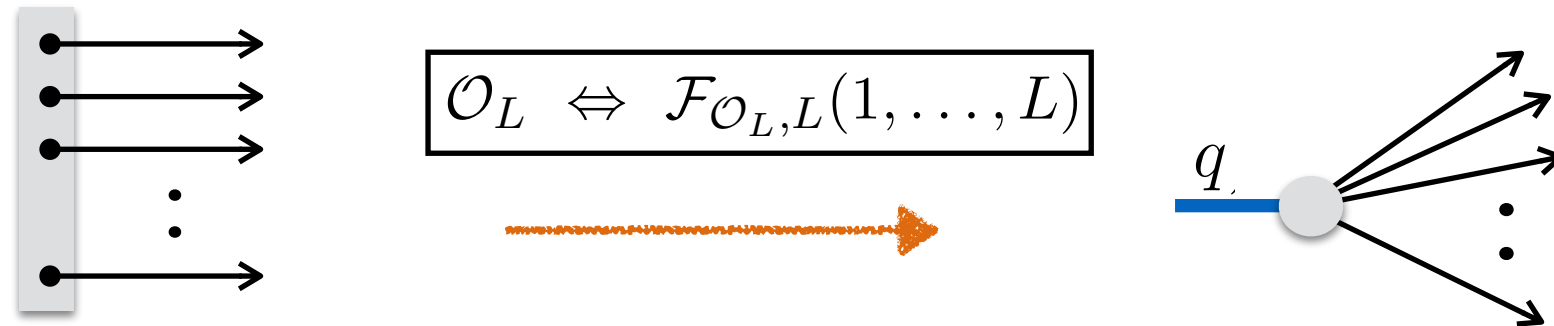
$$D_\mu \star = \partial_\mu + ig[A_\mu, \star], \quad [D_\mu, D_\nu] \star = ig[F_{\mu\nu}, \star] \quad F_{\mu\nu} = F_{\mu\nu}^a T^a, \quad [T^a, T^b] = if^{abc} T^c$$

D-dimensional on-shell methods using form factor formalism:

$$\mathcal{F}_n = \int d^4x e^{-iq \cdot x} \langle p_1, \dots, p_n | \mathcal{O}(x) | 0 \rangle$$

- Operator basis
- Two-loop renormalization and spectrum
- Two-loop EFT amplitudes

Minimal tree form factors



Dictionary for YM operators:

operator	$D_{\dot{\alpha}\alpha}$	$f_{\alpha\beta}$	$\bar{f}_{\dot{\alpha}\dot{\beta}}$
spinor	$\tilde{\lambda}_{\dot{\alpha}}\lambda_{\alpha}$	$\lambda_{\alpha}\lambda_{\beta}$	$-\tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}}$

4-dim

$$F_{\mu\nu} \rightarrow F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta}$$

operator	D_{μ}	$F_{\mu\nu}$
kinematics	p_{μ}	$p_{\mu}\epsilon_{\nu} - p_{\nu}\epsilon_{\mu}$

D-dim

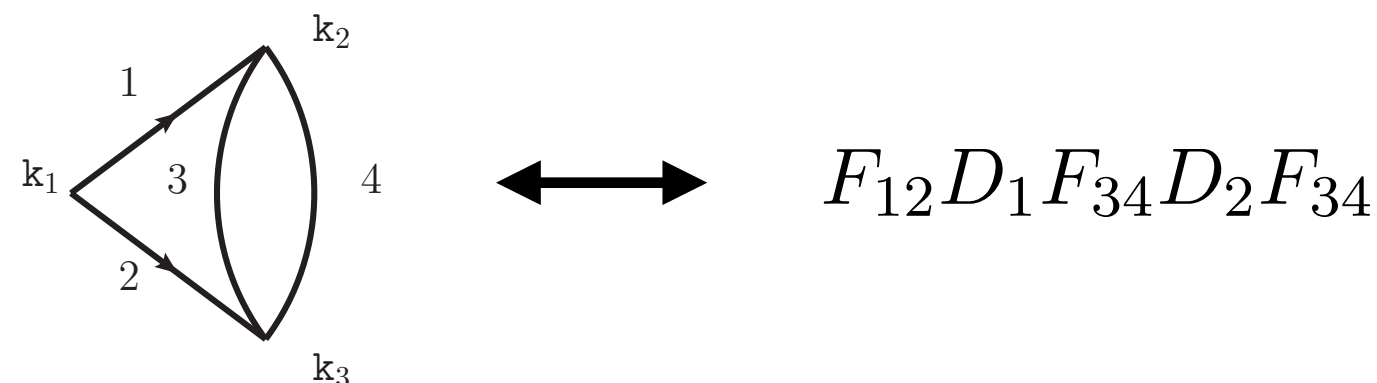
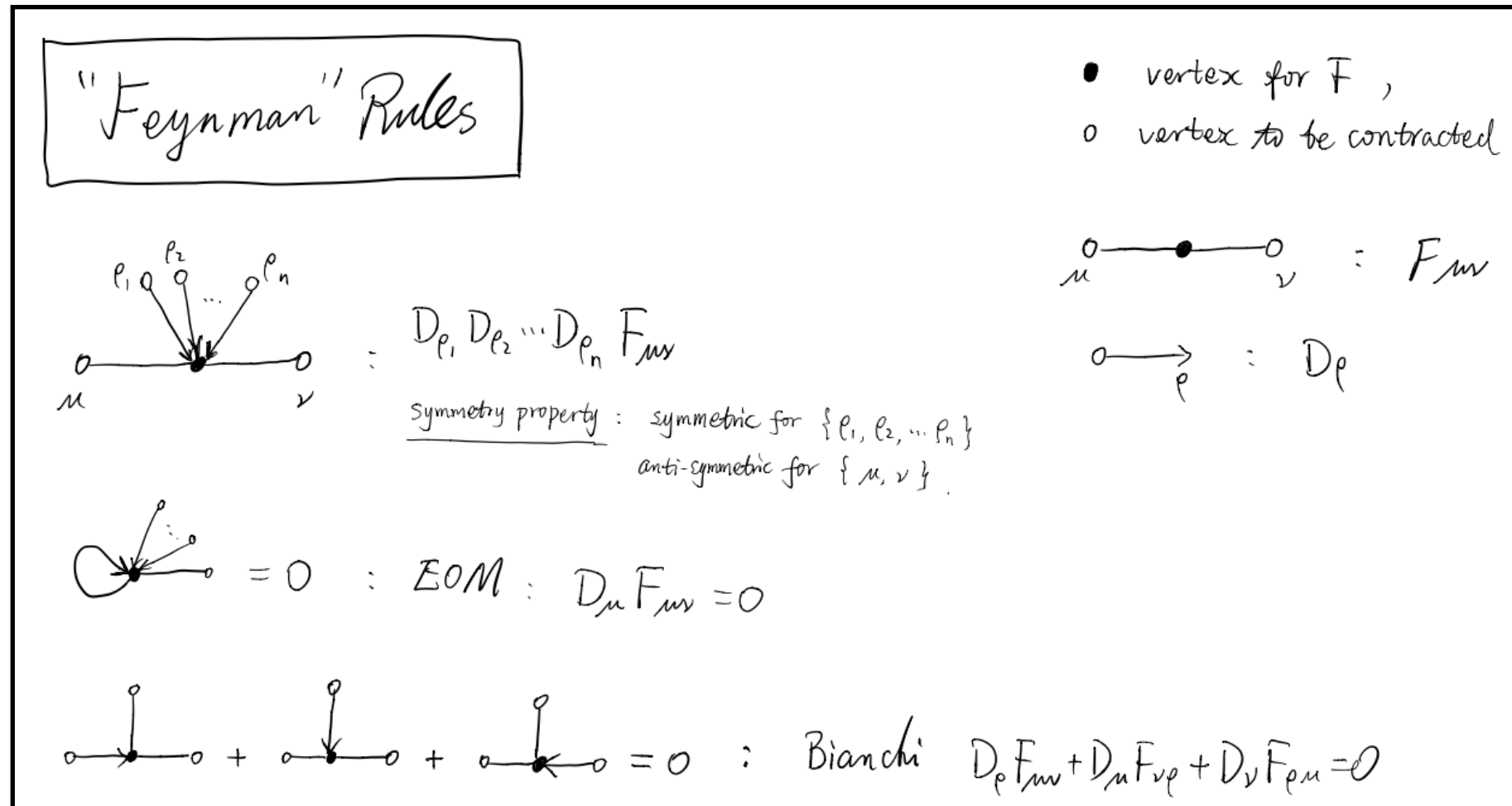
Used in N=4 SYM: Zwiebel 2011, Wilhelm 2014

Important for capturing
“Evanescent operators”

