Mellin Moment for Nucleon Structure: from Transversity to Threshold

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Outline

Base on following works:

- *Accessing nucleon transversity with one-point energy correlators.*
  M.S. Gao, Z.B. Kang, WL and D.Y. Shao, arXiv: 2509.15809.
- *N^4LL Threshold and TMD Joint Resummation in SIDIS.*
  S. Fang, WL and D.Y. Shao, (work in progress).
- *Probing fragmentation in the Mellin Space using Energy-Energy Correlators.*
  S. Fang, WL and D.Y. Shao, (work in progress).

Mellin Moment

- Mellin transformation of the function $f(z)$ yields

$$\tilde{f}(N) = \int_0^1 \mathrm{d}z\, z^{N-1}\, f(z)$$

with inverse Mellin transform

$$f(z) = \frac{1}{2\pi i} \int_C \mathrm{d}N\, z^{-N}\, \tilde{f}(N)$$

- Sum rule for unpolarized collinear fragmentation function (FF),

$$\sum_h \int_0^1 \mathrm{d}z\, z\, D_{q/h}(z,\mu) = 1,$$

- Mellin space representation of the threshold logarithm

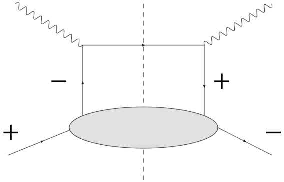
$$\int_0^1 \mathrm{d}z\, z^{N-1} \left(\frac{\log^m(1-z)}{1-z} \right)_+ = \frac{1}{m+1} \log^{m+1} N + \mathcal{O}(\log^{m-1} N).$$

Transversity Distribution

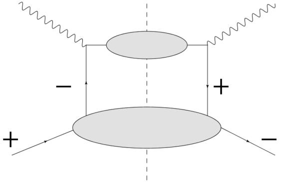
- Transversity distribution h_1^q is a fundamental, yet less known, parton distribution.
- Its chiral-odd nature prevents access via inclusive DIS.
- Nucleon tensor charge δq from h_1^q

$$\delta q \equiv \int_0^1 dx \left[h_1^q(x) - h_1^{\bar{q}}(x)\right]$$

  is essential for:
  - Nucleon spin structure,
  - Lattice QCD benchmarks,
    X. Gao, A. D. Hanlon, S. Mukherjee, P. Petreczky, Q. Shi, S. Syritsyn, and Y. Zhao, (2024).
  - BSM probes, e.g. neutron β-decay,
    J. D. Jackson, S. B. Treiman, and H. W. Wyld, (1957).



Transversity forbidden in Inclusive DIS



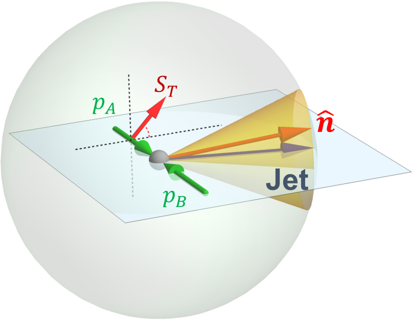
Semi-inclusive DIS (SIDIS)

One Point Energy Correlator (OPEC)

- We define the infrared-collinear (IRC) safe one-point energy correlator (OPEC) as

$$\Delta^q(z,\hat{\boldsymbol{n}}) = \sum_X \sum_{i\in J} \langle\Omega|\bar{\chi}_n \delta_{Q,\mathcal{P}_n} \delta^{(2)}(\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}}_i)|JX\rangle \frac{E_i}{E_J} \langle JX|\chi|\Omega\rangle .$$

- $\hat{\boldsymbol{n}}$ is the direction of the energy flow.
- The energy fraction of the jet carried by the hadron i is $z_i = \frac{E_i}{E_J}$.
- χ is the gauge-invarinat quark field.
- The state $|JX\rangle$ represents the final-state unobserved particles X and the jet J.



