On Canonicalising any Feynman Integrals

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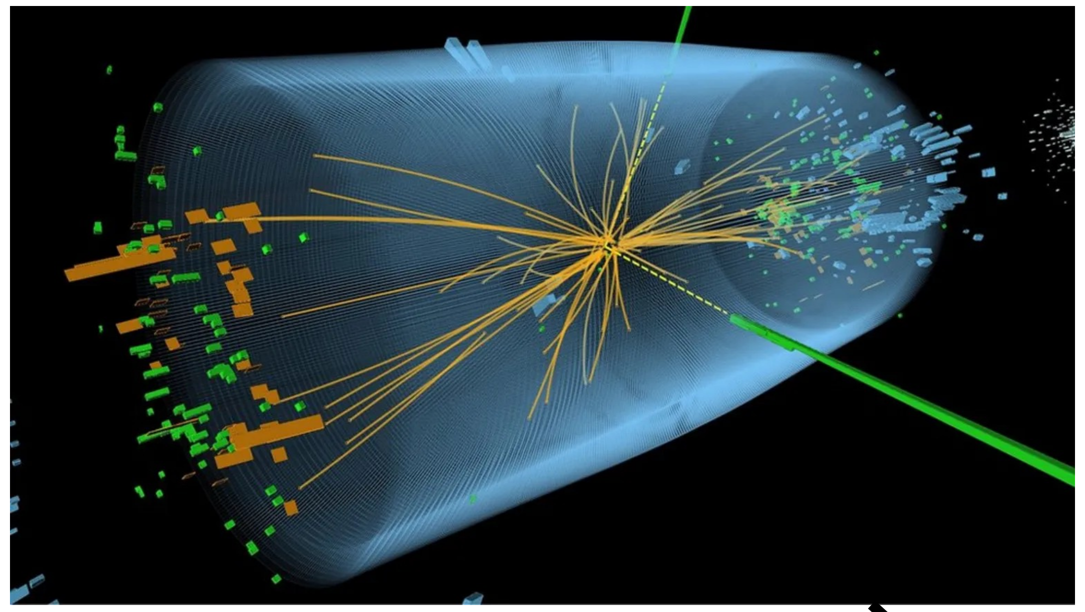
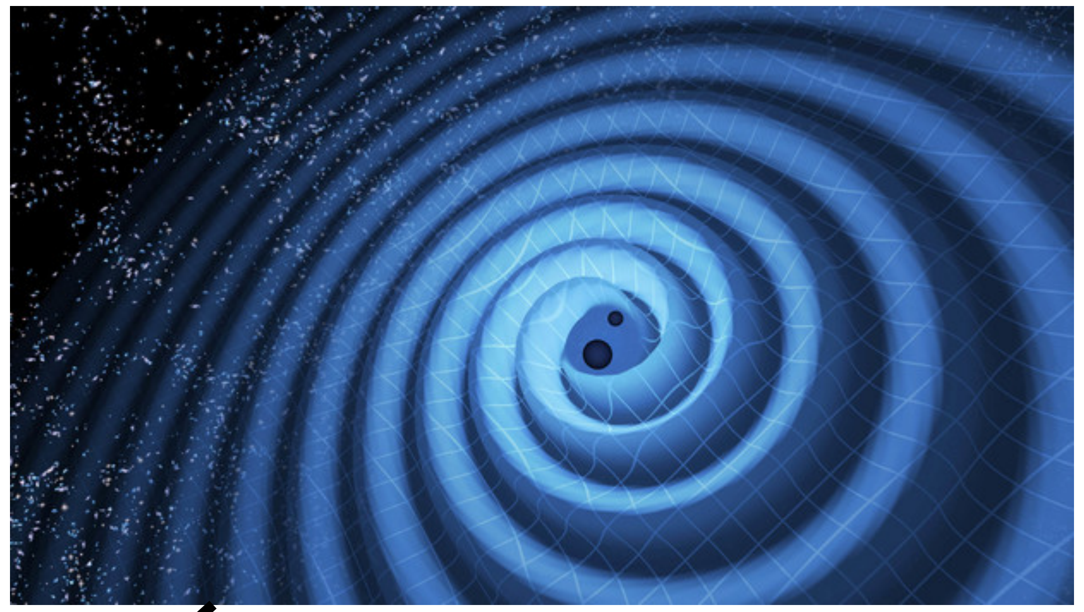
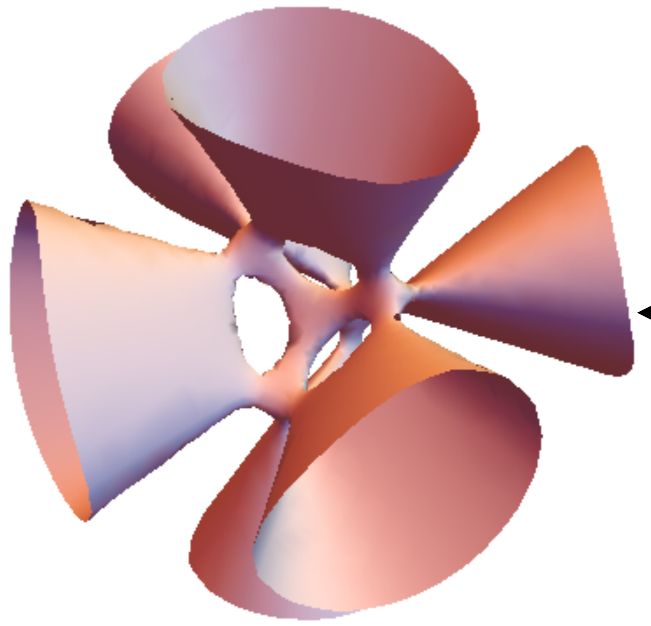
Motivation



precision prediction
with QFTs



Easy-to-(re)use analytic results are crucial.



mathematics

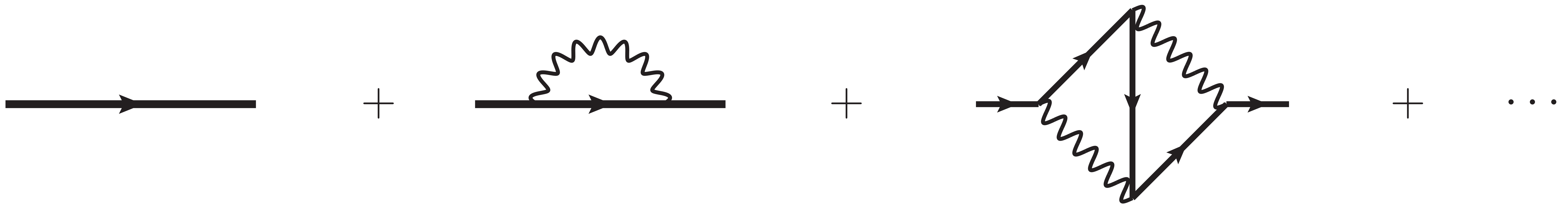
$$O\left(\alpha,\{x\}\right)=\sum_n c_n F_n$$

