Pentagonal Feynman Integrals and Wilson Loop with Lagrangian Insertion at Three-Loop

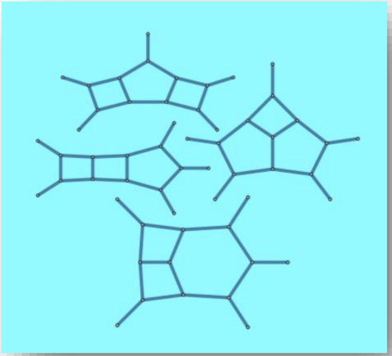
Base on:

Dmitry Chicherin, Yu Wu, Zihao Wu, Yongqun Xu, Shun-Qing Zhang, Yang Zhang [2511.XXXX]

Dmitry Chicherin, Johannes Henn, Yongqun Xu, Shun-Qing Zhang, Yang Zhang [2511.XXXX] (Stay Tune)

and

Yuanche Liu, Antonela Matijašić, Julian Miczajka, Yingxuan Xu, Yongqun Xu, Yang Zhang [2411.18697] (Phys. Rev. D 112, 016021)



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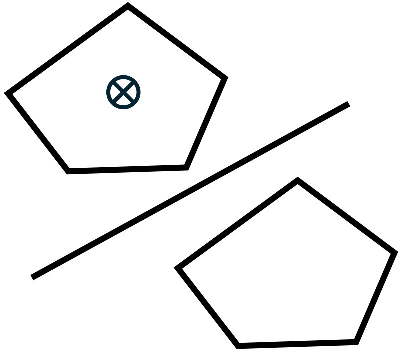
2025 Oct. 31

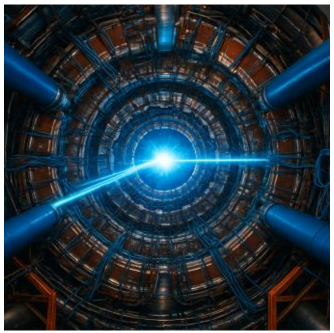


Pentagonal Feynman Integrals

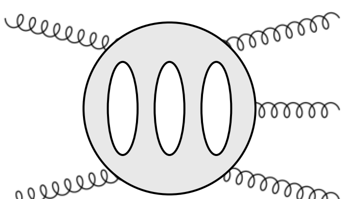
at Three-Loop

Background

Feynman Integrals as the basis building block of perturbative Quantum Field Theory



