Differential Equations for Energy Correlators in Any Angle

work with Jianyu Gong, Jingwen Lin, Kai Yan, Gang Yang and Yang Zhang
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Motivation

- From the phenomenological point of view, energy correlators can be used as jet observables for verify the standard model or find new physics.
- From the computability, energy correlators is perhaps the simplest infared safe observable to calculate analytically.

$$Q(q) \to P_1(p_1) + P_2(p_2) + P_3(p_3) + P_4(p_4)$$

Z boson, off-shell photon or Higgs q qbar g g q qbar q qbar g g g g

compute energy correlator analyticly

- n-point energy correlators are finite at LO, we can define a class of finite integrals
- This structure is similar to the Feynman-parameter representation of loop integrals

$$\frac{5x_1x_2x_3x_4\left(2\zeta_{12}x_2x_1+2\zeta_{13}x_3x_1+2\zeta_{14}x_4x_1+2\zeta_{23}x_2x_3+2\zeta_{34}x_3x_4-2x_1-x_2-2x_3-x_4+1\right)}{\left(\zeta_{12}x_2+\zeta_{13}x_3+\zeta_{14}x_4-1\right)\left(\zeta_{14}x_1+\zeta_{24}x_2+\zeta_{34}x_3-1\right)\left(\zeta_{13}x_1x_3+\zeta_{23}x_2x_3+\zeta_{34}x_4x_3+\zeta_{14}x_1x_4+\zeta_{24}x_2x_4-x_3-x_4\right)\cdots}$$

- we can develop a novel integration-by-parts technique that operates directly in the energy parameter (x_i) space, and hold the finite property meanwhile.

Content

Syzygy method for IBP of Feynman Integrals

Differential equations for 3-point energy correlator

4-point energy correlator

Traditional IBP

$$0 = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_k^\mu} \frac{v^\mu}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}} = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\frac{\partial v^\mu}{\partial l_k^\mu} - v^\mu \sum_{i=1}^n \frac{\partial D_i}{\partial l_k^\mu} \frac{\alpha_k}{D_i}}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

target integrals

```
{G[1, 1, 1, 1, 1, 1, 1, 1, 0, -4, 0], G[1, 1, 1, 1, 1, 1, 1, 1, 0, -3, -1],
 G[1, 1, 1, 1, 1, 1, 1, 1, 0, -2, -2], G[1, 1, 1, 1, 1, 1, 1, 1, 0, -1, -3],
 G[1, 1, 1, 1, 1, 1, 1, 1, 0, 0, -4], G[1, 1, -2, 1, 1, 1, 1, 1, 0, 0, 0], ...
```

redundant integrals G[1, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0], G[1, 1, 2, 1, 1, 1, 1, 1, 0, -1, 0] ...

master integrals

```
G[1, 1, 1, 1, 1, 1, 1, 1, 0, 0, -1], G[1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0],
G[1, 1, 1, 0, 1, 1, 1, 1, 0, -1, 0], G[1, 1, 1, 0, 1, 1, 1, 1, 0, 0, -2],
G[1, 1, 1, 0, 1, 1, 1, 1, 0, 0, -1], G[1, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0],...
```

redundant IBPs Time consuming! Memory consuming!

IBP syzygy method

IBP operator
$$O_{IBP} = \sum_{i=1}^{L} \frac{\partial}{\partial l_k^\mu} (v_k^\mu \cdot \,)$$

$$0 = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_k^\mu} \frac{v^\mu}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}} = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\frac{\partial v^\mu}{\partial l_k^\mu} - v^\mu \sum_{i=1}^n \frac{\partial D_i}{\partial l_k^\mu} \boxed{\frac{\alpha_k}{D_i}}}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

