Soft Photon, Gluon and Graviton Theorems in (A)dS from Conformal Invariance

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Background

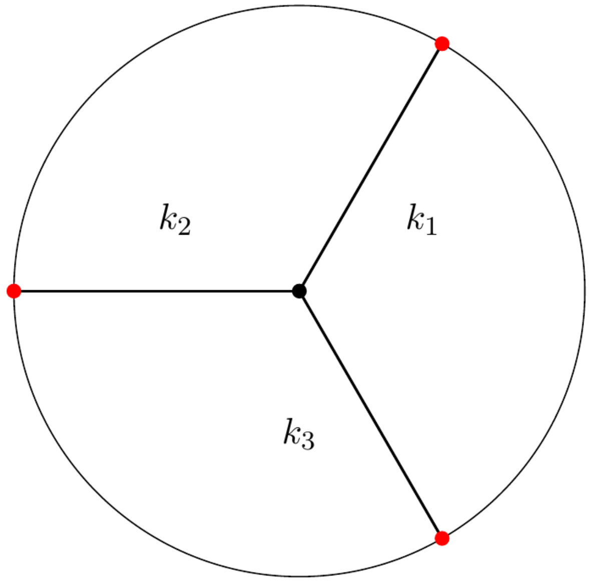
Flat-space
$$\lim_{q\to 0}\mathcal{M}_{n+1}\left(q;k_1,\ldots,k_n\right)=\left[S^{(0)}+S^{(1)}+S^{(2)}\right]\mathcal{M}_n\left(k_1,\ldots,k_n\right)+\mathcal{O}\left(q^2\right)$$

where

$$S_n^{(0)}\equiv\sum_{i=1}^{n}\frac{\epsilon_{\mu\nu}\,k_i^{\mu}k_i^{\nu}}{k_i\cdot q}\,,$$

$$S_n^{(1)}\equiv -i\sum_{i=1}^{n}\frac{\epsilon_{\mu\nu}\,k_i^{\mu}q_\rho J_i^{\nu\rho}}{k_i\cdot q}\,,$$

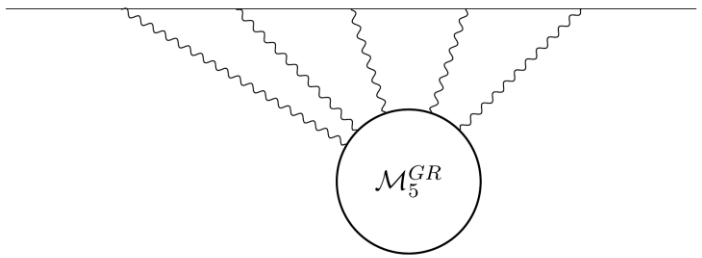
$$S_n^{(2)}\equiv -\frac{1}{2}\sum_{i=1}^{n}\frac{\epsilon_{\mu\nu}\,q_\rho J_i^{\mu\rho}\,q_\sigma J_i^{\nu\sigma}}{k_i\cdot q}\,.$$



$$\lim_{\boldsymbol{q}\to 0}\langle T(\boldsymbol{q})T(\boldsymbol{k}_1)\ldots T(\boldsymbol{k}_n)\rangle \stackrel{?}{=} S^{(0)}\langle T(\boldsymbol{k}_1)\ldots T(\boldsymbol{k}_n)\rangle+S^{(1)}\langle T(\boldsymbol{k}_1)\ldots T(\boldsymbol{k}_n)\rangle+\mathcal{O}(q^2)$$

[K. Hinterbichler, L. Hui, J. Khoury '13]
[P. McFadden '14]
[J. Maldacena '02]

