Symbols from Bi-projections

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Feynman Parameter Integrals

Feynman Parameter Integrals can be viewed as projective integrals

$$\int_\Delta \frac{\langle X \mathrm{d} X^{n-1} \rangle \, N[X^k]}{D[X^{n+k}]}.$$

- Δ: $(n-1)$-simplex, with vertices

$$U_i = [\underbrace{0:0:\cdots:0}_{i-1}:1:\underbrace{0:0:\cdots:0}_{n-i-1}], \qquad \forall i = 1, 2, \ldots, n.$$

- $N[X^k]$: some tensor of degree k.
- $D[X^{n+k}]$: Symanzik polynomial, with degree $L+1$ (L # of loops).

In this talk, we focus on one loop diagrams. D is a degree two polynomial.

Symbol

- The simplest results of Feynman Integrals are multiple polylogarithms
- A convenient way to study multiple polylogarithms (MPL)

$$G \longmapsto \mathcal{S}[G]. \tag{1}$$

- If

$$\mathrm{d}G = \sum_i G'_i \, \mathrm{d}\log R_i,$$

where R_is are algebraic functions, then

$$\mathcal{S}[G] = \sum_i S[G'_i] \otimes R_i.$$

Singularities and Symbols

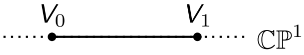
- Li_2: $\mathcal{S}[\mathrm{Li}_2(z)] = -(1-z) \otimes z$.
  - Branch points at $(1-z) = 0$ or ∞.
  - Discontinuity
    $$\mathrm{Disc}\,\mathrm{Li}_2(\mathrm{z}) = -2\pi i \log(z).$$
  - Symbol of Disc
    $$\mathcal{S}[\mathrm{Disc}\,\mathrm{Li}_2(\mathrm{z})] = -\cancel{(1-z)} \otimes z.$$
- In general, for some MPL I
  - Symbol
    $$\mathcal{S}[I] = f_1 \otimes f_2 \otimes \cdots \otimes f_t + \cdots .$$
  - Branch points at
    $$f_1 = 0 \text{ or } \infty.$$
  - Symbol of Disc
    $$\mathcal{S}[\mathrm{Disc}_{f_1} I] = f_2 \otimes f_3 \otimes \cdots \otimes f_t.$$

Calculation of Symbols

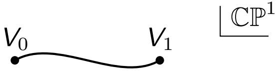
- Main Questions:
  - What is the first entry?
  - What discontinuity to calculate?
- Bi-projections:
  - Stratification of *Touching Configurations*
  - First entries from the jumps between configurations
  - Each jump contributes to an *Elementary Discontinuity*

Ambient Spaces and Touching Configurations

- Reason:
  - Analyticicities from the relative geometric configuration between contour and integrand singularities
  - Complex space → Contour Deformation
  - Relative geometric configuration will limit the deformation
- *Ambient Spaces*: The space where the contour can deform.
- *Touching Configurations*: The ambient spaces of one or more genuine faces of the contour simplex are contained in the singularity hypersurface of the integrand.



(1)



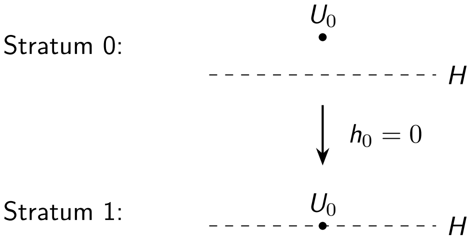
(2)

Stratification of Touching Configurations: Hyperplane

- Assume the hyperplane is

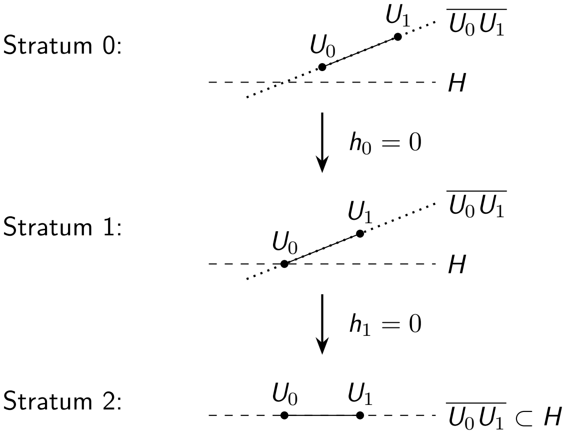
$$\mathcal{G}(X) = HX \equiv \sum_{i=0}^{d} h_i x_i$$

- Stratifying touching configurations of a 0-face in relation to a hyperplane singularity H.