Collider Physics



$$\mathcal{A}_5^{(3)}(1,2,3,4,5)$$

Real world Amplitudes

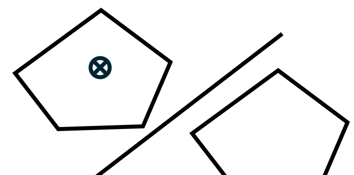
$$pp \to \gamma\gamma\gamma$$

$$pp \to jjj$$

@NNNLO

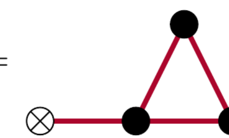
Future Collider:
CEPC, FCC, HL-LHC…

Observables in formal theory



Duality in Scattering Amplitudes

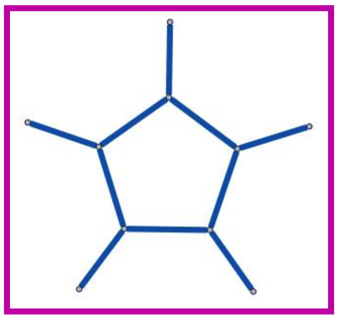
$I_{3,1}(z) =$



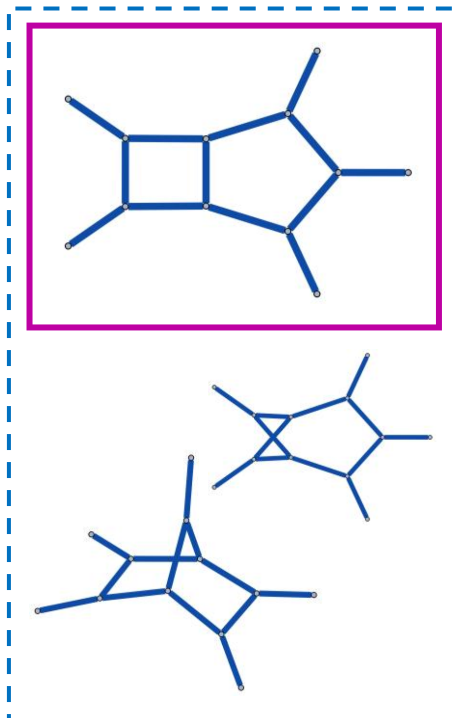
Negative geometries expansion

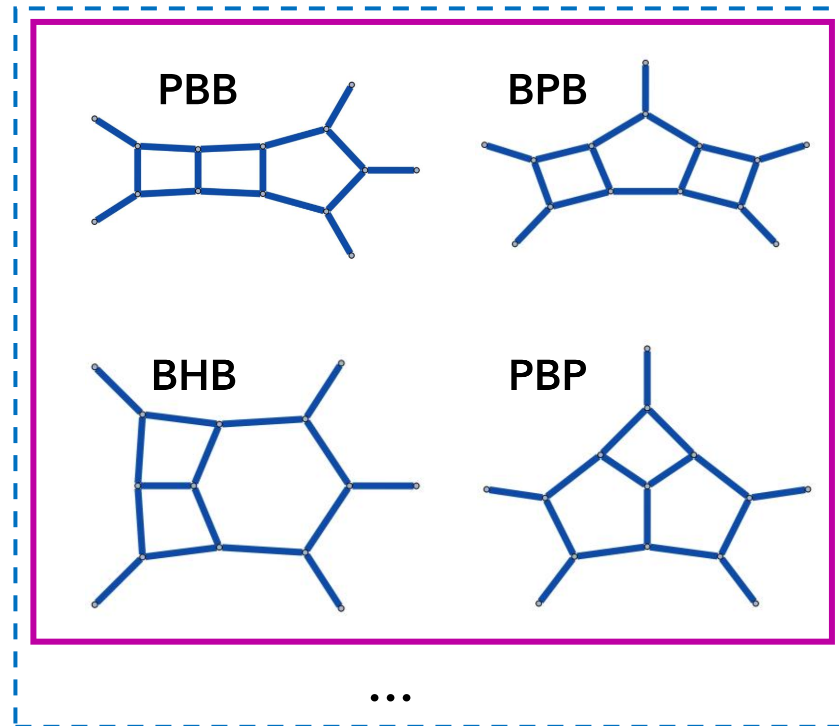
Pentagonal Feynman Integrals at Three-Loop

Planar Part



[Bern, Dixon, Kosower ' 92]
[A. Kniehl, V. Tarasov '10]
[Gehrmann, Henn, Lo Presti '18]
[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18]
[Abreu, Page, Zeng '18]
[Liu, Matijašić, Miczajka, Xu, Xu, Zhang '24]
and so on…