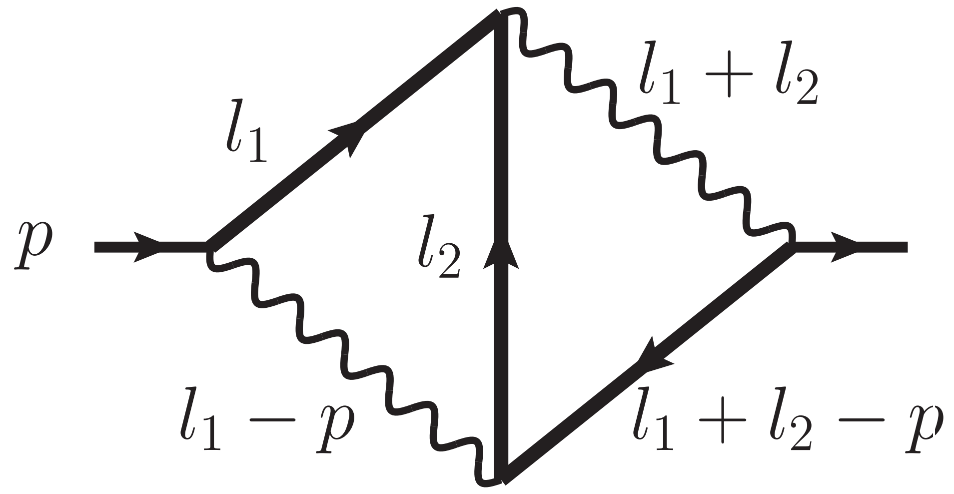
F_n: **Feynman Integrals**

$\mathcal{N}=4$ sYM,
string…

formal theory

Feynman Integrals are Inevitable in QFTs

$P_{e^-\to e^-} =$ 



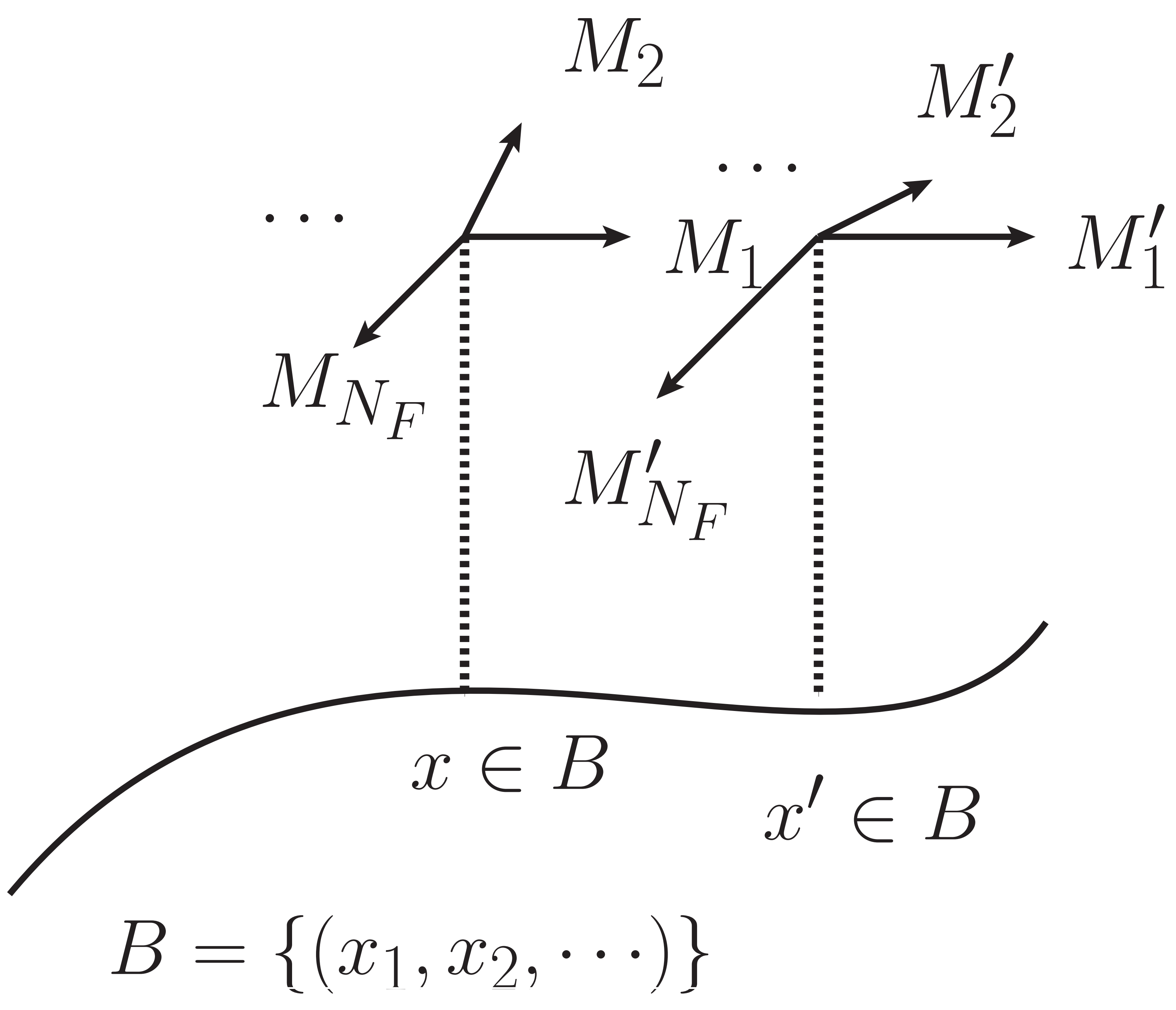
$$= \underbrace{\int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \int \frac{\mathrm{d}^D l_2}{i\pi^{D/2}} \frac{e^{2\varepsilon\gamma_E} \cdot (-p^2)^{|\nu|-D} \cdot \mathrm{Num}(\{l\})}{\left[l_1^2 - m^2\right]^{\nu_1} \left[l_2^2 - m^2\right]^{\nu_2} \left[(l_1+l_2-p)^2 - m^2\right]^{\nu_3} \left[(l_1+l_2)^2\right]^{\nu_4} \left[(l_1-p)^2 - m^2\right]^{\nu_5}}}_{I_{\nu_1\nu_2\nu_3\nu_4\nu_5}}$$

Each Feynman diagram corresponds to a **Feynman Integral**, of which the complexity exploits **factorially**.

Kinematics Dependence

Kinematics vary$\longrightarrow$ **differential equations (w. IBPs)** of FIs (MIs).

The **primary** method for analytic and (semi-)numerical calculation of FIs.



$$\mathrm{d}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{N_F} \end{pmatrix} = A_{N_F \times N_F}(\varepsilon, \{x\}) \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{N_F} \end{pmatrix}$$

[Kotikov '91; Remiddi '97; Gehrmann, Remiddi '00]

Solving this linearly coupled PDE system is non-trivial!

ε-factorisation (canonicalisation)

ε-factorisation:

Via **rotations** of basis (i.e., $\vec{M} \to \vec{K}$) and **variable change**, ***ε-dependence factorises in the connection matrix, with suitable boundary conditions.*** [Henn '13]

$$\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} = \varepsilon\, B_{N_F\times N_F}(\{q(x)\}) \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} \Longrightarrow \quad \vec{K} = \sum \varepsilon^n \vec{K}^{(n)} \longrightarrow \vec{K}^{(n+1)} = \int B_{N_F\times N_F} \vec{K}^{(n)} + \text{boundary}$$

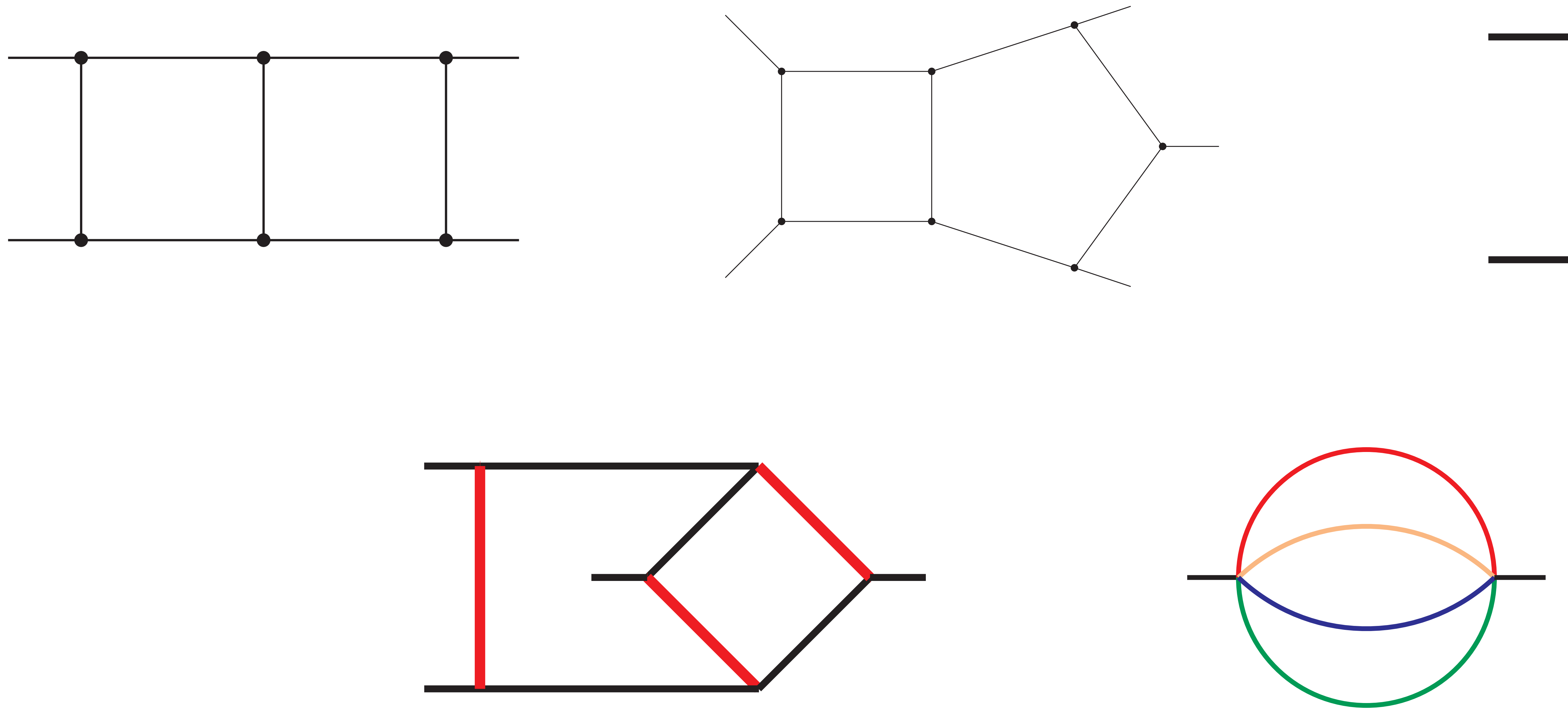
$$\hookrightarrow \quad \vec{K} = \mathbf{P}\exp\left(\varepsilon \int B_{N_F\times N_F}\right) \vec{K}_{\text{boundary}}$$

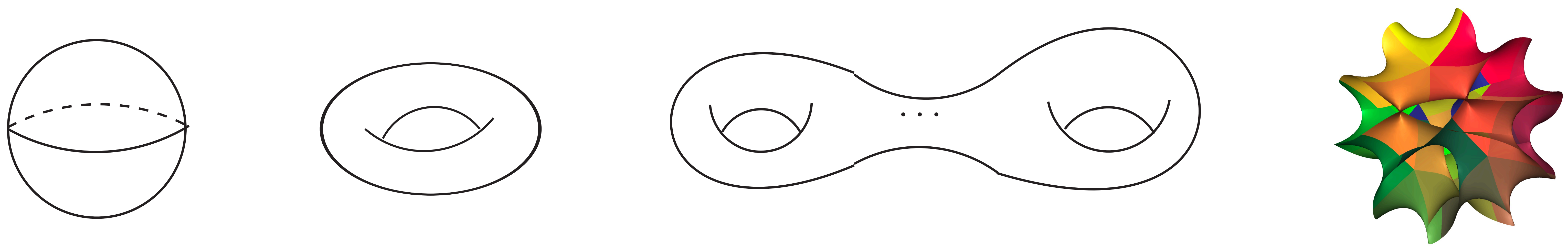
MIs can be written as Chen's iterated integrals [Chen '77].

Once the ε-factorised (canonical) form is derived, FIs (MIs) are viewed as solved.

Very Hard! Lots of progress recently, e.g., Baune, Bönisch, Broedel, ε-collaboration, Dlapa, Duhr, Frellesvig, Görges, Henn, Klement, Jiang, Maggio, Nega, Porkert, Sauer, Sohnle, Stawinski, Tancredi, Wager, Wilhelm, Yan, Yang, Zhang, Zhu + many more…

Feynman Integrals are Hard





$+\cdots$

[Snowmass review 2203.07088],

[a book by Weinzierl, '22]

Takehome Message

A unified algorithm towards deriving the ε-factorised (canonical) form of any Feynman integral, inspired by Hodge theory.

The algorithm cracks the complexity to the minimum.

figure generated by ChatGPT

- One can derive ε-factorisation (canonicalisation) step by step ***without*** prior knowledge of geometry;
- A unified framework!

This Talk: ε-collaboration

based on **2506.09124 and 2510.xxxxx** by the **ε-collaboration**


Irís Brée


Federico Gasparotto


Antonela Matijašić


Pouria Mazloumi


Dmytro Melnichenko


Sebastian Pögel


Toni Teschke


Xing Wang


Stefan Weinziel


Konglong Wu


Xiaofeng Xu

and **2507.23594** by **a subset of** the **ε-collaboration**

Outline

I. Baikov Representation and Twisted Cohomology

II. Step 1 of the Algorithm and an Example

III. Step 2 of the Algorithm and an Example(s)

Baikov Representation as Variable Change

- $\mathrm{d}^D l \to \mathrm{d}P_1(l)\mathrm{d}P_2(l)\cdots$, i.e., propagators, P_i's, as integration variables: $z_i = P_i/\mu^2$;
  ↪ non-trivial "Jacobian", **twist**:

extra variables

$$I_{\nu_1\cdots\nu_n} = C_{\text{Baikov}} \int_{\mathscr{C}} \underbrace{u(z_1, \cdots, z_n; \cdots, z_{n+N_v})}_{\text{twist}} \frac{1}{z_1^{\nu_1} z_2^{\nu_2} \cdots z_n^{\nu_n}} \mathrm{d}^n z \wedge \boxed{\mathrm{d}^{N_V} z}$$

Packages: [*Baikovletter*, Jiang, Yang; *BaikovPackage*, Frellesvig; *SOFIA*, Correia, Giroux, Mizera]

- Not for calculating FIs, but rather for studying the structures!
- It translates FIs to twisted cohomology language.
- Perfect for (onshell) propagator cuts: $\mathrm{cut}_i = \mathrm{res}_{z_i=0}$.