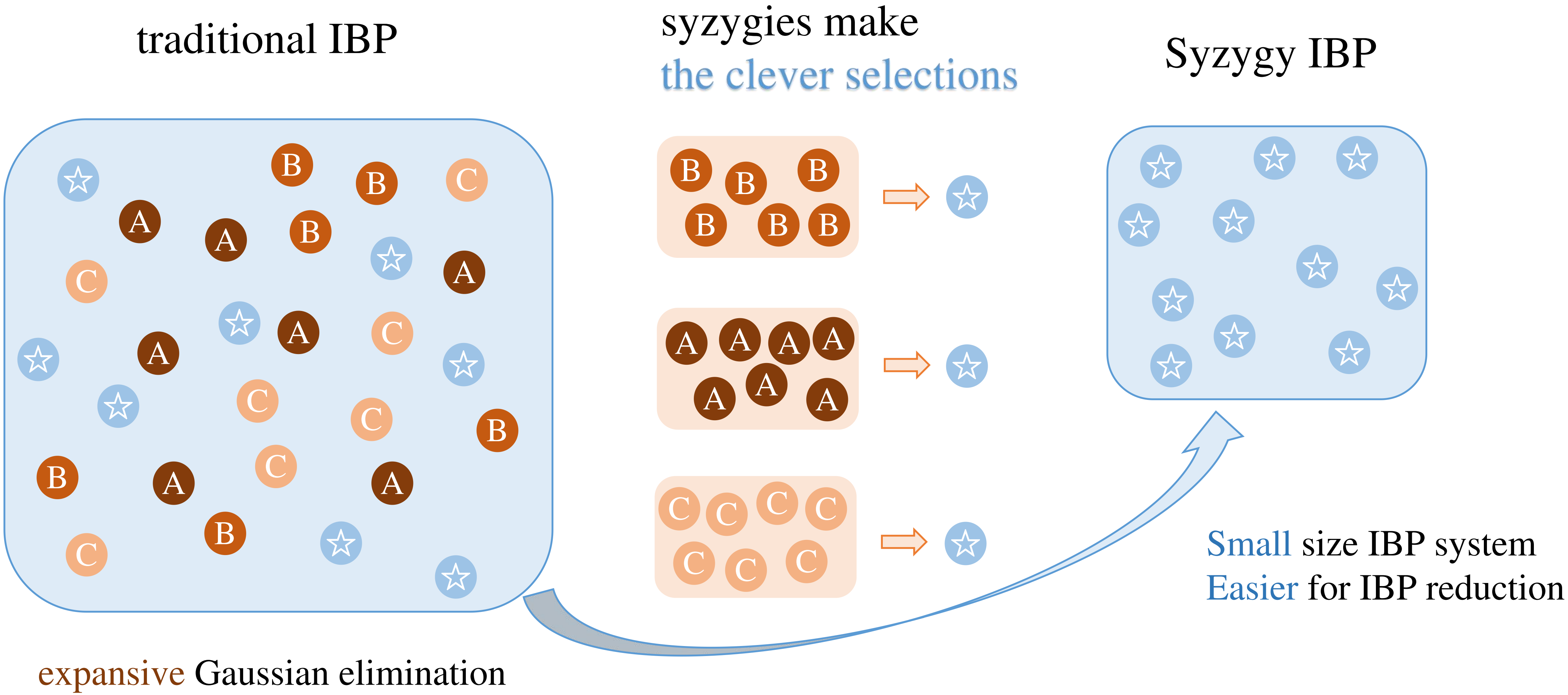
redundant integrals

avoid increasing propagators' degree

Syzygy equation
$$\sum_{k=1}^{L} v_k^\mu \frac{\partial D_i}{\partial l_k^\mu} = g_i D_i \qquad i \in \{j \,|\, \alpha_j > 0\}$$

Janusz Gluza, Krzysztof Kajda, David A. Kosower: Phys.Rev.D 83 (2011) 045012

The Magic of Syzygy



Content

Syzygy methoed for IBP

Differential equations for 3-point energy correlator

4-point energy correlator

n-point energy correlator

$$E^n C(\theta_{ij}) = \int \prod_{i=1}^{n} d\Omega_{\vec{n}_i} \prod_{i\neq j} \delta(\vec{n}_i \cdot \vec{n}_j - \cos(\theta_{ij})) \frac{\int d^4 x e^{iqx} \langle 0|\mathcal{O}^{+}(x)\mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_n)\mathcal{O}(0)|0\rangle}{(q^0)^n \int d^4 x e^{iqx} \langle 0|\mathcal{O}^{+}(x)\mathcal{O}(0)|0\rangle}$$

create the final state X

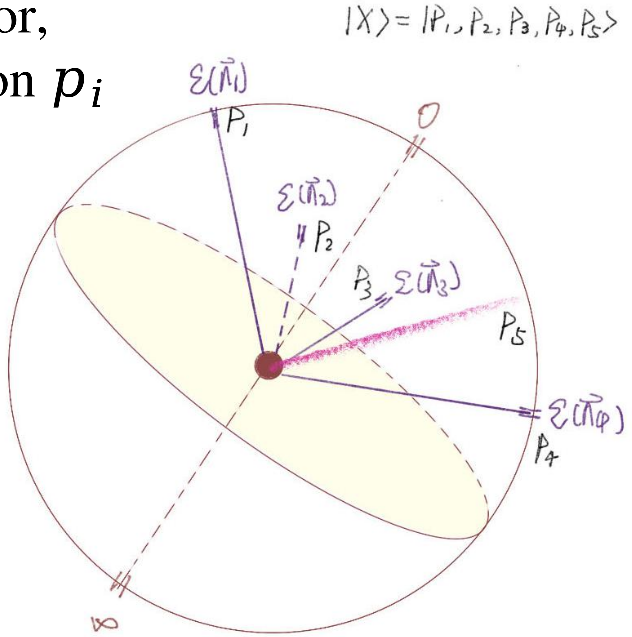
$$\Downarrow \quad \int d^4 x\, e^{iqx} \langle X|\mathcal{O}(x)|0\rangle \equiv (2\pi)^4 \delta^4(q - q_X) F_X$$

form factor, depends on p_i

$$\sim \sum_{(n_1,\cdots,n_n)\in X} \int d\Pi_X \left(\prod_{i=1}^{n} \delta^2(\vec{n}_i - \hat{p}_{n_i}) \frac{E_i}{q^0} \right) |F_X|^2$$

where $d\Pi_X$ is onshell phase-space of the finial state.

