Tame multi-leg Feynman integrals beyond one loop

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Based on works: L.H. Huang, R.J. Huang, Y.Q. Ma, arXiv: 2412.21053

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Outline

Era of precision physics

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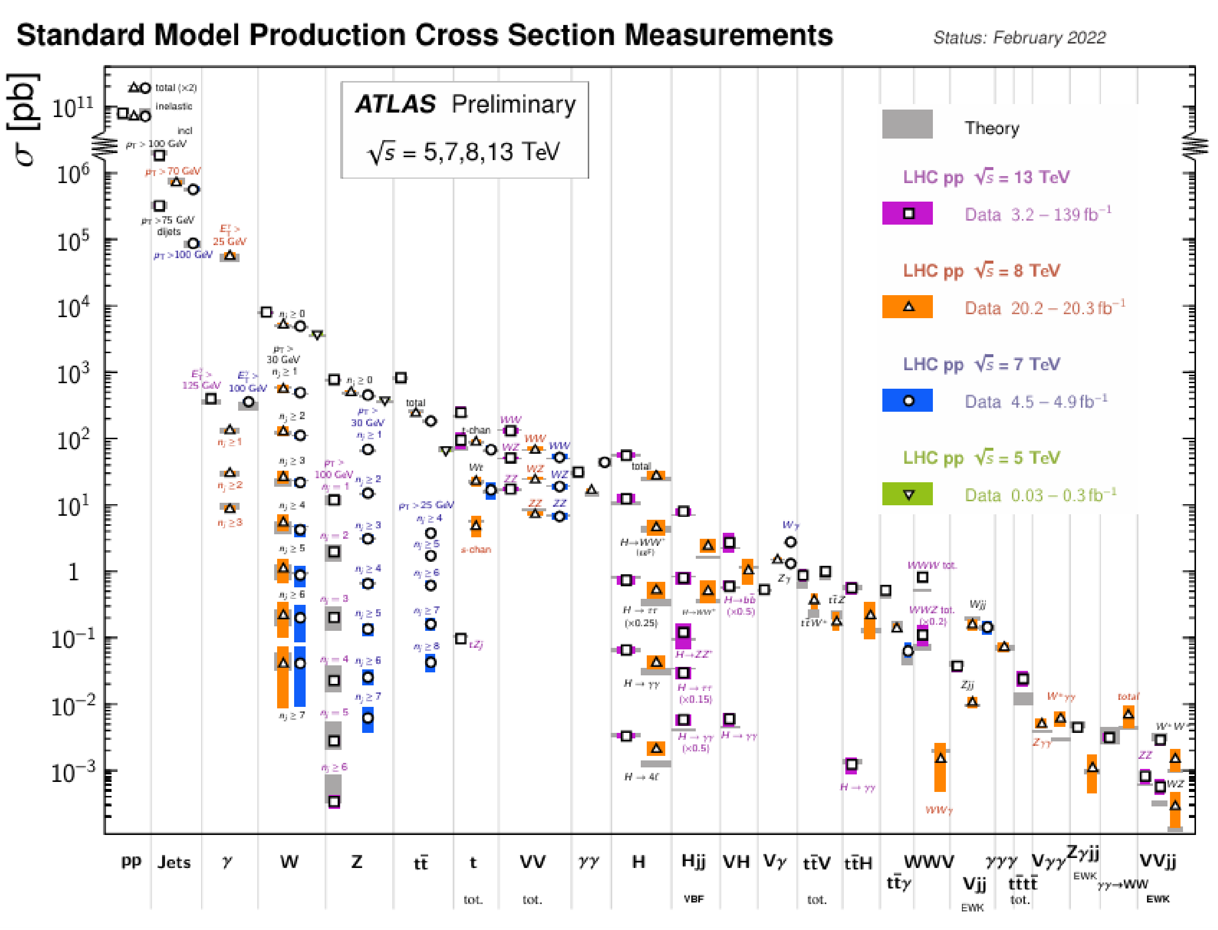
01 Run III of LHC (22-25)