Setup: Twist

MIs in a given sector share the same (minimal) twist ($b_i, b_j \in \mathbb{Z}$):

$$u(z_1, z_2, \cdots, z_{N_V}) = \prod_{i \in I_{\text{odd}}} [p_i(z)]^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j \in I_{\text{even}}} [p_j(z)]^{\frac{1}{2}b_j\varepsilon}$$

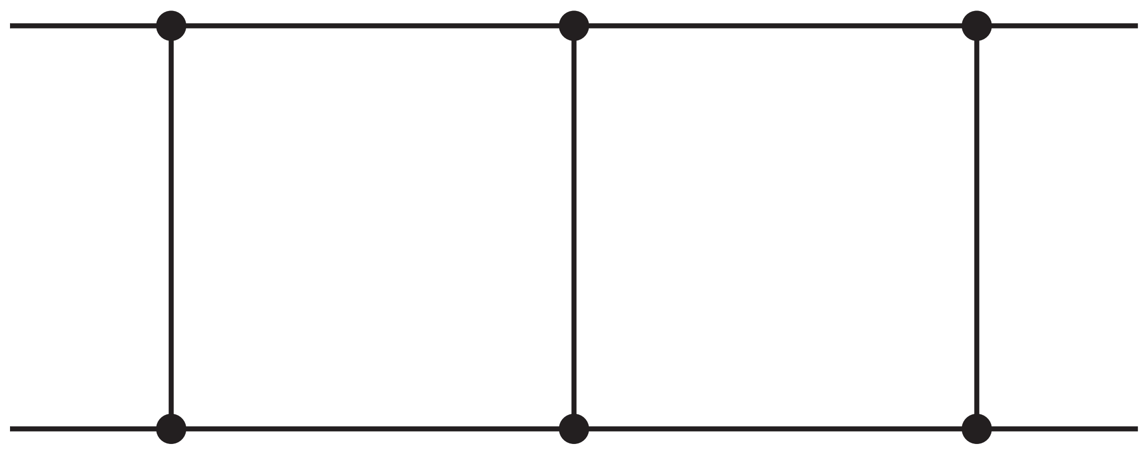
- Odd polynomials $\longrightarrow$ geometry;
- Even polynomials $\longrightarrow$ possible residues (punctures) to take;
- Different MIs $\longrightarrow$ different rational parts.

$$M_i = C_{\text{Baikov}} \int_{\text{MC}} u(z) \frac{q_i(z)}{\prod_{j \in \text{all}} [p_j(z)]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1, \qquad \mu_j \in \mathbb{Z}$$

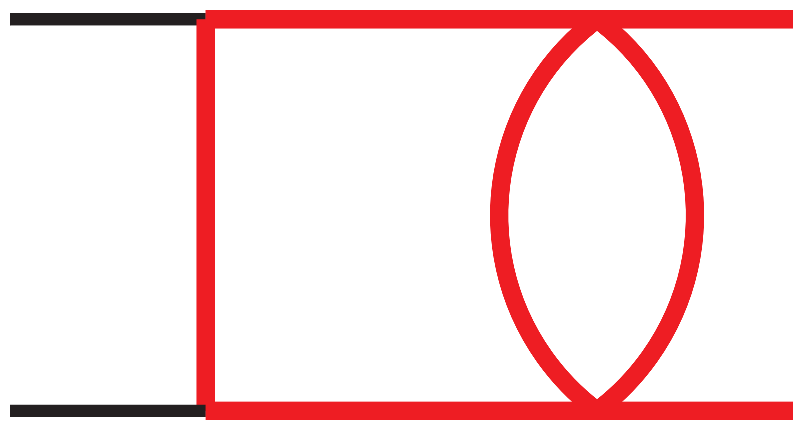
maximal cut

remaning variables after MC

Twist: Examples



$$I^{\text{MaxCut}}_{111111100} = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} \frac{\mathrm{d}z_1}{2\pi i} \overbrace{z_1^{-2\varepsilon} \left(z_1 - 1\right)^{-\varepsilon} \left(z_1 - x - 1\right)^{\varepsilon}}^{u(z_1)} \cdot 1$$



$$I^{\text{MaxCut}}_{111200100} = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} \frac{\mathrm{d}z_1}{2\pi i} \left[p_1(z_1)\right]^{-\frac{1}{2}} \left[p_2(z_1)\right]^{-\frac{1}{2}-\varepsilon} \left[p_3(z_1)\right]^{-\frac{1}{2}-\varepsilon} \cdot 1, \quad \text{w.}$$

$$p_1 = z_1 - x_2, \;\; p_2 = z_1 + 4 - x_2, \quad p_3 = \left(z_1 + 1\right)^2 - 4\left[x_2 + \frac{\left(1 - x_2\right)^2}{x_1}\right].$$

Setup: Twisted Cohomology

$$M_i = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} \underbrace{u(z) \frac{q_i(z)}{\prod_{j \in \text{all}} \left[p_j(z)\right]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1}_{\hat{\phi}_i}, \qquad \mu_j \in \mathbb{Z}$$

- Given a M_i, there is a differential form $\hat{\phi}_i$ ($H_\omega^{N_V} \twoheadrightarrow V^{N_V}$).
- One needs to mod out IBP relations $\hat{\phi}_i \sim \hat{\phi}_i + \nabla_u \hat{\eta}$, which leads us to the twisted cohomology [See e.g., Mastrolia and Mizera, '18].
- Study the differential forms to represent the corresponding MIs.
- Consider $\{\infty\}$ as well: $u \to U, \hat{\phi} \to \hat{\Phi}$ using homo. coord. $\left[z_0 : z_1 : \cdots : z_{N_V}\right]$.

Step 1 of the Algorithm

For each element in $H_\omega^{N_V}$={differential forms} mod IBPs, i.e., **{integrands}**, we define a prefactor C_ε:

$$H_\omega^{N_V} = \left\{ \Psi_{\mu_0 \ldots \mu_{N_D}}[Q] = C_\varepsilon(\{\mu\})\, C_{\text{Baikov}}\, U(z)\, \hat{\Phi}_{\mu_0 \ldots \mu_{N_D}}[Q]\, \eta \right\} \text{ modulo IBPs}$$

vector space
w./ fin. dim.

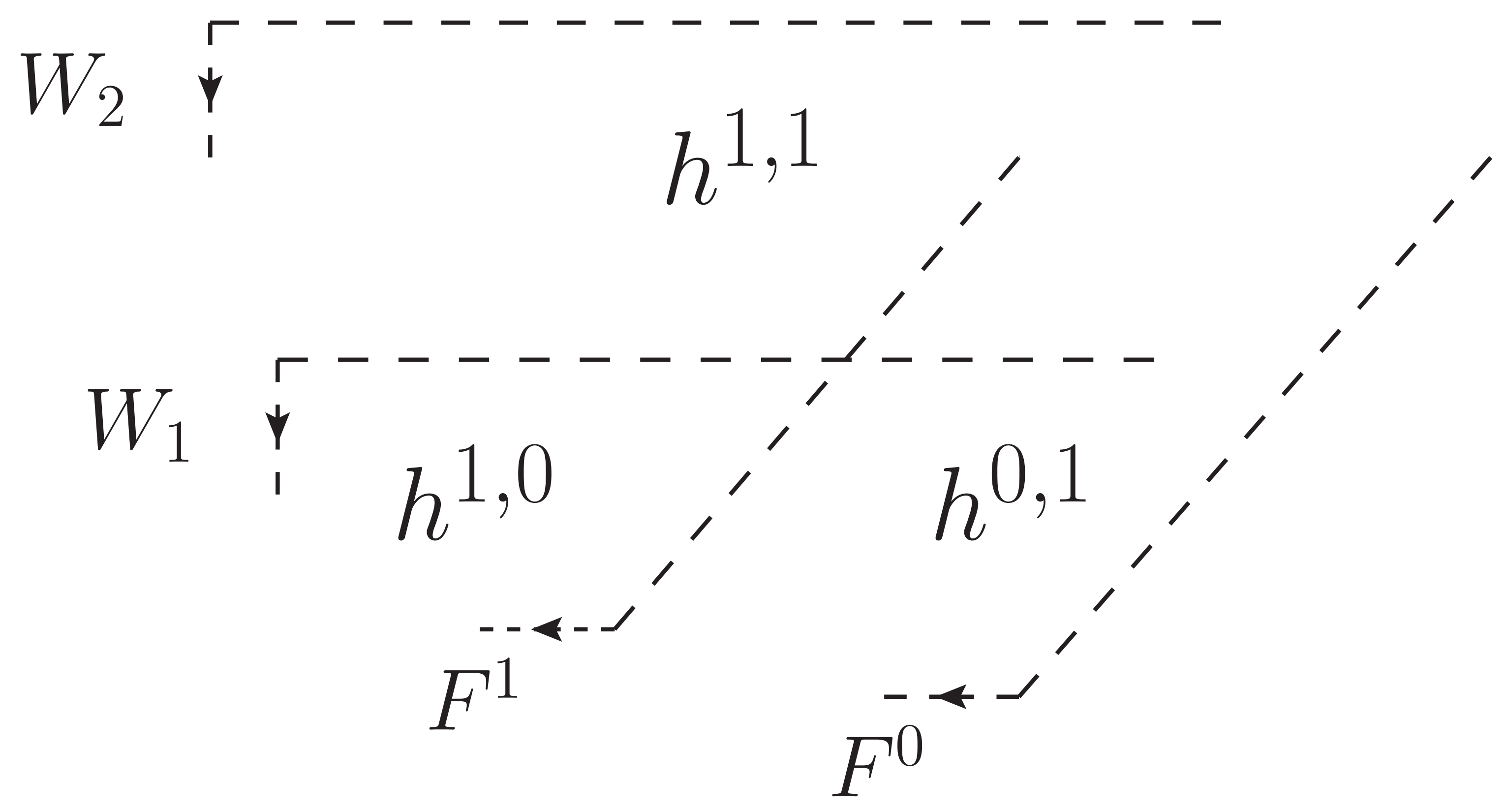
$$C_\varepsilon = C_{\text{abs}} \cdot \underbrace{\varepsilon^{-|\mu|}}_{C_{\text{clutch}}} \times \underbrace{\prod_{i \in I_{\text{odd}}} \left(-\frac{1}{2} + \frac{1}{2} b_i \varepsilon \right)_{\mu_i} \prod_{i \in I_{\text{even}}} \left(\frac{1}{2} b_i \varepsilon \right)_{\mu_i}}_{C_{\text{rel}}} \qquad (a)_n = \frac{\Gamma(a+1)}{\Gamma(a+1-n)}$$

This pre-factor is entirely determined by the integrand, and it simplifies the ε-complexity.

