Symbolic Reduction of Multi-loop Feynman Integrals via Generating Functions

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based on work with Xiang Li, Yuanche Liu, Yan-Qing Ma, Yang Zhang, arXiv:2509.21769 and paper coming soon

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The motivation

In this talk, we will present our recent work about finding **complete symbolic reduction rules** for general multi-loop integrals using the generating function.

- Scattering amplitude is one of central object in QFT. It connects the theoretical prediction with the experiment. Thus its computation is extremely important.
- The tree-level amplitudes give the classical effect, so to probe the quantum correction, we must compute the loop amplitudes.
- With the efforts of whole community, the one-loop integrals have been solved with the on-shell program.
- Currently, the frontiers are the computation of higher loop integrals. Especially with the huge data collected by experiments, precise computation become more and more important.

A very important step of multi-loop integral computation is the **reduciton**, i.e.,

$$I = \sum_i c_i I_i$$

where c_i is the rational function of external information, while I_i is called (master integrals).

For higher loop, there are a lot of proposal to do the reduction. One of very successful method is the IBP method (plus its various generalization). [Chetyrkin, Tkachov, 1981] [Tkachov, 1981] [Laporta, 2001]

However, with the increasing of complexities of problem, i.e., higher number of prapagators, higher tensor, higher powers of propagators etc, the IBP method faces serious challenge, especially with the exponential increasing of to-be-solved linear equations. Thus improving the efficiency of reduction becomes critical.

Setup

For L-loop integrals with E independent external momenta, a family of scalar FIs can be represented as

$$I(\vec{\nu}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}}, \tag{1}$$

The IBP method is to establish relations between different $I(\vec{\nu})$ and solve them by some basic integrals (master integrals).

We can group different $I(\vec{\nu})$ using **generating function**

$$G_{\vec{\mu}}(\eta) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} e^{\sum_{j=1}^{N}(1-\mu_j)\eta_j s_0^{-1}\mathcal{D}_j} \frac{1}{\prod_{i=1}^{N}(\mathcal{D}_i - s_0\eta_i)^{\mu_i}}, \tag{2}$$

Expand it we see that

$$G_{\vec{\mu}}(\eta) \quad = \quad \sum_{\vec{n}\geq 0} F_{\vec{n}} \vec{\eta}^{\vec{n}} \tag{3}$$

where

$$F_{\vec{n}} \sim I(n_1+1, ..., n_K+1, -n_{K+1}, ..., -n_N) \tag{4}$$

The general differential equations for generating functions have the form

$$\sum_t c_t \vec{\eta}^{\vec{a}_t} \frac{\partial^{\vec{b}_t}}{\partial \eta^{\vec{b}_t}} G_{\mu_1}(\vec{\eta}) = B \,, \tag{5}$$

where B is the function of generating functions of subsector of the sector $\vec{\mu}_1$ and

$$\frac{\partial^{\vec{b}_t}}{\partial \eta^{\vec{b}_t}} = \prod_{i=1}^{N} \frac{\partial^{(\vec{b}_t)_i}}{\partial \eta_i (\vec{b}_t)_i} \tag{6}$$

Putting the expansion (3) to equation (5) and collecting the coefficient of $\vec{\eta}^{\vec{n}}$, we get

$$\sum_t c_t \left(\prod_{j=1}^{N} \prod_{p=1}^{(\vec{b}_t)_j} (n_j - (\vec{a})_j + p) \right) F_{\vec{n}+\mathbf{b_t}-\mathbf{a_t}} = \widetilde{B} \tag{7}$$

where for simplicity, we have made the convention that

$$F_{\vec{n}} = 0, \quad \exists i, \;\; such \;\; (\vec{n})_i < 0 \tag{8}$$

- The recurrence relation connects $F_{\vec{n}}$ at the different lattice points $\vec{n}$.
- If we define the **degree of lattice point** $\vec{n}$ as $\sum_{i=1}^{N} n_i$, it is naturally to see that when we could express $F_{\vec{n}}$ at the higher lattice points by these $F_{\vec{n}}$ at the lower lattice points $\vec{n}$, we will get a nice recurrence relation or reduction rule.
- Since the choice $\vec{n}$ can be arbitrary as long as the reduction rule can be applied, we get a **symbolic reduction rule**