02 HL-LHC: Many observables probed at precent level precision, 30 times more data

High-precision data



ATL-PHYS-PUB-2022-009

QCD corrections requirement ideally:

- Most processes: N2LO
- Many processes: N3LO
- Some processes: N4LO

Current status of perturbative calculation

Amplitude generation → **Feynman integrals calculation** → **Phase Space integration**

IR Subtraction
Reverse Unitarity

A key obstacle in high-order computation

Mainstream method:

- Integration-by-parts: Reduce loop integrals to a set of basis (Master Integrals)

$$\int \prod_{i=1}^{L} \mathrm{d}^D \ell_i \frac{\partial}{\partial \ell_j^\mu} [v^\mu \mathcal{I}(\vec{v})] = 0$$

- Compute MIs

Legs / Order	2→1	2→2	2→3	2→4	2→5	2→6
NLO	★★★	★★★	★★★	★★★	★★★	★★★
N2LO	★★★	★ ★	★	●	?	?
N3LO	★ ★	★	●			
N4LO	★	?				
N5LO	?					

Planar two-loop six-particle: Johannes Henn, Yang Zhang, et al. PRL2025
See also Yang Zhang's talk

Planar Three–Loop Five–Point: Yong-Qun Xu, Yang Zhang, et al. PRD2025 See also Yong-Qun Xu's talk

Integration-by-parts reduction: the bottleneck!

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$$\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu}jk}(D,\vec{s})I_{\vec{\nu}'}(D,\vec{s})=0$$

The state-of-the-art IBP method: very challenging

4-loop DGLAP kernel cannot be obtained

H+t $t$bar production: exact two-loop contribution is missing

Improvements for IBPs

Syzygy equations

Gluza , Kajda, Kosower, PRD2011
Böhm , Georgoudis, Larsen, Schulze, Zhang, PRD2018
NeatIBP : Wu, et al. CPC2024

Block-triangular form

Liu, YQM, PRD2019
Guan, Liu, YQM, CPC2020
Blade : Guan, Liu YQM, Wu, CPC2025

Ways to bypass IBP

Intersection number

Asymptotic expansion

Iterative reduction

…

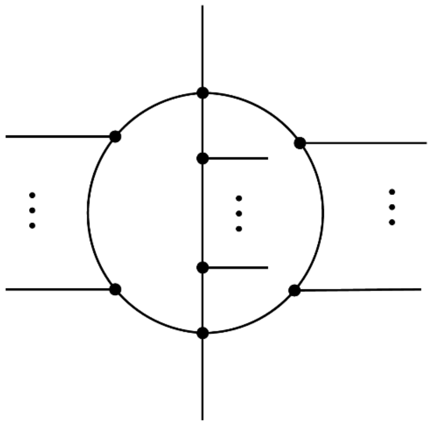
Too many integration variables

Outline

A new representation

Feynman parametrization

$$J(\vec{\nu}; D) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\Gamma(\nu_1)\cdots\Gamma(\nu_K)} \int \prod_{i=1}^{K} (x_i^{\nu_i - 1} \mathrm{d}x_i) \delta(1 - X) \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}}$$



U： homogeneous polynomial of degree L， independent of kinematics variables

F： homogeneous polynomial of degree L+1

Would it be simpler if we fix U unintegrated?

yes

$$J(\vec{\nu}; D) = \int [\mathrm{d}\mathbf{X}] \prod_{a=1}^{B} X_a^{\nu_a - 1} \mathcal{U}^{\nu - \frac{(L+1)D}{2}} \boxed{I_{\vec{\nu}}^{\frac{LD}{2}}(\vec{X}).}$$

Fixed-Branch Integrals (FBIs)

2 loop： B-1=2

3 loop： B-1=5

To fix U, we fix X_a, which is the summation of Feynman parameter for the a-th branch.