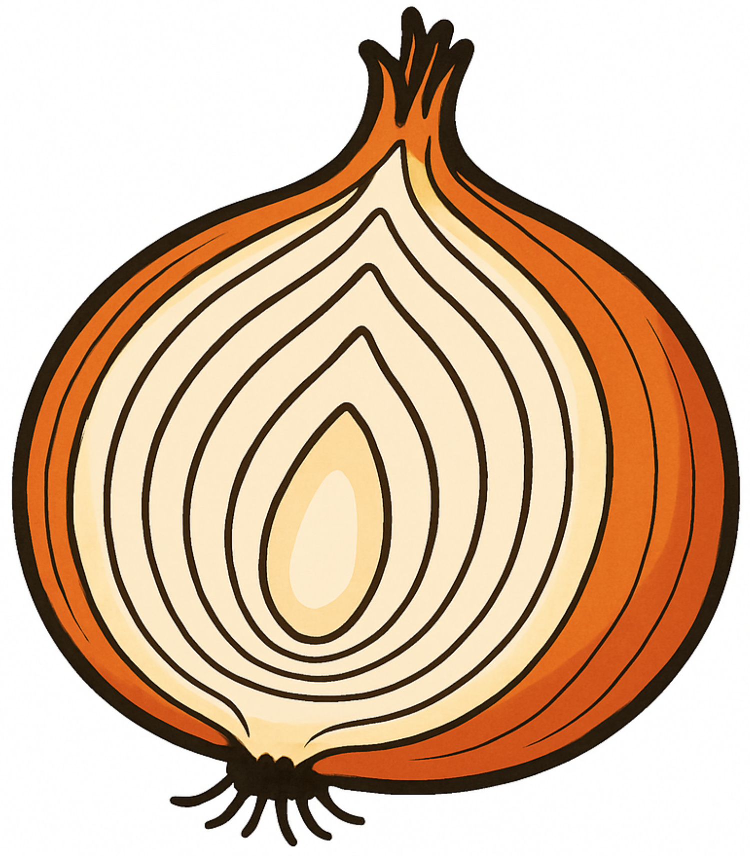
Step 1 of the Algorithm

"Layered" decomposition (**filtration, 滤**) of $H_\omega^{N_V}$ into subspaces! （剥洋葱）

$$\cdots \subseteq F^{p+1} H_\omega^{N_V} \subseteq F^{p} H_\omega^{N_V} \subseteq \cdots$$
$$\cdots \supseteq W_w H_\omega^{N_V} \supseteq W_{w-1} H_\omega^{N_V} \supseteq \cdots$$



two filtrations cut an "onion" into 3 subspaces in this example

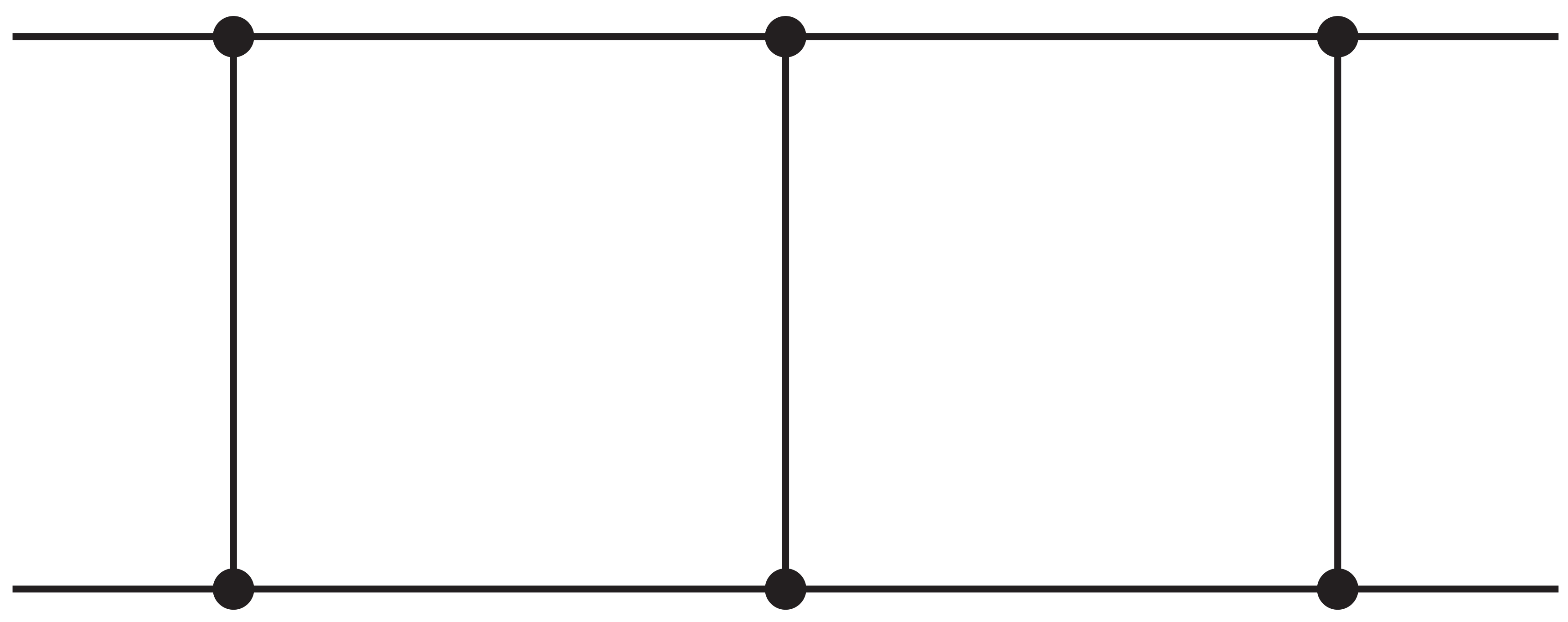


one filtration of onion

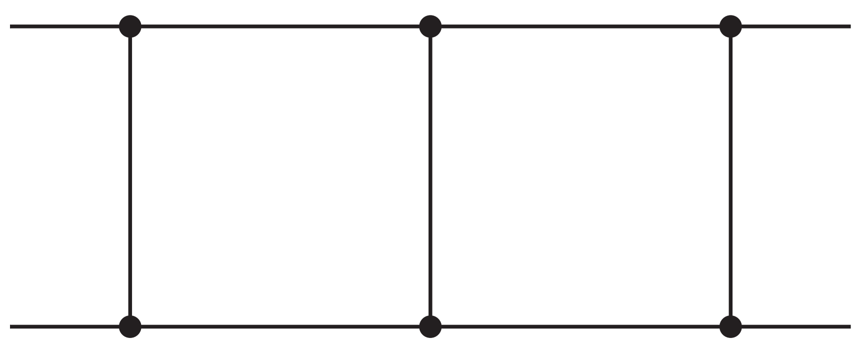
Given $\Psi_{\mu_0\cdots\mu_{N_D}}[Q]$, two more ordering criteria (two **filtrations**): pole order o, and the number of non-zero residues r . *These two integers indicate "layer" numbers (两种剥洋葱的指标).*

$$p = N_V - o + r; \qquad q = o; \qquad w = p + q = N_V + r\,.$$

Appetizers Example



Appetizers Example: Twist



$$I_{111111100} = C_{\text{Baikov}} \int_{\mathscr{C}_{\text{MC}}} \frac{\mathrm{d}z_1}{2\pi i} \overbrace{z_1^{-2\varepsilon} \left(z_1 - 1\right)^{-\varepsilon} \left(z_1 - x - 1\right)^{\varepsilon}}^{u(z_1)}$$

- Setting $C_{\text{abs}} = \varepsilon^4 x^2$ makes $C_{\text{abs}} \cdot C_{\text{Baikov}}$ pure.
- The minimal twist in the projective space reads: $U(z_0, z_1) = z_0^{2\varepsilon} z_1^{-2\varepsilon} (z_1 - z_0)^{-\varepsilon} \left[z_1 - (1+x) z_0\right]^{\varepsilon}$.
- All four polynomials are even $\Longrightarrow$ four possible (localisations) to take non-zero resides.

$$[z_0 : z_1] \in \left\{[0:1], \quad [1:0], \quad [1:1], \quad [1:1+x]\right\}$$

Appetizers Example: Howto

$$H^1_\omega \ni \Psi_{\mu_0\ldots\mu_3}[Q] = C_\varepsilon(\{\mu\})\, U(z)\, \hat{\Phi}_{\mu_0\ldots\mu_3}[Q]\, \eta; \qquad \eta = z_0 \mathrm{d}z_1 - z_1 \mathrm{d}z_0\,.$$

$$U(z_0, z_1) = z_0^{2\varepsilon} z_1^{-2\varepsilon} (z_1 - z_0)^{-\varepsilon} \left[z_1 - (1+x) z_0\right]^{\varepsilon} \Longrightarrow [z_0 : z_1] \in \left\{[0:1], \quad [1:0], \quad [1:1], \quad [1:1+x]\right\}$$

- $\deg U = 0, \deg \eta = 2 \Longrightarrow \deg \hat{\Phi} = -2.$
- Ψ should localise on those 4 points after taking one residue $\Longrightarrow$ 4 candidates.

$$\Psi_{1100}[1] = -4\varepsilon^4 x^2 C_{\mathrm{Baikov}} U \frac{\eta}{z_0 z_1}, \qquad \Psi_{1010}[1] = -2\varepsilon^4 x^2 C_{\mathrm{Baikov}} U \frac{\eta}{z_0\left(z_1 - z_0\right)},$$

$$\Psi_{1001}[1] = 2\varepsilon^4 x^2 C_{\mathrm{Baikov}} U \frac{\eta}{z_0\left[z_1 - (1+x) z_0\right]}, \qquad \Psi_{0110}[1] = 2\varepsilon^4 x^2 C_{\mathrm{Baikov}} U \frac{\eta}{z_1\left(z_1 - z_0\right)}.$$

$$2\varepsilon^4 x^2 = \underbrace{\varepsilon^4 x^2}_{C_{\mathrm{abs}}} \cdot \underbrace{\varepsilon^{-2}}_{C_{\mathrm{clutch}}} \cdot \underbrace{(2\varepsilon \cdot \varepsilon)}_{C_{\mathrm{rel}}}$$