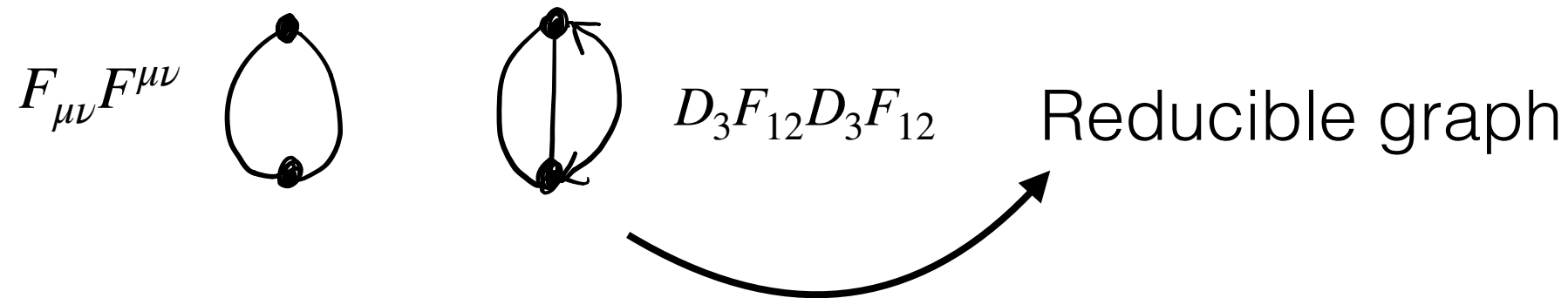
One can translate any local operator into “on-shell” kinematics.

A graphic representation of operators

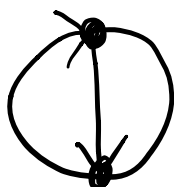
Jin, Ren, GY, Yu 2025



A graphic representation of operators



$$D_1F_{23}D_2F_{31}$$


 seems not reducible, but it is

$$\text{since } \text{circle with two vertices and two arcs} = -\left(\text{circle with two vertices and two arcs} + \text{circle with two vertices and two arcs}\right) \Rightarrow \text{circle with two vertices and two arcs} = -\frac{1}{2} \text{circle with two vertices and two arcs}$$

The operator basis construction is equivalent to the problem of finding a basis for irreducible graphs.

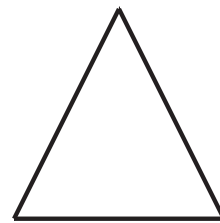
Primitive configurations

Length-2:

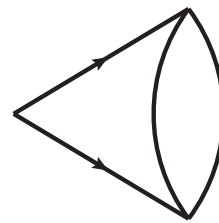


$$F_{12}F_{12}$$

Length-3:



$$F_{12}F_{23}F_{31}$$



$$F_{34}(D_3F_{12})(D_4F_{12})$$

All higher-dimensional operators of length-2 and length-3 are straightforward to obtain.

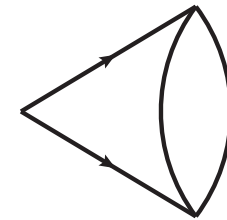
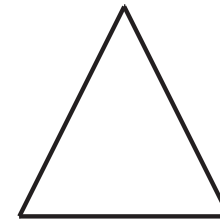
Length Dimension	2	3
4		/
6		
8		
10		
12		

Primitive configurations

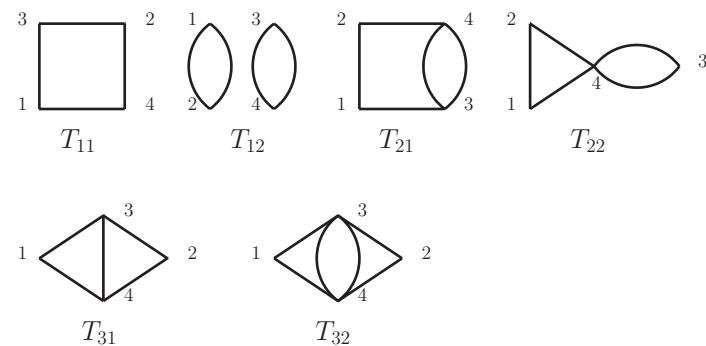
Length-2:



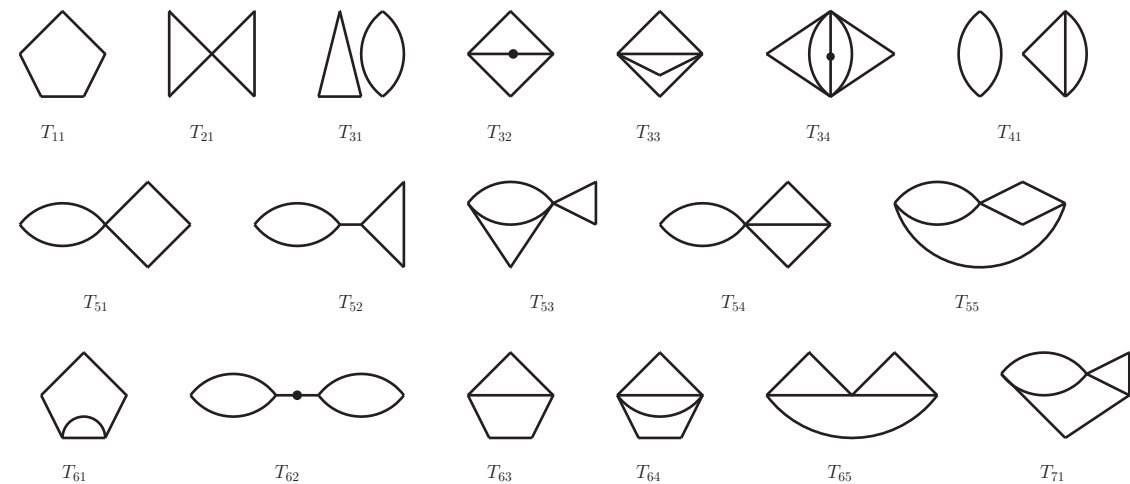
Length-3:



Length-4:



Length-5:



Explicit construction up to length-7.

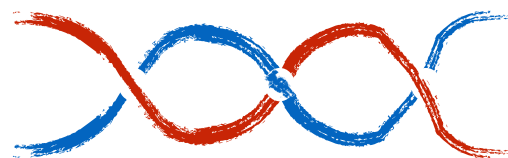
Jin, Ren, GY, Yu 2025

Unitarity-IBP strategy

D-dimensional unitarity-cut:

$$\mathcal{F}^{(l)}|_{\text{cut}} = \prod (\text{tree blocks}) = \text{cut integrand} = \sum_i c_i M_i|_{\text{cut}}$$

On-shell unitarity



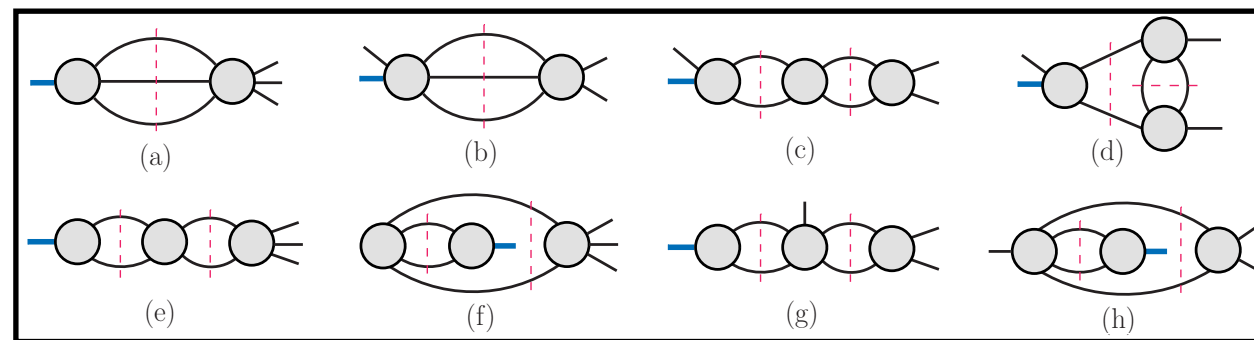
(cut) IBP reduction

Jin, GY 2018 Boels, Jin, Luo 2018

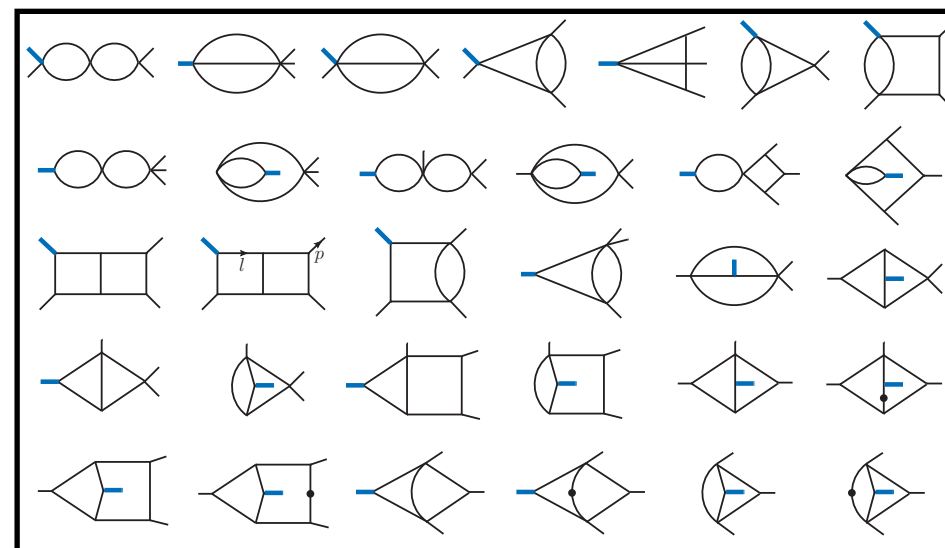
Numerical unitarity: Abreu, Cordero, Ita, Jaquier, Page, Zeng 2017

Unitarity cuts and master integrals

All cuts that are needed:



Master integrals are known in terms of 2d Harmonic polylogarithms.