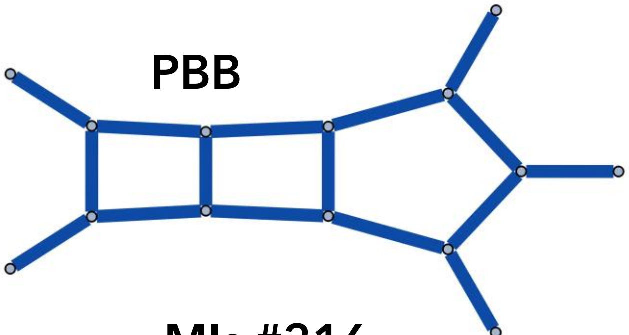




Integration-by-parts (IBP) reduction

The current bottleneck in perturbative calculation

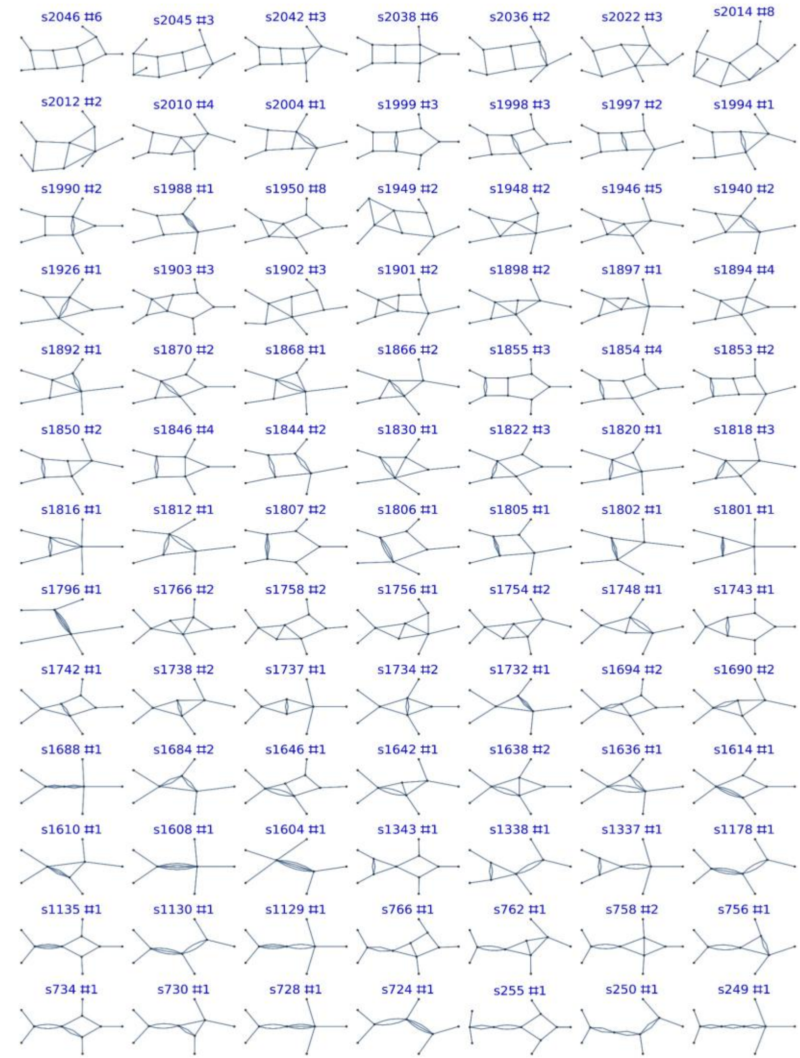
PBB



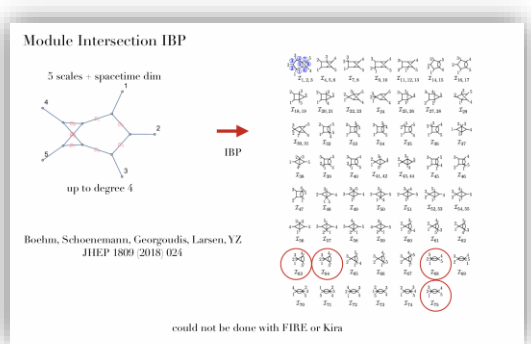
MIs #316

NeatIBP

~85,000 IBPs



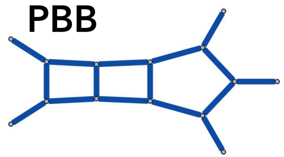
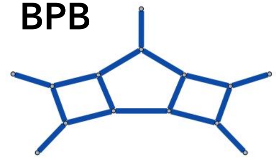
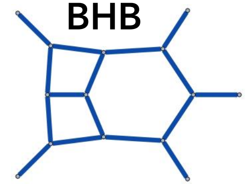
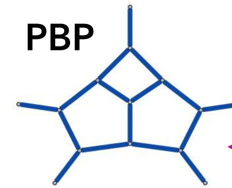
Picture created by Azurite 1.1.0



Similar Stories at Two Loop
Picture from Yang Zhang

Could not be done with FIRE or Kira

Algebraic Geometry Accelerating IBP

Family	PBB 	BPB 	BHB 	PBP 
Master Integrals	316	367	431	734
Five Point Integrals	120	164	169	342
# IBPs	~85,000	~183,000	Spanning Cuts	

Degree 4 and
Degree 3 + Dot 1

PRD Editor's Suggestion
[Liu, Matijašić, Miczajka, Xu, Xu, Zhang '24]

NeatIBP

Powered by 1.1.0.3

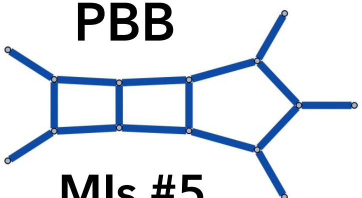
[Wu, Böhm, Ma, Xu, Zhang '23]
[Wu, Böhm, Ma, Usovitsch, Xu, Zhang '25]

Loop ++
Leg ++
Scale ++

UT Integrals: Leading Singularities Analysis

Some expressions..

PBB



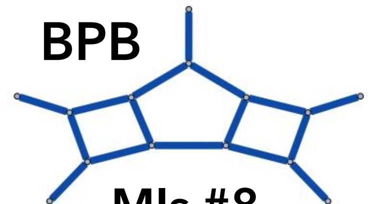
MIs #5

$$\mathcal{N}_1 = s_{12}s_{23}s_{45}^2(\ell_1 + k_5)^2$$

$$\mathcal{N}_3 = \frac{s_{45}^2}{\epsilon_5}G(\ell_1, k_1, k_2, k_3, k_4)$$

$$\mathcal{N}_4 = \frac{s_{45}^2}{\epsilon_5}G\begin{pmatrix}\ell_1, k_1, k_2, k_3, k_4 \\ \ell_2, k_1, k_2, k_3, k_4\end{pmatrix}$$

BPB



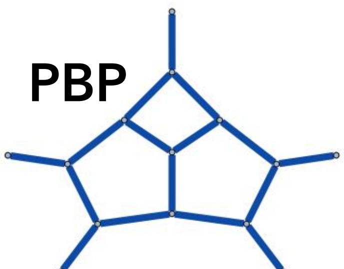
MIs #8

$$\mathcal{N}_1 = s_{12}s_{34}s_{45}^2(\ell_2 - k_1)^2$$

$$\mathcal{N}_5 = \frac{s_{45}s_{12}}{\epsilon_5}G(\ell_2, k_1, k_2, k_3, k_4)$$

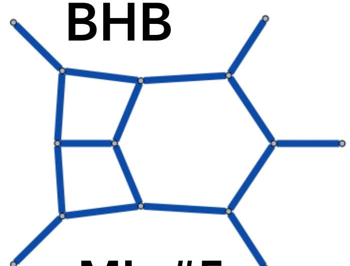
$$\mathcal{N}_6 = \frac{s_{45}s_{12}}{\epsilon_5}G\begin{pmatrix}\ell_1, k_1, k_2, k_3, k_4 \\ \ell_2, k_1, k_2, k_3, k_4\end{pmatrix}$$