Factorization with OPEC in Inclusive Jet Production

- The energy-weighted unpolarized and transversely polarized structure functions Z_{UU} and Z_{UT} admit the following factorized forms:

$$\begin{aligned} Z_{UU} =& \frac{\alpha_s^2}{s} p_T^2 \theta_n \sum_{a,b,c} \int \frac{\mathrm{d}x_1}{x_1} f_{a/A}(x_1,\mu) \int \frac{\mathrm{d}x_2}{x_2} f_{b/B}(x_2,\mu) \\ & \times \mathcal{J}^c(\theta_n, Q) H^{\mathrm{U}}_{ab\to c}(\hat{s},\hat{t},\hat{u}) \delta(\hat{s}+\hat{t}+\hat{u}) \,, \\ Z_{UT} =& \frac{\alpha_s^2}{s} p_T^2 \theta_n \sum_{a,b,c} \int \frac{dx_1}{x_1} h_1^a(x_1,\mu) \int \frac{dx_2}{x_2} f_{b/B}(x_2,\mu) \\ & \times p_T \theta_n \mathcal{J}^c_{1,\perp}(\theta_n, Q) H^{\mathrm{Collins}}_{ab\to c}(\hat{s},\hat{t},\hat{u}) \delta(\hat{s}+\hat{t}+\hat{u}) \,. \end{aligned}$$

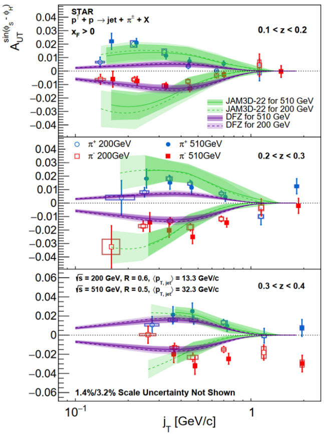
- Hard factors H for the partonic subprocess $ab \to c$ for UU and UT.
- The transveristy PDF h_1^a and unpolarized PDF f.
- $\mathcal{J}^c$ is the unpolarized energy weighted fragmenting jet function (FJF) and $\mathcal{J}^c_{1,\perp}$ is the transversely polarized energy weighted FJF.

$j_\perp$ or θ_n?

- Conventional non-weighted TMD analysis uses Collins function with hadron-in-jet transverse momentum $j_\perp$, related to the polar angle θ_n by
$$j_\perp \simeq p_T z_h \theta_n$$
in the collinear limit.

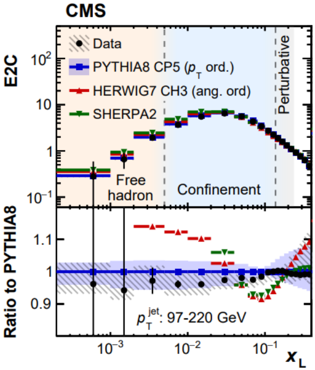
Why switch to θ_n?

- In one STAR configuration,
  - $\sqrt{s} = 510\,\text{GeV}$, $p_T = 32.3\,\text{GeV}$,
  - at lower cut $z = 0.1$,
  - $j_\perp$ can be measured in about (0.1, 1) GeV.
- $j_\perp$ only corresponds to θ_n about $(0.03, 0.3)$ rad.

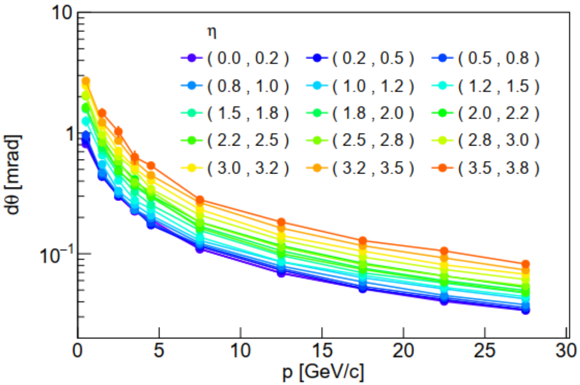


STAR collaboration arXiv.2507.16355

$j_\perp$ or θ_n?