Outline

Definition & Reduction relations for FBIs

$$I_{\vec{\nu}}^{\Delta}(\mathbf{X}) = \frac{(-1)^{\nu}\Gamma(\nu-\Delta)}{\prod_{\alpha=1}^{N}\Gamma(\nu_\alpha)} \int [\mathrm{d}\mathbf{y}] \frac{\prod_{\alpha=1}^{N} y_\alpha^{\nu_\alpha-1}}{\left(\frac{1}{2}\mathbf{y}^T\cdot R\cdot\mathbf{y} - \mathrm{i}0^+\right)^{\nu-\Delta}},$$

$$S = \begin{pmatrix} 0_{3\times 3} & \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & R \end{pmatrix}.$$

Generalized Gram matrix if $B = 3$ and $(n_1, n_2, n_3) = (2, 1, 1)$,

Introduce z_0 = 0 or 1 depending on generalized Gram determinant det$S$ = 0 or not

Also introduce other parameters determined by $S \cdot (C_1, \cdots, C_B, z_1, \cdots, z_N)^T = (z_0, \cdots, z_0, 0, \cdots, 0)^T$.

Recursion relation

$$S\cdot(t_1, \cdots, t_B, \nu_1 I_{\vec{\nu}+\vec{e}_1}^{\Delta}, \cdots, \nu_N I_{\vec{\nu}+\vec{e}_N}^{\Delta})^T = (-I_{\vec{\nu}}^{\Delta-1}, \cdots, -I_{\vec{\nu}}^{\Delta-1}, I_{\vec{\nu}-\vec{e}_1}^{\Delta-1}, \cdots, I_{\vec{\nu}-\vec{e}_N}^{\Delta-1})^T$$

Dimension-shift relation

$$C I_{\vec{\nu}}^{\Delta-1} = (2\Delta - \nu - B)\, z_0 I_{\vec{\nu}}^{\Delta} + \sum_{\alpha=1}^{N} z_\alpha I_{\vec{\nu}-\vec{e}_\alpha}^{\Delta-1}$$

Reduce to corner FBIs,
at most one MIs for each sector.

Compute master integrals of FBIs

Using auxiliary mass flow method

$$\mathcal{I}^{\Delta}_{\vec{\nu}}(\eta)=\frac{(-1)^{\nu}\Gamma(\nu-\Delta)}{\prod_{\alpha=1}^{N}\Gamma(\nu_\alpha)}\int[\mathrm{d}\mathbf{y}]\,\frac{\prod_{\alpha=1}^{N}y_\alpha^{\nu_\alpha-1}}{\left(\frac{1}{2}\mathbf{y}^T\!\cdot\! R\!\cdot\!\mathbf{y}+\eta\right)^{\nu-\Delta}},$$

We have differential equations for η

$$(2z_0\eta-C)\frac{\mathrm{d}}{\mathrm{d}\eta}\mathcal{I}^{\Delta}_{\vec{\nu}}(\eta)=(2\Delta-\nu-B)\,z_0\mathcal{I}^{\Delta}_{\vec{\nu}}(\eta)+\sum_{\alpha=1}^{N}z_\alpha\mathcal{I}^{\Delta-1}_{\vec{\nu}-\vec{e}_\alpha}(\eta).$$

Solve it with $\eta\to\infty$ as boundary condition

Using Dimension-Change Transformation to obtain desired FBIs

$$I^{\Delta+\delta}_{\vec{\nu}}=\frac{1}{\Gamma(\delta)}\int_{-\mathrm{i}0^+}^{-\mathrm{i}\infty}\mathrm{d}\eta\,\eta^{\delta-1}\mathcal{I}^{\Delta}_{\vec{\nu}}(\eta),$$

Rui-Jun Huang, Jian, YQM, Mu, Wu,PRD2025

Canonical form

Chen, Feng, Zhang, JHEP2025

$$d\mathcal{I}_{2m}=c_{2m\to 2m}\mathcal{I}_{2m}+\sum_i c_{2m\to 2m-1;i}\mathcal{I}^{(i)}_{2m-1}+\sum_{i\neq j}c_{2m\to 2m-2;ij}\mathcal{I}^{(ij)}_{2m-2}$$