Main Result



Soft graviton theorem:

$$\boxed{\lim_{\boldsymbol{q}\to 0}\langle T(\boldsymbol{q})T(\boldsymbol{k}_1)\dots T(\boldsymbol{k}_n)\rangle = S^{(0)}\langle T(\boldsymbol{k}_1)\dots T(\boldsymbol{k}_n)\rangle + S^{(1)}\langle T(\boldsymbol{k}_1)\dots T(\boldsymbol{k}_n)\rangle + \mathcal{O}(q^2)}$$

$$\boxed{\begin{aligned} S^{(0)} &:= -\frac{1}{2}\sum_{a=1}^{n}\varepsilon_{\mu\nu}k_a^{\mu}\partial_{k_a}^{\nu}, \\ S^{(1)} &:= \frac{1}{4}\sum_{a=1}^{n}\varepsilon_{\mu\nu}q_{\rho}(k_a^{\rho}\partial_{k_a}^{\mu}\partial_{k_a}^{\nu} - 2k_a^{\mu}\partial_{k_a}^{\nu}\partial_{k_a}^{\rho} - 2\partial_{k_a}^{\mu}S_a^{\nu\rho}) \end{aligned}}$$

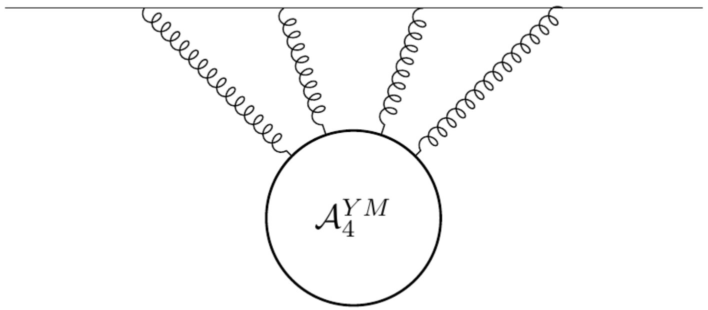
$$\mathcal{S}^{(0)}\Big|_{\epsilon^{\mu\nu}\to\eta^{\mu\nu}} = \sum_{a=1}^{n}D_a,$$

$$\mathcal{S}^{(1)}\Big|_{\epsilon^{\mu\nu}\to\eta^{\mu\nu}} = \sum_{a=1}^{n}q_{\rho}K_a^{\rho},$$

Checked with existing full 4 point result $\lim_{\vec{q}\to 0}\langle TTTT\rangle$

[J. Bonifacio, H. Goodhew, A. Joyce, E. Pajer, D. Stefanyszyn '22],
[C. Armstrong, H. Goodhew, A. Lipstein, J. Mei '23]

Main Result



Soft photon theorem:

$$\lim_{\boldsymbol{q}\to 0}\langle J(\boldsymbol{q})J(\boldsymbol{k}_1)\dots J(\boldsymbol{k}_n)\rangle = -\tfrac{1}{2}\sum_h e_h \varepsilon_q \cdot \partial_{k_h}\langle J(\boldsymbol{k}_1)\dots J(\boldsymbol{k}_n)\rangle + \mathcal{O}(q)$$

Soft gluon theorem:

$$\lim_{\boldsymbol{q}\to 0}\langle J(\boldsymbol{q})J(\boldsymbol{k}_1)\dots J(\boldsymbol{k}_n)\rangle = \tfrac{1}{2}(\varepsilon_q \cdot \partial_{k_n} - \varepsilon_q \cdot \partial_{k_1})\langle J(\boldsymbol{k}_1)\dots J(\boldsymbol{k}_n)\rangle + \mathcal{O}(q)$$

Checked with 5-point datas $\langle JJJJJ\rangle$ $\langle JOJOJ\rangle$.

Method

Solving Special Conformal Ward Identity (SCWI)

For correlators, the SCWI acts as

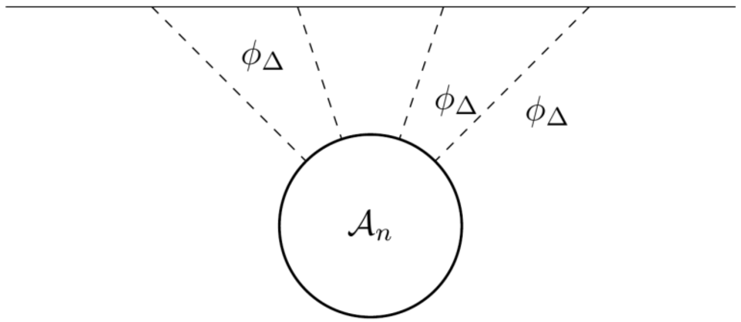
$$\sum_{a=1}^{n} K_a^{\mu} \langle \mathcal{O}_1 \ldots \mathcal{O}_a \ldots \mathcal{O}_n \rangle \stackrel{?}{=} 0,$$

where K is the special conformal generator. The right-hand side (RHS) is generally nonzero, up to local terms.