Stratification of Touching Configurations: Hyperplane

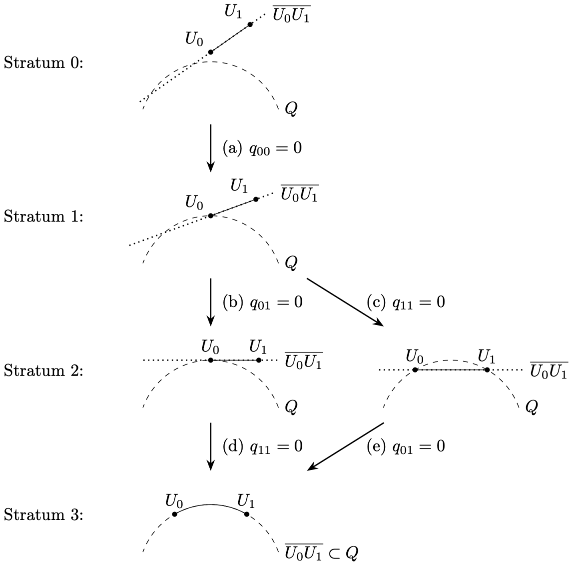
- Stratifying touching configurations of a 1-face.



It's sufficient to study $\mathbb{CP}^0$ ambient spaces.

Stratification of Touching Configurations: Quadric

- Assume the quadric is $\mathcal{G}(X) = Q(X) \equiv \sum_{0 \le i \le j \le d} q_{ij} x_i x_j$



First Entry Conditions

- Question 1: What is the first entry?
- Symbol:

$$\mathcal{S}[I] = f_1 \otimes f_2 \otimes \cdots \otimes f_t + \cdots .$$

Branch points at

$$f_1 = 0 \text{ or } \infty.$$

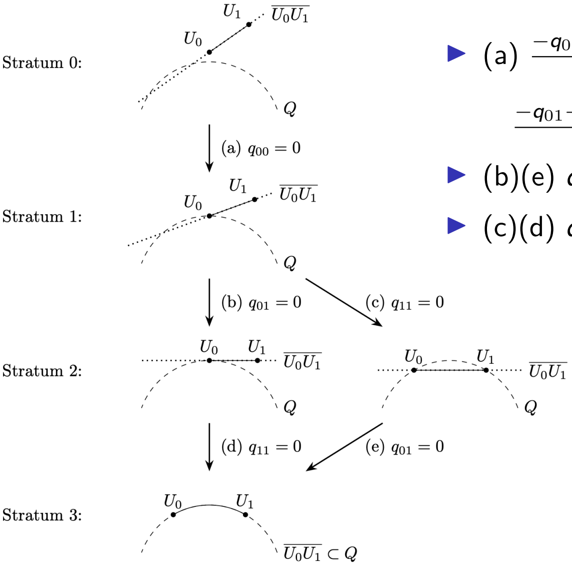
- An inverse understanding:

When the first entry becomes 0 or ∞, the geometries become more singular!

- First Entry: An expression makes the geometric configuration more singular when it becomes 0 or ∞

A jump occurs!

First Entry of Touching Configuration

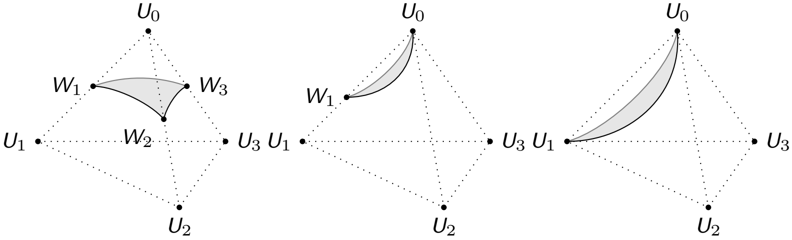


- (a) $\dfrac{-q_{01}+\sqrt{q_{01}^2-4q_{00}q_{11}}}{2q_{11}}$

  $\dfrac{-q_{01}-\sqrt{q_{01}^2-4q_{00}q_{11}}}{2q_{11}}$
- (b)(e) q_{01}
- (c)(d) q_{11}

Elementary Discontinuities

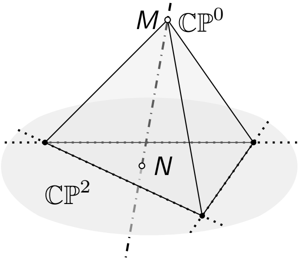
- Question 2: What discontinuity should be computed?
- **Track the contour when kinematics wind around the branch point.** The discontinuity contour should be the intersection of integral domain and singularity.