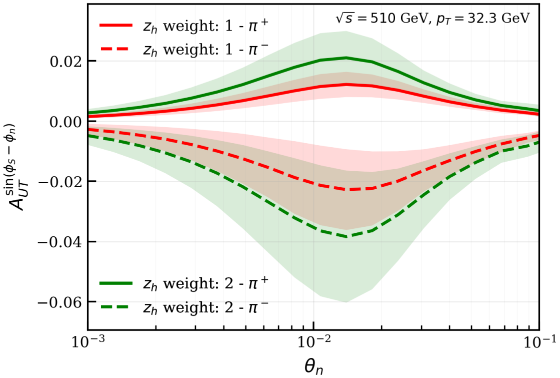
CMS (2024)



J. Arrington et al., EIC (2021)

- However, modern colliders could achieve angular resolution at 1 mrad!

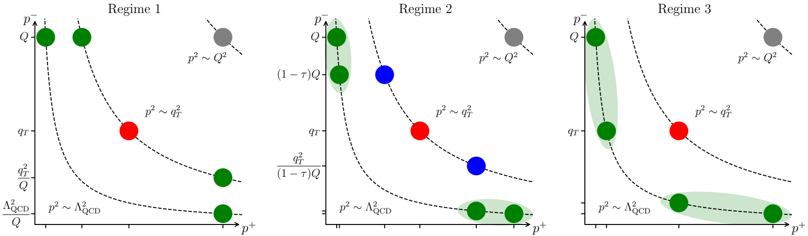
θ_n Distribution of the OPEC Collins Asymmetry for Different Weights



- Consistently enhancement of the Collins asymmetry for a higher power of the weight z_h (i.e. higher Mellin moment).

N^{4}LL threshold and TMD joint resummation

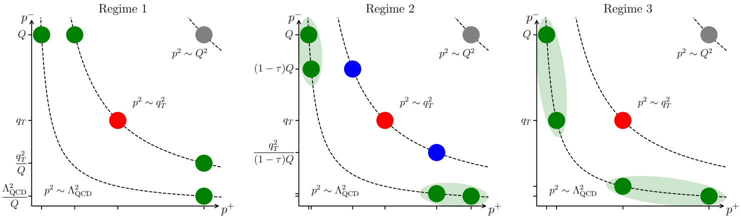
- With threshold parameter τ, threshold-TMD (TTMD) resummation involves the scales Q, $(1-\tau)Q$, q_T in following hierarchies:
  - **Regime 1:** $Q \sim (1-\tau)Q \gg q_T$, TMD factorization (TMD);
  - **Regime 2:** $Q \gg (1-\tau)Q \gg q_T$, TTMD factorization (TTMD);
  - **Regime 3:** $Q \gg (1-\tau)Q \sim q_T$, threshold factorization (T).



- Modes: collinear, collinear-soft, soft, hard.

- Consistency between these 3 regimes:
  - **Regime** $2 \to 1$,
    Rapidity: $\zeta_B^{\mathrm{TTMD}} = (1-\tau)^2 Q^2 \quad \longrightarrow \quad \zeta_B^{\mathrm{TMD}} = Q^2$
  - **Regime** $2 \to 3$,
    Rapidity: $\zeta_B^{\mathrm{TTMD}} = (1-\tau)^2 Q^2 \quad \longrightarrow \quad \zeta_B^{\mathrm{T}} = q_T^2$
- Proposed by G. Lustermans, W.J. Waalewijn and L. Zeune, Phys. Lett. B 762 (2016) 447, the matching of the three regimes reads

$$\frac{\mathrm{d}\sigma^{\mathrm{TMD}x}}{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}^2\boldsymbol{q}_T} = \frac{\mathrm{d}\sigma^{\mathrm{TTMD}}}{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}^2\boldsymbol{q}_T} + \left(\frac{\mathrm{d}\sigma^{\mathrm{TMD}}}{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}^2\boldsymbol{q}_T} - \frac{\mathrm{d}\sigma^{\mathrm{TTMD}}}{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}^2\boldsymbol{q}_T}\bigg|_{\zeta_B^{\mathrm{TTMD}} \to \zeta_B^{\mathrm{TMD}}} \right) + \left(\frac{\mathrm{d}\sigma^{\mathrm{T}}}{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}^2\boldsymbol{q}_T} - \frac{\mathrm{d}\sigma^{\mathrm{TTMD}}}{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}^2\boldsymbol{q}_T}\bigg|_{\zeta_B^{\mathrm{TTMD}} \to \zeta_B^{\mathrm{T}}} \right),$$



