

q_T -slicing method and factorization structures for multi-jet processes

Rong-Jun Fu (符荣峻)

Fudan University

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Collaborators: Rudi Rahn, Ding Yu Shao, Wouter J. Waalewijn, Bin Wu

第五届量子场论及其应用研讨会

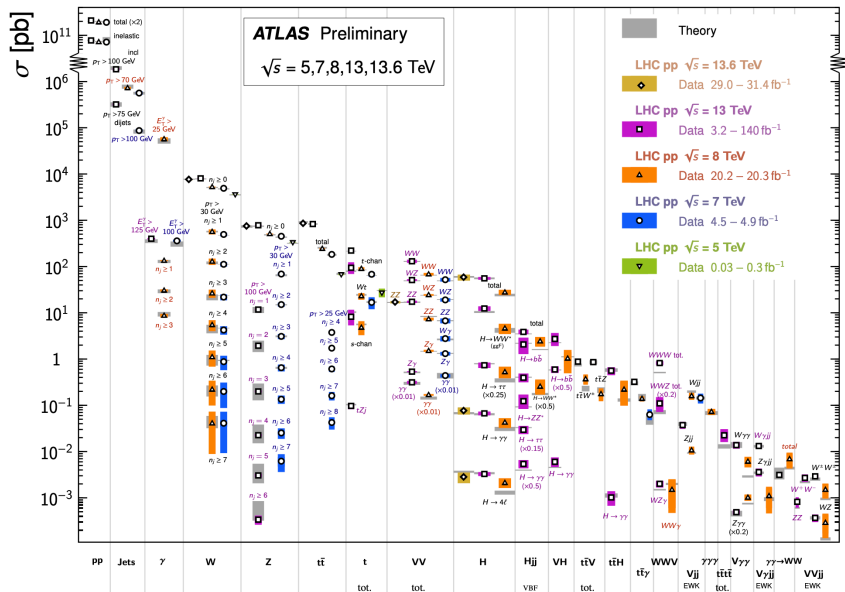
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Standard Model Summary Plots June 2024 [ATLAS Collaboration '24]

Standard Model Production Cross Section Measurements

Status: June 2024



$p_T > 100 \text{ GeV}$, $n_j = 1$

$n_j = 2$

$n_j = 3$

$n_j = 4$

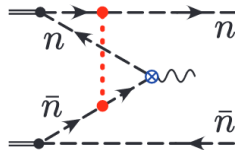
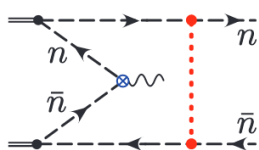
$n_j = 5$

$n_j \geq 6$

Z

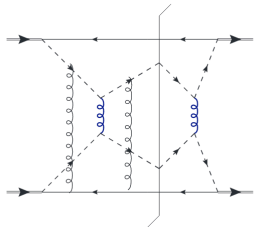
Factorization-violation

All proton collisions include forward proton remnants (spectators).



dashed red line: Glauber
[Rothstein, Stewart, '16]

Absence of factorization-violation due to Glauber gluons is important element of factorization proof for Drell-Yan process. [Bodwin '85; Collins, Soper, Sterman '85] e.g. Transverse momentum dependent (TMD) factorization is violated in dijet production [Collins, Qiu '07; Collins '07, Vogelsang, Yuan '07; Rogers, Mulders 10, ...]



The exchange of two extra gluons, as in this graph, will tend to give non-factorization in unpolarized cross sections.

Infrared cancellation (KLN theorem)

$$\begin{aligned}
 \sigma_{\text{tot}} &= \left| \text{tree} + \boxed{\text{virtual}} + \boxed{\text{real correction}} \right|^2 \\
 &= \sigma_0 \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2) \right\}
 \end{aligned}$$

$e^+e^- \rightarrow q\bar{q}$
 $e^+e^- \rightarrow q\bar{q}g$

Real and virtual corrections suffer from soft and collinear infrared divergences, e.g.,

$$\begin{aligned}
 \sigma_{\text{real}} &= \sigma_0 \frac{\alpha_s(\mu)}{2\pi} C_F \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{57}{6} - \frac{7\pi^2}{6} \right) \\
 \sigma_{\text{virtual}} &= \sigma_0 \frac{\alpha_s(\mu)}{2\pi} C_F \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7\pi^2}{6} \right)
 \end{aligned}$$

in $d = 4 - 2\epsilon$. Divergences cancel in the sum! [Kinoshita 1962; Lee & Nauenberg 1964]