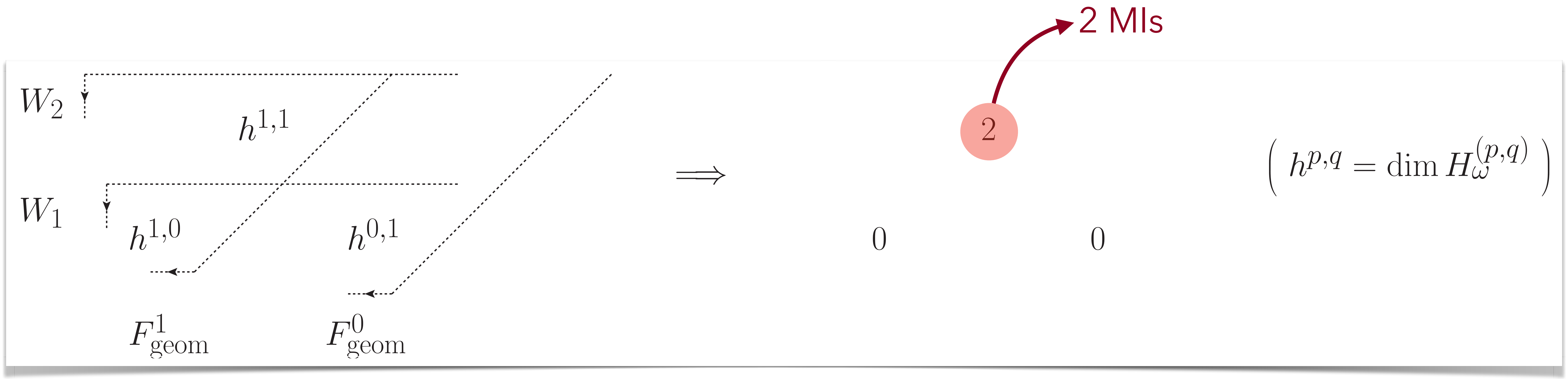
All have $(p, q) = (1,1)$.

Appetizers Example: MIs

There are IBP relations between these four candidates:

$$\left\{ \Psi_{1100}[1], \Psi_{1010}[1], \Psi_{1001}[1], \Psi_{0110}[1] \right\} \Longrightarrow \left\{ \Psi_{0110}[1], \Psi_{1010}[1] \right\}$$

$$\begin{aligned} \Psi_{0110}[1] \quad &\Longleftrightarrow \quad 2\varepsilon^4 x^2 I_{111111100} = K_1, \\ \Psi_{1010}[1] \quad &\Longleftrightarrow \quad -2\varepsilon^4 x^2 I_{1111111(-1)0} = K_2\,. \end{aligned} \quad \Longrightarrow \quad \dim V^1 = \dim H^1_\omega = 2.$$



This derived basis is already the ε-factorised: $\mathrm{d}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \varepsilon\, \mathbf{A}_{\mathrm{MC}}(x) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$

Step 2 of the Algorithm

We pick elements in $H_\omega^{N_V}$ by the filtration criteria, translate back to MIs, defining a new basis $\vec{J}$:

$$\mathrm{d}\,\vec{J} = \left[\frac{1}{\varepsilon^{N_V}}\mathbf{B}^{(-N_V)}(x) + \frac{1}{\varepsilon^{N_V-1}}\mathbf{B}^{(-N_V+1)}(x) + \cdots + \mathbf{B}^{(0)}(x) + \boxed{\varepsilon\,\mathbf{B}^{(1)}(x)}\right]\vec{J}$$

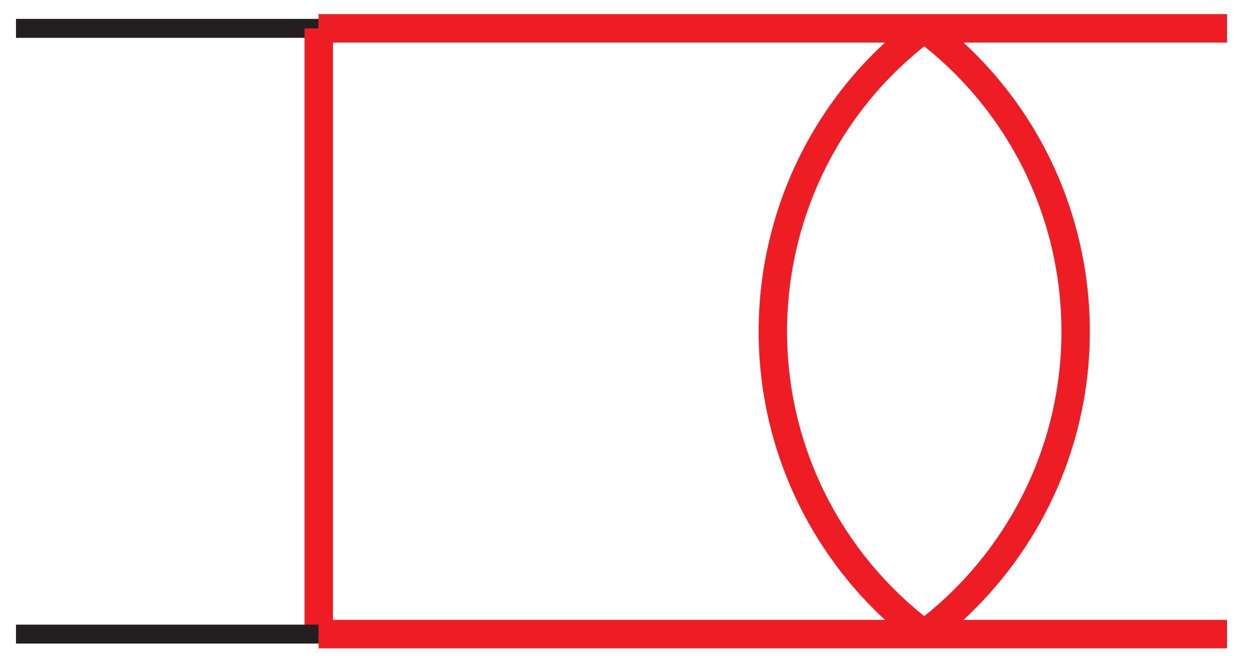
$$\vec{M} = \mathbf{R}_1\,\vec{J}$$

$\mathbf{B}^{(-N_V)}(x), \cdots, \mathbf{B}^{(0)}(x)$ are in a good block lower-triangular form!

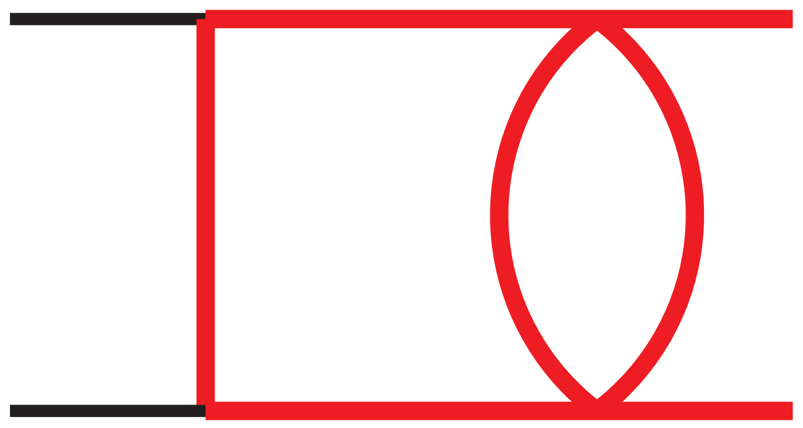
$$\mathbf{R}_2^{(-N_V)}\mathbf{R}_2^{(-N_V+1)}\cdots\mathbf{R}_2^{(0)}\,\vec{K} = \vec{J}$$

- Rotate away $\mathbf{B}^{(i)}$ step by step. It is systematic (the existence of ε-factorisation);
- $\mathbf{R}_2^{(i)}$ is determined by (*simpler*) PDEs. In particular, $\mathbf{R}_2^{(-N_V)}$ relates to periods of geometry.

Dessert Example



Dessert Example: Twist



$$I_{11121,\mathrm{MC}} = C_{\mathrm{Baikov}} \int_{\mathscr{C}_{\mathrm{MC}}} \frac{\mathrm{d}z_1}{2\pi i} \left[p_1(z_1)\right]^{-\frac{1}{2}} \left[p_2(z_1)\right]^{-\frac{1}{2}-\varepsilon} \left[p_3(z_1)\right]^{-\frac{1}{2}-\varepsilon}$$

$$H^1_\omega \ni \Psi_{\mu_0\ldots\mu_3}[Q] = C_\varepsilon(\{\mu\})\, U(z)\, \hat{\Phi}_{\mu_0\ldots\mu_3}[Q]\ \eta; \qquad \eta = z_0 \mathrm{d}z_1 - z_1 \mathrm{d}z_0\,.$$

- Setting $C_{\mathrm{abs}} = \varepsilon^3 x_1$ makes $C_{\mathrm{abs}} \cdot C_{\mathrm{Baikov}}$ pure.
- The minimal twist in the projective space reads ($P_0 = z_0$): $U(z_0, z_1) = P_0^{3\varepsilon} P_1^{-\frac{1}{2}} P_2^{-\frac{1}{2}-\varepsilon} P_3^{-\frac{1}{2}-\varepsilon}$.
- $\deg U = -2, \deg \eta = 2 \Longrightarrow \deg \hat{\Phi} = 0$.
- 3 odds, and 1 even $\Longrightarrow$ one possibility (localisations) to take non-zero residue: $[z_0 : z_1] \in \left\{[0:1]\right\}$

Dessert Example: Filtration

$h^{1,1} = 1$

$w = 1 + 1 \longrightarrow$ 1 residue upon the only even polynomial.

$\hookrightarrow \ \hat{\Phi} = \frac{z_1}{z_0}$, only one choice.