Regime 2 TTMD formalism

- Formulated in G. Lustermans, W. J. Waalewijn and L. Zeune (2016),
  Y. Li, D. Neill and H. X. Zhu, (2020),
  Z.B. Kang, K. Samanta, D.Y. Shao and Y.L. Zeng, (2023),
  the refactorization of the TMD function in the Mellin space at large N limit is given by

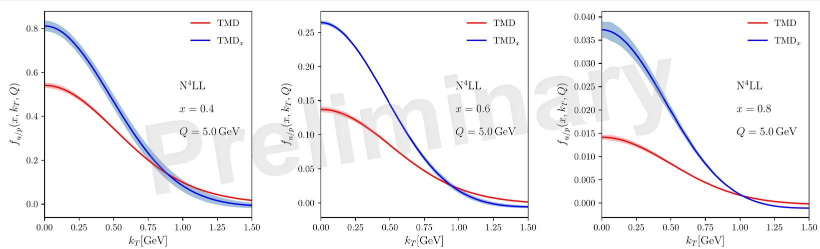
$$\tilde{f}_{i/h}^{\rm TMD}(N, b_T, \mu, \zeta) \xrightarrow{N\to\infty} \underbrace{\tilde{S}_c^{\rm unsub}(b_T, \mu, \zeta_N/\nu^2)\sqrt{S(b_T,\mu,\nu)}}_{\equiv\ \tilde{S}_c(b_T,\mu,\zeta_N)} \tilde{f}_{i/h}(N,\mu) + \mathcal{O}(b_T^2\Lambda_{\rm QCD}^2),$$

with anomalous dimension of the threshold PDF/FF

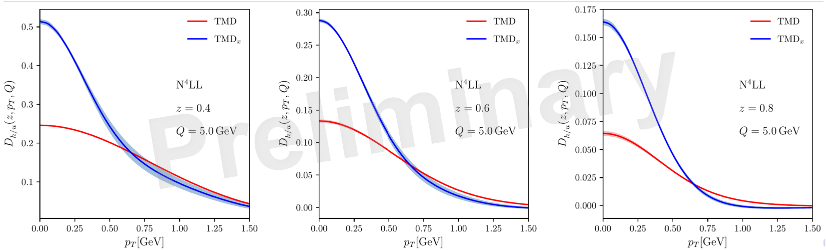
$$\gamma_\mu^{f_q}(N,\mu) = -C_F\gamma_{\rm cusp}(\alpha_s)\ln\bar{N}^2 + \gamma_{f_q}(\alpha_s)\,,$$

where $\bar{N} = Ne^{\gamma_E}$ and γ_E is the Euler Gamma constant.

- After the matching between the regimes, we derive the TMD PDF,



and the TMD FF



Probing fragmentation in the Mellin Space using EEC

- In e^+e^- annihilation, we study the energy–energy correlator (EEC) with an energy fraction weight $\left[\frac{E_i E_j}{(Q/2)^2}\right]^{N-1}$ in the back-to-back limit:

$$\mathrm{EEC}^{(N)}(\chi) = \int \mathrm{d}z_1\, \mathrm{d}z_2\, (z_1 z_2)^{N-1} \frac{\mathrm{d}\sigma}{\mathrm{d}^2 q_T} \delta\left(\chi - \frac{q_T^2}{Q^2}\right)$$
$$= \sigma_0 \frac{Q^2}{4} \sum_q e_q^2\, H(Q,\mu) \int db\, b\, J_0(bQ\sqrt{\chi})\, J_q^{(N)}(b,\mu)\, J_{\bar{q}}^{(N)}(b,\mu),$$

  with EEC jet function (with higher energy fraction weight power)

$$J_q^{(N)}(b,\mu) = \int \mathrm{d}z\; z^{N-1}\; \bar{D}_{q/h}(z,b,\mu)$$

- At leading order, the OPE matching coefficient is $\delta(1-z)$, hence

$$J_q^{(N)}(b,\mu) \simeq \tilde{d}_{q/h}(N,\mu) = \int \mathrm{d}z\; z^{N-1}\, d_{q/h}(z,\mu),$$

  i.e. the EEC jet function corresponds to the Mellin moments of the collinear FF.