Slicing approach

$$\frac{d\sigma_{\text{N}^k\text{LO}}^{(m)}}{dX} = \int_0^\delta dq_T \frac{d\sigma_{\text{N}^k\text{LO}}^{(m)}}{dX dq_T} + \int_\delta^\infty dq_T \frac{d\sigma_{\text{N}^{k-1}\text{LO}}^{(m+1)}}{dX dq_T}$$

- δ (or q_T^{cut}) plays a role as slicing variable here;
- The factorization formula, obtained e.g. through Soft-Collinear Effective Theory (SCET), can be used to handle the cancellation of infrared (IR) divergences in

$$\frac{d\sigma_{\text{N}^k\text{LO}}^{(m)}}{dq_T} \equiv \frac{d\sigma_{\text{SCET}}}{dq_T} [1 + \mathcal{O}(\delta^p)],$$

whereas

$$\frac{d\sigma_{\text{N}^{k-1}\text{LO}}^{(m+1)}}{dq_T} \equiv \frac{d\sigma_{\text{QCD}}}{dq_T}$$

is numerically easier to compute.

- Large cancellations between singular and regular terms.

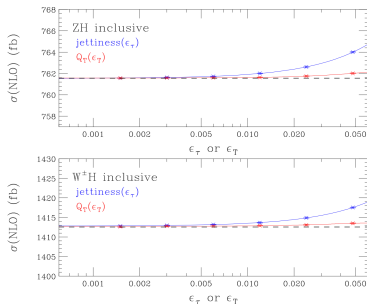
Modern slicing variables

1. Transverse momentum of a colorless or colored (but massive) system. [Catani, Grazzini ' 07]

For processes with jets, such as $pp \rightarrow V + \text{jets}$ or $pp \rightarrow 2\text{jets}$, q_T is **unsuitable** because $q_T = 0$ for radiation emitted inside jets.

2. Jettiness [Boughezal, Focke, Liu, Petriello ' 15; Gaunt, Stahlhofen, Tackmann, Walsh ' 15]

$$\tau_N = \sum_i \min_{j=a,b,\dots,N} \left\{ \frac{2q_j k_i}{Q^2} \right\}$$



3. For most color-singlet processes, q_T performs better than 0-jettiness as slicing variable [Campbell, Ellis, Seth ' 22], which motivates the exploration of extending q_T to processes involving jets.

- We propose two generalizations of q_T that can be used for jet processes. [RJF, Rahn, Shao, Waalewijn, Wu '24]

The key ingredient is the use of a recoil-free jet axis!
Utilizing the Winner-Take-All (WTA) scheme to define slicing variables.

- Azimuthal decorrelation $\delta\phi = q_x/p_{T,1}$:

$$q_x = p_{x,1}^{\text{WTA}} + p_{x,2}^{\text{WTA}}$$

- Magnitude of total transverse momentum $|\vec{q}_T|$:

$$\vec{q}_T = \sum_{i=\text{jets}, V, \dots} \vec{p}_{T,i}^{\text{WTA}}$$

Why the use of WTA scheme enables slicing?

A jet definition includes a jet algorithm and a **recombination scheme**. The former defines how some particles are grouped into jets, while the latter specifies how a momentum is assigned to a jet.

- **Standard E-scheme** simply sums the four-momenta of all particles within the jet.
- p_T^n -**weighted recombination scheme**: [Banfi, Dasgupta, Delenda '08]

$$p_{T,r} = p_{T,i} + p_{T,j}, \quad \phi_r = \frac{\omega_i \phi_i + \omega_j \phi_j}{\omega_i + \omega_j}, \quad y_r = \frac{\omega_i y_i + \omega_j y_j}{\omega_i + \omega_j}, \quad \omega = p_T^n$$

