東南大學
SOUTHEAST UNIVERSITY

Classical Gravitational Wave Observables from Quantum Field Theory

刘正文

Shing-Tung Yau Center & School of Physics, Southeast University, Nanjing

with Christoph Dlapa, Gregor Kälin, Jakob Neef, Rafael A. Porto, Zixin Yang

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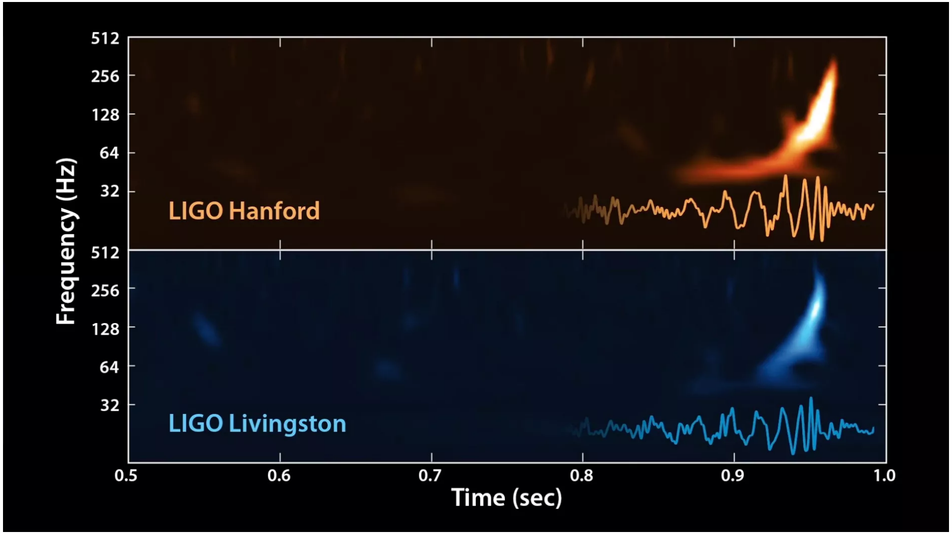
PLB 831 (2022) 137203 PRL 125 (2020) 261103 PRD 102 (2020) 124025 arXiv:2506.20665

第五届量子场论及其应用研讨会

Gravitational wave science era has begun!

止於至善



Opened a new window on the Universe!

Gravitational wave science era has begun!

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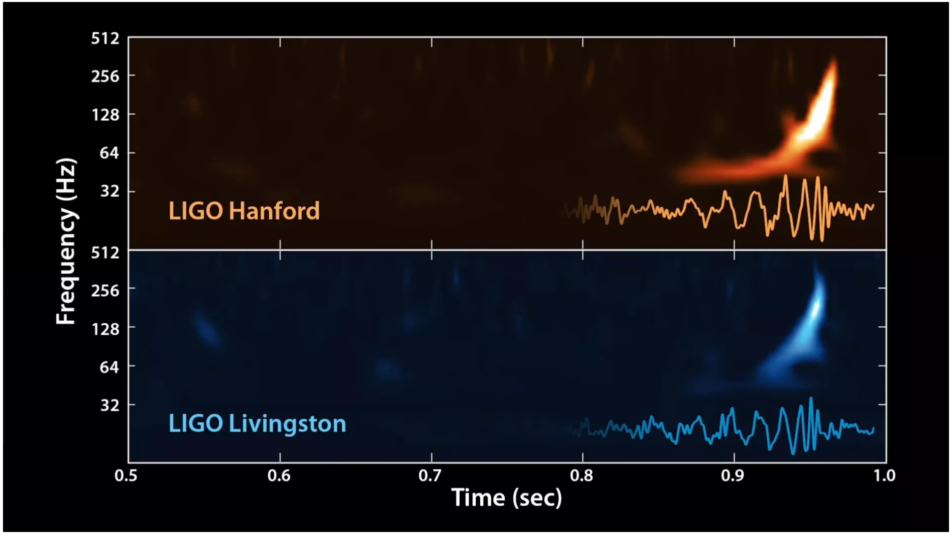

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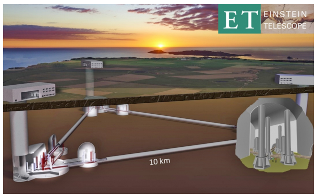
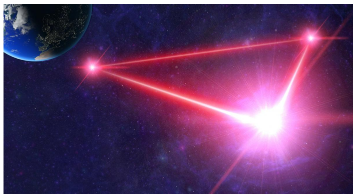
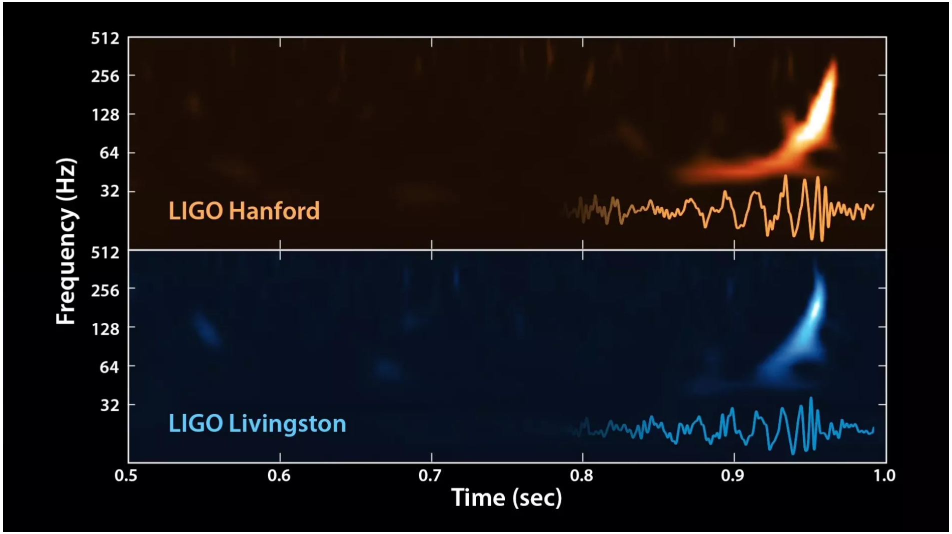
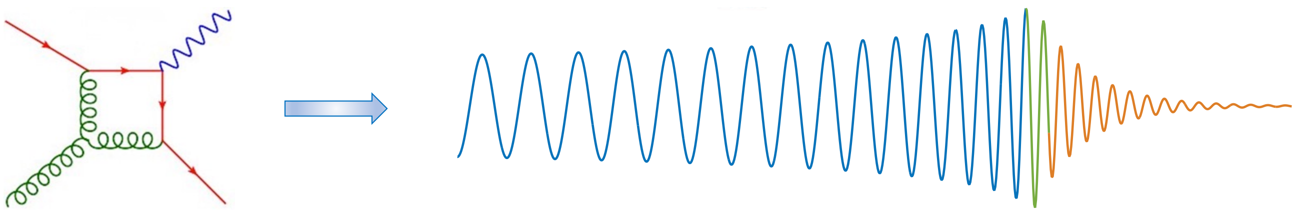




Gravitational wave science era has begun!