$$\frac{2\left(2q^4 \left(p_1+p_2+p_4+p_5\right)\cdot\left(p_2+p_3+p_4+p_5\right)+\cdots-q^4\left(p_2+p_3+p_4+p_5\right)\cdot\left(p_1+p_2+p_3+p_4+p_5\right)+q^6\right)}{\left(p_3+p_4\right)^2\left(p_1+p_5\right)^2\left(p_1+p_4+p_5\right)^2\left(p_1+p_2+p_4+p_5\right)^2\left(p_3+p_4+p_5\right)^2\left(p_2+p_3+p_4+p_5\right)^2}$$



Kai Yan and Xiaoyuan Zhang. Three-Point Energy Correlator in N=4 Supersymmetric Yang-Mills Theory. *Phys. Rev. Lett.*, 129(2):021602, 2022.

Lift Differential Equations: Overview

E³ C/S³

⇩ Partial Fraction Decomposition

Classify simple finite integral families

⇩ Syzygy Equations

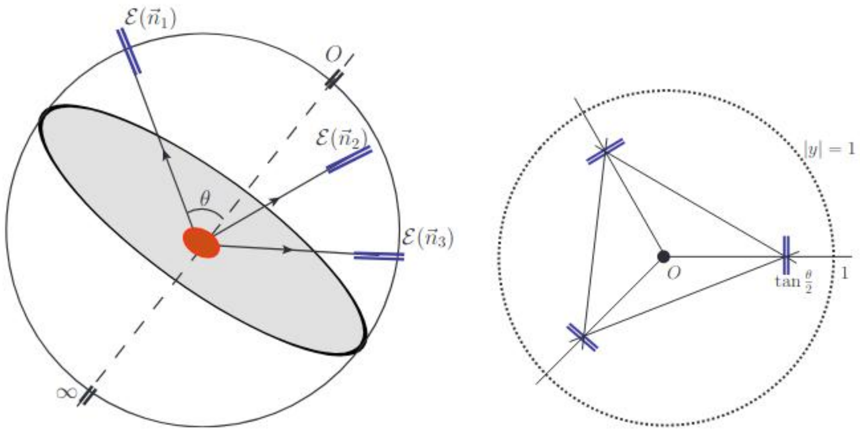
IBP and iterative boundary IBP

⇩ Lift Equations

Finite differential equations

⇩ Cubically nilpotent

Canonical differetial equations



$$\text{energy parameters: } x_i = \frac{2\,q\cdot p_i}{q^2}\,, \quad i = 1,\cdots,n$$

$$\text{angle parameters: } \zeta_{ij} = \frac{q^2\,p_i\cdot p_j}{2\,q\cdot p_i\,q\cdot p_j}\,, \qquad i,j = 1,\cdots,n$$

E^{3}C Divergent Region

- Propagators $$\mathcal{D}_1 = \frac{s_{134}}{q^2} = -1 + x_2,\ \mathcal{D}_2 = \frac{s_{124}}{q^2} = -1 + x_3,\ \mathcal{D}_3 = \frac{s_{123}}{q^2} = -1 + x_1 + x_2 + x_3,$$ $$\mathcal{D}_4 = \frac{s_{34}}{q^2 x_3} = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ \mathcal{D}_5 = \frac{s_{24}}{q^2 x_2} = -1 + x_1\zeta_{12} + x_3\zeta_{23}.$$

- Consider delta function measure

$$\delta(1 - x_1 - x_2 - x_3 + \zeta_{12}x_1x_2 + \zeta_{23}x_2x_3 + \zeta_{13}x_1x_3)$$

$$\begin{array}{ll} \text{region 1} & \{\, x_1 \to 1 + (\zeta_{12} + \zeta_{13} - 2)cx_1,\ x_2 \to cx_2,\ x_3 \to cx_3 \,\}|_{c\to 0}\,, \\ \text{region 2} & \{\, x_1 \to cx_1,\ x_2 \to 1 + (\zeta_{12} + \zeta_{23} - 2)cx_2,\ x_3 \to cx_3 \,\}|_{c\to 0}\,, \\ \text{region 3} & \{\, x_1 \to cx_1,\ x_2 \to cx_2,\ x_3 \to 1 + (\zeta_{13} + \zeta_{23} - 2)cx_3 \,\}|_{c\to 0}\,. \end{array}$$

Potentially Divergent Region

- Power counting

$$\{\mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{D}_3,\, \mathcal{D}_4,\, \mathcal{D}_5\} \xrightarrow[\text{counting}]{\text{power}} \begin{cases} \text{region 1} & \{0,\, 0,\, 1,\, 0,\, 0\} \\ \text{region 2} & \{1,\, 0,\, 1,\, 0,\, 0\}. \\ \text{region 3} & \{0,\, 1,\, 1,\, 0,\, 0\} \end{cases} \qquad \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\delta(\mathcal{D}_\delta) \xrightarrow[\text{counting}]{\text{power}} 2$$

Syzygy for finite IBP

SINGULAR

$$\text{Int}[n_1, n_2, n_3, 1] = \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \frac{\delta(D_\delta)}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} = \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_\delta} \Big|_{\text{cut}(D_\delta)}$$

Integration by parts (IBP) $\longrightarrow$ Syzygy Equations

$$\mathcal{O}_{\text{IBP}} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (a_i \cdot\ )$$

$$\sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0,$$

$$a_i,\ b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3]$$

for divergent propagators and D_δ

Not increase the power of divergent propagators!