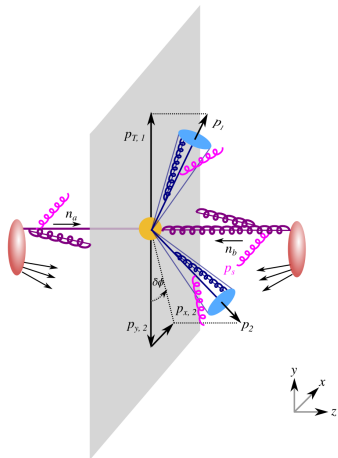
If $n \rightarrow \infty$: **Winner-take-all scheme**: [Bertolini, Chan, Thaler '13]

$$p_{T,r} = p_{T,i} + p_{T,j}, \quad \hat{n}_r = \begin{cases} \hat{n}_i, & p_{T,i} \geq p_{T,j} \\ \hat{n}_j, & p_{T,i} < p_{T,j} \end{cases}$$

- In the WTA scheme, soft radiation inside the jet can also influence the jet axis through momentum conservation, similar to radiation outside the jet, leading to a non-zero q_T .

Azimuthal decorrelation ($\delta\phi = q_x/p_{T,1}$)

- A simple factorization formula. [Chien, Rahn, Shao, Waalewijn & Wu ' 22 + Schrinder ' 21]
- Suitable for processes that are planar at Born level, such as $pp \rightarrow V + \text{jet}$, $pp \rightarrow 2\text{jets}$, $e^+e^- \rightarrow 3\text{jets}$, etc.



The $pp \rightarrow 2$ jet process. By using the WTA scheme, the transverse momentum perpendicular to the scattering plane q_x (equal to $p_{x,2}$ in the picture), or equivalently the azimuthal decorrelation $\delta\phi$, is a suitable slicing variable.

The factorization ingredients are: the **hard** scattering (yellow), collinear **initial-** (purple) and **final-state** (blue) radiation, and **soft** radiation (pink).

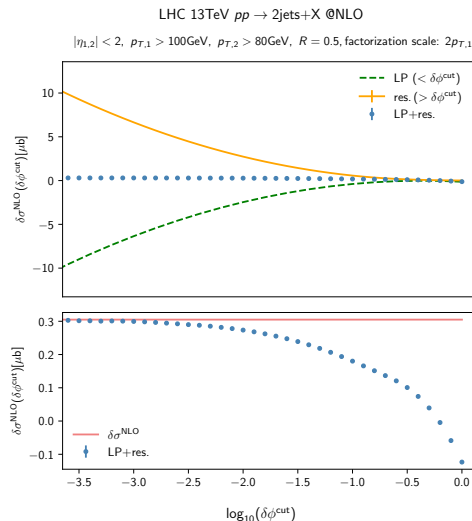
Azimuthal decorrelation

- Factorization formula ($pp \rightarrow 2\text{jets}$):

$$\frac{d\sigma_{\text{SCET}}}{d p_{T,1} d\eta_1 d\eta_2 dq_x} = \int \frac{db_x}{2\pi} e^{iq_x b_x} \sum_{i,j,k,\ell} B_i(x_a, b_x) B_j(x_b, b_x) \mathcal{J}_k(b_x) \mathcal{J}_\ell(b_x) \\ \times \text{tr} \left[\hat{\mathcal{H}}_{ij \rightarrow k\ell}(p_{T,1}, \eta_1 - \eta_2) \hat{S}_{ijkl}(b_x, \eta_1, \eta_2) \right].$$

- Factorization ingredients:
 - Standard** TMD PDFs $B_{i,j}$: known at N³LO; [Luo, Yang, Zhu, Zhu ' 19; Ebert, Mistlberger, Vita ' 20]
 - Soft function $\hat{S}_{ijkl}$: can directly be obtained from the **standard** TMD soft function at NNLO; [Gao, Li, Moul, Zhu ' 19]
 - TMD jet functions $\mathcal{J}_{k,\ell}$: partially known at NNLO. [Reyes, Scimemi, Waalewijn, Zoppi ' 19; Bell, Brune, Das, Wald ' 23; Fang, Gao, Li, Shao ' 24]

q_x -slicing for $pp \rightarrow 2\text{jets}$



- Slicing capitalizes on this,

$$\frac{d\sigma}{dX} = \int_0^\delta dq_x \frac{d\sigma_{\text{SCET}}}{dX dq_x} [1 + \mathcal{O}(\delta^p)] + \int_\delta^\infty dq_x \frac{d\sigma_{\text{QCD}}}{dX dq_x}.$$

- σ_{QCD} is obtained from NLOJET++. [Nagy, Trocsanyi '01]
- Jets are defined in a standard way with partons clustered by anti- k_T algorithm and momentum recombined by standard E-scheme.
- Large cancellation and nice convergence at NLO.