[Gehrmann, Remiddi 2001]

Mixing matrix and spectrum

Jin, Ren, GY 2020

Dim-16 length-3 operators at 2-loop:

$$Z_{\mathcal{O}_{16,f}}^{(2)} \Big|_{\frac{1}{\epsilon} - \text{part.}} = \frac{N_c^2}{\epsilon} \left(\begin{array}{c|cccccccccccccccc} -\frac{34}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{269}{72} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 0 & 0 & 0 & 0 \\ -\frac{209}{900} & -\frac{5579}{18000} & \frac{712}{125} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1493}{1200} & \frac{5}{36} & 0 & 0 & 0 \\ -\frac{180}{181} & \frac{3600}{60979} & -\frac{28800}{78487} & \frac{3575983}{2177} & \frac{9793}{704167} & 0 & 0 & 0 & 0 & 0 & \frac{13}{1229} & \frac{16877}{115501} & -\frac{7319}{9803} & 0 & 0 \\ -\frac{900}{523} & -\frac{36000}{2201287} & \frac{72000}{605939} & \frac{2000}{64128769} & \frac{72000}{3303367} & \frac{2000}{332422343} & \frac{72000}{6699071} & 0 & 0 & 0 & \frac{1200}{37547} & \frac{43200}{75071} & -\frac{43200}{497} & \frac{103}{1440} & 0 \\ -\frac{3920}{809} & \frac{29635200}{12166789} & \frac{1975680}{11202299} & -\frac{24696000}{73487} & \frac{9878400}{9182209} & \frac{29635200}{37249} & \frac{14817600}{26302879} & 0 & 0 & 0 & \frac{78400}{1613} & \frac{39200}{17401} & -\frac{576}{19} & \frac{1440}{1187} & 0 \\ -\frac{5600}{269} & -\frac{21168000}{125599} & \frac{7056000}{50369} & -\frac{36750}{98317} & -\frac{7056000}{73489} & \frac{156800}{8625329} & \frac{2116800}{97913} & \frac{90760559}{7408800} & \frac{25354501}{21168000} & \frac{40519}{56448} & \frac{3360}{184259} & \frac{6720}{65297} & -\frac{225}{420373} & \frac{2880}{248791} & -\frac{2747}{2747} \\ -\frac{2520}{19717} & \frac{10584000}{3374557} & \frac{1323000}{102465523} & -\frac{1176000}{5260289} & \frac{392000}{6201763} & -\frac{3528000}{115070197} & -\frac{756000}{10687837} & \frac{7408800}{6498287} & \frac{21168000}{1025255701} & \frac{56448}{25511} & \frac{1058400}{347437} & \frac{23520}{863371} & -\frac{211680}{230747} & \frac{235200}{938797} & -\frac{9408}{78243} \\ -\frac{176400}{19717} & \frac{7408800}{2733089} & -\frac{74088000}{88146899} & \frac{1764000}{5678651} & -\frac{4939200}{1966229} & -\frac{24696000}{17842339} & \frac{9261000}{6878309} & \frac{9261000}{58976629} & \frac{74088000}{8569667} & -\frac{493920}{179275483} & \frac{1764000}{28489} & \frac{302400}{54403} & -\frac{105840}{228689} & \frac{705600}{687461} & -\frac{196000}{485507} \\ -\frac{176400}{180} & \frac{9261000}{105840} & \frac{74088000}{15120} & -\frac{3528000}{35280} & -\frac{12348000}{35280} & \frac{18522000}{3528} & -\frac{4630500}{105840} & \frac{37044000}{14112} & \frac{9261000}{10584} & \frac{12348000}{10080} & \frac{661500}{151200} & \frac{14700}{15120} & -\frac{88200}{216} & \frac{264600}{1411200} & \frac{5292000}{4233600} \end{array} \right)$$

$$Z_{\mathcal{O}_{16,d}}^{(2)} \Big|_{\frac{1}{\epsilon} - \text{part.}} = \frac{N_c^2}{\epsilon} \left(\begin{array}{c|cccccccc} \frac{575}{144} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{23347}{14400} & \frac{46517}{5760} & 0 & 0 & 0 & 0 & \frac{487}{1800} & 0 \\ \frac{3883}{4032} & -\frac{171823}{37800} & \frac{36597791}{3024000} & -\frac{29581}{16800} & 0 & 0 & -\frac{1789}{4800} & 0 \\ -\frac{9271}{11200} & -\frac{35239}{50400} & \frac{74209}{168000} & \frac{188599}{18900} & 0 & 0 & \frac{2101}{4800} & 0 \\ \frac{3287}{3287} & -\frac{2048479}{1176000} & \frac{422283}{392000} & -\frac{2501309}{1764000} & \frac{49211483}{3528000} & \frac{293221}{392000} & \frac{2764807}{2116800} & -\frac{61}{20160} \\ \frac{84000}{947587} & -\frac{1176000}{1555357} & \frac{392000}{16831} & -\frac{1764000}{239641} & -\frac{3528000}{381527} & \frac{392000}{5839021} & -\frac{2116800}{5807} & \frac{118933}{118933} \\ \frac{1058400}{3349} & -\frac{705600}{2591} & \frac{29400}{0} & -\frac{75600}{0} & -\frac{2116800}{0} & \frac{423360}{0} & -\frac{201600}{150391} & \frac{1411200}{0} \\ -\frac{7200}{45083} & -\frac{2400}{16564} & 0 & 0 & 0 & 0 & \frac{14400}{1176541} & 0 \\ -\frac{44100}{11025} & \frac{5447}{117600} & \frac{380791}{176400} & \frac{1063}{29400} & -\frac{545189}{352800} & \frac{1176541}{1058400} & \frac{174229}{12600} \end{array} \right)$$

Mixing matrices and spectrum

Two-loop anomalous dimensions for length-3 operators up to dimension 16:

Jin, Ren, GY 2020

dim	4	6	8	10	12	14	16
$\gamma_{f,\alpha}^{(1)}$	$-\frac{22}{3}$	/	$\frac{7}{3}$	$\frac{71}{15}$	$\frac{241}{30}, \frac{101}{15}$	$\frac{61}{6}, \frac{172}{21}$	$\frac{331}{35}, \frac{1212 \pm \sqrt{3865}}{105}$
$\gamma_{f,\alpha}^{(2)}$	$-\frac{136}{3}$	/	$\frac{269}{18}$	$\frac{2848}{125}$	$\frac{49901119}{1404000}, \frac{8585281}{234000}$	$\frac{4392073141}{87847200}, \frac{685262197}{15373260}$	$\frac{231568398949}{4253886000}, \frac{355106171452034 \pm 95588158951\sqrt{3865}}{6576507756000}$
$\gamma_{f,\beta}^{(1)}$	$-\frac{22}{3}$	1	/	$\frac{17}{3}$	9	$\frac{43}{5}$	$\frac{67}{6}$
$\gamma_{f,\beta}^{(2)}$	$-\frac{136}{3}$	$\frac{25}{3}$	/	$\frac{2195}{72}$	$\frac{79313}{1800}$	$\frac{443801}{9000}$	$\frac{63879443}{1058400}$
$\gamma_{d,\alpha}^{(1)}$	/	/	/	$\frac{13}{3}$	$\frac{41}{6}$	$\frac{551 \pm 3\sqrt{609}}{60}$	$\frac{321 \pm \sqrt{1561}}{30}$
$\gamma_{d,\alpha}^{(2)}$	/	/	/	$\frac{575}{36}$	$\frac{46517}{1440}$	$\frac{5809305897 \pm 19635401\sqrt{609}}{131544000}$	$\frac{229162584707 \pm 225658792\sqrt{1561}}{4130406000}$
$\gamma_{d,\beta}^{(1)}$	/	/	/	/	9	/	$\frac{67}{6}$
$\gamma_{d,\beta}^{(2)}$	/	/	/	/	$\frac{150391}{3600}$	/	$\frac{174229}{3150}$