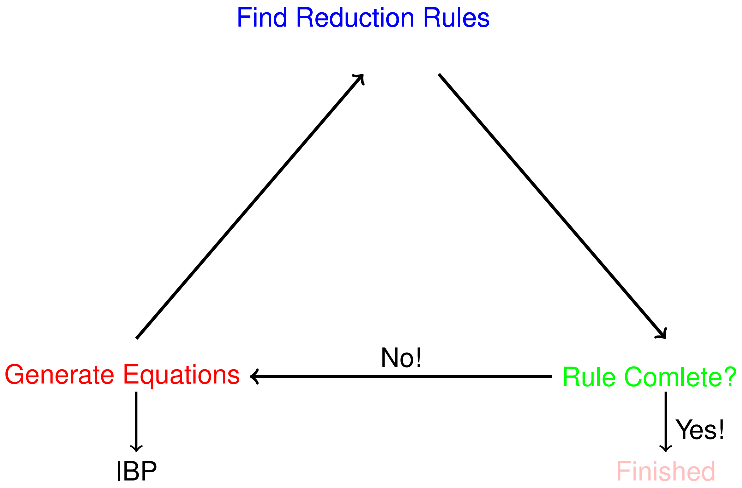
The expression (7) provide a nice mapping between operator $\widehat{O} = \vec{\eta}^{\vec{a}} \frac{\partial^{\vec{b}}}{\partial \eta^{\vec{b}}}$ and $F_{\vec{n}+\mathbf{b}_t-\mathbf{a}_t}$, thus

- We define the **finer index** by the pair of vectors $(\vec{a}, \vec{b})$ and the corresponding **index** as the vector $\vec{o} \equiv \vec{b} - \vec{a}$, thus the **degree** of a operator is given by $|\vec{o}| \equiv \sum_{i=1}^{N} (\vec{o})_i$
- *finding symbolic reduction rule is equivalent to* **express the higher degree operator as the sum of lower degree operators** *using the differential equations*

In this talk, I will present an algorithm to find complete symbolic reduction rules using differential equations of generating function!

The algorithm

The algorithm is given by the diagram:



Three type of equations:

- **First:** it contains one and only one highest degree operator

$$
\begin{aligned}
0 &= \left\{+2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}G_{111}\right\} \\
&\quad + \left\{-\frac{2m_1^2}{s_0}\frac{\partial}{\partial\eta_1}G_{111} - \frac{f_{+-+}}{s_0}\frac{\partial}{\partial\eta_3}G_{111}\right\} + \ldots. \qquad (9)
\end{aligned}
$$

- **Second:** it contains several highest degree operators, but all of them have the same operator index. For example, in the differential equation

$$
\begin{aligned}
0 &= \left\{-2\eta_5\frac{\partial^2}{\partial\eta_5^2}G - 2\eta_4\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_5}G + (6-3D)\frac{\partial}{\partial\eta_5}G\right\} \\
&\quad + \frac{f_{-+-}}{s_0}\eta_2\frac{\partial}{\partial\eta_4}G + \frac{-K^2\eta_4}{s_0}\eta_1\frac{\partial}{\partial\eta_5}G + \frac{-K^2(D-2)}{s_0}\eta_3 G \qquad (10)
\end{aligned}
$$

- **Third:** it contains several highest degree operators with different operator indices, for example,

$$
\begin{aligned}
0 &= -(D-2)\frac{\partial}{\partial\eta_4}G+(D-2)\frac{\partial}{\partial\eta_5}G \\
&+\frac{f_{+-+}\eta_1}{s_0}\frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_3}G+R \qquad (11)
\end{aligned}
$$

There is a partial ordering of operators to determine which operator should be solved!

One of the key concept: **descendant reduction rule**

Having a reduction rule, i.e., solved

$$\widehat{O}G = \sum_i c_i \widehat{O}_i G + B \tag{12}$$

where $\widehat{O}_i$ has lower degree comparing to $\widehat{O}$ and B are contributions from subsectors, we have

$$\widehat{O}_r \widehat{O} G = \sum_i c_i \widehat{O}_r \widehat{O}_i G + \widehat{O}_r B \tag{13}$$

which is a nice reduction rule for the operator $\widehat{O}_r \widehat{O}$ (it will be called "descendant operator")

The use of descendant reduction rules:

- Generate new equations: **acting $\frac{\partial}{\partial \eta_i}$ on reduction rules**
- Simplify equations: When and only when we use descendant reduction rules to clean the dust caused by the known reduction rules, **new operators can appear to provide new manifest reduction rules!!**

The general principle of solving:

- Linear combinations to give as many as possible first type equations, then second type, finally third type.
- Solving first type equations first, and then second type, and finally third type.