Modern Quantum Field Theory techniques are playing a crucial role in precision GW physics!



Gravitational waves from binary coalescences

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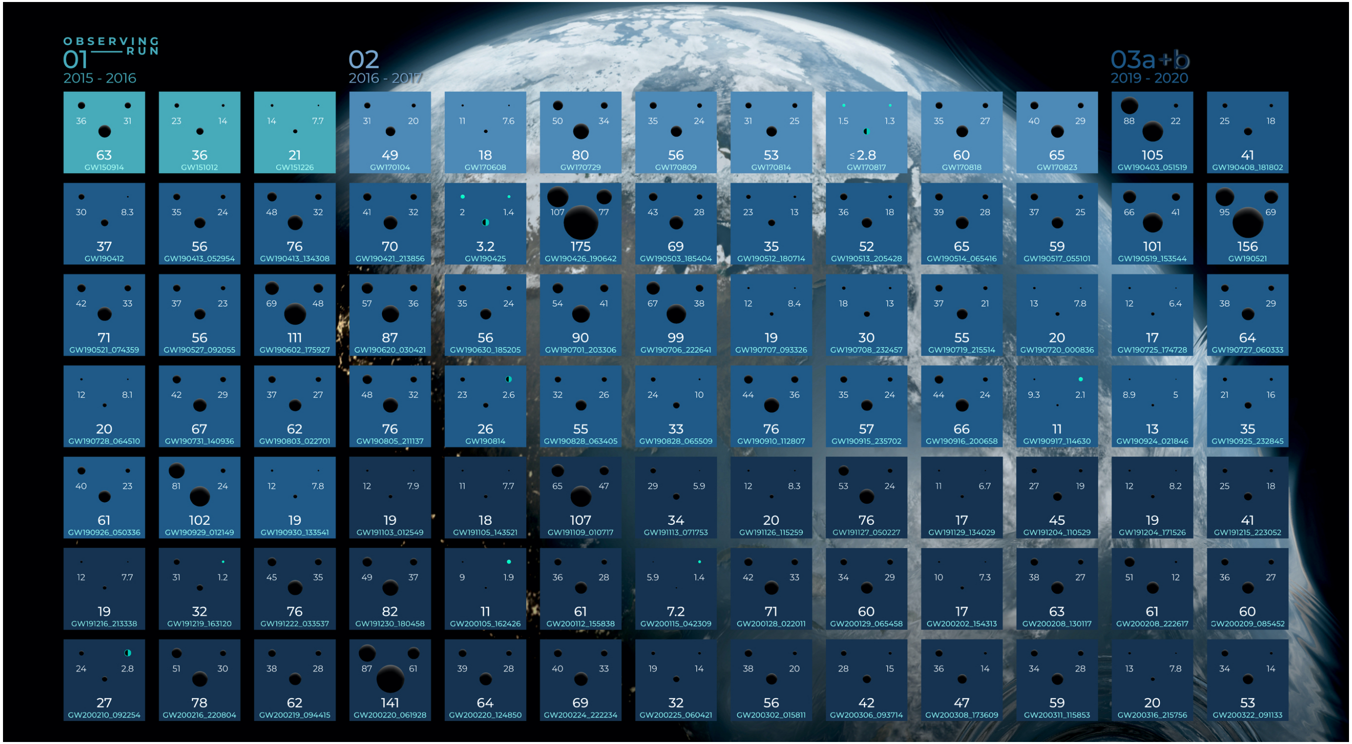


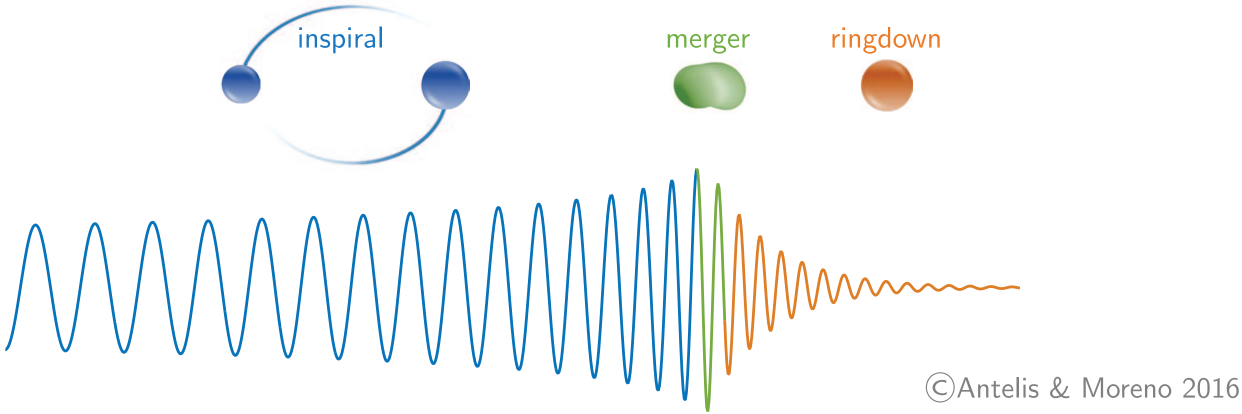
Credit: Carl Knox (OzGrav, Swinburne)

GWTC-3: 90 GW events – the majority are binary black holes (BH), but also several binary neutron stars (NS) and mixed NS-BHs.

gwosc.org

Gravitational waves from binary coalescences

止於至善



Merger: Numerical Relativity

Ringdown: black hole perturbation theory

Inspiral: the interaction between two bodies is weak

$$v^2 \sim \frac{GM}{r} \ll 1$$

- Numerical Relativity: accurately, but computationally expensive
- Analytic methods: corrections in v or G are perturbatively calculable

  Post-Newtonian/post-Minkowskian expansion

  - QFT methodology (EFT & Feynman loop techniques) shown great power!

Effective Field Theory

以於至善

- Gravitational binary system

$$S_{\mathrm{WL}} = \sum_{i=1,2} \left[-\frac{m_i}{2} \int dt\, g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu + \cdots \right] \qquad S_{\mathrm{GR}} = \frac{-1}{16\pi G} \int d^4x \sqrt{-g}\, R + \cdots$$

- Effective action for gravitational binary systems

Goldberger-Rothstein hep-th/0409156

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\boxed{e^{iS_{\mathrm{eff}}[x_a(\tau)]} = \int \mathcal{D}h_{\mu\nu}\, e^{iS_{\mathrm{WL}} + iS_{\mathrm{GR}}}}$$

- Post-Minkowskian expand in powers of G

$$L_{\mathrm{eff}} = L_0 + G L_1 + G^2 L_2 + \cdots \qquad L_0 = -\sum_i \frac{m_i}{2} \eta_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu$$

- The equations of motion for trajectories:

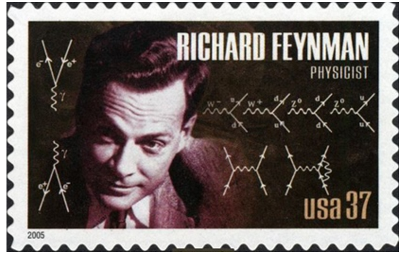
Kälin-Porto 2006.01184

$$m_i \ddot{x}_i^\mu = -\eta^{\mu\nu} \sum_{n=1}^{\infty} G^n \left(\frac{\partial L_n}{\partial x_i^\nu} - \frac{d}{d\tau_i} \frac{\partial L_n}{\partial \dot{x}_i^\nu} \right) \qquad x_i^\mu = b_i^\mu + u_i^\mu \tau_i + \delta x_i^\mu(\tau_i) + \cdots$$