Two-loop renormalization for higher length operators. Jin, Ren, GY, Yu 2022

Evanescent operators

Evanescent operator (“倏逝算符”):

Vanishing in 4 dimension but non-zero in $d = 4 - 2\epsilon$

$$\mathbf{F}_{\mathcal{O}_L^e, n \geq L}^{(0)}|_{4\text{-dim}} = 0, \quad \mathbf{F}_{\mathcal{O}_L^e, L}^{(0)}|_{d\text{-dim}} \neq 0.$$

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Four-fermion dimension-6 operators:

$$\mathcal{O}_{4\text{-ferm}}^{(n)} = \bar{\psi} \gamma^{[\mu_1} \dots \gamma^{\mu_n]} \psi \bar{\psi} \gamma_{[\mu_1} \dots \gamma_{\mu_n]} \psi, \quad n \geq 5.$$

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$$\mathbf{F}_{\mathcal{O}_L^e, n \geq L}^{(0)}|_{4\text{-dim}} = 0, \quad \mathbf{F}_{\mathcal{O}_L^e, L}^{(0)}|_{d\text{-dim}} \neq 0.$$

Gluonic evanescent operators (start to appear at dimension 10):

$$\mathcal{O}_e = \frac{1}{16} \delta_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \text{tr}(D_{\nu_5} F_{\mu_1 \mu_2} F_{\mu_3 \mu_4} D_{\mu_5} F_{\nu_1 \nu_2} F_{\nu_3 \nu_4})$$

$$\delta_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} = \det(\delta_{\nu}^{\mu}) = \begin{vmatrix} \delta_{\nu_1}^{\mu_1} & \dots & \delta_{\nu_n}^{\mu_1} \\ \vdots & & \vdots \\ \delta_{\nu_1}^{\mu_n} & \dots & \delta_{\nu_n}^{\mu_n} \end{vmatrix}$$

Length-4 basis counting

Δ_0	$N_+^p = N^p$	N_+^e	N_-^e
8	4	0	0
10	20	4	0
12	82	24	1
14	232	88	4
16	550	246	13

Two-loop renormalization

Jin, Ren, GY, Yu, 2022

Evanescent operators are important for renormalization beyond one-loop order.

$$\begin{pmatrix} Z_{\text{pp}}^{(1)} & Z_{\text{pe}}^{(1)} \\ 0 & Z_{\text{ee}}^{(1)} \end{pmatrix}, \quad \begin{pmatrix} Z_{\text{pp}}^{(l)} & Z_{\text{pe}}^{(l)} \\ Z_{\text{ep}}^{(l)} & Z_{\text{ee}}^{(l)} \end{pmatrix}, \quad l \geq 2$$

One can use finite renormalization scheme such that

$$\begin{pmatrix} \hat{\mathcal{D}}_{\text{pp}}^{(l)} & \hat{\mathcal{D}}_{\text{pe}}^{(l)} \\ 0 & \hat{\mathcal{D}}_{\text{ee}}^{(l)} \end{pmatrix}$$

but the lower-loop evanescent operator result are needed.

For example, $\hat{\mathcal{D}}_{\text{pp}}^{(2)}$ contains $(-2\epsilon \hat{Z}_{\text{pe}}^{(1)} \hat{Z}_{\text{ep}}^{(1)})$

Evanescent operators

- *Is Yang-Mills Theory Unitary in Fractional Spacetime Dimension?*

The answer is NO.

YM theory is non-unitary in non-integer spacetime dimensions, due to the existence of evanescent operators.

Evanescent operators

- *Is Yang-Mills Theory Unitary in Fractional Spacetime Dimension?*

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YM theory is non-unitary in non-integer spacetime dimensions, due to the existence of evanescent operators.

$$\begin{aligned}
 & \partial_\nu \partial_\rho \left[\delta_{3789\mu\rho}^{12456\nu} \left(\text{tr}(D_1 F_{23} F_{45} D_6 F_{78} F_{9\mu}) + \text{Rev.} \right) \right], \\
 & \partial_\nu \partial_\rho \left[\delta_4^1 \delta_{789\mu\rho}^{2356\nu} \left(\text{tr}(D_1 F_{23} F_{45} D_6 F_{78} F_{9\mu}) + \text{Rev.} \right) \right] \\
 & \partial_\nu \partial_\rho \left[\delta_4^1 \delta_{789\mu\rho}^{2356\nu} \left(\text{tr}(D_1 F_{23} D_4 F_{56} F_{78} F_{9\mu}) + \text{Rev.} \right) \right] \\
 & \partial_\nu \partial_\rho \left[\delta_4^1 \delta_{689\mu\rho}^{2357\nu} \left(\text{tr}(D_1 F_{23} D_4 F_{56} F_{78} F_{9\mu}) + \text{Rev.} \right) \right] \\
 & \partial_\nu \partial_\rho \left[\delta_4^1 \delta_{589\mu\rho}^{2367\nu} \left(\text{tr}(D_1 F_{23} F_{45} D_6 F_{78} F_{9\mu}) + \text{Rev.} \right) \right] \\
 & \partial_\nu \partial_\rho \left[\delta_4^1 \delta_{569\mu\rho}^{2378\nu} \left(\text{tr}(D_1 F_{23} D_4 F_{56} F_{78} F_{9\mu}) + \text{Rev.} \right) \right] \\
 & \partial_\nu \partial_\rho \left[\delta_5^1 \delta_{689\mu\rho}^{2347\nu} \left(\text{tr}(D_1 F_{23} D_4 F_{56} F_{78} F_{9\mu}) + \text{Rev.} \right) \right] \\
 & \partial_\nu \partial_\rho \left[\delta_4^2 \delta_{389\mu\rho}^{1567\nu} \left(\text{tr}(D_1 F_{23} F_{45} D_6 F_{78} F_{9\mu}) + \text{Rev.} \right) \right]
 \end{aligned}$$

Dim-12 evanescent operators

$$\begin{pmatrix}
 -\frac{38}{3\epsilon} & \frac{2}{\epsilon} & -\frac{13}{12\epsilon} & 0 & \frac{14}{3\epsilon} & 0 & \frac{14}{3\epsilon} & \frac{28}{3\epsilon} \\
 -\frac{1}{2\epsilon} & -\frac{85}{6\epsilon} & \frac{2}{\epsilon} & \frac{5}{6\epsilon} & -\frac{2}{3\epsilon} & -\frac{5}{12\epsilon} & -\frac{7}{3\epsilon} & -\frac{16}{3\epsilon} \\
 0 & -\frac{4}{\epsilon} & -\frac{22}{3\epsilon} & \frac{16}{3\epsilon} & 0 & -\frac{4}{3\epsilon} & 0 & \frac{16}{3\epsilon} \\
 0 & -\frac{4}{3\epsilon} & \frac{7}{3\epsilon} & -\frac{34}{3\epsilon} & 0 & -\frac{4}{3\epsilon} & 0 & 0 \\
 \frac{1}{12\epsilon} & -\frac{1}{12\epsilon} & -\frac{3}{8\epsilon} & \frac{1}{12\epsilon} & -\frac{44}{3\epsilon} & \frac{5}{8\epsilon} & \frac{1}{2\epsilon} & \frac{2}{\epsilon} \\
 0 & \frac{4}{3\epsilon} & \frac{2}{3\epsilon} & 0 & 0 & -\frac{18}{\epsilon} & 0 & -\frac{16}{3\epsilon} \\
 \frac{1}{6\epsilon} & \frac{3}{2\epsilon} & \frac{9}{16\epsilon} & -\frac{1}{2\epsilon} & \frac{29}{6\epsilon} & -\frac{5}{12\epsilon} & -\frac{49}{6\epsilon} & \frac{13}{3\epsilon} \\
 -\frac{5}{6\epsilon} & -\frac{1}{3\epsilon} & \frac{13}{32\epsilon} & -\frac{5}{6\epsilon} & \frac{3}{4\epsilon} & \frac{1}{4\epsilon} & \frac{5}{12\epsilon} & -\frac{91}{6\epsilon}
 \end{pmatrix}$$

One-loop mixing matrix

A pair of complex eigenvalues:

$$1.90386 \pm 0.181142i.$$

Jin, Ren, GY, Yu, 2023

“Color” evanescent operators

- *How about Yang-Mills Theory with fractional N_c (rank of gauge group)?*

In this case, there are also complex AD due to the existence of “color” evanescent operators.