This is due to the nature of the Ward identities, i.e.,

$$\partial^{\mu} \langle J_{\mu}(\vec{x}_1) O(\vec{x}_2) \cdots O(\vec{x}_n) \rangle = -\sum_{a=2}^{n} \delta(\vec{x}_1 - \vec{x}_a) \langle O(\vec{x}_2) \cdots \delta O(\vec{x}_a) \cdots O(\vec{x}_n) \rangle$$

Example: 4-point GR



[Bonifacio, Goodhew, Joyce, Pajer, Stefanyszyn '22]

$$\lim_{\vec{k}_4 \to 0} \langle\langle TTTT \rangle\rangle^{(s)} = -\frac{1}{2}\varepsilon_4^{ij} k_{3i} \partial_{k_{3j}} \langle\langle TTT \rangle\rangle$$

$$-\frac{\varepsilon_4 \cdot k_3 \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4}{2k_3^2} \left(\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot k_1 + \text{cyclic}\right)\left(k_1^2 - k_2^2\right)\left(\frac{1}{z_0} + \frac{k_1 k_2}{k_{12}} - k_1 - k_2\right)$$

boundary local terms

Flat space

Symmetry $\xrightarrow{\text{LSZ}}$ Soft theorem

$= 0$

$$\partial^i \langle J_i(\vec{x}_1) O(\vec{x}_2) \cdots O(\vec{x}_n) \rangle = -\cancel{\textstyle\sum_{a=2}^n \delta(\vec{x}_1 - \vec{x}_a) \langle O(\vec{x}_2) \cdots \delta O(\vec{x}_a) \cdots O(\vec{x}_n) \rangle}$$

Local terms pops up?

Way Out: LSZ

[C. Cheung, J. Parra-Martinez, A. Sivaramakrishnan '22, J. Mei '23]

Imposing the EoM on the bulk-to-boundary (Btb) propagator

$$\mathcal{D}_k^{\Delta}\phi_{\Delta}(k,z) = \left[z^2k^2 - z^2\partial_z^2 - (1-d)z\partial_z + \Delta(\Delta-d)\right]\phi_{\Delta}(k,z) = 0$$

By generalised LSZ, we can work with $\mathcal{A}_n$ within correlators

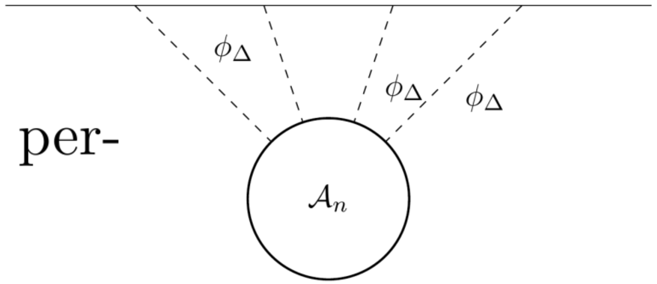
$$\left\langle \mathcal{O}\left(\vec{k}_1\right)\ldots\mathcal{O}\left(\vec{k}_n\right)\right\rangle = \int \tfrac{dz}{z^{d+1}}\mathcal{A}_n\left(z,\partial_z,\vec{k}_a,\vec{\varepsilon}_a\right)\prod_{a=1}^{n}\phi_{\Delta}\left(k_a,z\right),$$

Mellin transformation $\qquad \phi_{\Delta}(k,z) = \int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\, z^{-2s+d/2}\,\phi_{\Delta}(s,k).$

$$\langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle = \int[ds_i]\int\tfrac{dz}{z^{d+1}}\mathcal{A}_n(z\boldsymbol{k},s)\prod_{i=1}^{n}\phi(s_i,k_i)z^{-2s_i+d/2},$$

with $\int[ds_i] = \prod_{i=1}^{n}\int_{-i\infty}^{+i\infty}\frac{ds_i}{2\pi i}$.