- Physical observables:

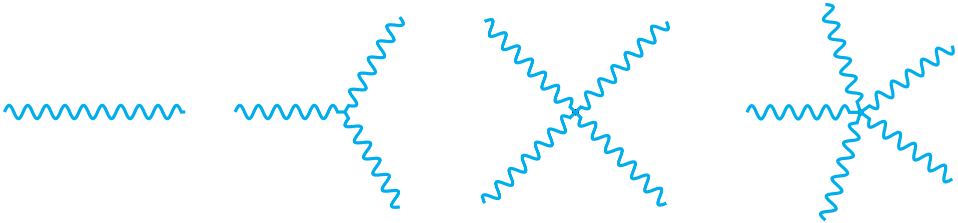
$$\Delta p_i^\mu = p_i^\mu(+\infty) - p_i^\mu(-\infty) = -\eta^{\mu\nu} \sum_{n=1}^{\infty} G^n \int_{-\infty}^{\infty} d\tau_i \left(\frac{\partial L_n}{\partial x_i^\nu} \right)$$

Effective Field Theory

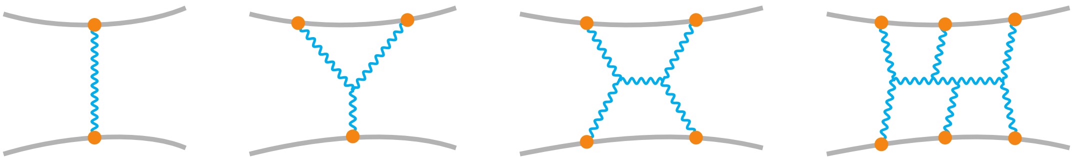


- Worldlines as classical sources in path integral:

- Hilbert-Einstein: $\mathcal{L}_{\text{HE}} = \mathcal{L}_{hh} + \mathcal{L}_{hhh} + \mathcal{L}_{hhhh} + \cdots$



- Classical physics: we use the saddle-point approximation in path integrals.



- Enjoy the advantages of quantum field theory methods and classical physics

  powerful and systematic & purely classical at all steps (simplicity)

Effective Field Theory

The in-in effective action is obtained by performing a closed-time-path integral

$$e^{iS_{\text{eff}}[x_{a,1},x_{a,2}]} = \int \mathcal{D}h_1 \mathcal{D}h_2\, e^{i(S_{\text{GR}}[h_1]-S_{\text{GR}}[h_2]+S_{\text{WL}}[h_1,x_{a,1}]-S_{\text{WL}}[h_2,x_{a,2}])}$$

It is convenient to use the Keldysh basis Galley PRL 110 (2013) 174301

$$h^-_{\mu\nu} = \frac{1}{2}(h_{1\mu\nu} + h_{2\mu\nu}) \qquad x^\alpha_{a,+} = \frac{1}{2}(x^\alpha_{a,1} + x^\alpha_{a,2})$$
$$h^+_{\mu\nu} = h_{1\mu\nu} - h_{2\mu\nu} \qquad x^\alpha_{a,-} = x^\alpha_{a,1} - x^\alpha_{a,2}$$

for which the matrix of (classical) propagators for gravitions becomes

$$i\begin{pmatrix} 0 & -\Delta_{\text{adv}}(x-y) \\ -\Delta_{\text{ret}}(x-y) & 0 \end{pmatrix}$$

The worldline equations of motion: Kälin-Neef-Porto JHEP 01 (2023) 140

$$m_i\ddot{x}^\mu_i(\tau) = -\eta^{\mu\nu}\frac{\delta S_{\text{eff, int}}[x_{a,\pm}]}{\delta x^\nu_{i,-}(\tau)}\bigg|_{\text{PL}} \qquad \Delta p^\mu_i = -\eta^{\mu\nu}\int_{-\infty}^{\infty} d\tau \frac{\delta S_{\text{eff, int}}[x_{a,\pm}]}{\delta x^\nu_{i,-}(\tau)}\bigg|_{\text{PL}}$$

Physical Limit (PL): $x_{a,-} \to 0$, $x_{a,+} \to x_a$.

Closed-time path integrals

EQUILIBRIUM AND NONEQUILIBRIUM FORMALISMS MADE UNIFIED

Kuang-chao CHOU, Zhao-bin SU,* Bai-lin HAO and Lu YU

Institute of Theoretical Physics, Academia Sinica, P.O. Box 2735, Beijing, China

Received 5 June 1984

Physics Reports

Volume 118, Issues 1–2, February 1985, Pages 1-131

Contents:

周光召

郝柏林

苏肇冰

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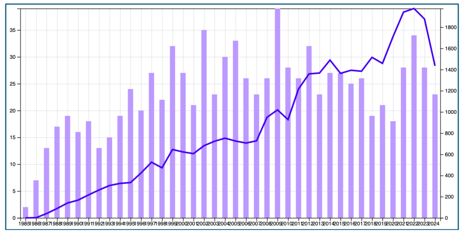


Effective Field Theory

以於至善

- Observables at $\mathcal{O}(G^N)$ Kälin-**ZL**-Porto PRL2020 Dlapa-Kälin-**ZL**-Porto PRL2022 JHEP2024

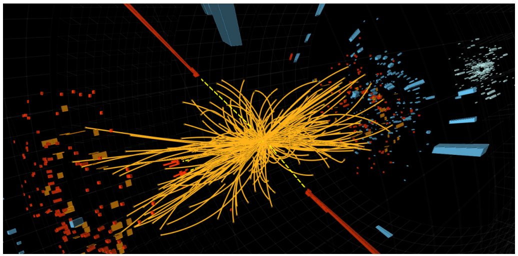
$$\Delta p_i^\mu \sim \int d^D q \frac{e^{iq\cdot b}\,\delta(q\cdot u_1)\delta(q\cdot u_2)}{|q^2|^\sharp} \int \left(\prod_{i=1}^{N-1} d^D \ell_i \frac{\delta(\ell_j\cdot u_a)}{(\ell_i\cdot u_b - i0)^{\nu_i}} \right) \frac{\mathcal{N}^\mu(q,u_a)}{D_1 D_2 D_3\cdots}$$

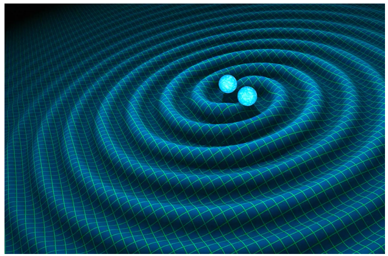
  - Graviton propagators:

$$\frac{1}{D_i} \longrightarrow \frac{1}{\ell^2 + i0} \quad \text{or} \quad \frac{1}{(\ell^0 \pm i0)^2 - \vec{\ell}^{\,2}}$$

  - Cut: always one delta function $\delta(\ell_i\cdot u_a)$ for each loop
  - Kinematics: $q\cdot u_a = 0,\ u_a^2 = 1,\ u_1\cdot u_2 = \gamma \implies$ single scale γ to all orders

- Multi-loop technology from QFT can be used to solve classical gravitational problems!