The determination of reducible and un-reducible lattice points:

- The reduction rule will be like

$$c(\vec{n})F_{\vec{n}+\vec{o}} = \sum_i c_i(\vec{n})F_i, \qquad \vec{n} \geq 0 \tag{14}$$

- Thus any lattice point $\vec{P}$, as long as $\vec{P} - \vec{o} \geq 0$, can use the result (14) to reduce
- Thus we have two sets

$$\mathcal{S}_{\vec{o}} = \{\vec{m} \in R^N | \vec{m} \geq 0 \ \& \ \vec{m} \geq \vec{o}\} \tag{15}$$

$$\mathcal{U}_{\vec{o}} = \{\vec{m} \in R^N | \vec{m} \geq 0 \ \& \ \exists i, (\vec{m} - \vec{o}_+)_i < 0\} \tag{16}$$

Criteria of completeness of a set of reduction rules: When and only when the number of un-reducible lattice points is equal to the number of master integrals in the given sector!

topsector of sunset

For sunset diagram, we have

$$\mathcal{D}_1 = (\ell_1^2 - m_1^2), \quad \mathcal{D}_2 = (\ell_2^2 - m_2^2), \quad \mathcal{D}_3 = ((\ell_1 + \ell_2 - K)^2 - m_3^2), \quad (17$$

and the numerators are

$$\mathcal{D}_4 = \ell_1 \cdot K, \quad \mathcal{D}_5 = \ell_2 \cdot K, \tag{18}$$

For sunset, among the $2^3 = 8$ possible generating function, only (remembering that for ISP, $\mu_4 = \mu_5 = 0$ so when we write down the index, we neglect them)

$$G_{111}, G_{110}, G_{101}, G_{011} \tag{19}$$

For massless case, some subsectors is just zero.

More explicitly we will have

$$
\begin{aligned}
G_{111} &= \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} e^{\sum_{j=4}^{5} \eta_j s_0^{-1} \mathcal{D}_j} \prod_{i=1}^{3} \frac{1}{\mathcal{D}_i - \eta_i s_0} \\
G_{110} &= \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} e^{\sum_{j=3}^{5} \eta_j s_0^{-1} \mathcal{D}_j} \prod_{i=1}^{2} \frac{1}{\mathcal{D}_i - \eta_i s_0} \\
G_{011} &= \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} e^{\sum_{j=1,4,5} \eta_j s_0^{-1} \mathcal{D}_j} \prod_{i=2}^{3} \frac{1}{\mathcal{D}_i - \eta_i s_0} \\
G_{101} &= \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} e^{\sum_{j=2,4,5} \eta_j s_0^{-1} \mathcal{D}_j} \prod_{i=1,3} \frac{1}{\mathcal{D}_i - \eta_i s_0}
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
0 &= \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_1} \cdot \left\{ \ell_1 e^{\sum_{j=4,5} \eta_j s_0^{-1} \mathcal{D}_j} \prod_{i=1}^{3} \frac{1}{\mathcal{D}_i - \eta_i s_0} \right\} \\
&= \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} D e^{\sum_{j=4,5} \eta_j s_0^{-1} \mathcal{D}_j} \prod_{i=1}^{3} \frac{1}{\mathcal{D}_i - \eta_i s_0} \\
&\quad + \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \eta_4 s_0^{-1} (\ell_1 \cdot K) e^{\sum_{j=4,5} \eta_j s_0^{-1} \mathcal{D}_j} \prod_{i=1}^{3} \frac{1}{\mathcal{D}_i - \eta_i s_0} \\
&+ \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} e^{\sum_{j=4,5} \eta_j s_0^{-1} \mathcal{D}_j} \frac{-2\ell_1^2}{\mathcal{D}_1 - \eta_1 s_0} \prod_{i=1}^{3} \frac{1}{\mathcal{D}_i - \eta_i s_0} \\
&+ \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} e^{\sum_{j=4,5} \eta_j s_0^{-1} \mathcal{D}_j} \frac{-2\ell_1 \cdot (\ell_1 + \ell_2 - K)}{\mathcal{D}_3 - \eta_3 s_0} \prod_{i=1}^{3} \frac{1}{\mathcal{D}_i - \eta_i}
\end{aligned} \tag{21}
$$