Lift for finite differential equation

SINGULAR

Derivative of a certain parameter $\longrightarrow$ Lift Differential Equations

$$\mathcal{O}_{\partial\zeta_{**}} = \frac{\partial}{\partial\zeta_{**}} + \mathcal{O}_{\mathrm{IBP}} \equiv \boxed{\frac{\partial}{\partial\zeta_{**}}} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i$$

Find the differential equations of a certain kinematic

May bring higher power of divergent propagators

$$\frac{\partial}{\partial\zeta_{**}} D_j + \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0,$$

$$a_i,\, b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3]$$

When the Lift equation is satisfied, the integral will be finite!

Boundary IBP

$$D_1 = -1 + x_3,\ D_2 = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ D_3 = -1 + x_1\zeta_{12} + x_3\zeta_{23},$$
$$D_\delta = 1 - x_1 - x_2 - x_3 + x_1x_2\zeta_{12} + x_2x_3\zeta_{23} + x_1x_3\zeta_{13}$$

$$\mathcal{O}_{\mathrm{IBP}}\,\mathrm{Int}[n_1, n_2, n_3, 1] = \sum_{i=1}^{3}(\mathrm{BT}_{x_i=1} - \mathrm{BT}_{x_i=0}).$$

subfamily 1 $\quad D_1 = -1 + x_1\zeta_{12},\ D_2 = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ D_\delta = 1 - x_1 - x_2 + x_1x_2\zeta_{12}\,;$

subfamily 2 $\quad D_1 = -1 + x_3\zeta_{13},\ D_2 = -1 + x_1\zeta_{13},\ D_\delta = 1 - x_1 - x_3 + x_1x_3\zeta_{13}\,;$

subfamily 3 $\quad D_1 = -1 + x_3\zeta_{23},\ D_2 = -1 + x_2\zeta_{23},\ D_\delta = 1 - x_2 - x_3 + x_2x_3\zeta_{23}\,.$

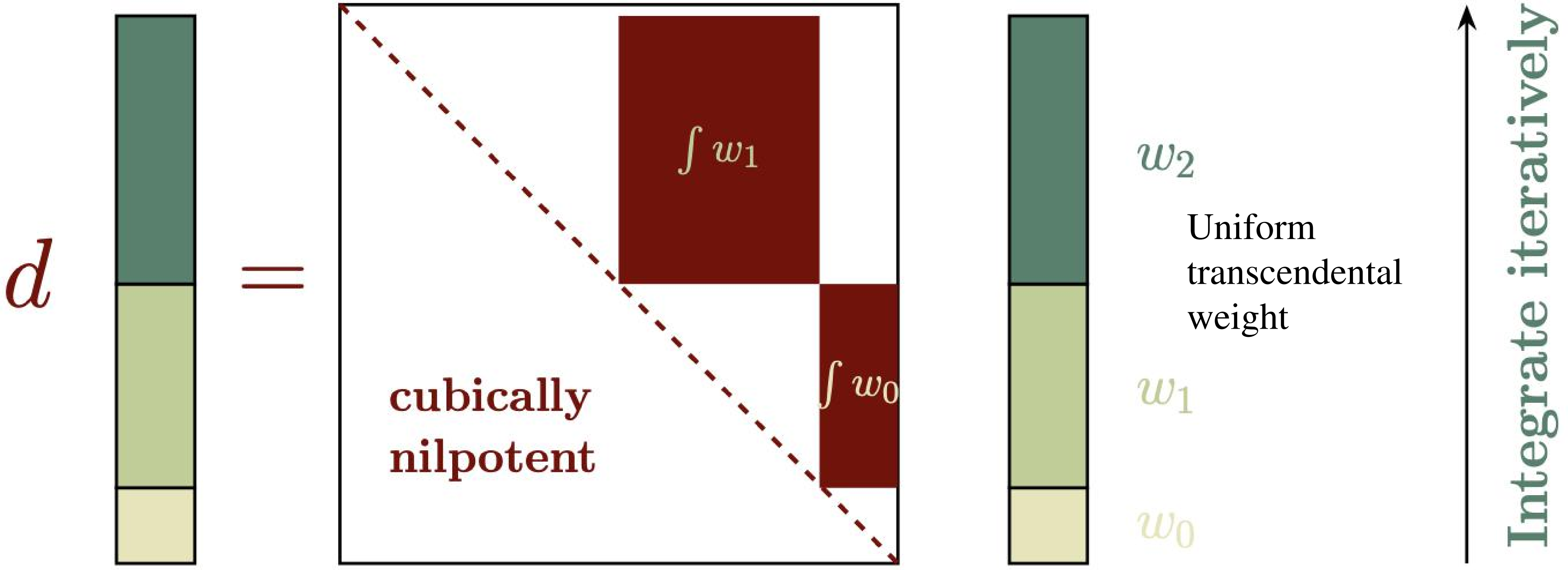
Master Integrals

$$\{\mathrm{Int}[1,1,-1,1],\ \mathrm{Int}[-1,1,1,1],\ \mathrm{Int}[1,1,0,1],\ \mathrm{Int}[0,1,1,1],\ \mathrm{Int}[1,0,0,1],$$
$$\mathrm{Int}_2[1,\{0,0,1\}],\ \mathrm{Int}_2[2,\{0,1,1\}],\ \mathrm{Int}_2[2,\{0,0,1\}],\ \mathrm{Int}_2[3,\{0,0,1\}],\ 1\}$$