$$d\mathcal{I}_{2m+1}=c_{2m+1\to 2m+1}\mathcal{I}_{2m+1}+\sum_i c_{2m+1\to 2m;i}\mathcal{I}^{(i)}_{2m}+\sum_{i\neq j}c_{2m+1\to 2m-1;ij}\mathcal{I}^{(ij)}_{2m-1}$$

analytical

Numerical

FBIs are as simple as one-loop FIs, thus a solved problem

Numerical method: Avoid IBP reduction

Integral with known integrand

$$\mathcal{M} = \int [\mathrm{d}\mathbf{X}]\, \hat{\mathcal{M}}\,(\mathbf{X})$$

Sector Decomposition +
Gaussian-Kronrod Integration Rule

Contour deformation to avoid divergences

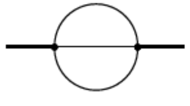
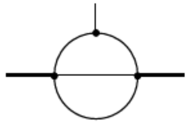
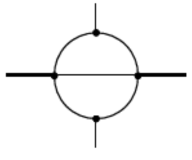
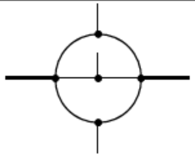
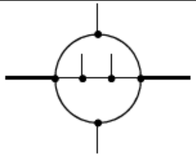
$$\tilde{X}_b = X_b + \mathrm{i}X_b(1 - X_b)G_b(\mathbf{X}),$$

$$G_b(\mathbf{X}) = \kappa \sum_j \lambda k_j \frac{\partial_{X_b} P_j}{P_j^2 + (\partial_{X_b} P_j)^2} \exp\left(-\frac{P_j^2}{\lambda^2 k_j^2}\right).$$

Adjust parameters

Subtract out boundary divergences

$$\tilde{\mathcal{M}}_{\mathrm{sub}} = \sum_{\mu(\epsilon),n_\mu} X_1^{\mu(\epsilon)} \log(X_1)^{n_\mu} \sum_{i=0}^{\lfloor -\mu(0)\rfloor} X_1^i \tilde{M}_1^{\mu,n_\mu,i}(X_2) + \sum_{\rho(\epsilon),n_\rho} X_2^{\rho(\epsilon)} \log(X_2)^{n_\rho} \sum_{j=0}^{\lfloor -\rho(0)\rfloor} X_2^j \tilde{M}_2^{\rho,n_\rho,j}(X_1)$$
$$- \sum_{\mu(\epsilon),n_\mu,\rho(\epsilon),n_\rho} X_1^{\mu(\epsilon)} \log(X_1)^{n_\mu} X_2^{\rho(\epsilon)} \log(X_2)^{n_\rho} \times \sum_{i=0,j=0}^{\lfloor -\mu(0)\rfloor,\lfloor -\rho(0)\rfloor} X_1^i X_2^j \tilde{M}_3^{\mu,n_\mu,i,\rho,n_\rho,j}.$$

graphs	# points	DCT time/points (ms)
	726	0.14
	726	0.19
	726	0.34
	726	0.76
	12826	2.91

Method	Precision	Time(h)
pySecDec	3	3
	5	108
AMFlow	20	1.7+2.7
FBI	6	0.01+

Analytical method: Reduction

Combining with series expansion

See also YQM's talk

e.g.: 1/D series, $1/\eta$ series, integral with variable limits Δ: Δ series $\int_{a(1-\Delta)}^{a+(1-a)\Delta} \mathrm{d}\,x$

2 loops: simplifying $\sim O(n^{10})$ to $\sim O(n^3)$

$n \sim O(100)$ is the terms to be obtained

Power: the number of integration parameters

More efficient

Combining with intersection theory

In collaboration with LLYang and ZWWang

Only 3 layers at two loop order

More efficient

Summary and outlook

Reveal a deep structure of Fis: simple integrand followed by integration over a few variables

2 for two-loop, and 5 for three-loop: independent of number of external legs!

The integrand (FBIs) can be fully solved, similar to one-loop FIs

All previous FIs techniques can be applied to resolve the remained integration

Either fully numerically, or via reduction + computing MIs

We look forward to potential collaborations to further explore the power of this new representation.

Thank you!