$h^{1,0} = 1$

$h^{0,1} = 1$

Forms without non-zero residues. There are 2 cases:

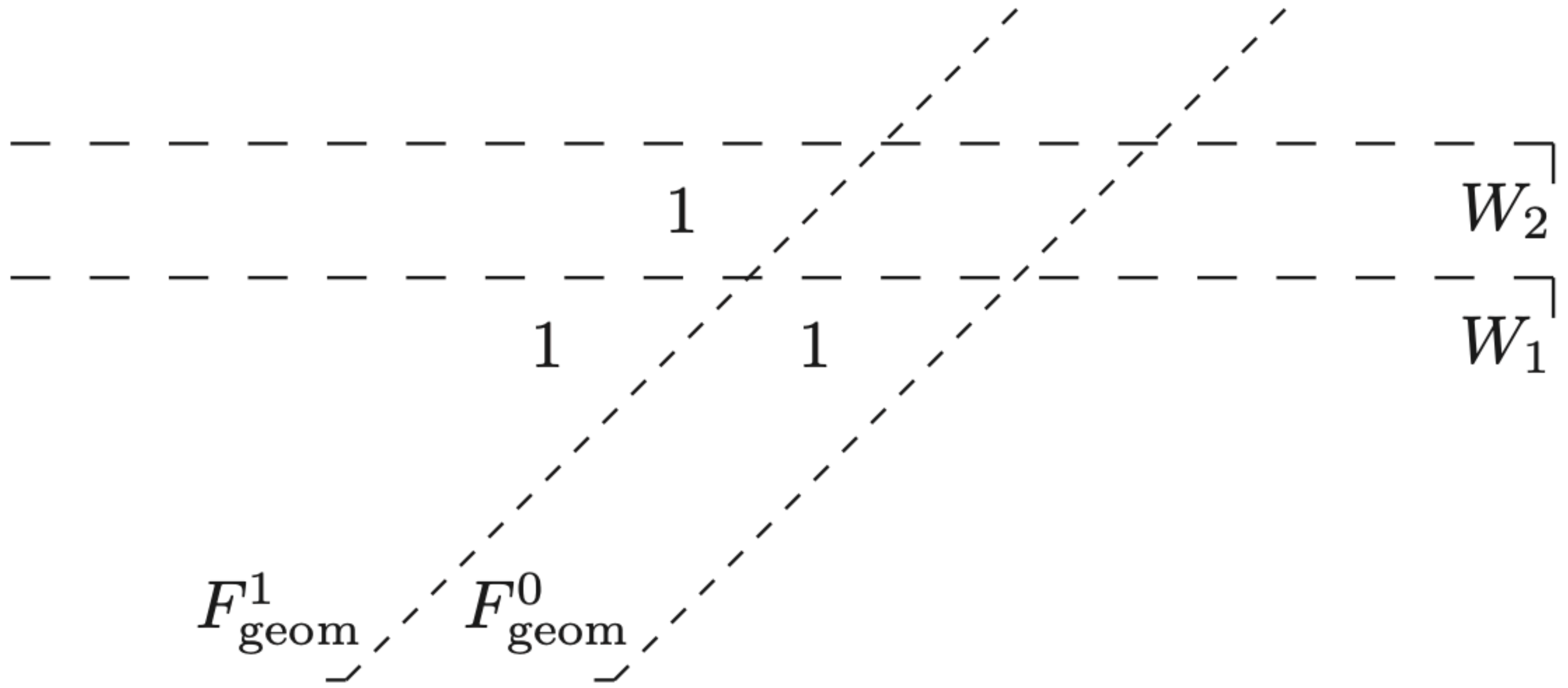
$\rightsquigarrow$ 1) no pole; 2) with pole, but no residue.

We order them by pole order o.

$\hat{\Phi} = 1$, only one choice

$\hat{\Phi} \in \left\{ \frac{z_0}{P_1}, \frac{z_0}{P_2}, \frac{z_0^2}{P_3} \right\}$, "three" choices.

But IBP relations $\rightsquigarrow$ only **one** choice.



Dessert Example: Basis $\vec{J}$ after Step 1

- Map the filtrated H^1 to the Feynman integral side to get the step-1 basis $\vec{J}$:

$$J_1 = j\big(\Psi_{0000}[1]\big) = \varepsilon^3 x_1 I_{111200100}$$

$$J_2 = j\big(\Psi_{1000}[z_1]\big) = 3\varepsilon^3 x_1 I_{11120001(-1)0}$$

$$J_3 = j\big(\Psi_{0100}[z_0]\big) = -\frac{1}{2}\varepsilon^2 x_1 \Big[c_1 I_{111200100} + c_2 I_{11120001(-1)0} + c_3 I_{21120001(-1)0}\Big]$$

- DEQ of the above basis is good enough:

$$\mathrm{d}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \left[\underbrace{\begin{pmatrix} B_{11}^{(-1)} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\varepsilon}B_{31}^{(-1)} & B_{32}^{(-1)} & B_{33}^{(-1)} \end{pmatrix}}_{\mathbf{B}^{(-1)}_{3\times3}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ B_{21}^{(0)} & 0 & 0 \\ B_{31}^{(0)} & 0 & 0 \end{pmatrix}}_{\mathbf{B}^{(0)}_{3\times3}} + \varepsilon \mathbf{B}^{(1)}_{3\times3}\right]\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix}$$

Dessert Example: Step 2

- We then need two rotations to remove $\mathbf{B}^{(-1)}$ and $\mathbf{B}^{(0)}$: $\mathbf{R}^{(-1)} \cdot \mathbf{R}^{(0)}$.

$$\mathbf{R}^{(-1)} = \begin{pmatrix} R_{11}^{(-1)} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\varepsilon} R_{31}^{(-1)} & R_{32}^{(-1)} & R_{33}^{(-1)} \end{pmatrix}, \qquad \mathbf{R}^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ R_{21}^{(0)} & 1 & 0 \\ R_{31}^{(0)} & 0 & 1 \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} K_1 &= \frac{J_1}{R_{11}^{(-1)}}, \\ K_2 &= 3J_2 - R_{21}^{(0)} K_1 \\ K_3 &= \cdots \end{aligned}$$

- All the elements are constrained by simpler differential operators;
- They are determined term by term. It is systematic.
- The matrix elements are, in fact, related to periods on the elliptic curve.
  But, we do not need to know this in advance!

Summary

A unified algorithm towards deriving the ε-factorised (canonical) form of any Feynman integral, inspired by Hodge theory.

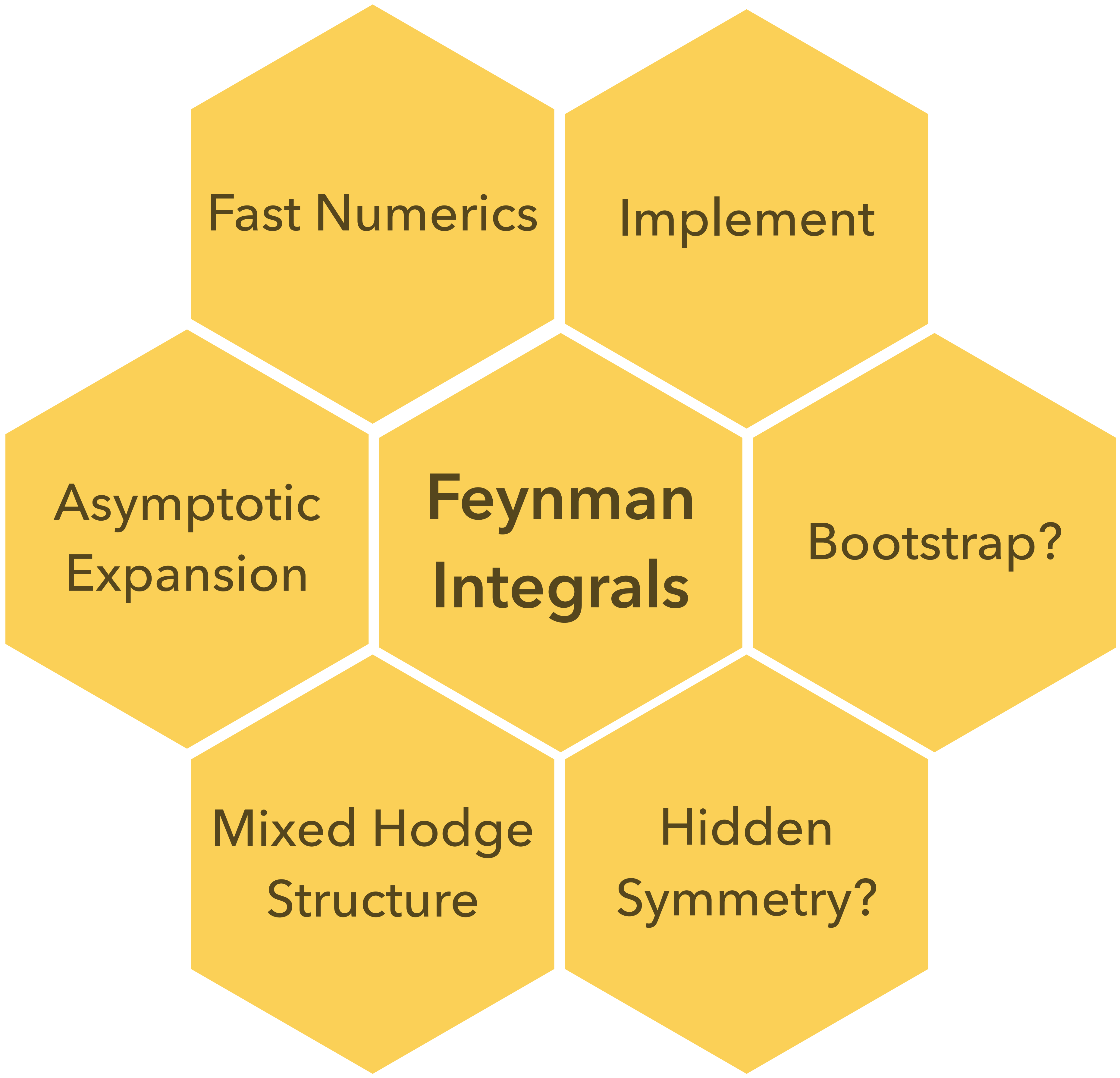
The algorithm cracks the complexity to the minimum.

figure generated by ChatGPT

- One can derive ε-factorisation step by step ***without*** prior knowledge of geometry;
- A unified framework!