Discussed in H. Chen, I. Moult, X. Zhang, and H. X. Zhu, (2020), and I. Moult, H.X. Zhu, (2025), jet shape variable is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}e} = \int d^4x\, e^{iQ\cdot x} \langle 0|\, \mathcal{O}(x)\delta(e-\hat{e})\mathcal{O}^\dagger(0)\, |0\rangle,$$

By expanding the δ function on its moment

$$\delta(e-\hat{e}) = \delta(e) + \hat{e}\delta^{(1)}(e) + \cdots + \frac{\hat{e}^n}{n!}\delta^{(n)}(e) + \ldots,$$

one can relate the nth moment of jet shape variable to the n-power energy weighted cross section

$$\int e^n \frac{\mathrm{d}\sigma}{\mathrm{d}e} = \int d^4x\, e^{iQ\cdot x} \langle 0|\, \mathcal{O}(x) e^n \mathcal{O}^\dagger(0)\, |0\rangle$$

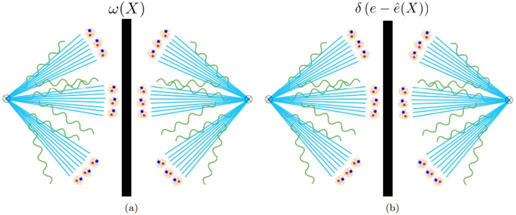
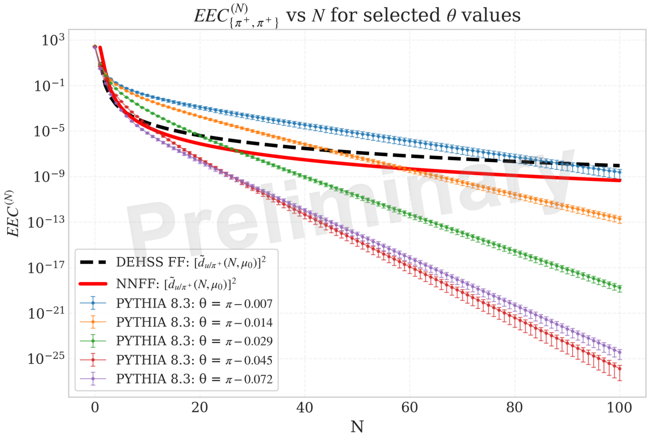


Figure from H. Chen, I. Moult, X. Zhang, and H. X. Zhu, (2020)

- Comparison between the squared fragmentation function $[\tilde{d}_{u/\pi^+}(N,\mu_0)]^2$ and PYTHIA 8.3 simulations with 40 million events:



- The small magnitude of the EEC values is a consequence of the strong weighting; it does not reflect an intrinsic measurement difficulty. The only challenge is achieving sufficient statistics on high z data.

- PYTHIA predicts an approximately exponential falloff of the EEC at large N, a feature likely driven by the weight factor z^{N-1}.
- The threshold resummation comes with anomalous dimension

$$\gamma_\mu^{f_q}(N,\mu) = -C_F\gamma_{\text{cusp}}(\alpha_s)\ln\bar{N}^2 + \gamma_{f_q}(\alpha_s)\,,$$

whose exponentiation leads to a power-law dependence in N.
- Typical parametrization of fragmentation function takes the form

$$d(z,\mu_0) = z^a(1-z)^b$$

with its Mellin transform

$$\tilde{d}(N,\mu_0) = \frac{\Gamma(1+b)\Gamma(a+N)}{\Gamma(1+a+b+N)},$$

which does not match the exponential behavior observed in PYTHIA.