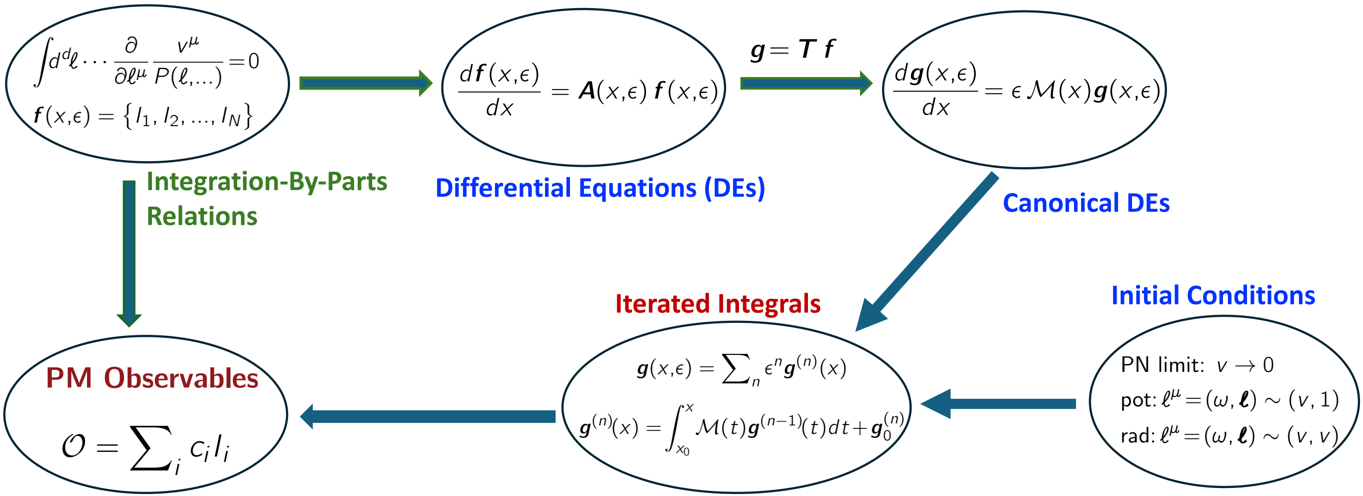


QFT toolbox

- Observables at $\mathcal{O}(G^N)$ Kälin-**ZL**-Porto PRL 2020 Dlapa-Kälin-**ZL**-Porto PRL 2022 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\Delta p_i^\mu \sim \int d^D q \, \frac{e^{iq\cdot b}\,\delta(q\cdot u_1)\delta(q\cdot u_2)}{|q^2|^\sharp} \int \left(\prod_{i=1}^{N-1} d^D\ell_i \, \frac{\delta(\ell_j\cdot u_a)}{(\ell_i\cdot u_b - i0)^{\nu_i}} \right) \frac{\mathcal{N}^\mu(q,u_a)}{D_1 D_2 D_3\cdots}$$

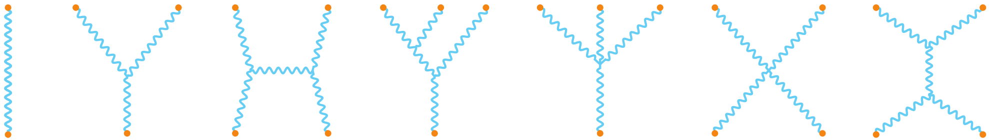
- Perturbative QFT toolbox: Dlapa-Kälin-**ZL**-Porto JHEP 2024



- Post-Minkowskian physics can be bootstrapped from Post-Newtonian data using DEs!

NNLO: 3PM

止於至善



- $\mathcal{O}(G^3)$: two-loop integrals Kälin-**ZL**-Porto PRL 2020 PRD 2020

$$\int \frac{d^D\ell_1 d^D\ell_2\, \delta(\ell_1\cdot u_1)\, \delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2]^{a_1}\, [\pm\ell_2\cdot u_1]^{a_2}} \frac{1}{[\ell_1^2]^{a_3}\, [\ell_2^2]^{a_4}\, [(\ell_1+\ell_2-q)^2]^{a_5}\, [(\ell_1-q)^2]^{a_6}\, [(\ell_2-q)^2]^{a_7}}$$

- The reduction and evaluation of integrals can be performed in standard techniques.

$$\partial_x \vec{f}(x,\epsilon) = \epsilon \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \right) \vec{f}(x,\epsilon)$$

- Conservative dynamics at $\mathcal{O}(G^3)$: Kälin-**ZL**-Porto PRL 2020

$$\begin{aligned}\Delta p_1^\mu = \frac{G^3 b^\mu}{|b^2|^2} \Bigg(& \frac{8m_1^2m_2^2(4\gamma^4-12\gamma^2-3)}{(\gamma^2-1)} \log\left(\gamma - \sqrt{\gamma^2-1}\right) - \frac{2m_1m_2(m_1^2+m_2^2)(16\gamma^6-32\gamma^4+16\gamma^2-1)}{(\gamma^2-1)^{5/2}} \\ & - \frac{4m_1^2m_2^2\gamma(20\gamma^6-90\gamma^4+120\gamma^2-53)}{3(\gamma^2-1)^{5/2}} \Bigg) + \frac{3\pi}{2} \frac{(2\gamma^2-1)\,(5\gamma^2-1)}{(\gamma^2-1)^2} \frac{G^3 m_1 m_2(m_1+m_2)}{|b^2|^{3/2}} \Big((m_1+\gamma m_2) u_2^\mu - (m_2+\gamma m_1) u_1^\mu \Big)\end{aligned}$$

- We provided the first confirmation for the result from a scattering amplitude clacualtion.

  Bern-Cheung-Roiban-Shen-Solon-Zeng 2019

- We computed quadrupolar and octupole tidal corrections at $\mathcal{O}(G^3)$. Kälin-**ZL**-Porto PRD 2020

NNNLO: 4PM

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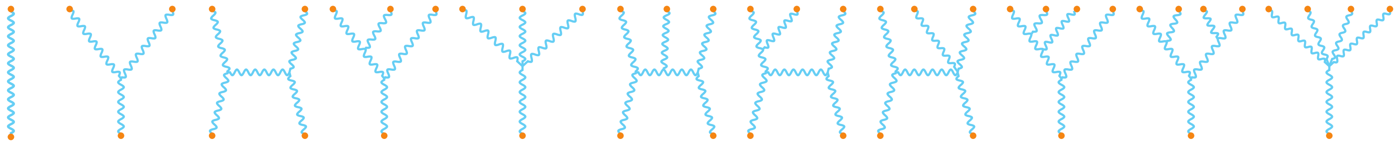