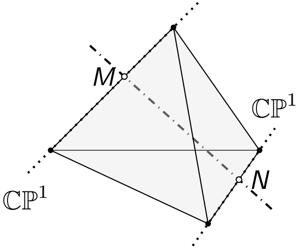
(i) (ii) (iii)

- Problem: How to compute?

Bi-projections



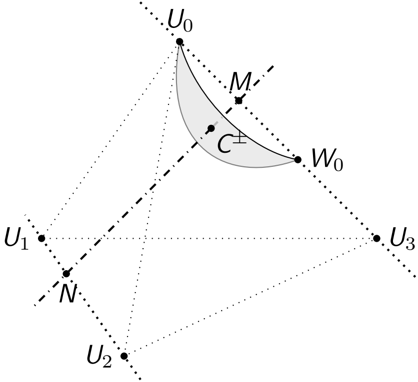
(1) $k = 0$



(2) $k = 1$

- (1) $\mathbb{CP}^n \to \mathbb{CP}^0 \otimes \mathbb{CP}^1 \otimes \mathbb{CP}^{n-1}$ (but not equivalent!)
- (2) $\mathbb{CP}^n \to \mathbb{CP}^1 \otimes \mathbb{CP}^1 \otimes \mathbb{CP}^{n-2}$
- ...
- (k-1) $\mathbb{CP}^n \to \mathbb{CP}^k \otimes \mathbb{CP}^1 \otimes \mathbb{CP}^{n-k-1}$

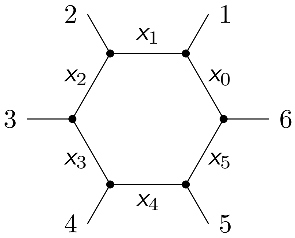
Bi-projections



- $M = m_1 U_0 + m_2 U_3, \qquad N = n_1 U_1 + n_2 U_2.$
- $C = c_1 M + c_2 N$
- Solve the intersection point $C_{\pm}$ and take the residue of the integral around this point.
- Integrate over m_1, m_2 from U_0 to W_0.

A $\mathbb{CP}^3$ Example

- Consider a massless hexagon

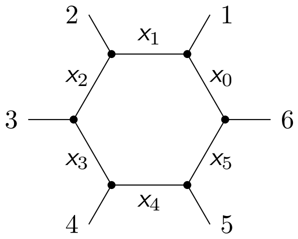


- Algebraically

$$I_{\text{hex}} = \sqrt{q_0} \int_{\nabla} \frac{\langle X \mathrm{d} X^5 \rangle}{(X Q_0 X)^3}, \qquad q_0 = -\det Q_0$$

$$\begin{aligned} X Q_0 X = & x_0 x_2 + x_0 x_3 + x_0 x_4 + u_1\, x_1 x_3 + x_1 x_4 \\ & + x_1 x_5 + u_2\, x_2 x_4 + x_2 x_5 + u_3\, x_3 x_5 . \end{aligned}$$

A $\mathbb{CP}^5$ Example



- Its symbol

$$\mathcal{S}[I_{\text{hex}}] = \sum_{\text{cyclic}} \left(u_1 \otimes (1 - u_1) \otimes r_0 - u_1 \otimes u_2 \otimes r_3 - u_1 \otimes u_3 \otimes r_2 \right).$$

$$r_0 = \frac{x_+}{x_-}, \qquad r_i = \frac{x_+(1 - u_i x_-)}{x_-(1 - u_i x_+)}, \quad i = 1, 2, 3.$$

$$x_\pm = \frac{-1 + u_1 + u_2 + u_3 \pm \sqrt{q_0}}{2 u_1 u_2 u_3},$$

Symbol Construction

- Consider the discontinuity of u_1

$$\mathrm{Disc}_{\overline{U_1 U_3}} I_{\mathrm{hex}} = \int_0^\infty \mathrm{d}n_1 \mathrm{d}n_2 \mathrm{d}n_3 \, \frac{\sqrt{q_0}\, u_1}{2(N\widetilde{Q}_1 N)^2},$$

Similar to a box diagram.

- Construction of Symbols

$$\underbrace{\frac{1}{u_1} \otimes \frac{1}{r_1 r_3} + \frac{u_3}{u_1} \otimes \frac{1}{r_1 r_2}}_{\text{(b)}} + \underbrace{\frac{u_3}{u_1} \otimes r_1}_{\text{(c)}} + \underbrace{\frac{1-u_1}{u_1} \otimes r_0 + u_2 \otimes \frac{1}{r_3}}_{\text{(e)}}$$

$$= (1-u_1) \otimes r_0 - u_2 \otimes r_3 - u_3 \otimes r_2 .$$

Thank you very much!

Questions & comments are welcome.