Kaluza-Klein AdS Virasoro-Shapiro and Veneziano Amplitudes

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BW, Di Wu , Ellis Ye Yuan, Phys. Rev. Lett.

BW, arXiv:2508.14968

w.i.p. w/ Xiaoyu Fa, Ellis Ye Yuan

Summary

1. Introduction
2. Virasoro-Shapiro amplitude in AdS
3. Kaluza-Klein AdS Viraoro-Shapiro amplitude
4. A general picture of KK AdS stringy amplitude
5. Discussion

Introduction

Motivation: the duality

AdS/CFT Correspondence

CFT on AdS boundary ⇔ String description in AdS

$$
\begin{tikzcd}
\mathcal{N}=4 \quad SYM \arrow[r, Leftrightarrow] \arrow[d] & Type\ IIB\ strings\ on\ AdS_5\times S^5 \arrow[d] \\
Correlator \arrow[r, Leftrightarrow] \arrow[d] & \boxed{stringy\ world-sheet\ in\ curved\ space?} \arrow[d, "?"] \\
\sum_{\mathcal{O}}\sum_{m=0}^{\infty}\frac{C_{\mathcal{O}}\mathcal{Q}_{\tau+4,m}(u)}{s-\tau-2m} \arrow[r, Leftrightarrow] & \int d^2z|z|^{-2S-2}|1-z|^{-2T-2}G(z,\bar{z};S,T)
\end{tikzcd}
$$

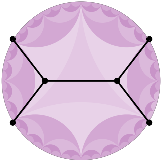
Motivation: a bootstrap problem

Keypoint is the contraints from this duality

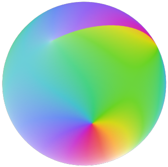
$$\boxed{\text{Blocks} \quad \Leftrightarrow \quad \text{world-sheet ansatz on AdS} \times \text{S} \quad \Rightarrow \quad \text{unique } \mathcal{A}^{(1)}}$$

- $\mathcal{A}^{(1)}$ is the first curvature correction of AdS VS amplitude
- Focus on $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle$ case first
- Generalize it to *arbitrary* Kaluza-Klein case
- Prediction OPE data and anomalous dimensions

Motivation: a bootstrap problem



$$\Longleftrightarrow \quad \int d^2 z \quad \text{(single valued)}$$



- OPE from the CFT side and OPE from the world-sheet side

$$\boxed{\sum_{\mathcal{O}} \sum_{m=0}^{\infty} \frac{f_{\mathcal{O}} \mathcal{Q}_{\tau+4,m}(u)}{s-\tau-2m}} \dashrightarrow \boxed{\int d^2 z |z|^{-2S-2} |1-z|^{-2T-2} \times ?}$$

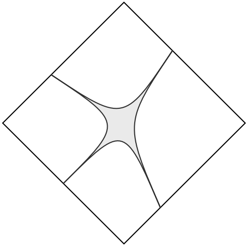
$$\boxed{\text{Input: flat space data}} \longrightarrow \boxed{\sum_{\mathcal{O}} \sum_{m=0}^{\infty} \frac{f_{\mathcal{O}} \mathcal{Q}_{\tau+4,m}(u)}{s-\tau-2m}}$$

$$\boxed{\text{Input: flat space data}} \dashrightarrow \boxed{\text{World-sheet correlator (perturbatively)}}$$

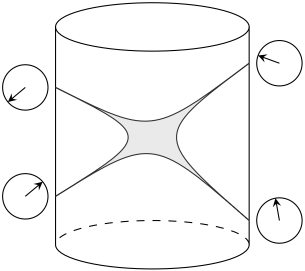
$$\boxed{\int d^2 z |z|^{-2S-2} |1-z|^{-2T-2} \times ?} \longrightarrow \boxed{\text{World-sheet correlator (perturbatively)}}$$

Figure: Solving AdS Virasoro-Shapiro as a bootstrap problem.