$$\delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} T_{i_1 j_1}^{a_1} T_{i_2 j_2}^{a_2} T_{i_3 j_3}^{a_3} = \text{tr}(123) + \text{tr}(132) = d^{a_1 a_2 a_3}$$

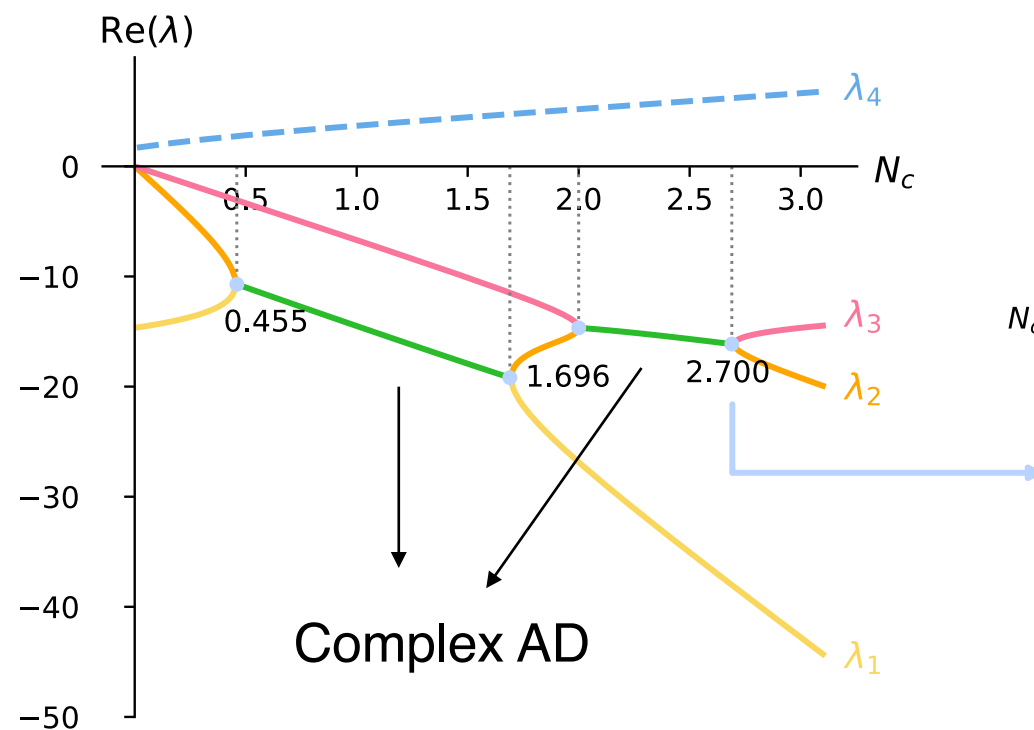
Mathematically, we make analytical continuation for N_c and consider AD as a function of N_c .

“Color” evanescent operators

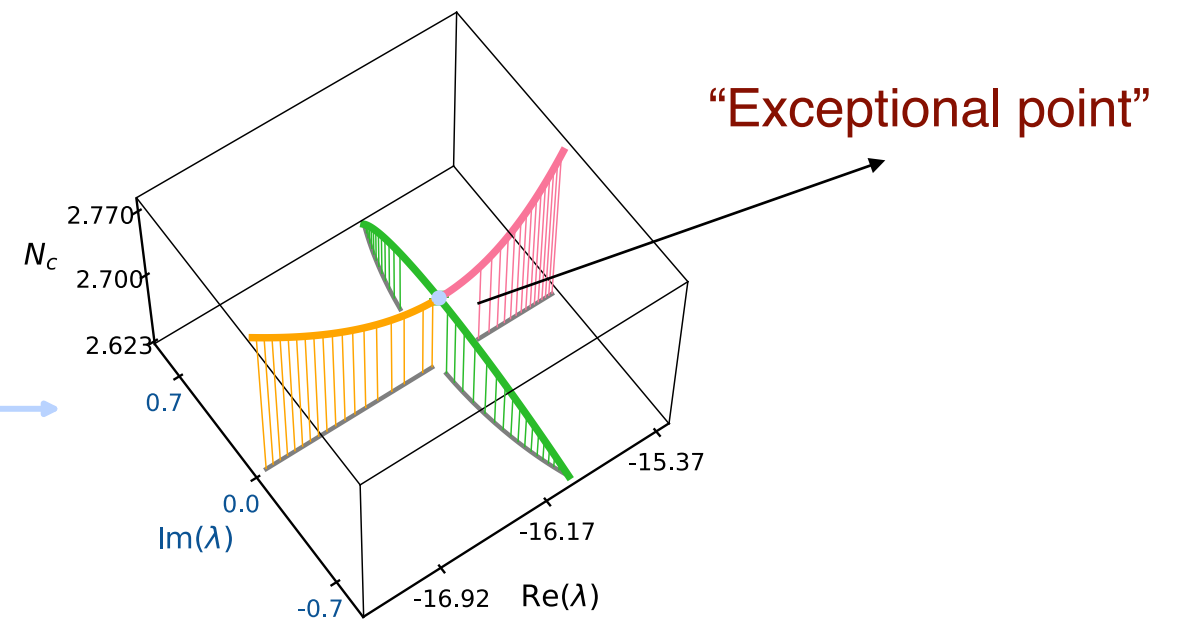
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Four Dim-8 operators