Collecting according to the degree we have **initial differential equation**

$$
\begin{aligned}
0 \;=\; & \left\{+2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}G_{111}\right\}+\left\{-\frac{2m_1^2}{s_0}\frac{\partial}{\partial\eta_1}G_{111}-\frac{f_{+-+}}{s_0}\frac{\partial}{\partial\eta_3}G_{111}\right\} \\
& +\left\{+(D-3)G_{111}+\eta_4\frac{\partial}{\partial\eta_4}G_{111}-2\eta_1\frac{\partial}{\partial\eta_1}G_{111}\right. \\
& \left.-(\eta_3+\eta_1-\eta_2)\frac{\partial}{\partial\eta_3}G_{111}\right\} \\
& +\left\{-\frac{1}{s_0}\frac{\partial}{\partial\eta_3}G_{011}|_{\eta_1=0}+\frac{1}{s_0}\frac{\partial}{\partial\eta_3}G_{101}|_{\eta_2=0}\right\} \qquad (22)
\end{aligned}
$$

where for simplicity we have defined

$$f_{abc}=am_1^2+bm_2^2+cm_3^2-K^2,\, a,b,c=\pm 1 \qquad (23)$$

Solving and finding reduction rules: For six IBP equations, the matrix of highest degree operators is given as

$$\begin{array}{c}
\begin{array}{cccccc}
\frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_4} & \frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_4} & \frac{\partial}{\partial\eta_3}\frac{\partial}{\partial\eta_4} & \frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_5} & \frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_5} & \frac{\partial}{\partial\eta_3}\frac{\partial}{\partial\eta_5}
\end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 2 & I_1 \\
-2 & 0 & -2 & -2 & 0 & 0 & I_2 \\
-2 & 0 & -2 & 0 & 0 & -2 & I_3 \\
0 & -2 & 0 & 0 & -2 & -2 & I_4 \\
0 & 0 & 2 & 0 & 0 & 0 & I_5 \\
0 & 0 & -2 & 0 & -2 & -2 & I_6
\end{pmatrix}
\end{array} \tag{24}$$

After the Gauss elimination we obtain

$$
\begin{pmatrix}
\frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_4} & \frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_4} & \frac{\partial}{\partial\eta_3}\frac{\partial}{\partial\eta_4} & \frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_5} & \frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_5} & \frac{\partial}{\partial\eta_3}\frac{\partial}{\partial\eta_5} & \\
0 & 0 & 0 & 0 & 0 & 2 & l_1 \\
-2 & 0 & 0 & 0 & 0 & 0 & l_3+l_5+l_1 \\
0 & 0 & 0 & -2 & 0 & 0 & l_2-l_3-l_1 \\
0 & -2 & 0 & 0 & 0 & 0 & l_4-l_5-l_6 \\
0 & 0 & 2 & 0 & 0 & 0 & l_5 \\
0 & 0 & 0 & 0 & -2 & 0 & l_6+l_5+l_1
\end{pmatrix}
$$

Now solving (25), we obtain reduction rules for following six degree two operators. For example, the first one is

$$
\begin{aligned}
& 2\frac{\partial}{\partial\eta_5}\frac{\partial}{\partial\eta_3}G_{111} \\
= \quad & \left\{-2m_1^2 s_0^{-1}\frac{\partial}{\partial\eta_1}G_{111} - f_{+--+}s_0^{-1}\frac{\partial}{\partial\eta_3}G_{111}\right\} \\
& + \left\{+(D-3)G_{111} + \eta_4\frac{\partial}{\partial\eta_4}G_{111} - 2\eta_1\frac{\partial}{\partial\eta_1}G_{111}\right. \\
& \left. -(\eta_3+\eta_1-\eta_2)\frac{\partial}{\partial\eta_3}G_{111}\right\} \qquad (26)
\end{aligned}
$$

Checking the un-reducible points: The solved operators is

$$\left\{\frac{\partial}{\partial\eta_1},\frac{\partial}{\partial\eta_2},\frac{\partial}{\partial\eta_3},\right\}\times\left\{\frac{\partial}{\partial\eta_4},\frac{\partial}{\partial\eta_5}\right\} \tag{27}$$

the set of points can not be reduced by above six reduction rules are

$$\mathcal{U}_1=(0,0,0,n_4,n_5),\qquad \mathcal{U}_2=(n_1,n_2,n_3,0,0) \tag{28}$$

Since these two sets contain infinity number of lattice points, we need to generate new equations to solve new reduction rules.