PBP



MIs #19

BHB



MIs #5

$$\begin{aligned}\mathcal{N}_3 = &- 2\frac{s_{12}s_{15}s_{23}s_{34}s_{45}}{\epsilon_5}s_{15}\ell_2^2 \\ &+ \frac{s_{23}s_{34}\left(s_{12}s_{15} + s_{45}s_{15} - s_{12}s_{23} + s_{23}s_{34} - s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^4 \\ &+ \frac{s_{34}s_{45}\left(s_{12}s_{15} - s_{45}s_{15} + s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2 - k_1\right)^2 \\ &- \frac{s_{15}s_{45}\left(s_{12}s_{15} - s_{45}s_{15} - s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2 - k_1 - k_2\right)^2 \\ &+ \frac{s_{12}s_{15}\left(s_{12}s_{15} - s_{45}s_{15} - s_{12}s_{23} + s_{23}s_{34} + s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2 - k_1 - k_2 - k_3\right)^2 \\ &- \frac{s_{12}s_{23}\left(s_{12}s_{15} - s_{45}s_{15} - s_{12}s_{23} + s_{23}s_{34} - s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2 - k_1 - k_2 - k_3 - k_4\right)^2\end{aligned}$$

$$\mathcal{N}_6 = \frac{\varepsilon}{1 + 2\varepsilon}\frac{s_{12} - s_{45}}{\epsilon_5}G\begin{pmatrix}\ell_1, \ell_2, k_1, k_2, k_3, k_4 \\ \ell_1, \ell_2, k_1, k_2, k_3, k_4\end{pmatrix}$$

By leading singularities analysis in d-dimensional Baikov Rep.

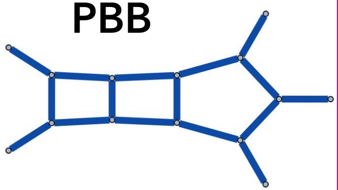
3L4P+1 off-shell leg from
[Di Vita, and Mastrolia, and Schubert, and Yundin '14]
[Henn, Lim, Bobadila'23]

DlogBasis [Henn, Mistlberger, Smirnov, Wasser '20]

Many thanks to my incredible collaborators!

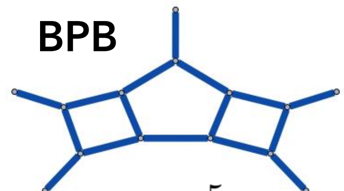
Three Loop Planar Pentagon Alphabet

PBB



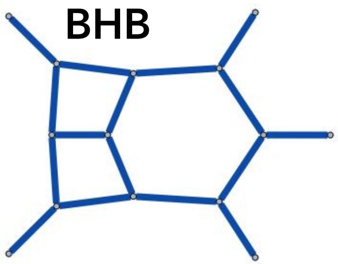
$\mathbb{A}_{\text{pl}}^{\text{2-loop}}$

BPB

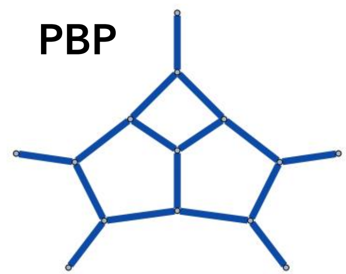


$\left\{\widehat{W}_i\right\}_{i=1}^{5}$

BHB



PBP

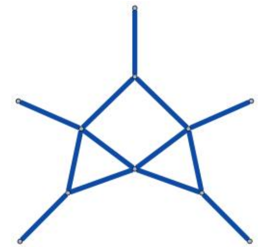


$\left\{\widetilde{W}_i\right\}_{i=16}^{20} \cup \left\{\widetilde{W}_i\right\}_{i=41}^{55} \cup \left\{\widetilde{W}_i\right\}_{i=76}^{80}$

Alphabet Letters: DNA of Feynman integrals

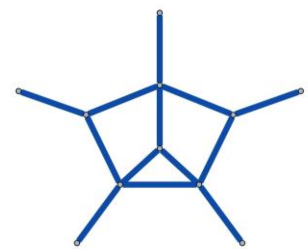
Conjectured in [Chicherin, Henn, Trnka, Zhang '24]

Originate from two topologies



$$\left\{\widehat{W}_i\right\}_{i=1}^{5}$$

$$\widehat{W}_1 = s_{23}s_{34} - s_{34}s_{45} + s_{45}s_{15}$$



New Square Root!

$$\left\{\sqrt{\Delta_4^{(i)}}\right\}_{i=1}^{5}$$

$$\begin{aligned}\Delta_4^{(1)} = & s_{15}^2 s_{12}^2 + s_{23}^2 s_{12}^2 - 2s_{15}s_{23}s_{12}^2 - 2s_{23}^2 s_{34}s_{12} \\ & + 2s_{15}s_{23}s_{34}s_{12} + 2s_{15}s_{34}s_{45}s_{12} + 2s_{23}s_{34}s_{45}s_{12} \\ & + s_{23}^2 s_{34}^2 + s_{34}^2 s_{45}^2 - 2s_{23}s_{34}^2 s_{45}\end{aligned}$$

56 letters

= 26 at two-loop.