- Some other parametrization candidates:

$$d(z, \mu_0) = e^{-az} z^b$$

and its Mellin moment is

$$\begin{aligned}\tilde{d}(N, \mu_0) &= a^{-b-N} \left(\Gamma(b+N) - \Gamma(b+N, a)\right) \\ &\simeq \frac{e^{-a}}{N} + \frac{(a-b)e^{-a}}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right),\end{aligned}$$

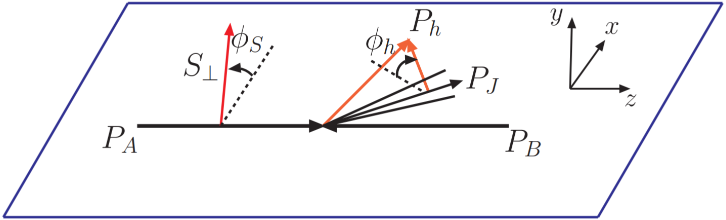
still some power-like function of N.

Conclusion

- We propose that using one-point energy correlator to probe the transversity distribution in transversely polarized $p^{\uparrow}p$ collision.
- Matching contributions from different regimes, we investigate N^{4}LL threshold and TMD joint resummation and its correction on the TMD PDF/FFs.
- We further suggest exploring the fragmentation process in Mellin space through the EEC with higher energy weight, as a new window into fragmentation process.

Back-ups...

Jet Production in Transversely Polarized $p^\uparrow p$ Collision



- Proposed by F. Yuan, Phys.Rev.Lett. 100, 032003 (2008), the differential cross section comes with an azimuthal modualtion,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathcal{PS}} = F_{UU} + \sin\left(\phi_s - \phi_h\right) F_{UT}.$$

- The reaction plane is spanned by the two incoming protons and the jet axis.
  - ϕ_s: the azimuthal angle of transverse polarization vector $S_\perp$, with respect the to reaction plane.
  - ϕ_h: the azimuthal angle of hadron h in jet, with respect the to reaction plane.

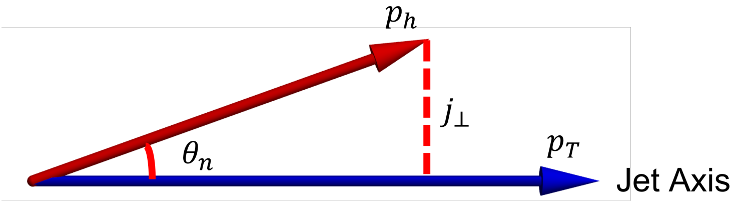
Decomposition of OPEC

- Ignoring irrelevant helicity and transverse components, we decompose OPEC into two parts:

$$\Delta^q = \frac{\not{\bar{n}}}{2}\mathcal{J}^q + \frac{\epsilon_T^{ij}\hat{\boldsymbol{n}}_{T,j}\theta_n p_J^-}{2}\frac{i\bar{n}_\mu\sigma^{i\mu}\gamma_5}{2}\mathcal{J}_{1,\perp}^q .$$

- $\mathcal{J}^q$ is the unpolarized OPEC fragmenting jet function (FJF).
- $\mathcal{J}_{1,\perp}^q$ is the transversely polarized OPEC FJF.
- θ_n is the energy flow polar angle.

Kinematics of OPEC Hadron in Jet



- The geometry yields

$$p_h^z = |\boldsymbol{p}_J| z_h = |\boldsymbol{p}_h| \cos\theta_n,$$

- In the collinear limit, we further see:

$$\theta_n \approx \sin\theta_n = \frac{j_\perp}{p_h^z} = \frac{j_\perp}{p_T z_h},$$

- Note in the centra rapidity, jet $|\boldsymbol{p}_J| = p_T$.