Total transverse momentum ($|\vec{q}_T|$)

- We can still use the magnitude of the total transverse momentum $q_T = |\vec{q}_T|$ of the color-singlets and jets as a slicing variable, when utilizing the WTA scheme.
- q_T -slicing can be applied to processes with non-planar kinematics, such as $pp \rightarrow Z + 2\text{jets}$, $pp \rightarrow 3\text{jets}$, etc.
- Factorization formula:

$$\begin{aligned} \frac{d\sigma_{\text{SCET}}}{dp_{T,1} d\eta_1 d\eta_2 dq_T} &= q_T \int \frac{d^2 \vec{b}_T}{2\pi} J_0(q_T |\vec{b}_T|) \sum_{i,j,k,\ell} B_i(x_a, \vec{b}_T) B_j(x_b, \vec{b}_T) \mathcal{J}_k(b_x) \\ &\quad \times \mathcal{J}_\ell(b_x) \text{tr} \left[\hat{\mathcal{H}}_{ij \rightarrow k\ell}(p_{T,1}, \eta_1 - \eta_2) \hat{S}_{ijkl}(\vec{b}_T, \eta_1, \eta_2, R) \right]. \end{aligned}$$

- **Only soft function is new:** Outside the jet (b_T), inside the jet (b_x).

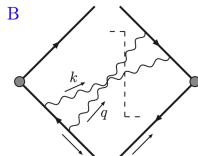
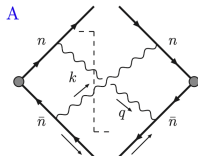
Refactorization of soft function

The q_T -soft function refactorizes into **global** and **collinear-soft** contributions in small- R limit, while the remaining power corrections appear as a series in R^{2n} ,

$$\hat{S}_{ij\ell k}(\vec{b}_T, \eta_1, \eta_2, R) = \hat{S}_{ij\ell k}^{\text{global}}(\vec{b}_T, \eta_1, \eta_2, R) S_k^{\text{cs}}(\vec{b}_T, \eta_1, R) S_\ell^{\text{cs}}(\vec{b}_T, \eta_2, R) + \mathcal{O}(R^{2n})$$

We find that, after boosting along the jet axis, the collinear-soft function reduces to the hemisphere soft function, with an in-cone dependence on $b_\perp$ and an out-of-cone dependence on b_- .

$$S_j^{\text{cs}}(b_\perp, b_-) = 1 + \frac{Z_\alpha \alpha_s(\mu)}{4\pi} S_j^{\text{in}}(b_\perp) + \frac{Z_\alpha \alpha_s(\mu)}{4\pi} S_j^{\text{out}}(b_-) \\ + \left(\frac{Z_\alpha \alpha_s(\mu)}{4\pi} \right)^2 S_j^{\text{cs},(2)}(b_\perp, b_-) + \mathcal{O}(\alpha_s^3)$$



.....

[Kelley, Schwartz, Schabinger, Zhu, '11]