Motivation: a UV completion



Virasoro-Shapiro
(all l_{s})



AdS Virasoro-Shapiro
(all l_{s} and R_{AdS})

VS + curvature correction
(all l_{s}, all KK)

Virasoro-Shapiro amplitude in AdS

Borel transform and large twist

▷ OPE from the CFT side and OPE from the world-sheet side

$$
\begin{tikzcd}
\boxed{\displaystyle\sum_{\mathcal{O}}\sum_{m=0}^{\infty}\frac{f_{\mathcal{O}}\mathcal{Q}_{\tau+4,m}(u)}{s-\tau-2m}} \arrow[r, dashed] \arrow[d] & \boxed{\displaystyle\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}\times ?} \arrow[d] \\
\boxed{\displaystyle\lambda^{-\frac{k}{2}}\sum_{i=1}^{3k+1}\frac{R_i^{(k)}(T,\delta)}{(S-\delta)^i}} \arrow[r, dashed, red] & \boxed{\displaystyle\lambda^{-\frac{k}{2}}\sum_{i=1}^{3k+1}\frac{X_i^{(k)}(T,\delta)}{(S-\delta)^i}}
\end{tikzcd}
$$

Figure: Pole structures from Borel transform and the world-sheet integral.

Hints from the Virasoro-Shapiro amplitude

The integrand: single-valued multiple polylogarithms

The single-valued multiple polylogarithms(SVMPL) satisfy the differential relations [Brown '20],

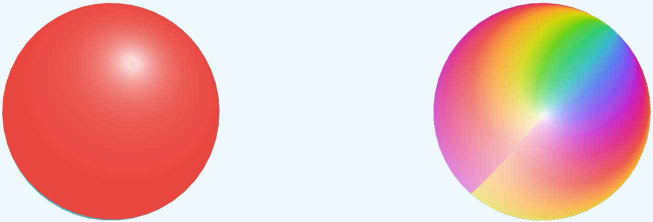
$$\partial_z \mathcal{L}_{0w}(z) = \frac{1}{z}\mathcal{L}_w(z) \quad \partial_z \mathcal{L}_{1w}(z) = \frac{1}{z-1}\mathcal{L}_w(z)$$

such that $\mathcal{L}_{\emptyset}(z) = 1$ and $\mathcal{L}_{0^n}(z) = \frac{\log^n |z|^2}{n!}$. The length of w is called transcendental weight $|\omega|$. Here we consider w is filled with 0 or 1.

- $\mathcal{L}_w$: finite $(2^{|\omega|})$ for every fixed transcendental weight $|\omega|$
- $\mathcal{L}^{+/-}$: combinations of pure transcendental SVMPLs $\mathcal{L}$
- $\mathcal{L}^{+/-}$: symmetric/antisymmetric under $z \leftrightarrow 1-z$

Hints from the Virasoro-Shapiro amplitude

single-valued multiple polylogarithms V.S. multiple polylogarithms



(a) A single-valued MPL: $\log|z|^3$

(b) A MPL: $\log(z)^3$

Figure: Phase (argument) visualization

Basic structure of AdS VS amplitude

- The AdS VS amplitude admits a curvature expansion in $\lambda^{-\frac{1}{2}}$ around flat space [Alday, Hansen, '23],

$$A(S,T) = \sum_{k=0}^{\infty} \lambda^{-\frac{k}{2}} A^{(k)}(S,T)\,.$$

- The corresponding integrand reads

$$G^{(k)}(z,\bar{z}) = \sum_i \frac{h_i^{(2k)}(S,T)}{U^2} \mathcal{L}_i^{(3k)}(z,\bar{z})$$

  - $h_i^{(2k)}(S,T)$: polynomials in S,T with homogeneous degree $2k$
  - $\mathcal{L}_i^{(3k)}(z,\bar{z})$: single-valued functions of transcendental weight $3k$
  - Finite parameter space for every k

Example: $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_p \mathcal{O}_p \rangle$

$$
\begin{tikzcd}
\boxed{\sum_{\mathcal{O}} \sum_{m=0}^{\infty} \frac{f_{\mathcal{O}} \mathcal{Q}_{\tau+4,m}(u)}{s-\tau-2m}} \arrow[r, dashed] \arrow[d] & \boxed{\int d^2 z \times KN \times G^{(1)}(z,\bar{z};x_j)} \arrow[d] \\
\boxed{\lambda^{-\frac{1}{2}} \sum_{i=1}^{4} \frac{R_i^{(1)}(T,\delta)}{(S-\delta)^i}} \arrow[r, dashed, red] & \boxed{\lambda^{-\frac{1}{2}} \sum_{i=1}^{4} \frac{X_i^{(1)}(T,\delta;x_j)}{(S-\delta)^i}}
\end{tikzcd}
$$

Figure: Pole structures from Borel transform and the world-sheet integral.