Factorization

- We consider the OPEC in inclusive jet production in $p^{\uparrow} + p \to J + X$

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\theta_n \mathrm{d}\phi_n \mathrm{d}\eta\, \mathrm{d}p_T} = \sum_{h\in J} \int_0^1 \mathrm{d}z_h \int \mathrm{d}^2\Omega_{\boldsymbol{h}}\, \delta\left(\phi_n - \phi_h\right) \times \delta\left(\theta_n - \theta_h\right)\, z_h \frac{\mathrm{d}\sigma}{\mathrm{d}z_h\, \mathrm{d}^2\Omega_h \mathrm{d}\eta\, \mathrm{d}p_T}$$

- z_h is the weight factor.
- Integral on z_h converts the final-state z_h distribution into a number!
- The jet is characterized by the rapidity η and transverse momentum p_T.
- The azimuthal dependent OPEC inclusive jet production can be described by

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\theta_n \mathrm{d}\phi_n \mathrm{d}\eta\, \mathrm{d}p_T} = Z_{UU} + \sin\left(\phi_s - \phi_n\right) Z_{UT}.$$

TMD Evolution and Operator Product Expansion

- In the perturbative limit, the OPEC FJFs can be matched onto their collinear counterparts using the operator product expansion (OPE) and TMD evolution

$$\begin{aligned}\mathcal{J}^q(\theta_n, Q) &= \sum_h \int_0^1 \mathrm{d}z_h z_h \int_0^\infty \frac{\mathrm{d}b\, b}{2\pi} J_0\left(p_T\theta_n b\right) \\ &\times \hat{C}^D_{i\leftarrow q} \otimes D_{h/i}(z_h, \mu_b) e^{-\frac{1}{2}S_{\mathrm{pert}}(Q,b)}, \\ p_T\theta_n \mathcal{J}^q_{1,\perp}(\theta_n, Q) &= \sum_h \int_0^1 \mathrm{d}z_h z_h \int_0^\infty \frac{\mathrm{d}b\, b^2}{2\pi} J_1\left(p_T\theta_n b\right) \\ &\times \delta\hat{C}^{\mathrm{collins}}_{i\leftarrow q} \otimes \hat{H}^{T(1)}_{1\,h/i}(z_h, \mu_b) e^{-\frac{1}{2}S_{\mathrm{pert}}(Q,b)},\end{aligned}$$

- The usual convolution $\otimes$ is defined as

$$\hat{C}_{i\leftarrow q} \otimes F_{h/i} = \sum_i \int_{z_h}^1 \frac{\mathrm{d}z'_h}{z'_h} F_{h/i}(z'_h, \mu_b) \hat{C}_{i\leftarrow q}\left(\frac{z_h}{z'_h}, \mu_b, R\right).$$

Phenomenology Study on Energy-Weighted $\pi^\pm$ Production in Jet

- The OPEC Collins azimuthal asymmetry is defined as

$$A_{UT}^{\sin(\phi_s-\phi_n)} = \frac{Z_{UT}}{Z_{UU}}.$$

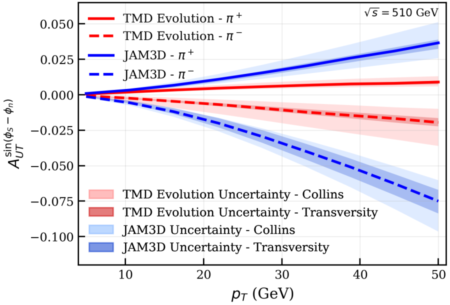
- We consider pion production in jets for two kinematic settings extensively studied by the STAR Collaboration:
    - (a) $\sqrt{s} = 510\,\text{GeV}$, $p_T = 32.3\,\text{GeV}$;
    - (b) $\sqrt{s} = 200\,\text{GeV}$, $p_T = 13.3\,\text{GeV}$.

Evaluation Frameworks

We present evaluation in two approaches:

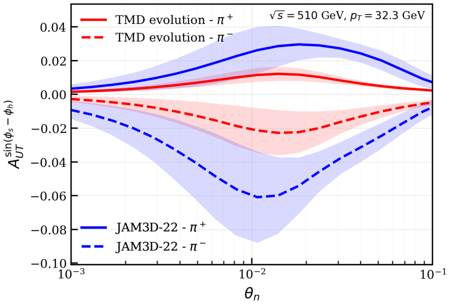
- A full TMD evolution framework with parametrization of the transversity PDF, Collins functions, and non-perturbative Sudakov factor from Kang, Prokudin, Sun, Yuan (2016) ,
  - The extraction is done for SIDIS and e^+e^- annhilation.
  - The application on $p^{\uparrow}p$ collision can be a complementary test of the TMD universality.
- The JAM3D-22 global QCD analysis JAM (2022), where the TMD functions are modeled by Gaussian ansätze, while the corresponding collinear components evolve with only DGLAP.