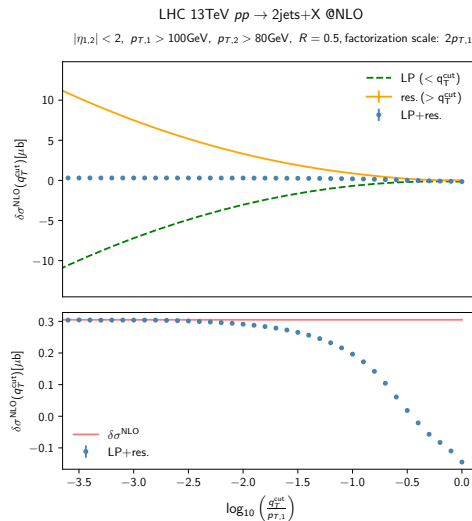
Soft function

$$\begin{aligned}
 \hat{S}_{\text{finite}}^{(1)}(q_T^{\text{cut}}, \eta_1, \eta_2, R, \mu, \nu) = & \frac{\alpha_s(\mu)}{4\pi} \left\{ -4L_\mu^2 \sum_i \mathbf{T}_i^2 + L_\mu \left[4 \ln \frac{\mu^2}{\nu^2} \sum_i \mathbf{T}_i^2 + 8 \sum_{i < j} \mathbf{T}_i \cdot \mathbf{T}_j \ln \frac{n_i \cdot n_j}{2} \right. \right. \\
 & \left. \left. - 8 \ln 2 (\mathbf{T}_a + \mathbf{T}_b) \cdot (\mathbf{T}_1 + \mathbf{T}_2) - 16 \ln 2 \mathbf{T}_1 \cdot \mathbf{T}_2 \right] - \frac{\pi^2}{6} \sum_i \mathbf{T}_i^2 \right. \\
 & + [(\mathbf{T}_a + \mathbf{T}_b) \cdot (\mathbf{T}_1 + \mathbf{T}_2) + 2 \mathbf{T}_1 \cdot \mathbf{T}_2] \left(4 \ln 2 \ln \frac{\mu^2}{\nu^2} + \frac{\pi^2}{3} + 4 \ln^2 \frac{R}{2} \right) \\
 & + \sum_{j \in \text{jets}} (\mathbf{T}_a + \mathbf{T}_b) \cdot \mathbf{T}_j \, 8 \ln 2 \ln (2 \cosh \eta_j) \\
 & + \mathbf{T}_1 \cdot \mathbf{T}_2 \left[8 \ln 2 \ln (4 \cosh \eta_1 \cosh \eta_2) - 2 \ln^2 (2 + 2 \cosh(\eta_1 - \eta_2)) + 2(\eta_1 - \eta_2)^2 \right] \\
 & \left. - \sum_{i < j} \mathbf{T}_i \cdot \mathbf{T}_j S_{ij}^{\text{corr}}(\eta_1, \eta_2, R) \right\}
 \end{aligned}$$

R -correction for jet-jet dipole:

$$\begin{aligned}
 S_{12}^{\text{corr}}(\eta_1, \eta_2, R) = & -2R^2 \ln \frac{R}{2} \tanh^2 \left(\frac{\eta_1 - \eta_2}{2} \right) + R^2 \left[\frac{7}{3} - \frac{6}{1 + \cosh(\eta_1 - \eta_2)} \right] \\
 & + R^4 \left[\frac{49}{720} - \frac{e^{\eta_1 + \eta_2} (3e^{2\eta_1} + 3e^{2\eta_2} - 8e^{\eta_1 + \eta_2})}{2(e^{\eta_1} + e^{\eta_2})^4} - \ln \left(\frac{R}{2} \right) \frac{(e^{2\eta_1} + e^{2\eta_2} - 10e^{\eta_1 + \eta_2})^2}{36(e^{\eta_1} + e^{\eta_2})^4} \right] + \mathcal{O}(R^6)
 \end{aligned}$$

q_T -slicing for $pp \rightarrow 2\text{jets}$

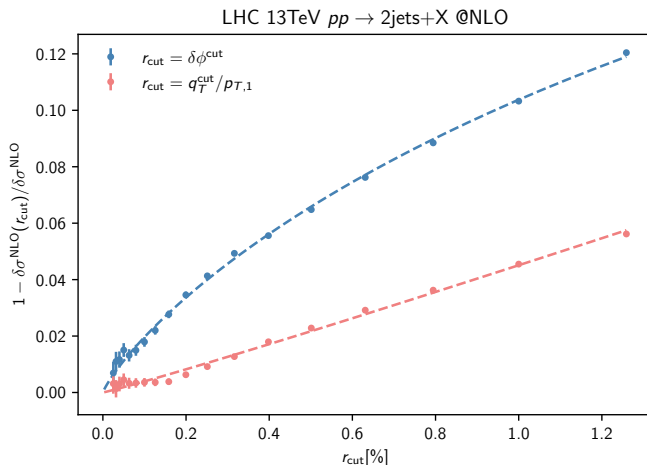


- Slicing capitalizes on this,

$$\frac{d\sigma}{dX} = \int_0^\delta dq_T \frac{d\sigma_{\text{SCET}}}{dX dq_T} [1 + \mathcal{O}(\delta^p)] + \int_\delta^\infty dq_T \frac{d\sigma_{\text{QCD}}}{dX dq_T}.$$

- We include finite terms up to $\mathcal{O}(R^4)$, achieving a 1% precision for the cross section at $R = 0.5$.