+30 more at three-loop

$$\mathrm{d}\mathbf{J} = \varepsilon \sum_{i=1}^{56} \mathbf{a}_i \mathrm{dlog} W_i \cdot \mathbf{J}$$

$$\mathbf{J} \in \{\text{PBB,BPB,BHB,PBP}\}$$

Were found by `Baikovletter`

[Jiang, Liu, Xu, Yang '24]

Solve
Analytic Boundary Values
by imposing absence of spurious singularities to Differential Equation

$$\mathrm{Res}\left(\frac{\mathrm{d}\widetilde{\mathbf{A}}}{\mathrm{d}x}\right)\Bigg|_{y_i} \cdot \mathbf{I}^{(n)}(y_i) = 0 \quad \Rightarrow \quad \mathrm{I}_i^{(n)}(x_0) = c_i \mathrm{I}_1^{(n)}(x_0) + \sum_{j,k,l} d_{ijk} \int_\gamma \mathrm{d}\widetilde{A}_{jl} \cdot \mathrm{I}_l^{(n-1)}(x)$$

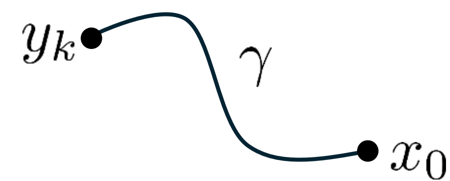
spurious singularities:

$$y_1 = \{-2, -1, -1, -1, -1\}$$

and its cyclic permutation

$$y_6 = \{1/4, -1, -1, -1, -1\}$$

Line Integration



Symmetric Point in Euclidean Region

$$x_0 = y_0 = \{-1, -1, -1, -1, -1\}$$

Solve the boundary values up to weight-six

Written in special values of GPL functions

[Gehrmann, Henn, Lo Presti '18] etc.

- **Fast numerical evaluation**

👉 See also: talk by Li-Hong Huang

more than 6,000 CPU hours by `pySecDec` for one top sector integral

➡ Tens of minutes for one family by DiffExp [Hidding '20]

- **Symbol solution up to weight-six**

$$[W_{i_1}, W_{i_2} \cdots, W_{i_n}] = \int W_{i_n}(t_{i_n}) \int \cdots \int W_{i_2}(t_2) \int W_{i_1}(t_1) \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_{i_n}$$

Three-loop topologies contribute only 2220 weight-six symbol $\ll 5 \times 56^5$

➡ Bootstrapping Wilson loop with Lagrangian insertion

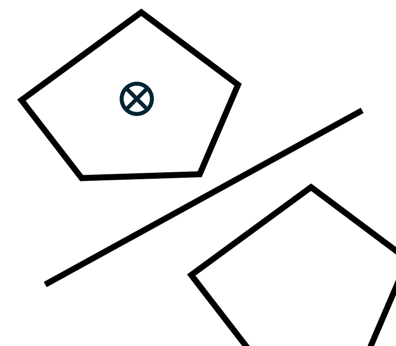
👉 See also: talk by Yang Zhang

- **Pentagon functions** in Euclidean region up to ...

Pentagonal Wilson Loop with Lagrangian Insertion at Three-Loop

$\mathcal{N}=4$ **Super Yang-Mills in planar limit**

$$F_5 = \frac{\langle W[x_1, x_2, x_3, x_4, x_5]\mathscr{L}(x_0)\rangle}{\langle W[x_1, x_2, x_3, x_4, x_5]\rangle} =$$



$$W[x_1, \cdots, x_n] = \mathrm{tr_F P} \exp\left(\mathrm{i} g_{\mathrm{YM}} t^a \oint A^a_\mu \mathrm{d}x^\mu\right)$$

n-cusp null Wilson loop with a Lagrangian insertion normalized by Wilson loop without Insertion

Theoretical Importance

Easy to compute

closely related to scattering amplitudes

- Null Wilson Loop ~ amplitudes in N=4 sYM
- Natural in AdS/CFT
- Duality with all-plus pure Yang-Mills amplitudes
- Positivity and geometry
- …

[Alday, Maldacena '07]

[Drummond, Korchemsky, Sokatchev '07]

[Arkani-Hamed, Henn, Trnka '21]

[Chicherin, Henn '22] [Chicherin, Henn '22]

[Chicherin, Henn, Trnka, Zhang '24] [Brown, Henn, Mazzucchelli, Trnka '24] and so on.