$\mathcal{O}(G^4)$: three-loop integrals Dlapa-Kälin-**ZL**-Porto PRL 2022 PLB 2022 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\int d^D\ell_1 d^D\ell_2 d^D\ell_3 \frac{\delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_2)\,\delta(\ell_3\cdot u_2)}{[\ell_1\cdot u_2]^{\alpha_1}\,[\ell_2\cdot u_1]^{\alpha_2}\,[\ell_3\cdot u_1]^{\alpha_3}} \frac{D_8^{-\nu_8} D_9^{-\nu_9}}{D_1^{\nu_1} D_2^{\nu_2}\cdots D_7^{\nu_7}}$$

$\{\ell_1^2, \ell_2^2, (\ell_1-q)^2, (\ell_2-q)^2, (\ell_3-q)^2, \ell_3^2, (\ell_1-\ell_2)^2, (\ell_2-\ell_3)^2, (\ell_3-\ell_1)^2\}$

IBP reduction:

conservative: $\mathcal{O}(10^2)$ master integrals full: $\mathcal{O}(10^3)$ master integrals

Differential Equations Dlapa-Kälin-**ZL**-Porto PRL 2022 PLB 2022

$$\frac{d\vec{f}(x,\epsilon)}{dx} = \epsilon\,\mathcal{M}(x)\,\vec{f}(x,\epsilon)$$

- The majority can be solved in terms of multiple polylogarithms.
- Elliptic integrals appear in post-Minkwskian gravity for the first time.

NNNLO: 4PM

止於至善

The full impulse at $\mathcal{O}(G^4)$: Dlapa-Kälin-**ZL**-Porto PRL 2022 JHEP 2023 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\Delta p_1^\mu\Big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

$$\begin{aligned}
\frac{c_b}{\pi} = &-\frac{3h_{34}m_2m_1(m_1^3+m_2^3)}{64v_\infty^5} + \frac{m_1^2m_{12}m_2^2}{4}\Bigg[\frac{3h_6\mathrm{K}^2(w_2)}{4v_\infty^3} - \frac{3h_8\mathrm{K}(w_2)\mathrm{E}(w_2)}{4v_\infty^3} + \frac{21h_5w_3\mathrm{E}^2(w_2)}{8v_\infty^3} - \frac{\pi^2h_{16}v_\infty}{4(\gamma+1)} + \frac{3\gamma h_{10}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{w_3v_\infty^2}\\
&+\log(v_\infty)\left(\frac{h_{32}}{2v_\infty^3} - \frac{3h_{14}\log(\frac{w_3}{2})}{v_\infty} - \frac{3\gamma h_{22}\log(w_1)}{2v_\infty^4}\right)\Bigg] + m_2^2m_1^3\Bigg[\frac{h_{52}}{48v_\infty^6} - \frac{h_{63}}{768\gamma^9w_3v_\infty^5} - \frac{3v_\infty(h_{40}\mathrm{Li}_2(w_2)+2w_3h_{33}\mathrm{Li}_2(-w_2))}{64w_3}\\
&+\frac{3h_{14}\log(\frac{w_3}{2})\log(w_3)}{4v_\infty} + \frac{\gamma h_{39}\log(w_1)}{8w_3^3v_\infty^2} + \frac{3\gamma h_{22}\log(w_3)\log(w_1) - h_{35}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{h_{56}\log(2) - h_{57}\log(w_3) + 2\gamma h_{55}\log(\gamma)}{32v_\infty^5} - \frac{\gamma h_{51}\log(w_1)}{16v_\infty^7}\Bigg]\\
&+m_1^2m_2^3\Bigg[\frac{h_{58}}{192\gamma^7v_\infty^5} + \frac{h_{53}}{48v_\infty^6} + \frac{\gamma h_{49}\log(w_1)}{16v_\infty^6} - \frac{2\gamma h_{50}\log(w_1) + 3\gamma^2h_{13}\log^2(w_1)}{32v_\infty^7} - \frac{h_{41}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{3\gamma\log(w_1)\big(5h_{26}\log(2)+8h_{12}\log(w_3)\big)}{8v_\infty^4}\\
&-\frac{h_{36}\log(w_3)}{4v_\infty^3} + \frac{\gamma h_{30}\log(\gamma)}{2v_\infty^3} + \frac{h_{37}\log(2)}{8v_\infty^3} + \frac{3(h_{17}w_3\mathrm{Li}_2(w_2) - 2h_{23}\mathrm{Li}_2(-w_2) + h_{15}\log^2(w_3) - h_9\log^2(2))}{8v_\infty} - \frac{3h_7\log(2)\log(w_3)}{v_\infty}\Bigg]
\end{aligned}$$

$$c_1 = m_1m_2^2\left(\frac{2h_{46}m_{12s}}{v_\infty^6} + \frac{9\pi^2h_1m_{12}^2}{32v_\infty^2}\right) + m_1^2m_2^3\left(\frac{4\gamma h_{47}}{3v_\infty^6} - \frac{8\gamma h_2\log(w_1)}{v_\infty^6} + \frac{16h_{25}\log(w_1)}{v_\infty^3} - \frac{8h_3}{3v_\infty^5}\right)$$

$$\begin{aligned}
c_2 = &-m_1^4m_2\left(\frac{9\pi^2h_1}{32v_\infty^2} + \frac{2h_{46}}{v_\infty^6}\right) + m_2^2m_1^3\Bigg[+\frac{h_{60}}{705600\gamma^8v_\infty^5} - \frac{4\gamma h_{48}}{3v_\infty^6} + \frac{3h_{38}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2})) - \gamma h_{21}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{16v_\infty^4}\\
&+\frac{3\gamma h_{31}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{8v_\infty^4} + \frac{h_{62}\log(w_1)}{6720\gamma^9v_\infty^6} + \frac{32\gamma^2h_{44}\log^2(w_1)}{v_\infty^7} + \frac{8\gamma\big(2h_4\log(2) - h_{27}\log(w_1)\big)\log(w_1)}{v_\infty^4} - \frac{32h_{29}\log(w_1)}{3v_\infty^3} + \frac{\pi^2h_{42}}{192v_\infty^4}\Bigg]\\
&+m_2^3m_1^2\Bigg[\frac{h_{59}}{1440\gamma^7v_\infty^5} - \frac{h_{19}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{8v_\infty^4} + \frac{h_{43}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{32v_\infty^4} - \frac{h_{20}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{4v_\infty^4}\\
&-\frac{h_{61}\log(w_1)}{480\gamma^8v_\infty^6} - \frac{16\gamma^2h_{11}\log^2(w_1)}{v_\infty^4} - \frac{32\gamma h_{45}\log^2(w_1)}{v_\infty^7} + \frac{16\gamma h_{28}\log(w_1)}{5v_\infty^3} - \frac{32h_{24}\log(2)\log(w_1)}{v_\infty^4} - \frac{\pi^2h_{18}}{48v_\infty^4} - \frac{2h_{54}}{45v_\infty^6}\Bigg]
\end{aligned}$$

with $\gamma \equiv u_1\cdot u_2$, $v_\infty = \sqrt{\gamma^2-1}$, $w_1 = \gamma - v_\infty$, $w_2 = \frac{\gamma-1}{\gamma+1}$, $w_3 = \gamma+1$, $h_i =$ polynomial in γ.

$$\mathrm{Li}_2(z) \equiv \int_0^z dx\,\frac{\log(1-x)}{-x}$$

$$K(z) \equiv \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}$$

$$E(k) \equiv \int_0^1 dx\,\frac{\sqrt{1-zx^2}}{\sqrt{1-x^2}}$$

NNNLO: 4PM

The full impulse at $\mathcal{O}(G^4)$: Dlapa-Kälin-**ZL**-Porto JHEP 2023 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\Delta p_1^\mu\Big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

- We obtained the full dynamics of binary inspirals at $\mathcal{O}(G^4)$ for the first time.
- Conservative part agrees perfectly with previous derivations.