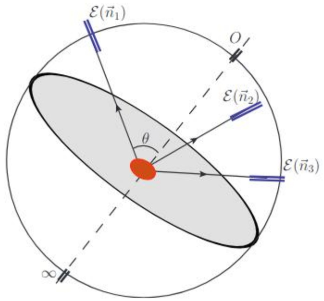
Canonical Differential Equations

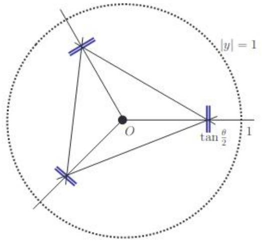
Differential Equations include boundary integrals

$$\frac{\partial}{\partial \zeta_{**}} \mathrm{Int}[n_1, n_2, n_3, 1] - \mathcal{O}_{\partial \zeta_{**}} \mathrm{Int}[n_1, n_2, n_3, 1] = \sum_{i=1}^{3} (\mathrm{BT}_{x_i=0}).$$



Analytic result





$$\zeta_{12} = -\frac{s(1-x_2)^2}{(1+s)^2 x_2},\ \zeta_{23} = -\frac{s(1-x_1x_2)^2}{(1+s)^2 x_1 x_2},\ \zeta_{13} = -\frac{s(1-x_1)^2}{(1+s)^2 x_1}.$$

letters

family 1 $\quad s,\ x_1,\ x_2,\ 1+s,\ 1-x_1,\ 1-x_2,\ s+x_1,\ s+x_2,$
$1+sx_1,\ s+x_1x_2,\ 1+sx_1x_2,\ 1-x_1x_2,\ 1-x_1^2x_2\,;$

family 2 $\quad s,\ x_1,\ x_2,\ 1+s,\ 1-x_1,\ 1-x_2,\ s+x_1,\ s+x_2,$
$1+sx_1,\ s+x_1x_2,\ 1+sx_1x_2,\ 1-x_1x_2\,;$

family 3 $\quad s,\ x_1,\ x_2,\ 1+s,\ 1-s,\ 1-x_1,\ 1-x_2,\ s+x_1,\ s+x_2,$
$1+sx_1,\ 1+sx_2,\ s+x_1x_2,\ 1+sx_1x_2,\ 1-x_1x_2.$

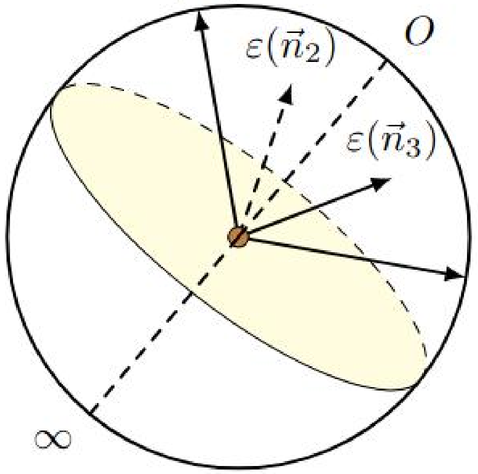
Content

Syzygy methoed for IBP

Differential equations for 3-point energy correlator

4-point energy correlator

Four Point Energy Correlator



one term in form factor

$$\frac{2\left(2q^4\left(p_1+p_2+p_4+p_5\right)\cdot\left(p_2+p_3+p_4+p_5\right)+\cdots-q^4\left(p_2+p_3+p_4+p_5\right)\cdot\left(p_1+p_2+p_3+p_4+p_5\right)+q^6\right)}{\left(p_3+p_4\right)^2\left(p_1+p_5\right)^2\left(p_1+p_4+p_5\right)^2\left(p_1+p_2+p_4+p_5\right)^2\left(p_3+p_4+p_5\right)^2\left(p_2+p_3+p_4+p_5\right)^2}$$

$$\frac{5x_1x_2x_3x_4\left(2\zeta_{12}x_2x_1+2\zeta_{13}x_3x_1+2\zeta_{14}x_4x_1+2\zeta_{23}x_2x_3+2\zeta_{34}x_3x_4-2x_1-x_2-2x_3-x_4+1\right)}{\left(\zeta_{12}x_2+\zeta_{13}x_3+\zeta_{14}x_4-1\right)\left(\zeta_{14}x_1+\zeta_{24}x_2+\zeta_{34}x_3-1\right)\left(\zeta_{13}x_1x_3+\zeta_{23}x_2x_3+\zeta_{34}x_4x_3+\zeta_{14}x_1x_4+\zeta_{24}x_2x_4-x_3-x_4\right)\cdots}$$

Dmitry Chicherin, Ian Moult, Emery Sokatchev, Kai Yan, and Yunyue Zhu. The Collinear Limit of the Four-Point Energy Correlator in $\mathcal{N}=4$ Super Yang-Mills Theory. 1 2024.