Here, for convenience in performing IBP with respect to ∂_z, we perform a Mellin transform of the bulk-to-boundary propagator:



Exact Conformal Invariance

Punchline

By generalised LSZ, we can work with $\mathcal{A}_n$ within correlators

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \int [ds_i] \int \tfrac{dz}{z^{d+1}} \mathcal{A}_n(z\boldsymbol{k}, s) \prod_{i=1}^{n} \phi(s_i, k_i) z^{-2s_i + d/2},$$

[J. Mei,Y. Mo '24]

Working with Mellin-momentum amplitudes

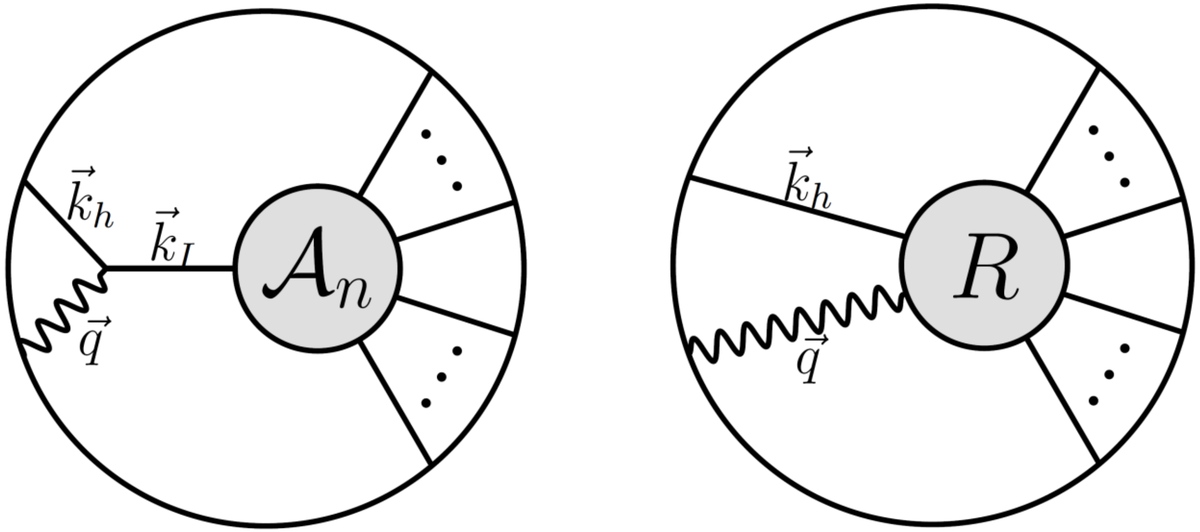
$$\sum_{a=1}^{n} \mathcal{K}_a^{\mu} \mathcal{A}_n = 0\,.$$

The RHS of the SCWI is **precisely zero** by EoM and IBP.

Example: Deriving Soft Photon Theorem

We have the following SCWI constraint:

$$\left(\sum_{a=1}^{n} \mathcal{K}_a^\mu + \mathcal{K}_q^\mu\right) \mathcal{A}_{n+1} = 0\,.$$



Ansatz for $\mathcal{A}_{n+1}$:

$$\mathcal{A}_{n+1}\left(\boldsymbol{q}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\right) = \sum_h e_h z \varepsilon_q \cdot k_h G\left(\boldsymbol{k}_I, z, z'\right) \mathcal{A}_n\left(z', \boldsymbol{k}_1, \ldots, \boldsymbol{k}_I, \ldots, \boldsymbol{k}_n\right)$$
$$+ R\left(\boldsymbol{q}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_h, \ldots, \boldsymbol{k}_n\right)$$

where G is the bulk-to-bulk propagator, and $\vec{k}_I \equiv \vec{q} + \vec{k}_h$.