p_T Distribution of the OPEC Collins Asymmetry

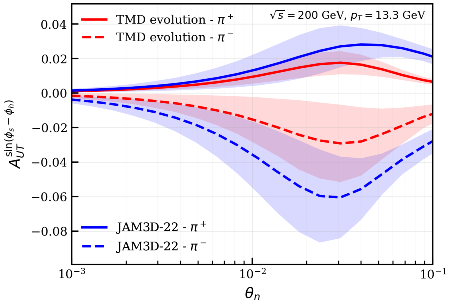


- JAM3D analysis gives overall larger Collins asymmetry than the TMD evolution framework.
- The error propagated from the parametrization of the Collins function is larger than from the transversity PDF.
- θ_n is integrated on (0, 0.1).

θ_n Distribution of the OPEC Collins Asymmetry



(a) $\sqrt{s} = 510$ GeV, $p_T = 32.3$ GeV



(b) $\sqrt{s} = 200$ GeV, $p_T = 13.3$ GeV

- Error is estimated from transversity PDF and Collins function combined.
- Two scenarios differ on the positions of their peak in θ_n.
- Current parametrizations do not yet allow us to resolve the effects of TMD evolution.

In Z.-B. Kang, K. Samanta, D.Y. Shao and Y.-L. Zeng, JHEP 11 (2023) 220, they consider the RG consistency

$$\begin{aligned}
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}H^{\mathrm{SIDIS}}\left(Q,\mu\right) &= \Gamma^{h}(\alpha_s)H^{\mathrm{SIDIS}}\left(Q,\mu\right),\\
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\tilde{S}_c\left(b_T,\mu,\zeta_{N,f}\right) &= \Gamma^{\tilde{S}_c}(\alpha_s,\zeta_{N,f})\tilde{S}_c\left(b_T,\mu,\zeta_{N,f}\right),\\
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\tilde{D}_q\left(N,\mu\right) &= \Gamma^{\tilde{D}_q}(\alpha_s)\tilde{D}_q\left(N,\mu\right),\\
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\tilde{f}_q\left(N,\mu\right) &= \Gamma^{\tilde{f}_q}(\alpha_s)\tilde{f}_q\left(N,\mu\right)
\end{aligned}$$

with corresponding anomalous dimensions

$$\begin{aligned}
\Gamma^{h}(\alpha_s) &= 2\,\Gamma_{\mathsf{cusp}}(\alpha_s)\ln\frac{Q^2}{\mu^2}+2\gamma_V(\alpha_s)\\
\Gamma^{\tilde{S}_c}(\alpha_s,\zeta_{N,f}) &= -\Gamma_{\mathsf{cusp}}(\alpha_s)\ln\frac{\zeta_{N,f}}{\mu^2}+\gamma_{\tilde{S}_c}(\alpha_s)\\
\Gamma^{\tilde{D}_q}(\alpha_s) &= -2\,\Gamma_{\mathsf{cusp}}(\alpha_s)\ln\bar{N}+2\gamma_{\tilde{D}_q}(\alpha_s)\\
\Gamma^{\tilde{f}_q}(\alpha_s) &= -2\,\Gamma_{\mathsf{cusp}}(\alpha_s)\ln\bar{N}+2\gamma_{\tilde{f}_q}(\alpha_s)
\end{aligned}$$

N^4LL Ingredients

Order	H, S, S_T, S_c	non-cusp γ	Γ_{cusp}	β
LL	LO		1-loop	1-loop
NLL	LO	1-loop	2-loop	2-loop
NNLL	NLO	2-loop	3-loop	3-loop
N^3LL	NNLO	3-loop	4-loop	4-loop
N^4LL	N^3LO	4-loop	5-loop	5-loop