Lift Differential Equations: Overview

4-point energy correlator

E^3 C/S^3

↓ Partial Fraction Decomposition

Classify simple finite integral families

↓ Syzygy Equations

IBP and iterative boundary IBP

↓ Lift Equations

Finite differential equations

↓ Cubically nilpotent

Canonical differetial equations

+ **Combine Divergent Integrals into Finite Integrals**

+ **polynomial reduction integrand reduction**

Difficulties of Higher Points

- Increasing number of integral variables
- Polynomial Propagators
- Complex Measurement because of the delta function

A part of letters of E^4C (SOFIA)

$\lambda, z, \bar{z}, w-z, \bar{z}-z, -1+z, \bar{w}-\bar{z}, -1+\bar{z}, 1+\lambda^2,\ w-\bar{w}z, \bar{w}-\bar{z}w, -1+\bar{z}z, 1+\lambda^2 z, 1+\bar{z}\lambda^2, \bar{z}w-\bar{w}z, \bar{w}w-\bar{z}z, 1+\bar{z}\lambda^2 z, -1+\bar{z}\lambda^4 z, \bar{w}w-\bar{z}w-\bar{w}z, -1+w+\lambda^2 w-\lambda^2 z, -1+\bar{z}+z+\bar{z}\lambda^2 z, \bar{z}+z-\bar{z}z+\bar{z}\lambda^2 z, -1+\bar{w}+\bar{w}\lambda^2-\bar{z}\lambda^2, -1+w+\lambda^2 w-\bar{z}\lambda^2 z, -1+\bar{w}+\bar{w}\lambda^2-\bar{z}\lambda^2 z, \bar{w}-w-\bar{z}\lambda^2 w+\bar{w}\lambda^2 z, -1+w+\bar{z}\lambda^2 w-\bar{z}\lambda^2 z, \bar{w}-z+\bar{w}\lambda^2 z-\bar{z}\lambda^2 z, -\bar{z}+w+\bar{z}\lambda^2 w-\bar{z}\lambda^2 z, -1+\bar{w}+\bar{w}\lambda^2 z-\bar{z}\lambda^2 z, -\bar{w}+\bar{z}+w-\bar{z}w-z+\bar{w}z, 1-\bar{w}w-\bar{w}\lambda^2 w+\bar{z}\lambda^2 z, 1-\bar{w}z-\bar{w}\lambda^2 z+\bar{z}\lambda^2 z, 1-\bar{w}-w+\bar{z}w+\bar{w}z-\bar{z}z, -1+\bar{z}w+\bar{z}\lambda^2 w-\bar{z}\lambda^2 z, -1+w+\lambda^2 w-2\lambda^2 z+\lambda^2 z^2, -1+\bar{w}+\bar{w}\lambda^2-2\bar{z}\lambda^2+\bar{z}^2\lambda^2, -\lambda^2 w-z+wz+2\lambda^2 wz-\lambda^2 z^2, \bar{z}-\bar{w}\bar{z}+\bar{w}\lambda^2-2\bar{w}\bar{z}\lambda^2+\bar{z}^2\lambda^2, -w+\bar{w}w-\bar{w}z+wz+\bar{w}\lambda^2 wz-\bar{w}\lambda^2 z^2, -\bar{z}\lambda^2 w-z+wz+2\bar{z}\lambda^2 wz-\bar{z}\lambda^2 z^2, \bar{z}-\bar{w}\bar{z}+\bar{w}\lambda^2 z-2\bar{w}\bar{z}\lambda^2 z+\bar{z}^2\lambda^2 z, -\bar{w}+\bar{w}\bar{z}+\bar{w}w-\bar{z}w+\bar{w}\bar{z}\lambda^2 w-\bar{z}^2\lambda^2 w, -\bar{z}w+\bar{w}\bar{z}w+\bar{w}z-\bar{w}\bar{z}z-\bar{w}wz+\bar{z}wz, -\bar{w}w+\bar{z}w+\bar{w}z-\bar{w}\bar{z}z-\bar{z}wz+\bar{w}\bar{z}wz,$

$1-\bar{w}-w+\bar{w}w+\bar{w}\lambda^2 w-\bar{z}\lambda^2 w-\bar{w}\lambda^2 z+\bar{z}\lambda^2 z,$

$1-\bar{w}-w+\bar{w}w-\bar{w}\lambda^4 w+\bar{z}\lambda^4 w+\bar{w}\lambda^4 z-\bar{z}\lambda^4 z,$