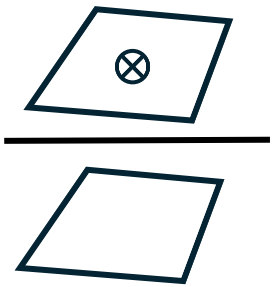
- Free of IR divergence
- Uniformly Transcendental of 2L
- Exact dual conformal symmetry
- All Loop Leading Singularity
- …

👉See also: talk by Dmitry Chicherin at Amplitudes 2022, Prague

➡ A good application of our Three-Loop Five-Point integrals

(pre) History: A Four-Point Example

Transcendental: Analytic functions
Rational: Leading singularities



$$\frac{\langle W[x_1,x_2,x_3,x_4]\mathscr{L}(x_5)\rangle}{\langle W[x_1,x_2,x_3,x_4]\rangle} = \frac{2}{\pi^2}\frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}F(x) + \mathcal{O}(\varepsilon)$$

$$p_i \equiv x_i - x_{i+1}$$

[Alday, Buchbinder, Tseytlin '11]
[Engelund, Roiban '11]

$$F(x) = -\frac{1}{4}g^2 + \left(\frac{1}{8}H_{0,0} + \frac{\pi^2}{16}\right)g^4 + \left(-\frac{\pi^2}{16}H_{0,0} - \frac{3}{16}H_{0,0,0,0} + \cdots\right)g^6 + \left(\frac{323\pi^4}{11520}H_{0,0} + \cdots\right)g^8 + \cdots$$

$$g^2 \equiv \frac{g_{\mathrm{YM}}^2 N_{\mathrm{c}}}{4\pi^2}$$

[Alday, Buchbinder, Tseytlin '11]
[Alday, Heslop, Sikorowski '12]
[Alday, Henn, Sikorowski '13]
[Henn, Korchemsky, Mistlberger '19]

Review of perturbative results

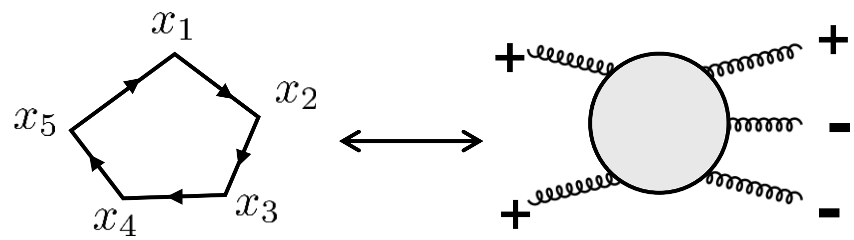
	Four-Point	Five-Point	Six-Point
Tree-Level	[Alday, Buchbinder, Tseytlin '11]	[Chicherin, Henn '22] n-point Tree-Level and One-Loop	
One-Loop	[Alday, Heslop, Sikorowski '12]		
Two-Loop	[Alday, Henn, Sikorowski '13]	[Chicherin, Henn '22]	[Carrôlo, Chicherin, Henn Yang, Zhang ' 25]
Three-Loop	[Henn, Korchemsky, Mistlberger ' 19]	$F_5^{(3)}$	

We are now Here

$$F_5 = \sum_{L\geq 1} (g^2)^{L+1} F_5^{(L)} = \lim_{x_0 \to \infty} \frac{\langle W[x_1, x_2, x_3, x_4, x_5] \mathscr{L}(x_0) \rangle}{\langle W[x_1, x_2, x_3, x_4, x_5] \rangle}$$

See also
[Alday, Buchbinder, Tseytlin '11] Four-Point @ Strong Coupling
[Arkani-Hamed, Henn, Trnka '21] Four-Point Negative geometries expansion

Integrand



planar N=4 sYM

from MHV amplitude Wilson Loop duality

[Alday, Maldacena '07]

[Drummond, Korchemsky, Sokatchev '07]

[Alday, Roiban '08] [Henn '09]

$$\log M_n + \mathcal{O}(1/N) \leftrightarrow \log\langle W_n\rangle$$

Loop expansion

MHV amplitudes integrand

Lagrangian insertion: $g^2\partial_{g^2}\langle W_n\rangle = -\mathrm{i}\int \mathrm{d}^d x_0 \langle W_n \mathscr{L}(x_0)\rangle$

$$-\mathrm{i}\int \mathrm{d}^d x_0 \frac{\langle W[x_1,\cdots,x_n]\mathscr{L}(x_0)\rangle}{\langle W[x_1,\cdots,x_n]\rangle} = g^2\partial_{g^2}\log\langle W_n\rangle \sim g^2\partial_{g^2}\log M_n$$

$W[x_1,\cdots,x_n] = \mathrm{tr}_{\mathrm{F}}\mathrm{P}\exp\left(\mathrm{i}g_{\mathrm{YM}}t^a\oint A^a_\mu \mathrm{d}x^\mu\right)$