Comparison between two slicing variables



The curves are obtained from fitting to $a r_{\text{cut}} \ln r_{\text{cut}} + b r_{\text{cut}}$.

q_T converges faster than q_x slicing at the expense of a more complicated soft function, and can also be extended to non-planar Born processes.

2-loop ingredient: NNLO collinear-soft function

$$\begin{aligned}
 \bar{S}_j^{cs,(2)}(b_T) = & \left(\frac{\mu b_T}{b_0} \right)^{4\epsilon} \left[\left(\frac{\nu b_T R_j}{b_0} \right)^\eta \omega^2 C_F C_A \bar{v}_A^{\text{in}} + R^{-4\epsilon} C_F C_A \bar{v}_A^{\text{out}} \right. \\
 & + \left(\frac{\nu b_T R_j}{b_0} \right)^\eta \omega^2 \left[\omega^2 \left(\frac{\nu b_T R_j}{b_0} \right)^\eta C_F^2 \bar{h}_{2F} + C_F C_A \bar{h}_A + C_F n_f T_F \bar{h}_f \right] \\
 & + \left(\frac{\nu b_T R_j}{b_0} \right)^\eta R^{-2\epsilon} \omega^2 C_F^2 \bar{p}_{2F} + R^{-4\epsilon} (C_F C_A \bar{p}_A + C_F n_f T_F \bar{p}_f) + R^{-2\epsilon} C_F C_A \bar{p}_{\text{NGL}} \\
 & \left. + R^{-4\epsilon} (C_F^2 \bar{g}_{2F} + C_F C_A \bar{g}_A + C_F n_f T_F \bar{g}_f) \right],
 \end{aligned}$$

$$\begin{aligned}
 \bar{v}_A^{\text{in}} = & -\frac{1}{\epsilon^4} + \frac{4}{\eta \epsilon^3} - \frac{16 \ln 2}{\eta \epsilon^2} + \frac{\frac{7\pi^2}{6} + 8 \ln^2 2}{\epsilon^2} + \frac{2(\pi^2 + 16 \ln^2 2)}{\eta \epsilon} - \frac{2(10\pi^2 \ln 2 + 32 \ln^3 2 + 34\zeta_3)}{3\epsilon} \\
 & - \frac{4(6\pi^2 \ln 2 + 32 \ln^3 2 + 46\zeta_3)}{3\eta} + \frac{1009\pi^4}{360} + \frac{52}{3} \pi^2 \ln^2 2 + 32 \ln^4 2 + 152\zeta_3 \ln 2 \\
 \bar{v}_A^{\text{out}} = & \frac{1}{\epsilon^4} - \frac{\pi^2}{2\epsilon^2} + \frac{8\zeta_3}{3\epsilon} + \frac{\pi^4}{120}
 \end{aligned}$$

2-loop ingredient: NNLO collinear-soft function

$$\begin{aligned}
 \bar{h}_A &= \frac{1}{\epsilon^4} + \frac{11}{6\epsilon^3} + \left(\frac{67}{18} - \frac{5\pi^2}{3} - 8\ln^2 2 \right) \frac{1}{\epsilon^2} + \left(\frac{211}{27} - \frac{121\pi^2}{36} + 8\pi^2 \ln 2 - \frac{44\ln^2 2}{3} + \frac{64\ln^3 2}{3} + \frac{35\zeta_3}{3} \right) \frac{1}{\epsilon} \\
 &\quad + \left[-\frac{4}{\epsilon^3} + \left(-\frac{22}{3} + 16\ln 2 \right) \frac{1}{\epsilon^2} + \left(-\frac{134}{9} - \frac{4\pi^2}{3} + \frac{88\ln 2}{3} - 32\ln^2 2 \right) \frac{1}{\epsilon} \right. \\
 &\quad \left. + \left(-\frac{808}{27} - \frac{55\pi^2}{9} + \frac{536\ln 2}{9} + \frac{16\pi^2 \ln 2}{3} - \frac{176\ln^2 2}{3} + \frac{128\ln^3 2}{3} + \frac{268\zeta_3}{3} \right) \right] \frac{1}{\eta} - 365.293(48), \\
 \bar{h}_f &= -\frac{2}{3\epsilon^3} - \frac{10}{9\epsilon^2} + \left(-\frac{74}{27} + \frac{11\pi^2}{9} + \frac{16\ln^2 2}{3} \right) \frac{1}{\epsilon} + \left[\frac{8}{3\epsilon^2} + \left(\frac{40}{9} - \frac{32\ln 2}{3} \right) \frac{1}{\epsilon} \right. \\
 &\quad \left. \left(\frac{224}{27} + \frac{20\pi^2}{9} - \frac{160\ln 2}{9} + \frac{64\ln^2 2}{3} \right) \right] \frac{1}{\eta} - 40.8677(14)
 \end{aligned}$$