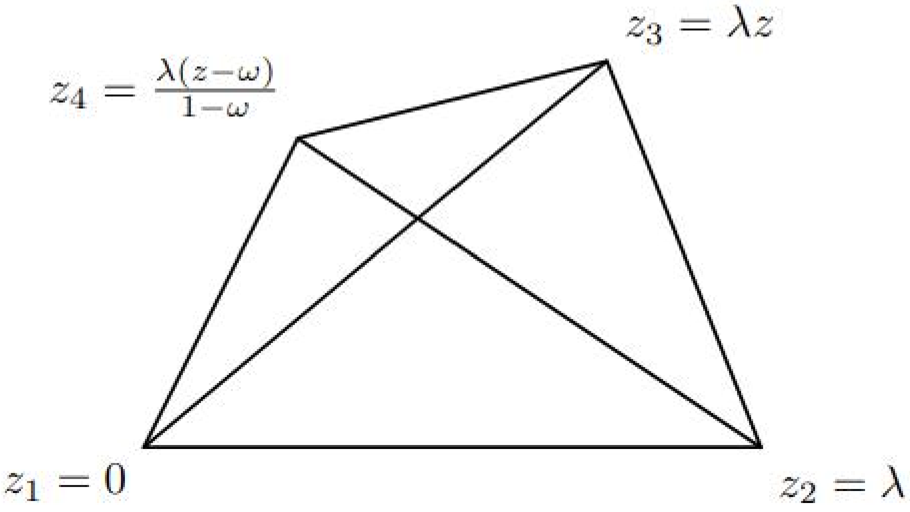
$\bar{z}-\bar{z}w-\bar{z}\lambda^2 w-z+\bar{z}\lambda^2 z+wz+\bar{z}\lambda^2 wz-\bar{z}\lambda^2 z^2,$

$-\bar{z}w+\bar{z}w^2+\bar{z}\lambda^2 w^2+z-wz-2\bar{z}\lambda^2 wz+\bar{z}\lambda^2 z^2,$

$1-w-\lambda^2 w-\bar{z}\lambda^2 w-\bar{z}\lambda^4 w+\lambda^2 z+\bar{z}\lambda^2 z+\bar{z}\lambda^4 z^2,$

$\bar{z}-\bar{w}\bar{z}-\bar{w}z+\bar{w}^2 z+\bar{w}^2\lambda^2 z-2\bar{w}\bar{z}\lambda^2 z+\bar{z}^2\lambda^2 z,$

$\bar{z}-\bar{w}\bar{z}-z+\bar{w}z+\bar{w}\lambda^2 z-\bar{z}\lambda^2 z-\bar{w}\bar{z}\lambda^2 z+\bar{z}^2\lambda^2 z,$



Four-Point Energy Correlators

$$\delta(1 - x_1 - x_2 - x_3 - x_4 + x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_1x_4\zeta_{14} + x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}).$$

$$D_1 = -1 + x_1\zeta_{14} + x_2\zeta_{24} + x_3\zeta_{34}\,, \quad D_2 = -1 + x_2\zeta_{12} + x_3\zeta_{13} + x_4\zeta_{14}\,,$$

$$D_3 = -1 + x_1\,, \quad D_4 = -1 + x_2\,, \quad D_5 = -1 + x_4\,, \quad D_6 = -1 + x_1 + x_2 + x_3 + x_4\,,$$

$$\boxed{\begin{aligned} &D_7 = -1 + x_1 + x_2 - x_1x_2\zeta_{12}\,, D_8 = -1 + x_2 + x_3 - x_2x_3\zeta_{23}\,, D_9 = -1 + x_3 + x_4 - x_3x_4\zeta_{34}\,, \\ &D_{10} = x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_2x_3\zeta_{23}\,, D_{11} = x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}\,. \end{aligned}}$$

$\{D_6, D_7\}$ $\{D_6, D_8\}$ $\{D_6, D_9\}$
$\{D_7, D_9\}$

elliptic

$\{D_{10}, D_{11}\}$

hyperelliptic g=2

Summary

syzygy IBP and lift DE the UT integrals of leading order E^3C in $N = 4$ SYM.

% + integrand relations the master integrals of leading order E^4C in $N = 4$ SYM.

syzygy + lift avoid divergent integrals, shrink IBP system.

E^3C Canonical differential equations for analytic result.

E^4C differential equations for function space.

differential equations may not suitable for higher point energy correlators

Thanks

Setup

energy parameters: $x_i = \frac{2\,q\cdot p_i}{q^2}\,, \quad i=1,\cdots,n$

angle parameters: $\zeta_{ij} = \frac{q^2\,p_i\cdot p_j}{2\,q\cdot p_i\,q\cdot p_j}\,, \qquad i,j=1,\cdots,n$

$$\mathrm{E^nC}(\vec{\zeta}_{ij})\big|_{\mathrm{LO}} \sim \int d^4p_{n+1}\delta^4(q-p_1-\cdots-p_{n+1})\,\delta_+(p_{n+1}^2)$$
$$\times\int_0^1 dx_1\cdots dx_n\,(x_1\cdots x_n)^2\left[|F_{n+1}^{(0)}|^2(p_1,\cdots,p_{n+1})+\mathrm{perm.}(1,\cdots,n+1)\right]$$

integrate out p_{n+1}

$$\mathrm{E^nC}(\vec{\zeta}_{ij})\big|_{\mathrm{LO}} \sim \int_0^1 dx_1\cdots dx_n\,(x_1\cdots x_n)^2\delta(1-Q_n)\,|F_{n+1}^{(0)}|^2_{\mathrm{sym.}}$$