Example: Deriving Soft Photon Theorem

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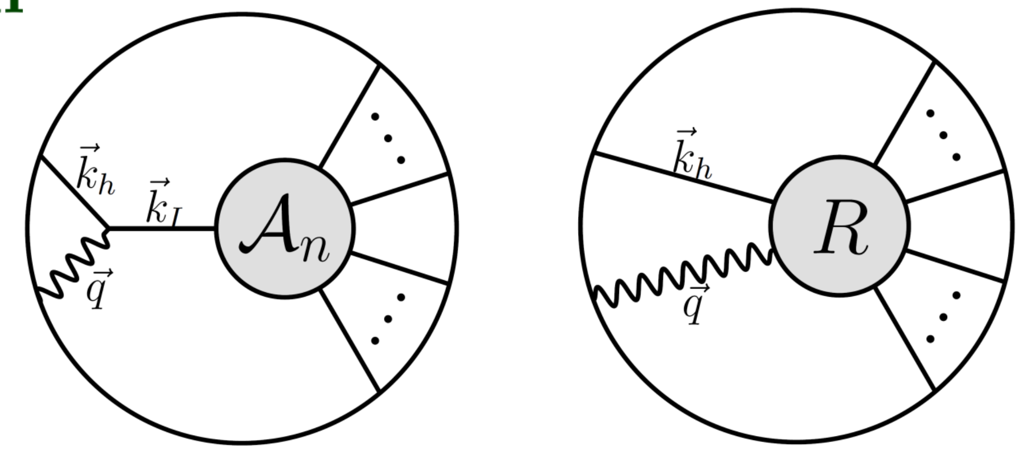


$$\left(\mathcal{K}_q^\mu + \mathcal{K}_h^\mu\right) \mathcal{A}_{n+1}\left(\boldsymbol{q}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\right)$$
$$= -\frac{4s_q + d - 2}{2zq^2}\left(\sum_h e_h \mathcal{A}_n\left(z, \boldsymbol{k}_1, \ldots, \boldsymbol{q} + \boldsymbol{k}_h, \ldots, \boldsymbol{k}_n\right) + 2z\,\boldsymbol{q}\cdot R\right)$$

$\mathcal{A}_n$ and R have expansions in q as follows:

$$\mathcal{A}_n(z, \boldsymbol{q} + \boldsymbol{k}_h) = \mathcal{A}_n(z, \boldsymbol{k}_h) + (\boldsymbol{q}\cdot\partial_{\boldsymbol{k}_h})\mathcal{A}_n(z, \boldsymbol{k}_h) + \cdots$$
$$R(z, \boldsymbol{q}, \{\boldsymbol{k}\}) = \varepsilon_q \cdot R^{(0)}(z, \{\boldsymbol{k}\}) + \varepsilon_q^\mu \boldsymbol{q}^\nu R^{(1)}_{\mu\nu}(z, \{\boldsymbol{k}\}) + \ldots$$