  Bern-Parra-Martinez-Roiban-Ruf-Shen-Solon-Zeng 2022 Dlapa-Kälin-**ZL**-Porto PRL 2022
- Perfect agreement with the state-of-the-art PN computations

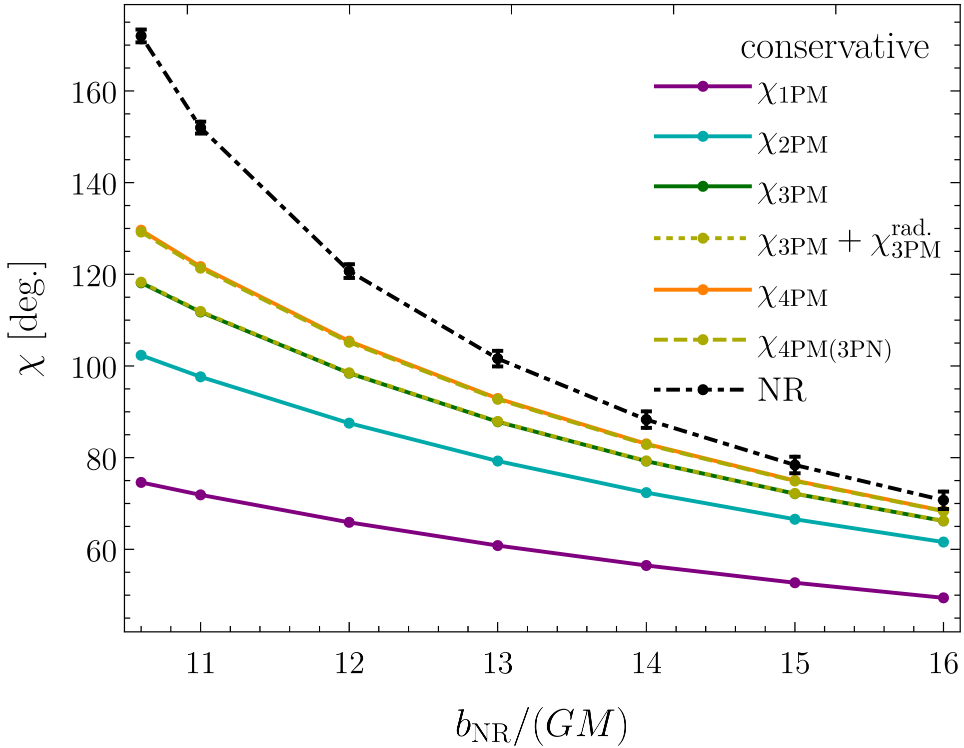
  Cho-Dandapat-Gopakumar 2021 Cho 2022 Bini-Geralico 2021 2022 Bini-Damour 2022
- Later, two new calculations confirmed our results.

  Damgaard-Hansen-Planté-Vanhove 2023 (exponentiation of amplitudes)

  Jakobsen-Mogull-Plefka-Sauer-Xu 2023 (worldline formalism)

Analytic vs Numerical Relativity

止於至善



Khalil-Buonanno-Steinhoff-Vines 2204.05047

NNNLO: local-in-time part

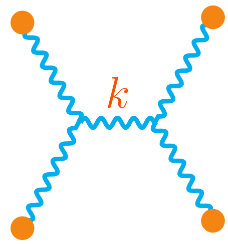
止於至善

- The full result does not describe generic elliptic-like motion due to nonlocal-in-time effects.

  Damour-Jaranowski-Schäfer 2014   Galley-Leibovich-Porto-Ross 2015   Cho-Kälin-Porto 2021

- Nonlocal-in-time radial action:

$$\mathcal{S}_r^{(\mathrm{nloc})} = -\frac{GE}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{dE}{d\omega}\log\left(\frac{4\omega^2}{\mu^2}e^{2\gamma_E}\right)$$



- The 4PM integrand can be built from 3PM diagrams. $\omega \equiv k\cdot u_{\mathrm{com}}$

$$\int d^D\ell_1 d^D\ell_2 \frac{\delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2][\ell_2\cdot u_1]}\frac{\log(\omega^2)}{[\ell_1^2][\ell_2^2][(\ell_1+\ell_2-q)^2][(\ell_1-q)^2][(\ell_2-q)^2]}$$

- We managed to compute the integrals and obtained nonlocalin-time contribution:

$$\frac{\nu}{(\gamma^2-1)^2}\Bigg[h_1 + \frac{\pi^2 h_2}{\sqrt{\gamma^2-1}} + h_3\log\frac{\gamma+1}{2} + \frac{h_4\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_5\log\frac{\gamma-1}{8} + h_6\log^2\frac{\gamma+1}{2} + \frac{h_8\log(2)\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}$$
$$+h_7\,\mathrm{arccosh}(\gamma)^2 + h_9\log\frac{\gamma-1}{8}\log\frac{\gamma+1}{2} + \frac{h_{10}\log\frac{\gamma^2-1}{16}\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_{11}\mathrm{Li}_2\frac{\gamma-1}{\gamma+1} + h_{12}\frac{\mathrm{arccosh}^2(\gamma)+4\mathrm{Li}_2(\sqrt{\gamma^2-1}-\gamma)}{\sqrt{\gamma^2-1}}\Bigg]$$

Coefficients h_i: exact-ν (iterated elliptic integrals) and SF-expanded (30SF) forms

Dlapa-Kälin-**ZL**-Porto PRL 2024

- Using 6PN results in the literature, we constructed an improved bound Hamiltonian.

Conclusion & Outlook

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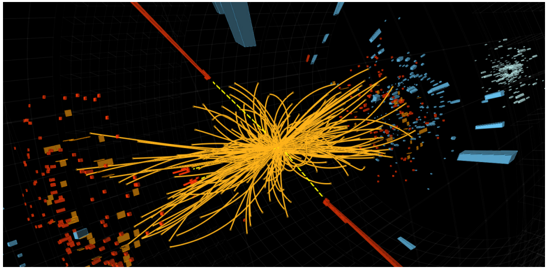
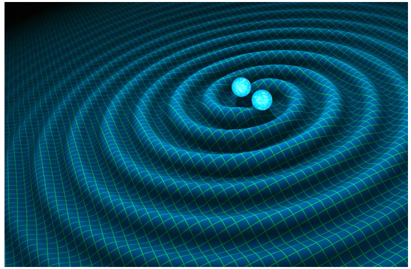
Modern techniques from Quantum Field Theory have already proven useful to improve theoretical predictions for gravitational-wave observables.





We have developed an efficient framework and made breakthroughs to NNNLO.