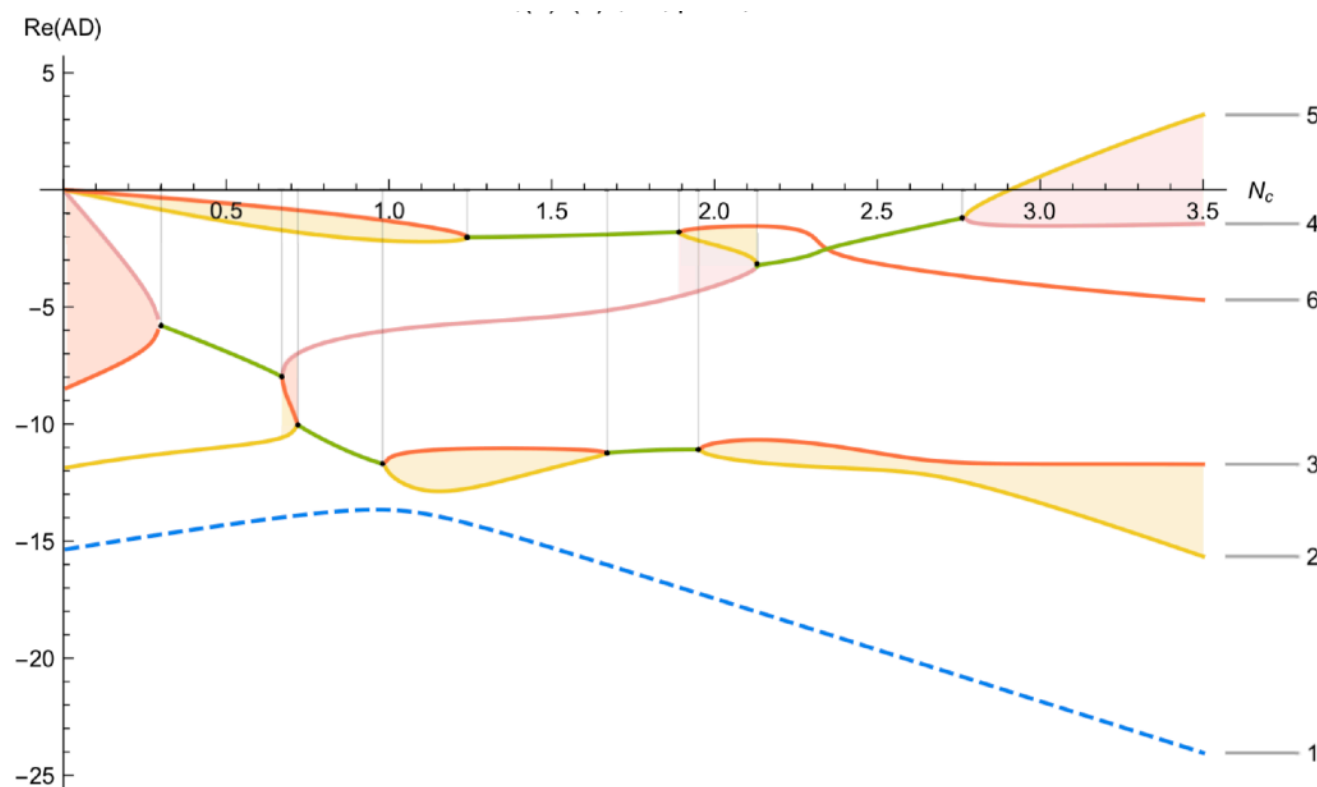


To appear with Q. Jin, K. Ren, R. Yu,

“Color” evanescent operators

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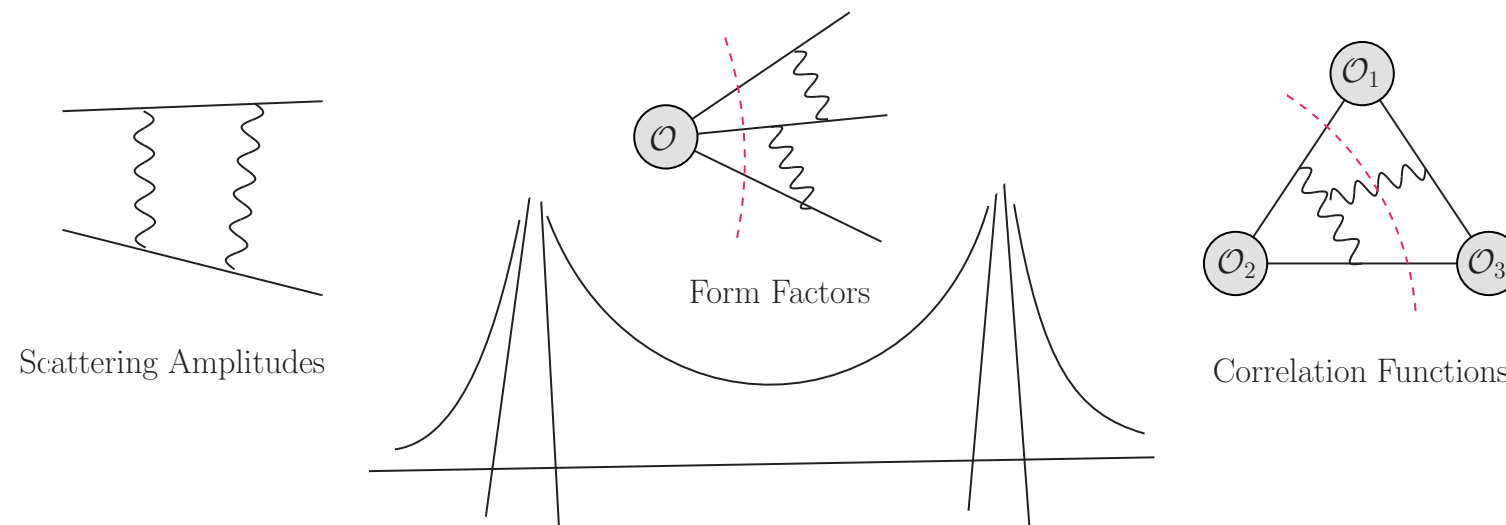
In this case, there are also complex AD due to the existence of “color” evanescent operators.



Dim-12 operators

To appear with Q. Jin, K. Ren, R. Yu,

Summary



- Form factors provide a framework to study many interesting physical quantities using **on-shell amplitude methods**:
 - IR divergences
 - UV renormalization
 - Finite remainder
- Other hidden structure of form factors
 - CK duality, double-copy of form factors, DDCl, FFOPE, etc.

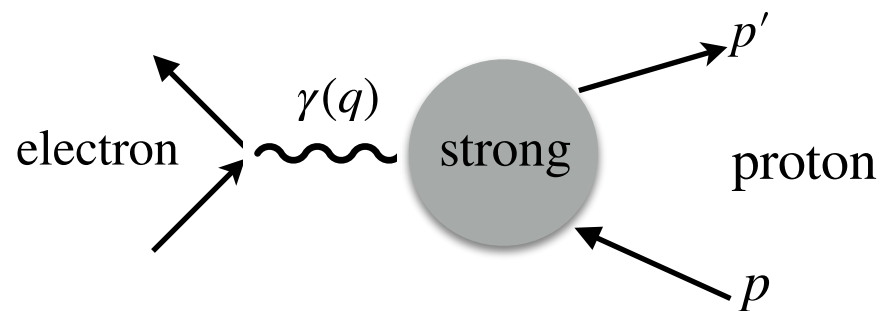
Thank you for your attention!

祝李重生老师八十华诞快乐，身体健康！

Backup slides

1) Nuclear “structure factor”

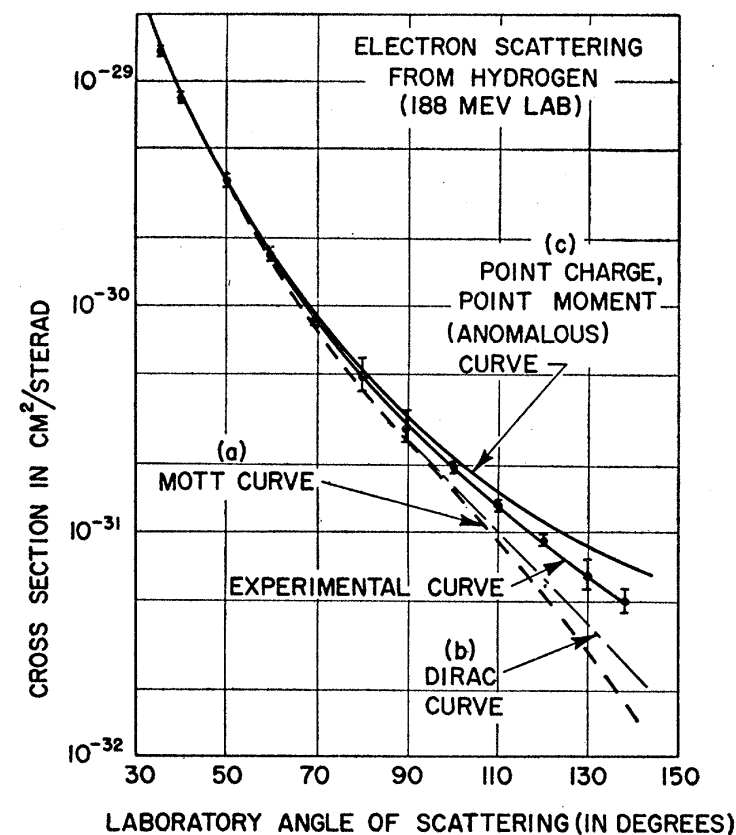
- Form factor characterizes the deviation from the point-particle picture.



$$\sigma_s(\theta) = \left(\frac{Ze^2}{2E} \right)^2 \frac{\cos^2 \frac{1}{2}\theta}{\sin^4 \frac{1}{2}\theta} \left[\int_0^\infty \rho(r) \frac{\sin qr}{qr} 4\pi r^2 dr \right]^2.$$

$$F = \frac{4\pi}{q} \int_0^\infty \rho(r) \sin(qr) r dr$$

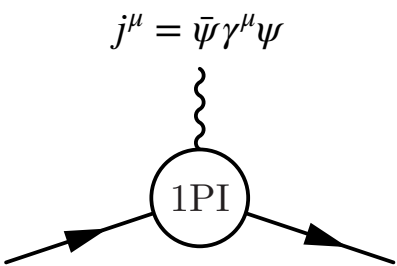
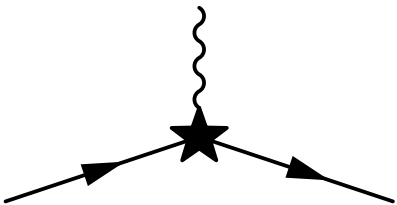
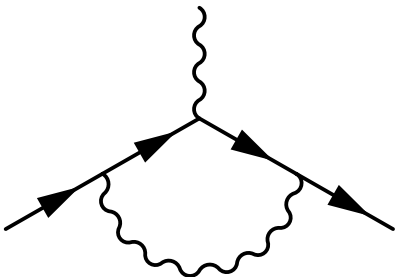
“form factor”



Robert Hofstadter
(1915 – 1990)
Nobel laureate 1961

McAllister and Hofstadter, Phys.Rev. (1956)

2) Form factor in text book

$-ie_R\Gamma^\mu =$

 $=$

 $+$


$$\Gamma^\mu(q) = \gamma^\mu \underbrace{F_1(q^2)} + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \underbrace{F_2(q^2)}$$

Form factors

$\bar{\psi}\gamma^\mu\psi \rightarrow \bar{\psi}\Gamma^\mu\psi$

Leading order: $F_1(p^2) = 1, \quad F_2(p^2) = 0$

One-loop order: $F_2(0) = \frac{\alpha}{2\pi} \longrightarrow g - 2 = 2F_2(0) = \frac{\alpha}{\pi}$

Anomalous magnet moment

3) Sudakov form factor

- Pioneer work by [Vladimir Sudakov](#) in 1954

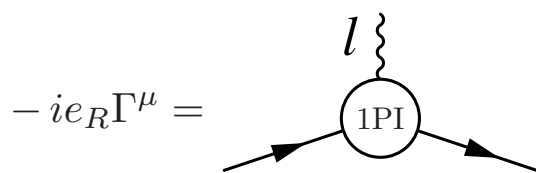
Vertex Parts at Very High Energies in Quantum Electrodynamics

V. V. SUDAKOV

(Submitted to JETP editor Nov. 4, 1954)

J. Exper. Theoret. Phys. USSR 30, 87-95 (January 1956)

A method is developed for calculating Feynman integrals with logarithmic accuracy, working to any order of perturbation theory. The method is applied to calculate the vertex part in quantum electrodynamics for a certain range of values of the momenta. The result is displayed as the sum of a perturbation series.



$$\Gamma_\sigma(p, q; l) = \gamma_\sigma \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{e^2}{2\pi} \ln \left| \frac{l^2}{p^2} \right| \ln \left| \frac{l^2}{q^2} \right| \right)^n$$

$$= \gamma_\sigma \exp \left\{ -\frac{e^2}{2\pi} \ln \left| \frac{l^2}{p^2} \right| \ln \left| \frac{l^2}{q^2} \right| \right\}.$$

A closed formula of summing up the [leading-logarithm terms](#).