L-loop Integrand

$$F_n^{(L)} \to \int \mathrm{d}^4 y_1 \cdots \int \mathrm{d}^4 y_L \Omega^{(L+1)}$$

With numerator which suppress all cusp divergence

[Chicherin, Henn '22]

[Chicherin, Henn, Trnka, Zhang '24]

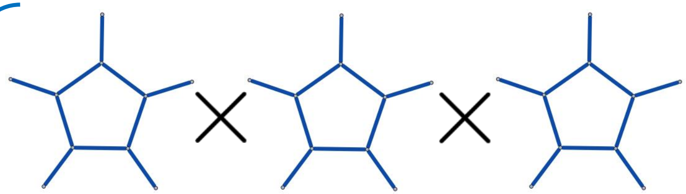
[Brown, Henn, Mazzucchelli, Trnka '24]

Direct Computation of $F_5^{(3)}$

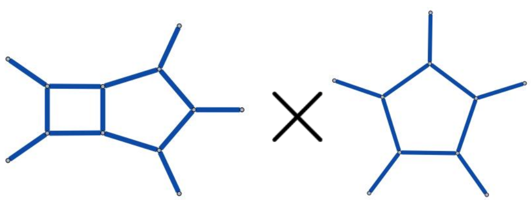
Step1: Separating Targets

$\mathscr{D}_1=(\ell_1-k_1)^2$	$\mathscr{D}_6=(\ell_2-k_1)^2$	$\mathscr{D}_{11}=(\ell_3-k_1)^2$
$\mathscr{D}_2=(\ell_1-k_1-k_2)^2$	$\mathscr{D}_7=(\ell_2-k_1-k_2)^2$	$\mathscr{D}_{12}=(\ell_3-k_1-k_2)^2$
$\mathscr{D}_3=(\ell_1-k_1-k_2-k_3)^2$	$\mathscr{D}_8=(\ell_2-k_1-k_2-k_3)^2$	$\mathscr{D}_{13}=(\ell_3-k_1-k_2-k_3)^2$
$\mathscr{D}_4=(\ell_1-k_5)^2$	$\mathscr{D}_9=(\ell_2-k_5)^2$	$\mathscr{D}_{14}=(\ell_3-k_5)^2$
$\mathscr{D}_5=\ell_1^2$	$\mathscr{D}_{10}=\ell_2^2$	$\mathscr{D}_{15}=\ell_3^2$
$\mathscr{D}_{16}=(\ell_1-\ell_2)^2$	$\mathscr{D}_{17}=(\ell_1-\ell_3)^2$	$\mathscr{D}_{18}=(\ell_2-\ell_3)^2$

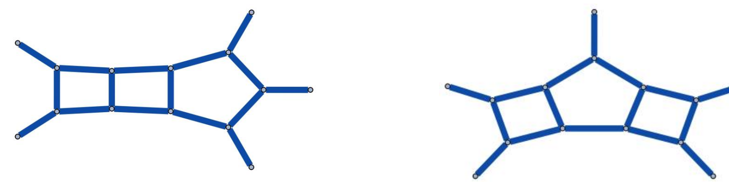
Loop momenta coupling



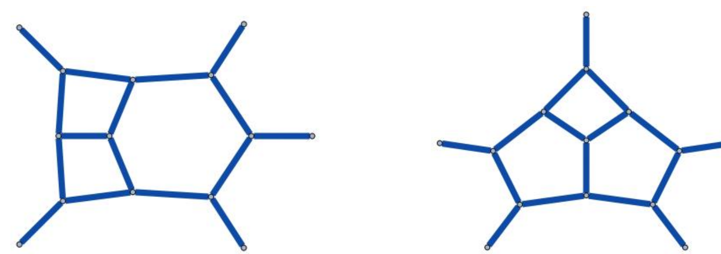
Free of $\mathscr{D}_{16}, \mathscr{D}_{17}, \mathscr{D}_{18}$



One of $\mathscr{D}_{16}, \mathscr{D}_{17}, \mathscr{D}_{18}$



Two of $\mathscr{D}_{16}, \mathscr{D}_{17}, \mathscr{D}_{18}$



All of $\mathscr{D}_{16}, \mathscr{D}_{17}, \mathscr{D}_{18}$

with all cyclic permutation S_5

Step 1: Separating targets in integrand $F_5^{(3)} \to \int \mathrm{d}^4 y_1 \int \mathrm{d}^4 y_2 \int \mathrm{d}^4 y_3 \Omega^{(4)}$

Step 2: Reduce all integrals into UT basis NeatIBP

Step 3: Assembling with numerator $\Rightarrow$ **Results** $F_5^{(3)} = r_0 g_0^{(3)} + \sum_{i=1}^{5} r_i g_i^{(3)}$

Rational Part: Leading Singularities

$$r_0 = \mathrm{tr}_-((p_1+p_2)(p_2+p_3)(p_3+p_4)(p_4+p_5))$$

$$r_i = \frac{s_{i+2,i+3}}{s_{i+1,i+4}} \mathrm{tr}_-(p_{i+1}p_{i+2}p_{i+3}p_{i+4}), \quad i = 1 \cdots 5$$