- Conservative spin & tidal effects at NLO JHEP 06 (2021) 012 PRD 102 (2020) 124025
- Conservative dynamics at NNLO PRL 125 (2020) 261103
- Conservative dynamics at NNNLO PLB 822 (2021) 136698 PRL 128 (2022) 161104 PRL 130 (2023) 101401
- Local-in-time & nonlocal-in-time separation to N^{4}LO PRL 132 (2024) 221401 arXiv:2506.20665
- Novel techniques to evaluate classical loop integrals JHEP 07 (2023) 181 JHEP 08 (2023) 109

Outlook: precision frontier

止於至善

GW science is just starting!   discovery potential = precise theoretical predictions!

$$
\begin{array}{lllll}
 & & \text{0PN} \quad \text{1PN} \quad \text{2PN} \quad \text{3PN} \quad \text{4PN} \quad \text{5PN} \quad \text{6PN} \quad \cdots & & \\
\text{1PM} & \left(\frac{GM}{r}\right) \times & \left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots \right) & = & \left(\frac{GM}{r}\right) \times (1) \\
\text{2PM} & \left(\frac{GM}{r}\right)^2 \times & \left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \cdots \right) & = & \left(\frac{GM}{r}\right)^2 \times (1) \\
\text{3PM} & \left(\frac{GM}{r}\right)^3 \times & \left(1 + v^2 + v^4 + v^6 + v^8 + \cdots \right) & = & \left(\frac{GM}{r}\right)^3 \times (1 + \nu) \\
\text{4PM} & \left(\frac{GM}{r}\right)^4 \times & \left(1 + v^2 + v^4 + v^6 + \cdots \right) & = & \left(\frac{GM}{r}\right)^4 \times (1 + \nu) \\
\text{5PM} & \left(\frac{GM}{r}\right)^5 \times & \left(1 + v^2 + v^4 + \cdots \right) & = & \left(\frac{GM}{r}\right)^5 \times (1 + \nu + \nu^2) \\
\text{6PM} & \left(\frac{GM}{r}\right)^6 \times & \left(1 + v^2 + \cdots \right) & = & \left(\frac{GM}{r}\right)^6 \times (1 + \nu + \nu^2) \\
\text{7PM} & \left(\frac{GM}{r}\right)^7 \times & \left(1 + \cdots \right) & = & \left(\frac{GM}{r}\right)^7 \times (1 + \nu + \nu^2 + \nu^3) \\
 & & & & \text{0SF} \quad \text{1SF} \quad \text{2SF} \quad \text{3SF} \quad \cdots
\end{array}
$$

Outlook: precision frontier

止於至善

GW science is just starting! discovery potential = precise theoretical predictions!

0PN 1PN 2PN 3PN 4PN 5PN 6PN $\cdots$

1PM $\left(\frac{GM}{r}\right) \times \left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots\right) = \left(\frac{GM}{r}\right) \times (1)$

2PM $\left(\frac{GM}{r}\right)^2 \times \left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \cdots\right) = \left(\frac{GM}{r}\right)^2 \times (1)$

3PM $\left(\frac{GM}{r}\right)^3 \times \left(1 + v^2 + v^4 + v^6 + v^8 + \cdots\right) = \left(\frac{GM}{r}\right)^3 \times (1 + \nu)$

4PM $\left(\frac{GM}{r}\right)^4 \times \left(1 + v^2 + v^4 + v^6 + \cdots\right) = \left(\frac{GM}{r}\right)^4 \times (1 + \nu)$

5PM $\left(\frac{GM}{r}\right)^5 \times \left(1 + v^2 + v^4 + \cdots\right) = \left(\frac{GM}{r}\right)^5 \times (1 + \nu + \nu^2)$

6PM $\left(\frac{GM}{r}\right)^6 \times \left(1 + v^2 + \cdots\right) = \left(\frac{GM}{r}\right)^6 \times (1 + \nu + \nu^2)$

7PM $\left(\frac{GM}{r}\right)^7 \times \left(1 + \cdots\right) = \left(\frac{GM}{r}\right)^7 \times (1 + \nu + \nu^2 + \nu^3)$

0SF 1SF 2SF 3SF $\cdots$

- 1SF: Driesse-Jakobsen-Mogull-Plefka-Sauer-Usovitsch 2024 D-J-Klemm-M-Nega-P-S-U 2024

Outlook: precision frontier

止於至善

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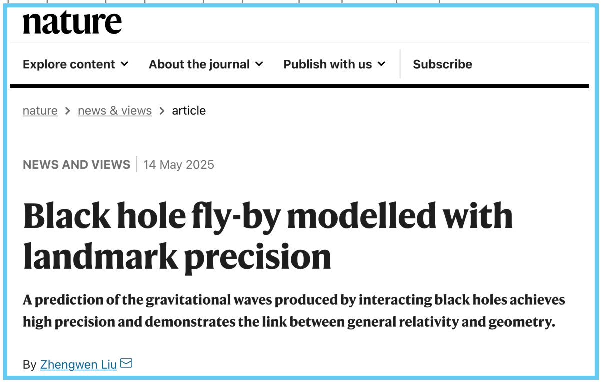
$$\begin{array}{lccccccccl} & 0\text{PN} & 1\text{PN} & 2\text{PN} & 3\text{PN} & 4\text{PN} & 5\text{PN} & 6\text{PN} & \cdots & \\ 1\text{PM} & \left(\frac{GM}{r}\right)\times(1 & + v^2 & + v^4 & + v^6 & + v^8 & + v^{10} & + v^{12} & + \cdots) & = \left(\frac{GM}{r}\right)\times(1) \\ 2\text{PM} & & \left(\frac{GM}{r}\right)^2\times(1 & + v^2 & + v^4 & + v^6 & + v^8 & + v^{10} & + \cdots) & = \left(\frac{GM}{r}\right)^2\times(1) \end{array}$$

$$\begin{array}{ll} 3\text{PM} & \left(\frac{GM}{r}\right)^3\times(1+\nu) \\ 4\text{PM} & \left(\frac{GM}{r}\right)^4\times(1+\nu) \\ 5\text{PM} & \left(\frac{GM}{r}\right)^5\times(1+\nu+\nu^2) \\ 6\text{PM} & \left(\frac{GM}{r}\right)^6\times(1+\nu+\nu^2) \\ 7\text{PM} & \left(\frac{GM}{r}\right)^7\times(1+\nu+\nu^2+\nu^3) \\ & \quad 0\text{SF} \quad 1\text{SF} \quad 2\text{SF} \quad 3\text{SF} \cdots \end{array}$$



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NEWS AND VIEWS | 14 May 2025

Black hole fly-by modelled with landmark precision

A prediction of the gravitational waves produced by interacting black holes achieves high precision and demonstrates the link between general relativity and geometry.