Generating new equations: We act $\frac{\partial}{\partial \eta_i}, i = 1, .., 5$ to above six reduction rules and get 30 equations of degree three. They belong to two types:

- **Trivially generated:** For example, $\frac{\partial}{\partial \eta_5}\left(\frac{\partial}{\partial \eta_5}\frac{\partial}{\partial \eta_3}\right)$
- **Non-trivially generated:** For example, $\frac{\partial}{\partial \eta_1}\left(\frac{\partial}{\partial \eta_3}\frac{\partial}{\partial \eta_5}\right)$ and $\frac{\partial}{\partial \eta_3}\left(\frac{\partial}{\partial \eta_1}\frac{\partial}{\partial \eta_5}\right)$. Using the integrability condition $[\frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_3}] = 0$, we can get a non-trivial new equation.

After using descendant reduction rules to simplify, the remaining degree two part is given by

$$\frac{\partial}{\partial\eta_1}\left\{-2m_1^2 s_0^{-1}\frac{\partial}{\partial\eta_1}G_{111}-f_{+-+}s_0^{-1}\frac{\partial}{\partial\eta_3}G_{111}\right\} -\frac{\partial}{\partial\eta_3}\left\{-\frac{f_{+-+}}{s_0}\frac{\partial}{\partial\eta_1}G_{111}-\frac{2m_3^2}{s_0}\frac{\partial}{\partial\eta_3}G_{111}\right\} \tag{29}$$

There are three new degree two operators $\frac{\partial^2}{\partial\eta_1^2}, \frac{\partial^2}{\partial\eta_3^2}, \frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_3}$, which are not descendant of 6 known reduction rules. **The appearance of these new operators is the key of the algorithm**, but as we remarked before, it comes out when and only when we use descendant reduction rules to clean the dust caused by the known reduction rules.

There are 9 new equations, but only 8 independent one. Their behaviors are different for massive and massless case:

- **Massive:** 3 of them, degree two and 5, degree one
- **Massless:** 5 of them, degree two and 3, degree one

For **massless** case:

- (a) The 3 degree two equations for first type. We can solve degree two operators: $\frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_2}$, $\frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_3}$ and $\frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_3}$
- (b) 2 degree one equations are second type and we solve

$$
\begin{aligned}
& 2\eta_4\frac{\partial^2}{\partial\eta_4^2}G_{111}+2\eta_5\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_5}G_{111}-(6-3D)\frac{\partial}{\partial\eta_4}G_{111} \\
= & \left\{\frac{K^2(D-2)}{s_0}G_{111}+\frac{K^2}{s_0}\eta_5\frac{\partial}{\partial\eta_5}G_{111}+\frac{K^2}{s_0}(\eta_4+\eta_5)\frac{\partial}{\partial\eta_4}G_{111}\right\} \\
& +\left\{-\frac{(K^2)^2\eta_5}{2s_0^2}G_{111}\right\}
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
& 2\eta_5\frac{\partial^2}{\partial\eta_5^2}G_{111}+2\eta_4\frac{\partial}{\partial\eta_4}\frac{\partial}{\partial\eta_5}G_{111}-(6-3D)\frac{\partial}{\partial\eta_5}G_{111} \\
= & \left\{\frac{K^2\eta_4}{s_0}\frac{\partial}{\partial\eta_4}G_{111}+\frac{K^2(\eta_5+\eta_4)}{s_0}\frac{\partial}{\partial\eta_5}G_{111}+\frac{K^2(D-2)}{s_0}G_{111}\right\} \\
& +\left\{-\frac{(K^2)^2\eta_4}{2s_0^2}G_{111}\right\}
\end{aligned}
\tag{31}
$$