- Infinite equations from a finite parameter space
- Applied to $A^{(2)}$ of $\langle 2222 \rangle$ [Alday, Hansen, '23] and $A^{(1)}$ of $\langle 22pp \rangle$ [Fardlli, Hansen, Silva, '23] $\rightarrow$ unique solution.

Kaluza-Klein AdS Viraoro-Shapiro amplitude

The AdS×S formalism

It is convenient to introduce the AdS×S Mellin formalism [Aprile, Vieira '20]

- Discrete Mellin transform on S^5

$$M(y_{ij}) = \sum_{n_{ij}} \mathcal{M}(n_{ij}) \times \prod_{i<j} \frac{y_{ij}^{n_{ij}}}{\Gamma(n_{ij}+1)}. \tag{1}$$

- y_{ij} is distance on the sphere S^5
- n_{ij} is discrete Mellin variable on the sphere S^5

Borel transform and AdS×S formalism

$$\begin{tikzcd}
\boxed{\mathcal{M}(s,t;n_{ij})} \arrow[r, dashed] \arrow[d] & \boxed{M(s,t;y_{ij})} \arrow[d] \\
\boxed{\mathcal{A}(S,T;n_{ij})} & \boxed{A(S,T;y_{ij})} \arrow[l, dashed]
\end{tikzcd}$$

Figure: Borel transformations and discrete Mellin transformations.

- Black solid line: Borel transformations [Aprile, Vieira, ‘20]
- Red dashed line: Discrete Mellin transformations [Alday, Hansen, Silva, ‘23]

Borel transform and AdS×S formalism

$$\begin{tikzcd} \boxed{\mathcal{M}(s,t;n_{ij})} \arrow[r, dashed] \arrow[d] & \boxed{M(s,t;y_{ij})} \arrow[d] \\ \boxed{\mathcal{A}(S,T;n_{ij})} & \boxed{A(S,T;y_{ij})} \arrow[l, dashed] \end{tikzcd}$$

Figure: Borel transformations and discrete Mellin transformations.

Uplifting specific KK configurations in to general AdS×S space.

[**BW**, Wu, Yuan, ‘25]

Results

The final result only depends on five linearly independent SVMPLs ($\mathcal{L}_i$),

$$\mathcal{G}^{(1)}(z,\bar{z};n_s,n_t) = (\Sigma - 2) \times \sum_{i=1}^{5} \mathcal{L}_i \mathcal{R}_i \,, \tag{2}$$

where $\Sigma = \frac{p_1+p_2+p_3+p_4}{2}$, and the rational prefactors $\mathcal{R}_i(S,T)$ are explicitly

$$\begin{aligned} \mathcal{R}_1 &= \frac{(n_s-n_t)(S-T)+2U}{12(S+T)} \,, \quad \mathcal{R}_2 = \frac{1}{2}\frac{1}{\Sigma-2} \,, \\ \mathcal{R}_3 &= \frac{n_s\,S+n_t\,T+n_u\,U}{12(S+T)} \,, \ \mathcal{R}_4 = \frac{n_s-n_t}{12} \,, \\ \mathcal{R}_5 &= \frac{n_sS-n_tT+(n_u+2)(T-S)}{12(S+T)} \,. \end{aligned} \tag{3}$$

A general picture of KK AdS stringy amplitude

Borel transform and AdS×S formalism

$$
\begin{tikzcd}
\mathcal{M}(s,t;n_{ij}) \arrow[r, dashed] \arrow[d] & M(s,t;y_{ij}) \arrow[d] \\
\mathcal{A}(S,T;n_{ij}) & A(S,T;y_{ij}) \arrow[l, dashed]
\end{tikzcd}
$$

Figure: Borel transformations and discrete Mellin transformations.