By Zhengwen Liu

- 1SF: Driesse-Jakobsen-Mogull-Plefka-Sauer-Usovitsch 2024 D-J-Klemm-M-Nega-P-S-U 2024

Outlook: precision frontier

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$$
\begin{array}{lcccccccc}
 & & 0\text{PN} & 1\text{PN} & 2\text{PN} & 3\text{PN} & 4\text{PN} & 5\text{PN} & 6\text{PN}\ \cdots
\end{array}
$$

1PM $\left(\frac{GM}{r}\right)\times\left(\boxed{1} + \boxed{v^2} + \boxed{v^4} + \boxed{v^6} + \boxed{v^8} + \boxed{v^{10}} + \boxed{v^{12}} + \cdots\right) = \left(\frac{GM}{r}\right)\times(1)$

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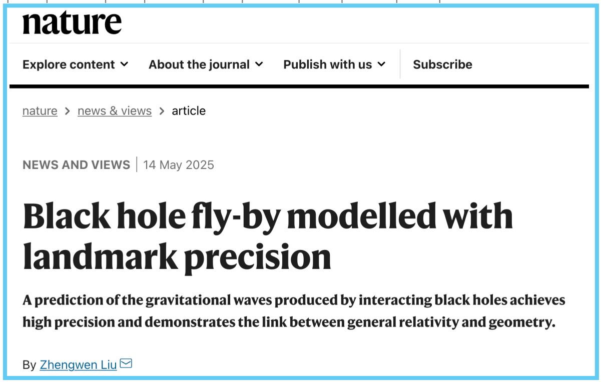
7PM $\left(\frac{GM}{r}\right)^7\times(1 + \nu + \nu^2 + \nu^3)$

0SF 1SF 2SF 3SF $\cdots$



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Outlook: precision frontier

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\text{3PM} & & \left(\frac{GM}{r}\right)^3 \times (1 + \nu) \\
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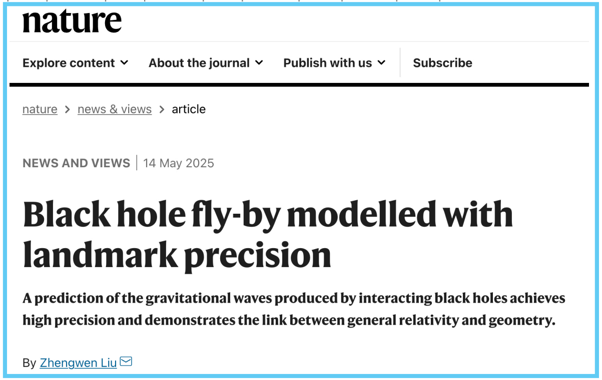
\text{5PM} & & \left(\frac{GM}{r}\right)^5 \times (1 + \nu + \nu^2) \\
\text{6PM} & & \left(\frac{GM}{r}\right)^6 \times (1 + \nu + \nu^2) \\
\text{7PM} & & \left(\frac{GM}{r}\right)^7 \times (1 + \nu + \nu^2 + \nu^3) \\
 & & \quad \text{0SF} \quad \text{1SF} \quad \text{2SF} \quad \text{3SF} \cdots
\end{array}
$$



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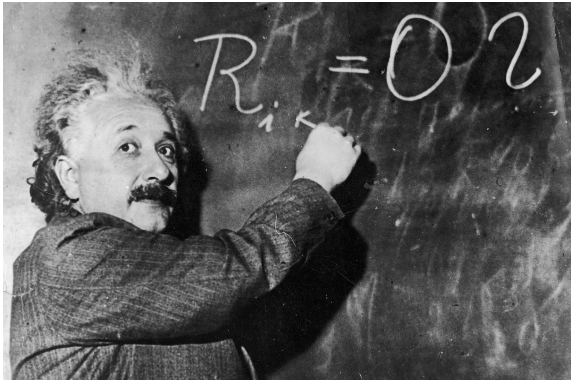
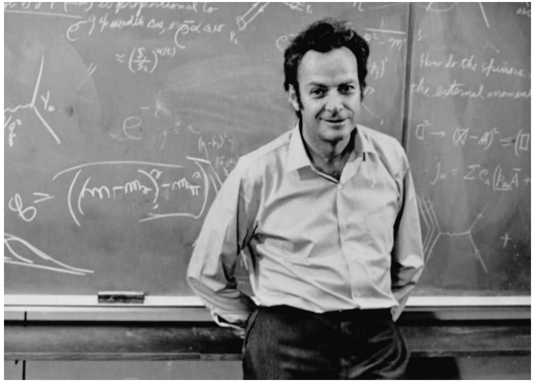
- 1SF: local-in-time conservative dynamics: Dlapa-Kälin-**ZL**-Porto 2506.20665
- 2SF: nonplanar diagram, more complicated functions ⋯

Feynman's integrals solve Einstein's equations!

谢谢！

Elliptic differential equations

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- From 4PM order, more complicated functions appear beyond polylogarithms.

- An elliptic example:

$$\frac{d}{dx}\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \frac{1-x^2}{2x(1+x^2)} & \frac{1+x^2}{4x(1-x^2)} & \frac{3x}{(1-x^2)(1+x^2)} \\ -\frac{1-x^2}{x(1+x^2)} & \frac{3(1+x^2)}{2x(1-x^2)} & -\frac{6x}{(1-x^2)(1+x^2)} \\ \frac{1-x^2}{x(1+x^2)} & -\frac{1+x^2}{2x(1-x^2)} & -\frac{1-4x^2+x^4}{x(1-x^2)(1+x^2)} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} + \mathcal{O}(\epsilon)$$

It can then be written as a third-order differential equation:

$$\left[\frac{d^3}{dx^3} - \frac{6x}{1-x^2}\frac{d^2}{dx^2} - \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\frac{d}{dx} - \frac{1+x^2}{x^3(1-x^2)}\right] f_1(x) = 0$$

It is easy to find the three solutions:

$$x\,\mathrm{K}^2\,(1-x^2), \qquad x\,\mathrm{K}(1-x^2)\mathrm{K}(x^2), \qquad x\,\mathrm{K}^2(x^2)$$

Complete elliptic integrals: $\mathrm{K}(x) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-xt^2)}}$

Elliptic differential equations

以於至善

With the knowledge of leading-ϵ solutions, one may transform the elliptic diagonal block into

$$\frac{d}{dx}\vec{g}(x,\epsilon) = \epsilon\,\tilde{D}_{\mathrm{ell}}(x)\,\vec{g}(x,\epsilon) + \ldots$$

with

$$\tilde{D}_{\mathrm{ell}} = \begin{pmatrix} -\frac{4\left(1+x^2\right)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)} & 0 \\ \frac{2\left(1+110x^2+x^4\right)\mathrm{K}^2\left(1-x^2\right)}{3\pi^2 x(1-x^2)} & -\frac{4\left(1+x^2\right)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)} \\ \frac{16\left(1+x^2\right)\left(1-18x+x^2\right)\left(1+18x+x^2\right)\mathrm{K}^4\left(1-x^2\right)}{27\pi^2 x(1-x^2)} & \frac{2\left(1+110x^2+x^4\right)\mathrm{K}^2\left(1-x^2\right)}{3\pi^2 x(1-x^2)} & -\frac{4\left(1+x^2\right)}{3x(1-x^2)} \end{pmatrix}$$

- Elliptic integrals appear in the transformation matrix: automated by INITIAL
- Higer $\mathcal{O}(\epsilon)$: Iterated integrals involving elliptic kernels.