[Generalization to non-leading logarithms in QED of Mueller 1979, Collins 1980, and in QCD Sen 1980](#)

IR divergences

Infrared structure of amplitudes:

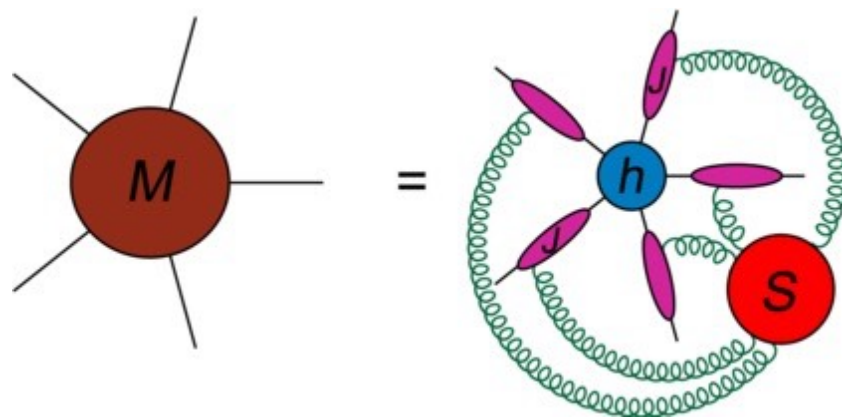


figure from L. Dixon 1105.0771

For modern dim-reg representation, see:
Magnea and Sterman 1990;
Catani 1998,
Sterman and Tejeda-Yeomans 2002
Bern, Dixon, Smirnov 2005

$$\mathcal{M}_n = \prod_{i=1}^n \left[\mathcal{M}^{[gg \rightarrow 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n(k_i, \mu, \alpha_s, \epsilon)$$

↓

Sudakov form factor = $\exp \left[-\frac{1}{4} \sum_{l=1}^{\infty} a^l \left(\frac{\mu^2}{-Q^2} \right)^{l\epsilon} \left(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon} \right) + \text{finite} \right]$

↓

Leading IR singularity -> Cusp anomalous dimension

Unitarity computation

Consider one-loop amplitudes:

$$\text{One-loop bubble diagram} = \sum \underline{d_i} \text{ [Square diagram]} + \sum \underline{c_i} \text{ [Triangle diagram]} + \sum \underline{b_i} \text{ [Cross diagram]}$$

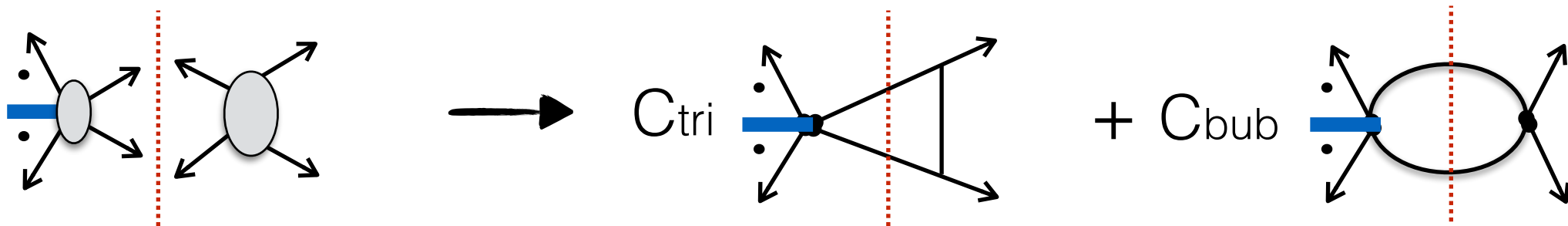
What we really want

Unitarity computation

$$F_2^{(1)} = C_{\text{tri}} \text{ (triangle diagram) } + C_{\text{bub}} \text{ (bubble diagram) }$$

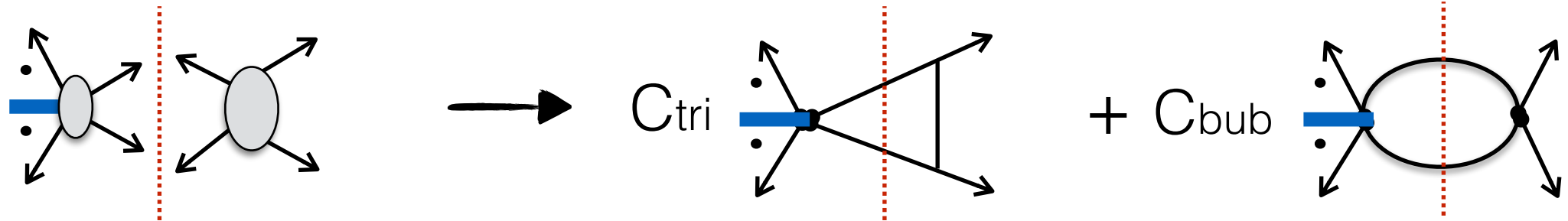
The diagrammatic equation shows $F_2^{(1)}$ as the sum of two terms. The first term is C_{tri} multiplied by a triangle diagram with a blue horizontal line on the left and two outgoing lines on the right. The second term is C_{bub} multiplied by a bubble diagram with a blue horizontal line on the left and two outgoing lines on the right.

The basis coefficient can be computed by cuts:



$$\mathcal{F}_2^{(1)}(1,2) \Big|_{s_{12}\text{-cut}} = \int d\text{PS}_2 \mathcal{F}_2^{(0)}(-l_1, -l_2) \mathcal{A}_4^{(0)}(1,2,l_2,l_1)$$

Unitarity computation



$$\begin{aligned}
 \mathcal{O} = \text{tr}(\phi_{12}^2) \quad & \mathcal{F}_2^{(1)}(1, 2)|_{s_{12}\text{-cut}} = \int d\text{PS}_2 \sum_{\text{helicity of } l_i} \mathcal{F}_2^{(0)}(-l_1, -l_2) \times \mathcal{A}_4^{(0)}(1, 2, l_2, l_1) \\
 & = \mathcal{F}_2^{(0)}(1, 2) i \int d\text{PS}_2 1 \times \frac{\langle l_1 l_2 \rangle \langle 12 \rangle}{\langle l_1 p_1 \rangle \langle l_2 2 \rangle} \\
 & = \mathcal{F}_2^{(0)}(1, 2) i \int d\text{PS}_2 \frac{-s_{12}}{(l_1 + p_1)^2} \\
 & = \mathcal{F}_2^{(0)}(1, 2)(-s_{12}) \rightarrow \text{triangle diagram}
 \end{aligned}$$

The triangle diagram shows a blue horizontal line entering from the left, meeting a vertical red dashed line, and then forming a triangle with two internal lines labeled p_1 and p_2 .

$$\longrightarrow \quad C_{\text{tri}} = -s_{12}, \quad C_{\text{bub}} = 0$$

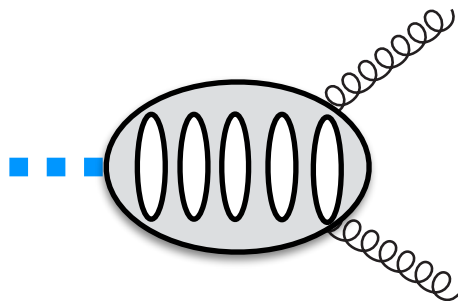
Four-loop non-planar cusp AD

[Boels, Huber, GY 2017]