- (c) 3 degree one equations are third type and we solve two of them

$$\begin{aligned} & \left\{(D-4)\frac{\partial}{\partial \eta_2} G_{111} - 2\eta_2 \frac{\partial^2}{\partial \eta_2^2} G_{111}\right\} \\ = & \left\{(D-4)\frac{\partial}{\partial \eta_3} G_{111} - 2\eta_3 \frac{\partial^2}{\partial \eta_3^2} G_{111}\right\} \end{aligned} \tag{32}$$

and

$$\begin{aligned} & \left\{(D-4)\frac{\partial}{\partial \eta_1} G_{111} - 2\eta_1 \frac{\partial^2}{\partial \eta_1^2} G_{111}\right\} \\ = & \left\{(D-4)\frac{\partial}{\partial \eta_3} G_{111} - 2\eta_3 \frac{\partial^2}{\partial \eta_3^2} G_{111}\right\} \end{aligned} \tag{33}$$

we find that these **un-reducible** points is the set

$$\mathcal{U}_3 = \{(0,0,n_3,0,0) | \forall n_3 \geq 0\} \tag{34}$$

Generating new equation again: Acting on $\frac{\partial}{\partial\eta_3}$ on four degree one reduction rules, we find using anyone we can generate a new equation to reduce n_3. For example, acting $\frac{\partial}{\partial\eta_3}$ at the both sides of the one reducing n_4 and simplifying, we have

$$\begin{aligned}&\left\{\frac{(D-4)K^2}{2s_0}\frac{\partial}{\partial\eta_3}G_{111}-\frac{K^2}{s_0}\eta_3\frac{\partial^2}{\partial\eta_3^2}G_{111}\right\}\\ =\quad&\left\{-\frac{(D-3)(3D-8)}{2}G_{111}+\frac{(5D-16)\eta_3}{2}\frac{\partial}{\partial\eta_3}G_{111}-\eta_3^2\frac{\partial^2}{\partial\eta_3^2}G_{111}\right\}\end{aligned} \tag{35}$$

Using the result (35), one can see that all nonzero n_3 can be reduced to $n_3=0$. So finally the only point can not be reduced using the reduction rule is the point $(0,0,0,0,0)$, which is the master integral for this case. **Done!!!**

For **massive** case:

- (a) 2 degree one equations are second type. Solving them we can reduce n_4, n_5 to zero.
- (b) 6 degree two equations are third type. Using 5 we can solve five operators $\frac{\partial^2}{\partial\eta_1^2}, \frac{\partial^2}{\partial\eta_2^2}, \frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_2}, \frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_3}$ and $\frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_3}$ by the operator $\frac{\partial^2}{\partial\eta_3^2}$
- (c) un-reducible points are

$$\mathcal{U}_3 = \{(0,0,n_3,0,0)|\forall n_3 \geq 0\} \tag{36}$$

and $(1,0,0,0,0), (0,1,0,0,0)$.

- (d) we act $\frac{\partial}{\partial \eta_3}$ on three degree one equations, the degree two part after the partial reduction is

$$\begin{aligned} & \frac{m_3^2(K^2+3m_1^2-3m_2^2-m_3^2)}{s_0^2}\frac{\partial^2}{\partial \eta_3^2} \\ & +\frac{m_2^2(K^2+3m_1^2-3m_3^2-m_2^2)}{s_0^2}\frac{\partial}{\partial \eta_2}\frac{\partial}{\partial \eta_3} \\ & +\frac{2m_1^2(m_1^2-K^2)}{s_0^2}\frac{\partial}{\partial \eta_1}\frac{\partial}{\partial \eta_3} \end{aligned} \tag{37}$$

  Adding it to above 5 degree two equations, we get 6 independent equations to solve 6 degree two operators.
- (e) Using it, the un-reducible points are only following four, i.e., $(1,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$ and $(0,0,0,0,0)$, which are exactly the master integrals. **Done!**

topsector of planar double box

First round:

- There are 9 Lorentz invariant combinations. For top-sector, there are 7 propagators and 2 ISP's.
- There are 10 initial IBP equations. With proper combinations, we will have 4 degree one first type equations and 6 degree two first type equations.
- Using the 4 degree one first type equations we can solve $\frac{\partial}{\partial \eta_i}, i = 1, 3, 4, 6$.
- Using the 6 degree one first type equations we can solve

$$\left\{ \frac{\partial}{\partial \eta_2}, \frac{\partial}{\partial \eta_5}, \frac{\partial}{\partial \eta_7}, \right\} \times \left\{ \frac{\partial}{\partial \eta_8}, \frac{\partial}{\partial \eta_9} \right\} \tag{38}$$

- The un-reducible lattice points are

$$\mathcal{U}_1 = (0, n_2, 0, 0, n_5, 0, n_7, 0, 0), \qquad \mathcal{U}_2 = (0, 0, 0, 0, 0, 0, 0, n_8, n_9)$$

Second round:

- Since the components n_1, n_3, n_4, n_6 have been fully reduced, we will not consider the action $\frac{\partial}{\partial \eta_i}, i = 1, 3, 4, 6$.
- Let us start with $\frac{\partial}{\partial \eta_i}, i = 2, 5, 7, 8, 9$ on four degree one reduction rules. We get 9 **non-trivial generated new equations**, but only 8 are independent
- Among 8, 3 are degree one third type equations, 3, degree two first type equations, 2 degree two second type equations.
- 3 degree two first type equations can be used to solve $\frac{\partial}{\partial \eta_2}\frac{\partial}{\partial \eta_5}$, $\frac{\partial}{\partial \eta_2}\frac{\partial}{\partial \eta_7}$ and $\frac{\partial}{\partial \eta_5}\frac{\partial}{\partial \eta_7}$

- 2 degree two second type equations and 1 degree one third type equation gives

$$
\begin{aligned}
-R_1 &= \frac{2s_0}{s}(n_8+1)(n_9+1)(D-4+n_8)F_{\vec{n}+\vec{e}_8+\vec{e}_9} + \frac{1}{2}(n_8+1)(D-4+n_8)F_{\vec{n}+\vec{e}_8} \\
&\quad + \frac{1}{2}(n_9+1)(n_9-2n_8)F_{\vec{n}+\vec{e}_9} \qquad (40) \\
-R_2 &= \frac{1}{2}(n_8+1)(D-4+n_8)F_{\vec{n}+\vec{e}_8} - \frac{1}{2}(n_9+1)(D-4+n_9)F_{\vec{n}+\vec{e}_9} \qquad (41) \\
R_3 &= \frac{2s_0}{s}(n_8+1)(n_9+1)(D-4+n_9)F_{\vec{n}+\vec{e}_8+\vec{e}_9} + \frac{1}{2}(n_8+1)(2(D-4)+n_8+2n_9)F_{\vec{n}+\vec{e}_8} \\
&\quad - \frac{1}{2}(n_9+1)(D-4+n_9)F_{\vec{n}+\vec{e}_9} \qquad (42)
\end{aligned}
$$

We can solve $F_{\vec{n}+\vec{e}_8+\vec{e}_9}, F_{\vec{n}+\vec{e}_8}, F_{\vec{n}+\vec{e}_9}$

- Remaining 2 degree one third type equation gives

$$\begin{aligned}
\frac{(D-6)}{2}\frac{\partial}{\partial\eta_7}G-\eta_7\frac{\partial^2}{\partial\eta_7^2}G &= \frac{(D-6)}{2}\frac{\partial}{\partial\eta_5}G-\eta_5\frac{\partial^2}{\partial\eta_5^2}G+R_6 \\
\frac{(D-6)}{2}\frac{\partial}{\partial\eta_2}G-\eta_2\frac{\partial^2}{\partial\eta_2^2}G &= \frac{(D-6)}{2}\frac{\partial}{\partial\eta_5}G-\eta_5\frac{\partial^2}{\partial\eta_5^2}G-R_7
\end{aligned} \tag{43}$$

- The unreduced lattice points are just $(0,0,0,0,n_5,0,0,0,0)$, so we need generate further equations.

third round:

- Acting $\frac{\partial}{\partial \eta_5}$ on four new found degree one reduction rules and simplifying it, one gives

$$\frac{(D-8)}{2}\frac{\partial^2}{\partial \eta_5^2}G - \eta_5 \frac{\partial^3}{\partial \eta_5^3}G$$
$$= \frac{(D-6)}{2}\frac{\partial}{\partial \eta_5}\frac{\partial}{\partial \eta_2}G - \eta_2 \frac{\partial^2}{\partial \eta_2^2}\frac{\partial}{\partial \eta_5}G + \frac{\partial}{\partial \eta_5}R_7 \quad (44)$$

  we get a perfect degree two reduction rule for $(0,0,0,0,n_5,0,0,0,0)$.
- Since the left hand side is $F_{\vec{n}+2\vec{e}_5}$, we can only reduce lattice points $(0,0,0,0,n_5,0,0,0,0)$ with $n_5 \geq 2$. Now the only points can not be reduced is $n_5 = 0, 1$ and we have found two master integrals. **Done!**

In this talk, I have present an algorithm to find complete symbolic reduction rules using generating functions. We have shown the efficiency using several examples. Although it has been checked conceptually, programming it will be important!

Thanks a lot of your attention !