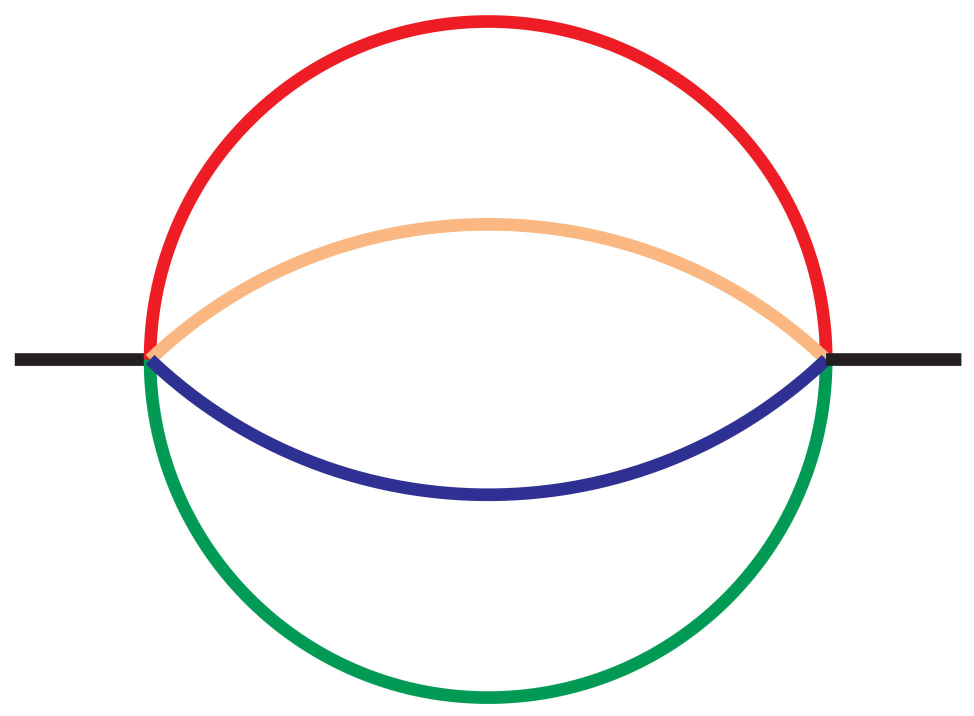
Thank you for listening.

Outlook



Fast Numerics
Implement
Asymptotic Expansion
Feynman Integrals
Bootstrap?
Mixed Hodge Structure
Hidden Symmetry?

Main Course Example



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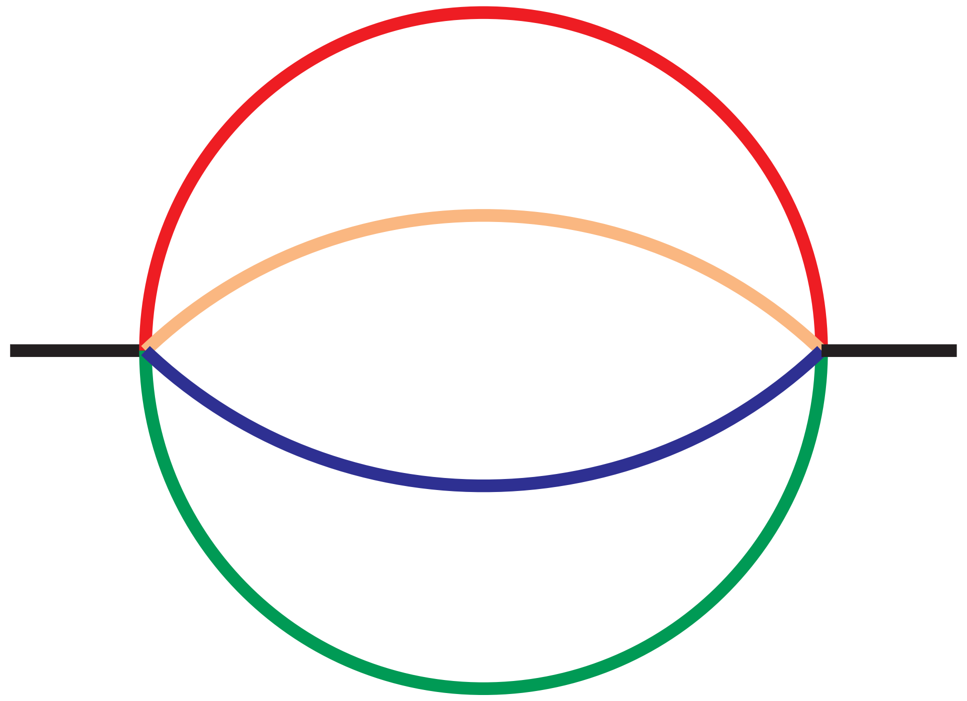
[See also *2507.23061* by Duhr, Maggio, Porkert, Semper and Stawinski]

Main Course Example: Twist



Highly non-trivial in several aspects:

- It involves 2 Baikov variables (3 after homo.), and is beyond elliptics.
- Non-trivial step-2 rotations.
- Super-sectors (id=31, 47) come into play.

$$H^2_\omega \ni \Psi_{\mu_0\ldots\mu_5}[Q] = C_\varepsilon(\{\mu\})\, U(z)\, \hat{\Phi}_{\mu_0\ldots\mu_5}[Q]\; \eta; \qquad \eta = z_0\, \mathrm{d}z_1 \wedge \mathrm{d}z_2 + \cdots,$$

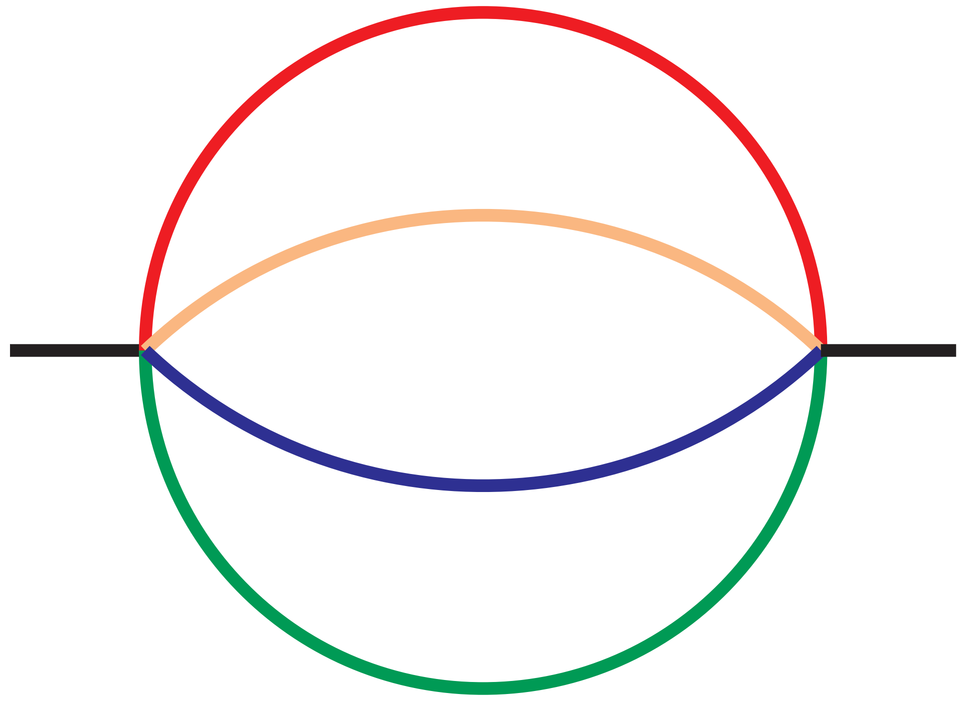
$$U(z_0, z_1, z_2) = z_0^{4\varepsilon}\, z_1^{\varepsilon}\, z_2^{\varepsilon}\, P_3^{-\frac{1}{2}-\varepsilon}\, P_4^{-\frac{1}{2}-\varepsilon}\, P_5^{-\frac{1}{2}-\varepsilon}\,.$$

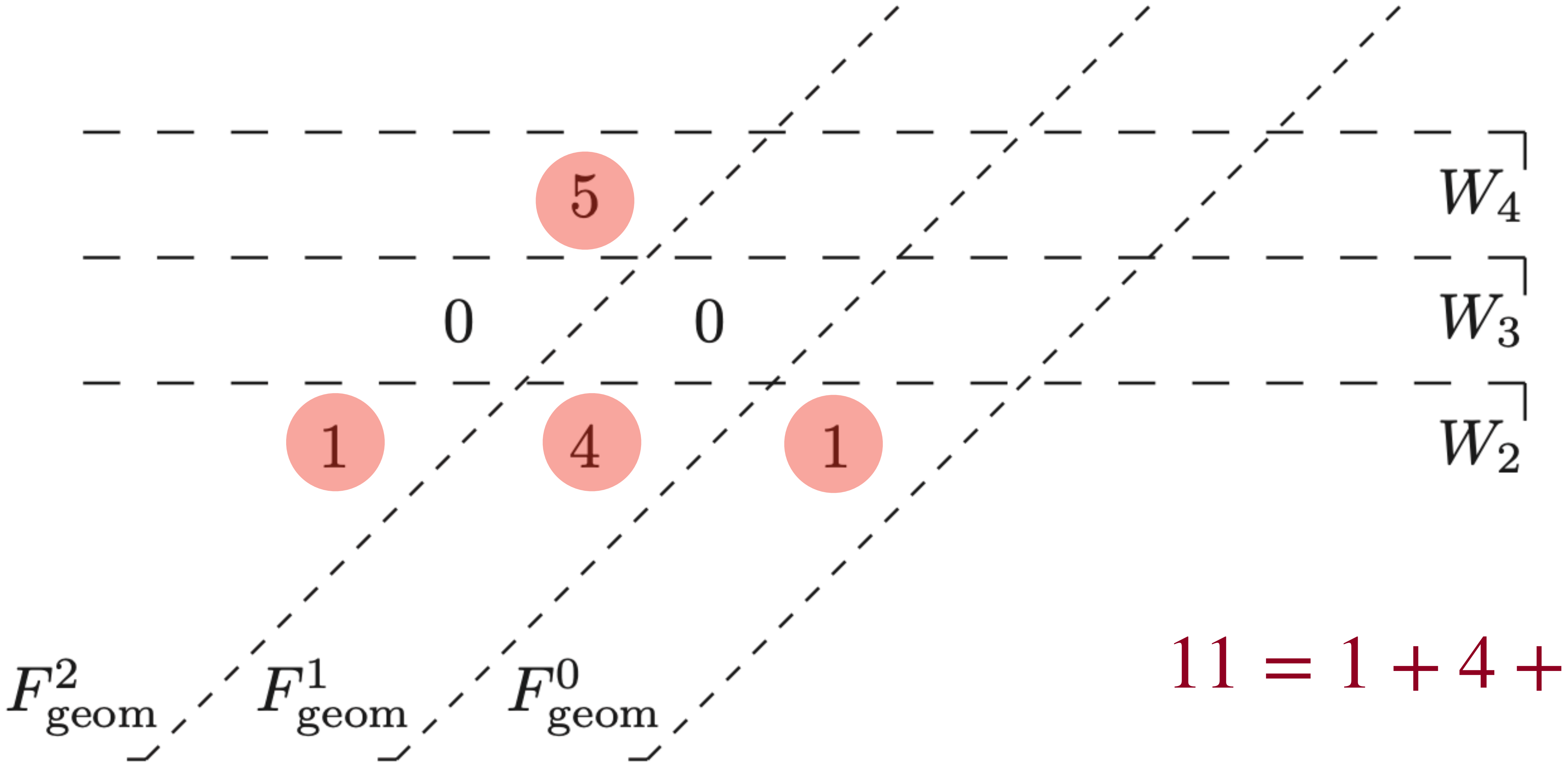
- $\deg U = -3, \deg \eta = 3 \implies \deg \hat{\Phi} = 0.$
- $C_{\text{Baikov}} \implies C_{\text{abs}} = \varepsilon^3.$
- Even polynomials: $\{z_0, z_1, z_2\}$; Odd polynomials: $\{P_3, P_4, P_5\}$, ($y^2 = P_3\, P_4\, P_5$ defines a K3).