where $Q_n = \sum_i x_i - \sum_{(ij)} x_i x_j \zeta_{ij}\,.$

Verify finite integrals by power counting

Finite integral: power counting of c is positive

$$I(x_0,y_0)=\int_0^{x_0}dx\int_0^{y_0}dy\frac{1}{x+y}\qquad\{x\to c, y\to c\}\quad \text{when } c\to 0$$
$$=(x_0+y_0)\log(x_0+y_0)-x_0\log(x_0)-y_0\log(y_0)$$

$$cPower[I(x_0,y_0)]=cPower[\int_0^{x_0}dcx\int_0^{y_0}dcy\frac{1}{cx+cy}]=1>0$$

$$D_1=-1+x_3,\ D_2=-1+x_1\zeta_{13}+x_2\zeta_{23},\ D_3=-1+x_1\zeta_{12}+x_3\zeta_{23},$$
$$D_\delta=1-x_1-x_2-x_3+x_1x_2\zeta_{12}+x_2x_3\zeta_{23}+x_1x_3\zeta_{13}$$

$$\text{Int}[1,1,-1,1]=\int dx_1dx_2dx_3\frac{D_3\,\delta(D_\delta)}{D_1D_2}$$

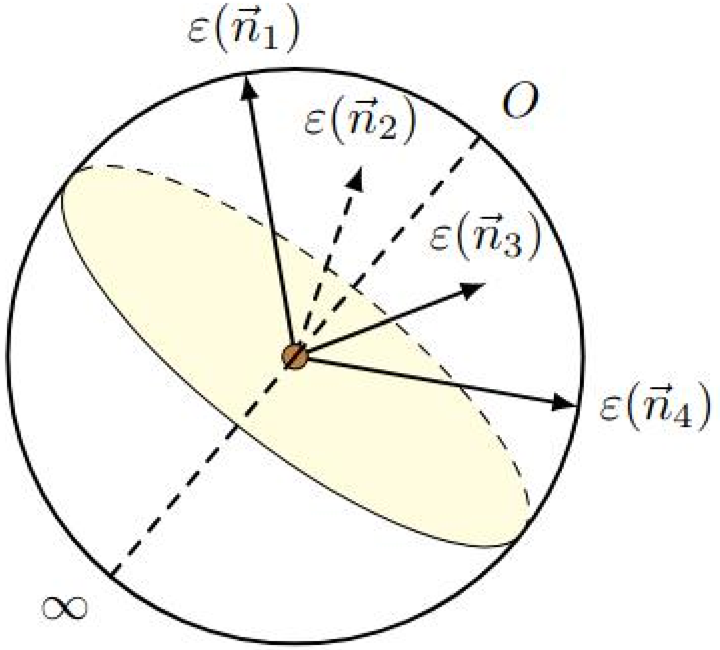
region: $\{x_1\to 0,\ x_2\to 0,\ x_3\to 1\}$

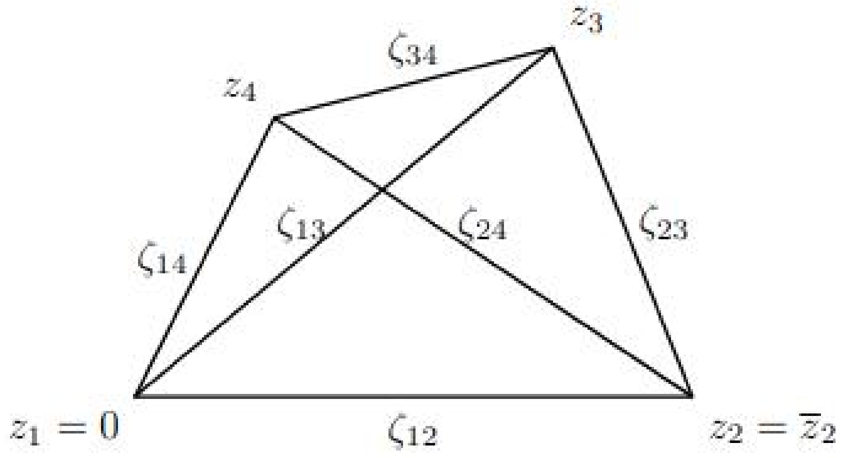
$$dx_1dx_2dx_3\delta(D_\delta)\xrightarrow[\text{counting}]{\text{power}}2$$

$$\{D_1,\ D_2,\ D_3\}\xrightarrow[\text{counting}]{\text{power}}\{1,0,0\}$$

Four point energy correlator

$$E^4 C(\vec{\zeta}_{ij})|_{LO} = \int_0^1 dx_1 \cdots dx_4 (x_1 \cdots x_4)^2 \delta(1 - Q_4) |F_5^{(0)}|^2_{sym}$$





$$\zeta_{ij} = \frac{|z_i - z_j|^2}{(1 + |z_i|^2)(1 + |z_j|^2)}, \quad z_1 = 0, \; \bar{z}_2 = z_2$$