$\mathcal{O}\left(q^{-2}\right)$ $\qquad \displaystyle\sum_h e_h = 0$

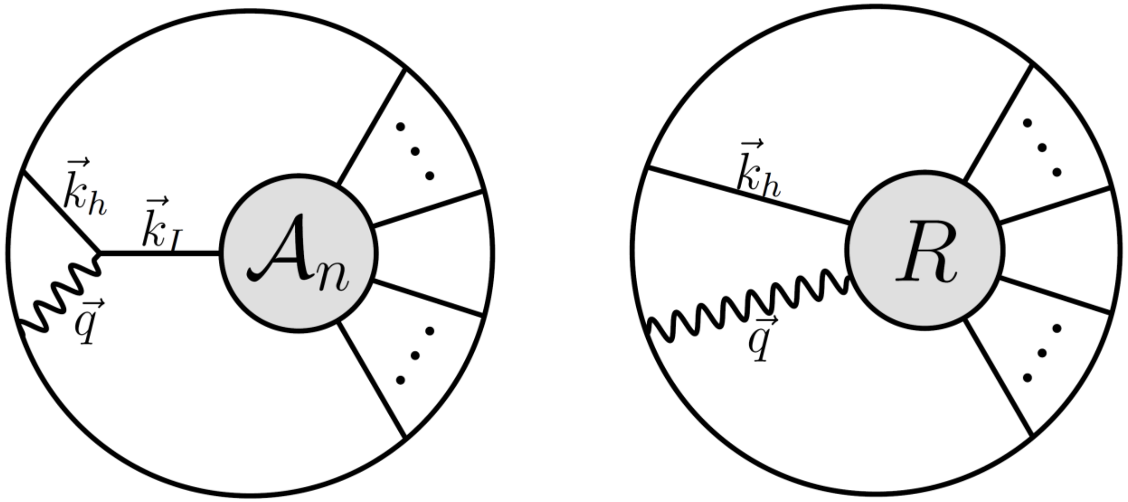
$\mathcal{O}\left(q^{-1}\right)$ $\qquad \displaystyle R^{(0)}_\mu = -\sum_h e_h \frac{1}{2z}\partial_{k_h^\mu}\mathcal{A}_n.$

Solving the SCWI Order by Order in q

Similar to [Z. Bern, S. Davies, P. Di Vecchia, J. Nohle '14]

Punchline

$$\left(\sum_{a=1}^{n} \mathcal{K}_a^\mu + \mathcal{K}_q^\mu\right) \mathcal{A}_{n+1} = 0\,.$$



So the result is

$$\lim_{\boldsymbol{q}\to 0} \mathcal{A}_{n+1} = \sum_h e_h \left(z' \varepsilon_q \cdot k_h G\left(\boldsymbol{k}_I, z, z'\right) - \frac{1}{2z} \varepsilon_q \cdot \partial_{k_h} \right) \mathcal{A}_n(z) + O(q)$$

Go back to boundary correlator space

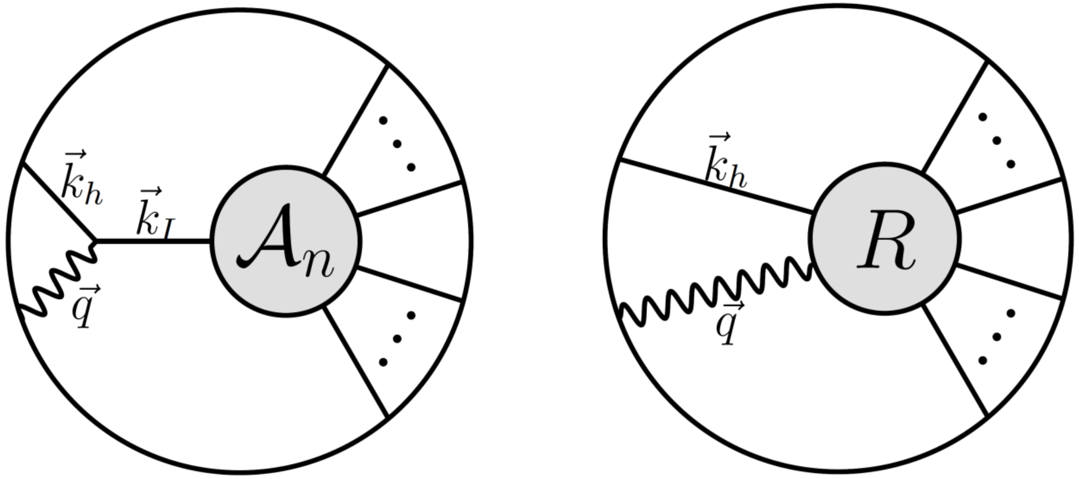
$$\int_0^\infty \frac{dz}{z^{d+1}}\, z\, \epsilon_q \cdot k_h\, \phi_{d-1}(z, k_h)\, \phi_{d-1}(z, q)\, G(k_h, z, z') = -\frac{\mathcal{N}_{d-1}}{2}\, \varepsilon_q \cdot \partial_{k_h} \phi_{d-1}(z', k_h),$$

$$\lim_{\boldsymbol{q}\to 0} \langle J(\boldsymbol{q}) J(\boldsymbol{k}_1) \ldots J(\boldsymbol{k}_n)\rangle = -\tfrac{1}{2} \textstyle\sum_h e_h \varepsilon_q \cdot \partial_{k_h} \langle J(\boldsymbol{k}_1) \ldots J(\boldsymbol{k}_n)\rangle + \mathcal{O}(q)$$