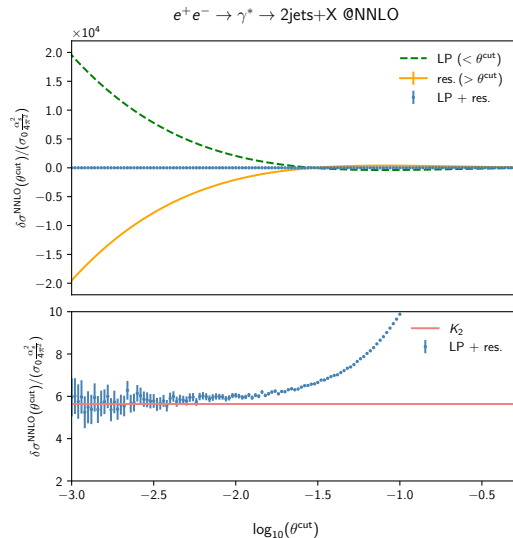
$$\begin{aligned}
 \bar{g}_A &= -\frac{1}{\epsilon^4} - \frac{11}{6\epsilon^3} + \frac{-67 + 6\pi^2}{18\epsilon^2} + \frac{-47840 - 2010\pi^2 + 1233\pi^4 - 75240\zeta_3}{3240} + \frac{-772 - 33\pi^2 + 468\zeta_3}{108\epsilon} \\
 \bar{g}_f &= \frac{2}{3\epsilon^3} + \frac{10}{9\epsilon^2} + \frac{38 + 3\pi^2}{27\epsilon} + \frac{238 + 15\pi^2 + 684\zeta_3}{81}.
 \end{aligned}$$

$$\begin{aligned}
 \bar{p}_f &= \frac{4(3 - 2\pi^2)}{9\epsilon} - \frac{68}{9} + \frac{64\pi^2}{27} - \frac{16\zeta_3}{3}, \quad \bar{p}_A = \left(-\frac{2}{3} + \frac{22\pi^2}{9} - 4\zeta_3 \right) \frac{1}{\epsilon} + \frac{40}{9} - \frac{134\pi^2}{27} + \frac{44\zeta_3}{3} + \frac{8\pi^4}{45}, \\
 \bar{p}_{\text{NGL}} &= \frac{2\pi^2}{3\epsilon^2} + \left(8\zeta_3 - \frac{4}{3}\pi^2 \ln 2 \right) \frac{1}{\epsilon} + \left(\frac{13\pi^4}{45} + \frac{4}{3}\pi^2 \ln^2 2 - 16\zeta_3 \ln 2 \right)
 \end{aligned}$$

Summary and outlook

1. IR divergences present a significant challenge in precision QCD calculations, especially for processes involving multi-jet final states.
2. Two novel extensions of transverse momentum slicing variables (q_x and q_T) are proposed, specifically tailored for jet processes.
 - q_x slicing only applies to planar Born processes, such as $pp \rightarrow 2$ jets, but offers a dramatic simplification of the soft function.
 - q_T slicing necessitates a more complex soft function but offers faster convergence and can be applied to non-planar cases.
 - Two slicing approaches are successfully demonstrated at NLO.
3. The NNLO collinear-soft function has been calculated and successfully passes both the RG and RRG consistency checks.
4. These developments offer a promising framework for tackling the challenges of multi-jet final states at NNLO, paving the way for further advancements in precision QCD calculations.

Back up: NNLO slicing in $e^+e^- \rightarrow 2 \text{ jet process}$



we define our slicing variable θ as

$$\theta = \arctan\left(\frac{2q_T}{Q}\right) \approx \frac{2q_T}{Q},$$

which is equal to π minus the angle between the leading and subleading jets.

$$K_2 = \frac{365}{6} - 44\zeta_3 + n_f \left(\frac{8}{3}\zeta_3 - \frac{11}{3} \right)$$