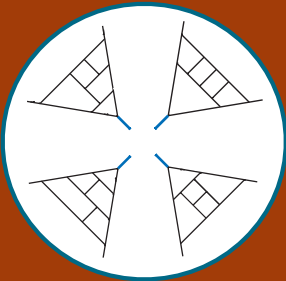
$$\begin{aligned}
 I_1^{(21)} &= \text{Diagram 1} \times [(\ell_3 - p_1)^2]^2 & I_2^{(22)} &= \text{Diagram 2} \times (\ell_3 - p_1)^2 [\ell_4^2 + \ell_6^2 - \ell_3^2 + (\ell_3 - \ell_4 + p_1)^2 + (\ell_3 - \ell_6 - p_1)^2] \\
 I_3^{(23)} &= \text{Diagram 3} \times [(\ell_3 - p_1)^2]^2 & I_4^{(24)} &= \text{Diagram 4} \times (\ell_3 - p_1)^2 [(q - \ell_3 - \ell_5)^2 + (\ell_5 + p_2)^2] \\
 I_5^{(25)} &= \text{Diagram 5} \times \left\{ [(p_1 - \ell_5)^2 + 2(\ell_4 - \ell_5)^2 + (\ell_3 - \ell_4)^2 - (\ell_3 - \ell_5)^2 - (p_1 - \ell_4)^2]^2 - 4(\ell_4 - \ell_5)^2 (p_1 - \ell_3 + \ell_4 - \ell_5)^2 \right\} \\
 I_6^{(26)} &= \text{Diagram 6} \times \left\{ [(\ell_3 - \ell_4 - \ell_5)^2 - (\ell_3 - \ell_4 - p_1)^2 - (\ell_6 - p_2)^2 - \ell_5^2] [\ell_5^2 - \ell_4^2 - \ell_6^2 + (\ell_4 - \ell_6)^2] + 4\ell_5^2 (\ell_6 - p_2)^2 + (\ell_4 - \ell_5)^2 (\ell_3 - \ell_4 + \ell_6 - p_2)^2 \right\} \\
 I_7^{(26)} &= \text{Diagram 7} \times \left\{ 4[(\ell_4 - \ell_5)(\ell_3 - \ell_4 + \ell_5 - p_1)][(\ell_4 - \ell_6)(\ell_3 - \ell_4 + \ell_6 - p_2)] - \ell_5^2 (\ell_6 - p_2)^2 - 4(\ell_4 - \ell_5)^2 (\ell_3 - \ell_4 + \ell_6 - p_2)^2 - \ell_6^2 (\ell_5 - p_1)^2 \right. \\
 &\quad \left. - (\ell_3 - \ell_4)^2 (\ell_5 + \ell_6 - \ell_4)^2 - \ell_4^2 (\ell_3 - \ell_4 + \ell_5 + \ell_6 - p_1 - p_2)^2 \right\} \\
 I_8^{(27)} &= \text{Diagram 8} \times \frac{1}{2} [\ell_3^2 - \ell_4^2 - (\ell_4 - \ell_3 - p_1)^2] [(\ell_3 - \ell_4 - \ell_5)^2 + (\ell_5 + p_2)^2] \\
 I_9^{(28)} &= \text{Diagram 9} \times (\ell_3 - \ell_4 - p_2)^2 [(\ell_3 - \ell_4)^2 - (\ell_3 - p_1)^2] & I_{10}^{(29)} &= \text{Diagram 10} \times \frac{1}{2} [\ell_3^2 - \ell_4^2 - (\ell_4 - \ell_3 - p_1)^2] [\ell_6 \cdot (\ell_6 - \ell_4 + \ell_3 - p_2)] \\
 I_{11}^{(30)} &= \text{Diagram 11} \times (\ell_3 - \ell_4 - p_2)^2 [(p_1 - \ell_4)^2 + (\ell_3 - \ell_4)^2 - (\ell_3 - p_1)^2]
 \end{aligned}$$

Plus 12 simpler 11- and 10-line integrals

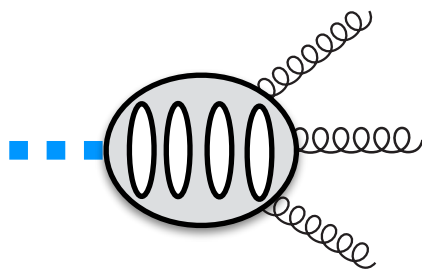
Full color form factors



<i>L</i> -loop	<i>L</i> =1	<i>L</i> =2	<i>L</i> =3	<i>L</i> =4	<i>L</i> =5	Four master graphs @ 5-loop:
# of topologies	1	2	6	34	306	
# of masters	1	1	1	2	4	



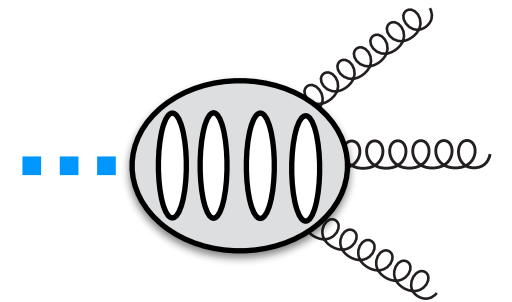
Boels, Kniehl, Tarasov, GY 2012
GY, 2016



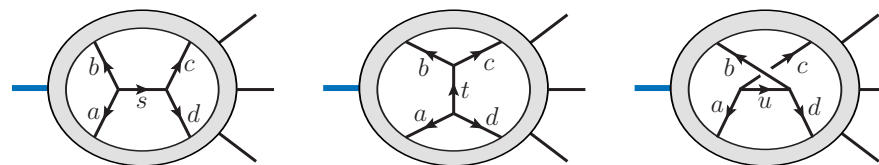
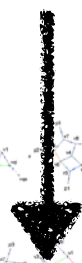
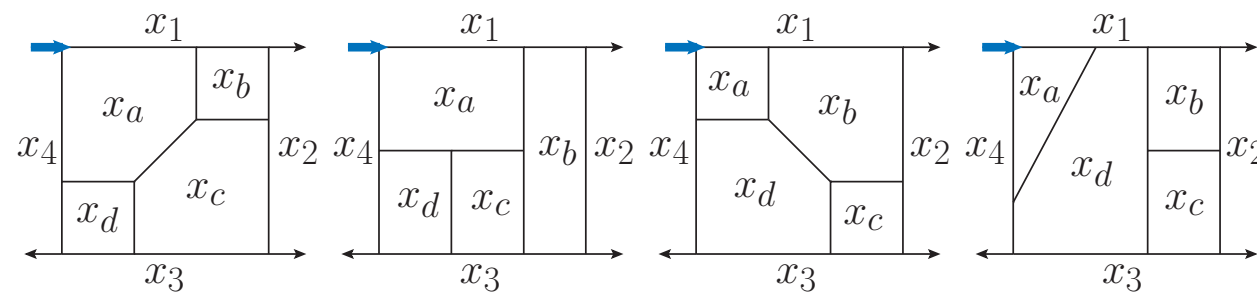
<i>L</i> loops	<i>L</i> =1	<i>L</i> =2	<i>L</i> =3	<i>L</i> =4
# of cubic graphs	2	6	29	229
# of planar masters	1	2	2	4
# of free parameters	1	4	24	133

Lin, GY, Zhang, 2021

Three-point form factors



Master graphs



$$\mathbf{F}_3^{(4)} = \sum_{\sigma_3} \sum_{i=1}^{229} \int \prod_{j=1}^4 d^D \ell_j \frac{1}{S_i} \sigma_3 \cdot \frac{\mathcal{F}_3^{(0)} C_i N_i}{\prod_{\alpha_i} P_{\alpha_i}^2}$$

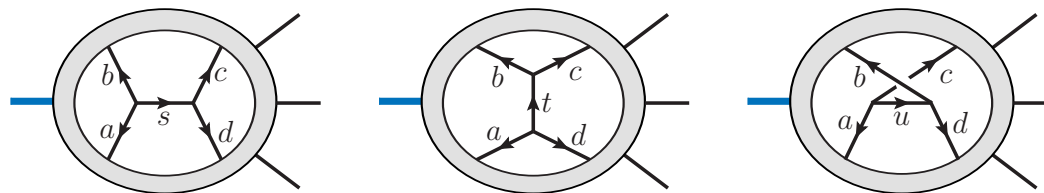
Strategy of loop computation

CK-duality

Conjecture !

$$\mathcal{F}^{(\ell)} \sim \sum_i \int \frac{C_i \times N_i}{\prod D}$$

**Compact ansatz of
the loop integrand**



$$C_s = C_t + C_u$$



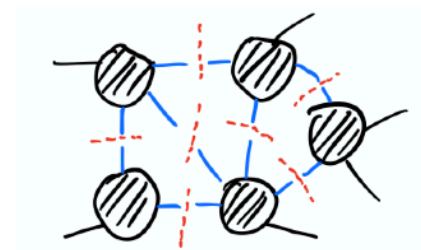
$$N_s = N_t + N_u$$

Strategy of loop computation

CK-duality

Conjecture !

Compact ansatz of
the loop integrand



Unitarity cuts

Loop-ansatz $|_{\text{cut}} = \prod \text{tree-blocks}$

Solving linear equations

Main challenge: it is a priori not known whether the solution exists