Summary

We have the following SCWI constraint:

$$\left(\sum_{a=1}^{n}\mathcal{K}_a^\mu+\mathcal{K}_q^\mu\right)\mathcal{A}_{n+1}=0\,.$$



Solving the SCWI Order by Order in q

Leading order in q contrains

Charge conservation $\sum_h e_h=0$

Equivalence principle $\kappa_i=\kappa$ for any i

Sub-leading order in q
Soft theorem

$$\lim_{\boldsymbol{q}\to 0}\langle J(\boldsymbol{q})J(\boldsymbol{k}_1)\dots J(\boldsymbol{k}_n)\rangle=\tfrac{1}{2}(\varepsilon_q\cdot\partial_{k_n}-\varepsilon_q\cdot\partial_{k_1})\langle J(\boldsymbol{k}_1)\dots J(\boldsymbol{k}_n)\rangle+\mathcal{O}(q)$$

$$\boxed{\lim_{\boldsymbol{q}\to 0}\langle T(\boldsymbol{q})T(\boldsymbol{k}_1)\dots T(\boldsymbol{k}_n)\rangle=S^{(0)}\langle T(\boldsymbol{k}_1)\dots T(\boldsymbol{k}_n)\rangle+S^{(1)}\langle T(\boldsymbol{k}_1)\dots T(\boldsymbol{k}_n)\rangle+\mathcal{O}(q^2)}$$

$$\boxed{\begin{aligned}S^{(0)}&:=-\frac{1}{2}\sum_{a=1}^{n}\varepsilon_{\mu\nu}k_a^\mu\partial_{k_a}^\nu,\\ S^{(1)}&:=\frac{1}{4}\sum_{a=1}^{n}\varepsilon_{\mu\nu}q_\rho(k_a^\rho\partial_{k_a}^\mu\partial_{k_a}^\nu-2k_a^\mu\partial_{k_a}^\nu\partial_{k_a}^\rho-2\partial_{k_a}^\mu S_a^{\nu\rho})\end{aligned}}$$

Checked

Thanks

Questions are welcome

Solving the SCWI Order by Order in q

Similar to [Z. Bern, S. Davies, P. Di Vecchia, J. Nohle '14]

At $\mathcal{O}\left(q^{-2}\right)$ we get charge conservation

$$-\sum_h e_h \frac{\varepsilon_q^\mu}{2zq^2}(4s_q + d - 2)\mathcal{A}_n = 0. \qquad \Longrightarrow \qquad \sum_h e_h = 0$$

At $\mathcal{O}\left(q^{-1}\right)$ we can constrain $R^{(0)}$

$$R_\mu^{(0)} = -\sum_h e_h \frac{1}{2z} \partial_{k_h^\mu} \mathcal{A}_n.$$

So the result is $$\mathcal{A}_{n+1} = \sum_h e_h \left(z' \varepsilon_q \cdot k_h G\left(\boldsymbol{k}_I, z, z'\right) - \frac{1}{2z} \varepsilon_q \cdot \partial_{k_h} \right) \mathcal{A}_n(z) + O(q)$$

Go back to boundary correlator space

$$\lim_{\boldsymbol{q}\to 0} \langle J(\boldsymbol{q})\phi(\boldsymbol{k}_1)\ldots J(\boldsymbol{k}_n)\rangle = -\tfrac{1}{2}\sum_h e_h \varepsilon_q \cdot \partial_{k_h} \langle \phi(\boldsymbol{k}_1)\ldots J(\boldsymbol{k}_n)\rangle + \mathcal{O}(q)$$