- ▷ $\mathrm{AdS}_5\times \mathrm{S}^5$ [**BW**, Wu, Yuan, '25]
- ▷ $\mathrm{AdS}_3\times \mathrm{S}^3$ [Jiang, Zhong, 2508.06039]

Borel transform and AdS×S formalism

$$
\begin{tikzcd}
\boxed{\mathcal{M}(s,t;n_{ij})} \arrow[r, dashed] \arrow[d] & \boxed{M(s,t;y_{ij})} \arrow[d] \\
\boxed{\mathcal{A}(S,T;n_{ij})} & \boxed{A(S,T;y_{ij})} \arrow[l, dashed]
\end{tikzcd}
$$

Figure: Borel transformations and discrete Mellin transformations.

- ▷ Test AdS Veneziano amplitude $\mathcal{A}^{(1)}$ [**BW**, arXiv:2508.14968]

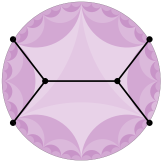
A general picture of KK AdS stringy amplitude

$$\begin{tikzcd}
\boxed{\displaystyle\sum_{\mathcal{O}}\sum_{m=0}^{\infty}\frac{f_{\mathcal{O}}\mathcal{Q}_{\tau+2,m}(u)}{s-\tau-2m}} \arrow[r, dashed] \arrow[d] & \boxed{\displaystyle\int dz z^{-S-1}(1-z)^{-T-1}\mathcal{G}^{(1)}(z;x_j)} \arrow[d] \\
\boxed{\displaystyle\mathcal{A}^{(1)}=\sum_{i=1}^{4}\frac{\mathcal{R}_i^{(1)}(T,\delta)}{(S-\delta)^i}} \arrow[r, dashed, red] & \boxed{\displaystyle\lambda^{-\frac{1}{2}}\sum_{i=1}^{4}\frac{\mathcal{X}_i^{(1)}(T,\delta;x_j)}{(S-\delta)^i}}
\end{tikzcd}$$

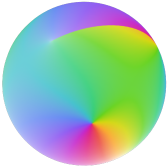
Figure: Pole structures from Borel transform and the world-sheet integral.

- $\mathcal{A}^{(\frac{1}{2})}=0 \ \Rightarrow$ gives $\langle\tau_1\rangle_{\delta,\ell}=-\ell$ and $\langle f_1\rangle_{\delta,\ell}=((\Sigma-2)\ell+(p_1-p_2)^2+(p_3-p_4)^2-1)/(2\sqrt{\delta})\langle f_0\rangle_{\delta,\ell}$
- Matching the known $\langle f_0\rangle_{\delta,\ell}$ *completely fixes* the ansatz, which further determines $\langle f_0\tau_2\rangle_{\delta,\ell}$ and $\langle f_2\rangle_{\delta,\ell}$!

A general picture of KK AdS stringy amplitude



$$\Longleftrightarrow \quad \int d^2 z \quad \text{(single valued)}$$



A general picture

- ✓ Works for Veneziano $\mathcal{A}^{(1)}$ in $\mathrm{AdS}_5\times\mathrm{S}^3$ [**BW**, 2508.14968].
- ✓ Virasoro-Shapiro $\mathcal{A}^{(1)}$ in $\mathrm{AdS}_5\times\mathrm{S}^5$ [**BW**, Fa, Yuan, w.i.p.].
- ✓ Virasoro-Shapiro $\mathcal{A}^{(1)}$ in $\mathrm{AdS}_3\times\mathrm{S}^3$ [Jiang, Zhong, 2508.06039].
- ☐ Unknown for Virasoro-Shapiro $\mathcal{A}^{(1)}$ in $\mathrm{AdS}_4\times\mathrm{CP}^3$ [Chester, Hansen, Zhong, '24].

Discussion

Discussion

- ▷ This work establishes a universal framework for computing curvature corrections to the AdS Virasoro-Shapiro amplitude with arbitrary KK modes using AdS$\times$S formulism.
- ▷ World-sheet ansatz easily constrained by CFT data for massive stringy operators at lower order, yielding predictions at higher order.
- ▷ Interesting to generalize our method to higher-order corrections ($\mathcal{A}^{(2)}$ in progress) as well as other AdS background.
- ▷ Consider the Regge behavior and other special limit.
- ▷ Consider the KLT/Monodromy relation in AdS space.

Thanks