Example: Filtrations

$$p = N_V - o + r; \qquad q = o; \qquad w = p + q = N_V + r\,.$$

- The top sector has 11 nontrivial MIs (subsector: 4 tadpoles).
- Step I: decompose 11 MIs into several subspaces according to complexity.





$$w = 2 + 2$$
$$w = 2 + 1$$
$$w = 2 + 0$$

$$11 = 1 + 4 + 1 + 5$$

Example: MIs after Step 1

$h^{2,2} = 5$

$w = 4 \longrightarrow$ 2 consecutive residues upon even polynomials

$$\hookrightarrow \quad \hat{\Phi} \in \left\{ \frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_0}{z_1}, \frac{z_2}{z_1}, \frac{z_0}{z_2}, \frac{z_1}{z_2}, \frac{z_2^2}{z_0 z_1}, \frac{z_1^2}{z_0 z_2}, \frac{z_0^2}{z_1 z_2} \right\}. \# - \text{IBPs} - \{\text{supersector}\} = 5.$$

$h^{2,1} = 0 \qquad h^{1,2} = 0$

No forms with just one non-zero residue.

$h^{2,0} = 1 \qquad h^{1,1} = 4 \qquad h^{0,2} = 1$

Forms without non-zero residues: there are 3 cases, and we order them by pole order.

$$\hat{\Phi} = 1 \rightsquigarrow \varepsilon^3 I_{111100000}$$

$$\hat{\Phi} \sim \frac{Q}{P_{3,4,5}} \rightsquigarrow \varepsilon^2 y_i \partial_i I_{111100000}, \; i = 1,2,3,4$$

$$\frac{1}{16}\varepsilon\Big(\sum_{i=1}^{4} y_i \partial_i\Big)^2 I_{111100000}$$

$$U(z_0, z_1, z_2) = z_0^{4\varepsilon} z_1^{\varepsilon} z_2^{\varepsilon} P_3^{-\frac{1}{2}-\varepsilon} P_4^{-\frac{1}{2}-\varepsilon} P_5^{-\frac{1}{2}-\varepsilon}$$

Example: Step 2 Setup

–	1	1	1	1	–	–	–	–	–	–
−1	0	0	0	0	1	1	1	1	1	1
−1	0	0	0	0	1	1	1	1	1	1
−1	0	0	0	0	1	1	1	1	1	1
−1	0	0	0	0	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1	–	–
0	1	1	1	1	1	1	1	1	–	–
0	1	1	1	1	–	1	1	–	1	–
0	1	1	1	1	1	1	–	1	1	–
0	1	1	1	1	–	–	1	1	1	–
−2	−1	−1	−1	−1	0	0	0	0	0	0

$\varepsilon^{-\max}$ terms in connection matrix elements **after step 1**

$$\mathbf{R}_2^{(-2)}\,\mathbf{R}_2^{(-1)}\,\mathbf{R}_2^{(-0)}\,\vec{K} = \vec{J}$$

$$\begin{bmatrix} 0 & - & - \\ \mathbf{-1} & \mathbf{0} & - \\ -2 & \mathbf{-1} & 0 \end{bmatrix} \qquad \begin{bmatrix} - & - & - \\ \mathbf{0} & \mathbf{0} & - \\ -1 & \mathbf{0} & - \end{bmatrix} \qquad \begin{bmatrix} - & - & - \\ - & - & - \\ 0 & - & - \end{bmatrix}$$

- We derive $\mathbf{R}_2^{(-2)}$ first, followed by $\mathbf{R}_2^{(-1)}$, and $\mathbf{R}_2^{(0)}$ at last, to remove ε-***non***-factorised terms;
- These three matrices are constrained by simplified differential operators.

Example: $\mathrm{R}_2^{(-2)}$ Structure

$$
\left[\begin{array}{c|cccccccc|cc}
R_{55}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{1}{\varepsilon}R_{65}^{(-2)} & R_{66}^{(-2)} & R_{67}^{(-2)} & R_{68}^{(-2)} & R_{69}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\varepsilon}R_{75}^{(-2)} & R_{76}^{(-2)} & R_{77}^{(-2)} & R_{78}^{(-2)} & R_{79}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\varepsilon}R_{85}^{(-2)} & R_{86}^{(-2)} & R_{87}^{(-2)} & R_{88}^{(-2)} & R_{89}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\varepsilon}R_{95}^{(-2)} & R_{96}^{(-2)} & R_{97}^{(-2)} & R_{98}^{(-2)} & R_{99}^{(-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{1}{\varepsilon^2}R_{F5}^{(-2)} & \frac{1}{\varepsilon}R_{F6}^{(-2)} & \frac{1}{\varepsilon}R_{F7}^{(-2)} & \frac{1}{\varepsilon}R_{F8}^{(-2)} & \frac{1}{\varepsilon}R_{F9}^{(-2)} & 0 & 0 & 0 & 0 & R_{FE}^{(-2)} & R_{FF}^{(-2)}
\end{array}\right]
\xrightarrow{\text{to remove}}
\begin{bmatrix} 0 & - & - \\ \mathbf{-1} & \mathbf{0} & - \\ -2 & \mathbf{-1} & 0 \end{bmatrix}
$$

- The index starts from 5, since tadpoles (4 MIs) are suppressed here;
- A posteriori, we can show that $R_{55}^{(-2)} = \psi$ is a period of the K3-surface.

Example: $\mathbf{R}_2^{(-2)}$ Details

- The constraints in the first column, in fact, generate the Picard-Fuchs ideal for ψ. Normally, we take ψ to be the holomorphic solution, ψ_0. Then it turns out:

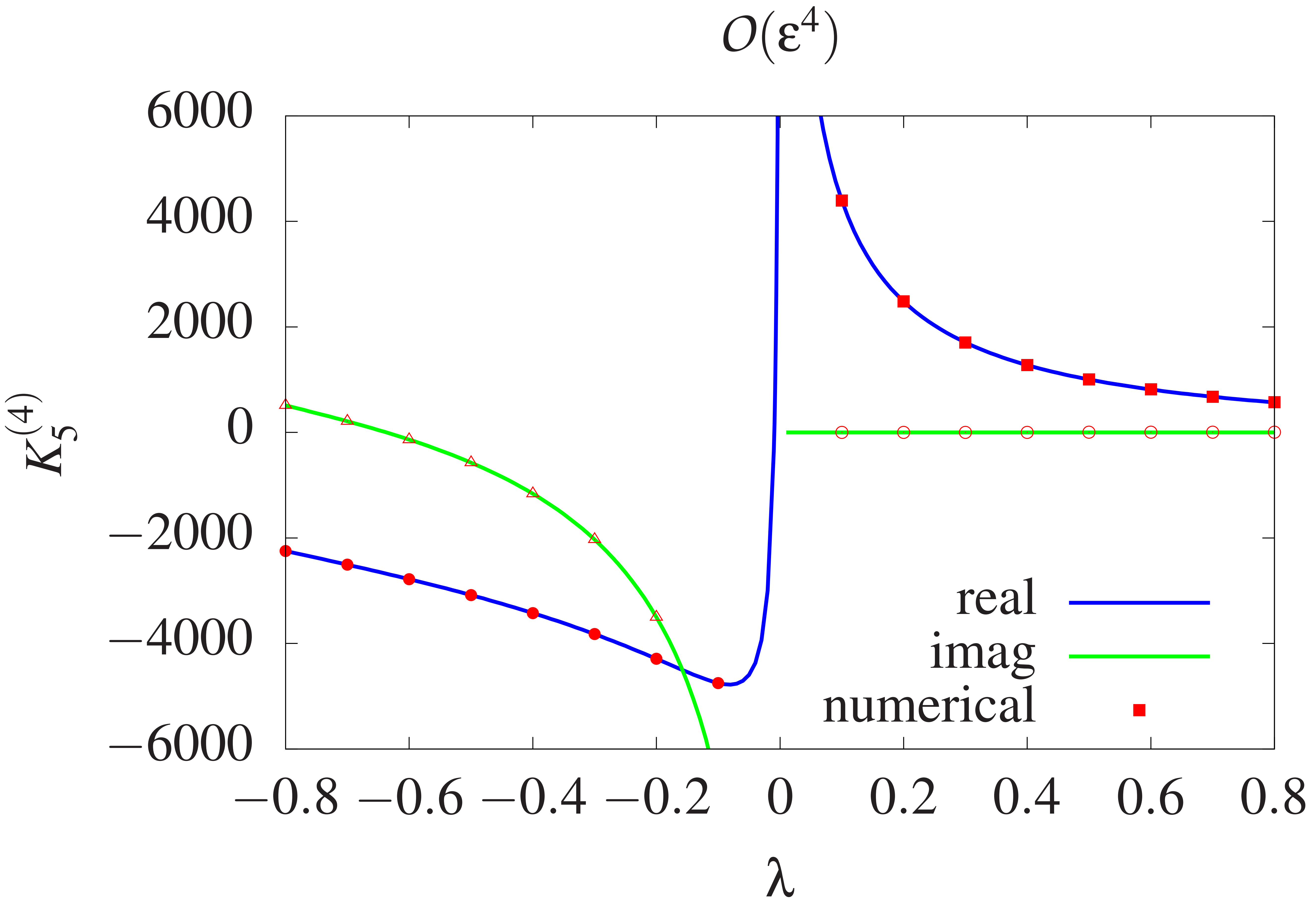
$$R^{(-2)}_{(i+5)5} = y_i \frac{\partial}{\partial y_i} \psi_0, \quad i = 1, 2, 3, 4.$$

$$R^{(-2)}_{F5} = -\frac{1}{64} \sum_{i=1}^{4} \left(y_i \frac{\partial^2}{\partial y_i^2} + \left(8 y_i + 1\right) \frac{\partial}{\partial y_i} + 1 \right) \psi_0$$

- To proceed, we need other (meromorphic) solutions by which we can define new kinematic variables (turns out to be the mirror map).

- Remark: One can ignore the concepts of periods and the mirror map and focus on the constraints step by step in general!

Example: Results



compare with AMFLOW

$$K_5 = \varepsilon^3 \left[16\zeta_3 + \cdots \right] + \varepsilon^4 K_5^{(4)} + O\left(\varepsilon^5\right)$$

$$\lambda = \frac{(m_W + m_Z + m_H + m_t)^2}{-p^2}$$