Transcendental Part: Weight-Six Symbol

$$g_i^{(3)} = \sum_{i_1,i_2,i_3,i_4,i_5,i_6} t_{i_1,i_2,i_3,i_4,i_5,i_6}[W_{i_1}, W_{i_2}, W_{i_3}, W_{i_4}, W_{i_5}, W_{i_6}]$$

$$[W_{i_1}, W_{i_2} \cdots, W_{i_n}] = \int W_{i_n}(t_{i_n}) \int \cdots \int W_{i_2}(t_2) \int W_{i_1}(t_1) \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_{i_n}$$

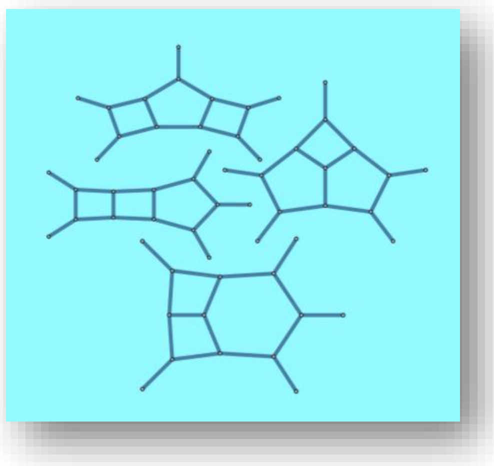
Cancellation between 3,107,425 terms
Results involves 1,373,140 terms…

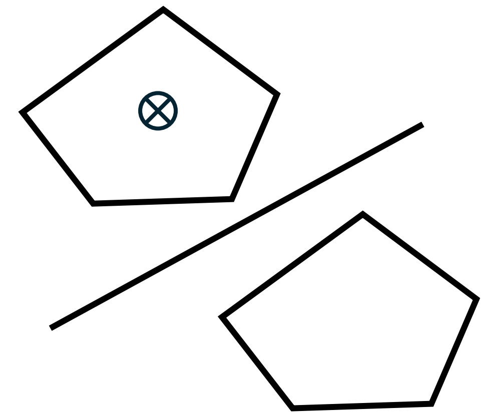
Dual to four-loop five-point all plus pure Yang-Mills amplitudes?

- ☑ IR finite: All weight 0 … 5 symbol vanished
- ☑ Free of new square roots
- ☑ Respect soft and collinear Limits
- ☑ …

[Wu, Böhm, Ma, Xu, Zhang '23]
[Wu, Böhm, Ma, Usovitsch, Xu, Zhang '25]
[Chicherin, Henn '22] [Chicherin, Henn '22]
[Chicherin, Henn, Trnka, Zhang '24]
[Brown, Henn, Mazzucchelli, Trnka '24]

Conclusion





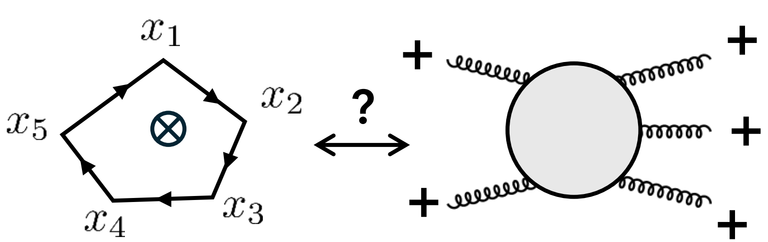
Into the Future

- Boundary value and pentagon function in physical region
- Function level Wilson Loop with Lagrangian insertion

  Positivity properties $(-1)^{(L+1)} f_5^{(L)}|_{\mathrm{Eucl}^+} > 0, L = 0, 1, 2$ [Chicherin, Henn '22]

  Complete monotonicity? $(-\partial_x)^n f(x) \geq 0, \forall n \geq 0$ [Henn, Raman '24]

- Pure Yang-Mills all plus amplitudes. Is it dual to $F_5^{(2)}$?



$\langle W[x_1, x_2, x_3, x_4, x_5] \mathscr{L}(x_0) \rangle$
$x_0 \to \infty$
N=4 sYM

$\mathcal{A}^{(3)}_{\mathrm{YM},5}(1^+, 2^+, 3^+, 4^+, 5^+)$
All plus pure Yang Mills

The expression (6.7) is a special case of the n-particle two-loop duality relation that we verified in [25]. However, eq. (6.8) is new, and it predicts the maximally transcendental part of the five-particle three-loop all-plus planar amplitude.

It would be extremely interesting to test our conjecture about the maximally transcendental part of the planar five-particle all-plus amplitude by an explicit three-loop Feynman graph calculation. Furthermore, a three-loop calculation of f_5 will provide a conjecture for the maximally transcendental part of the planar four-loop five-particle all-plus amplitude.

[Chicherin, Henn '22]

- Many more other applications…

Thank You!