Francesco Becattini
University of Florence and INFN

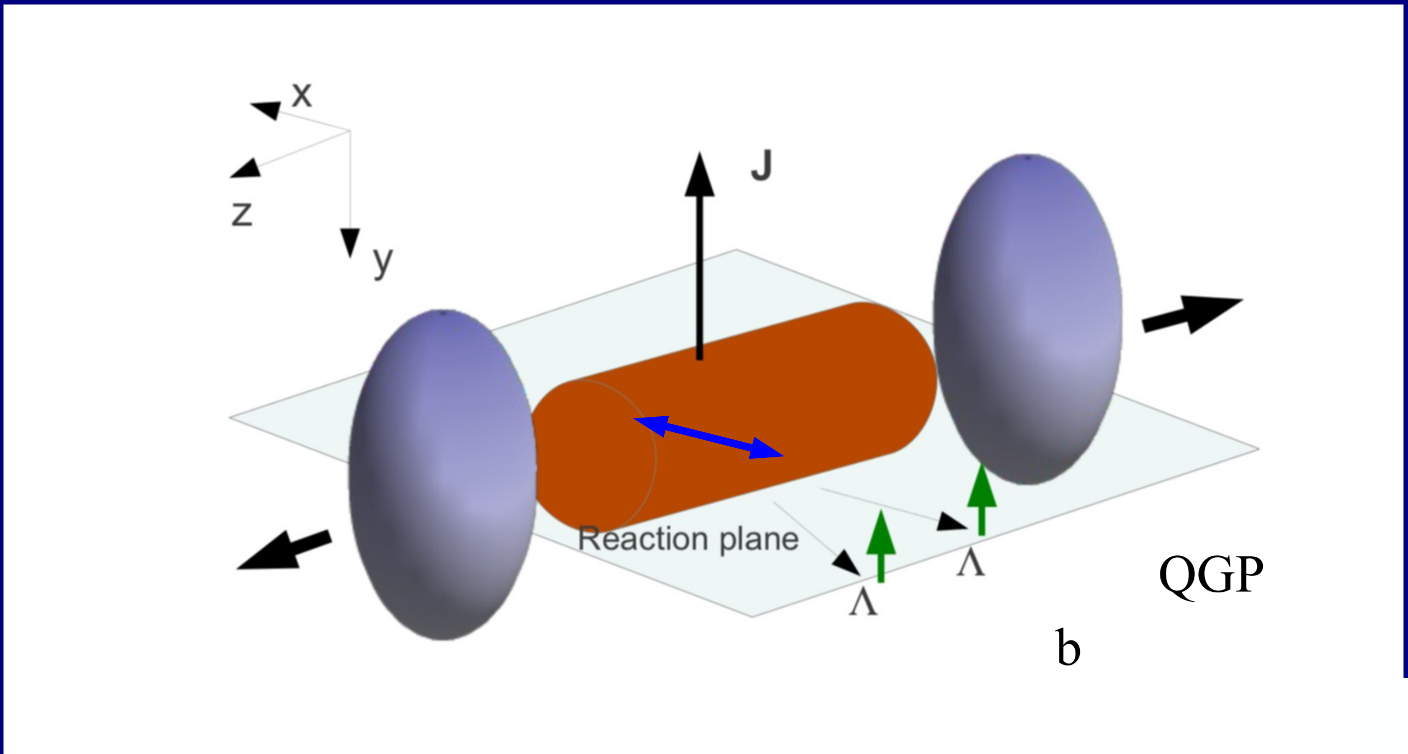
An improved formula for spin polarization at local thermodynamic equilibrium

X. L. Sheng, F.B., D. Roselli, 2509.xxxx

OUTLINE

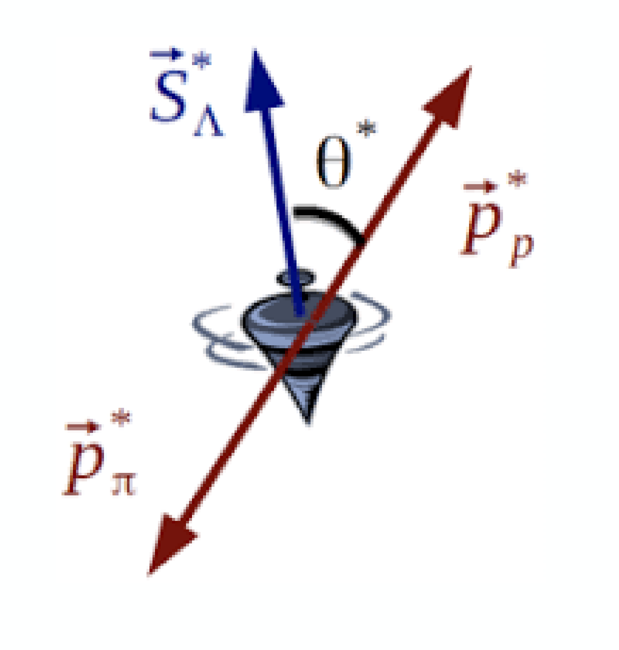
- Introduction and motivations
- Wigner function, density operator, local equilibrium
- A new method to calculate the corrections to classical local equilibrium
- Results and discussion

Introduction



In peripheral heavy ion collisions, particles are expected to be polarized

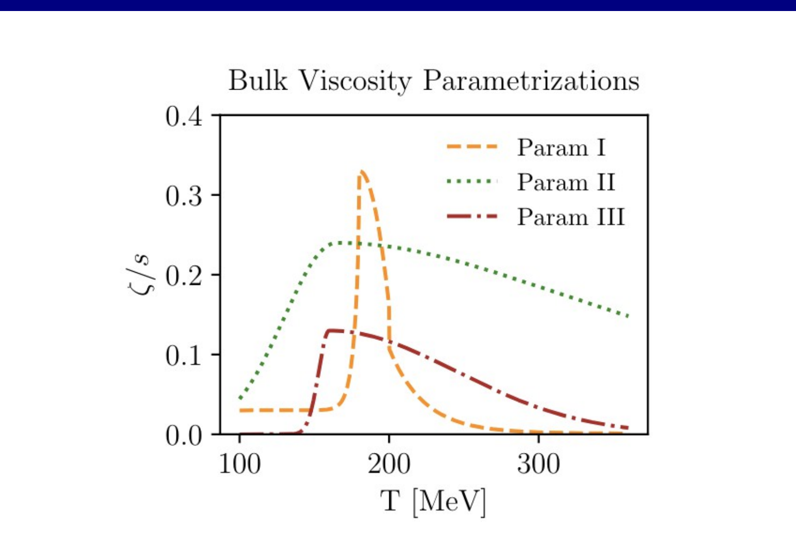
Z. T. Liang, X. N. Wang, Phys. Rev. Lett. 94 (2005) 102301
F. B., F. Piccinini, J. Rizzo, Phys. Rev. C 77 (2008) 024906

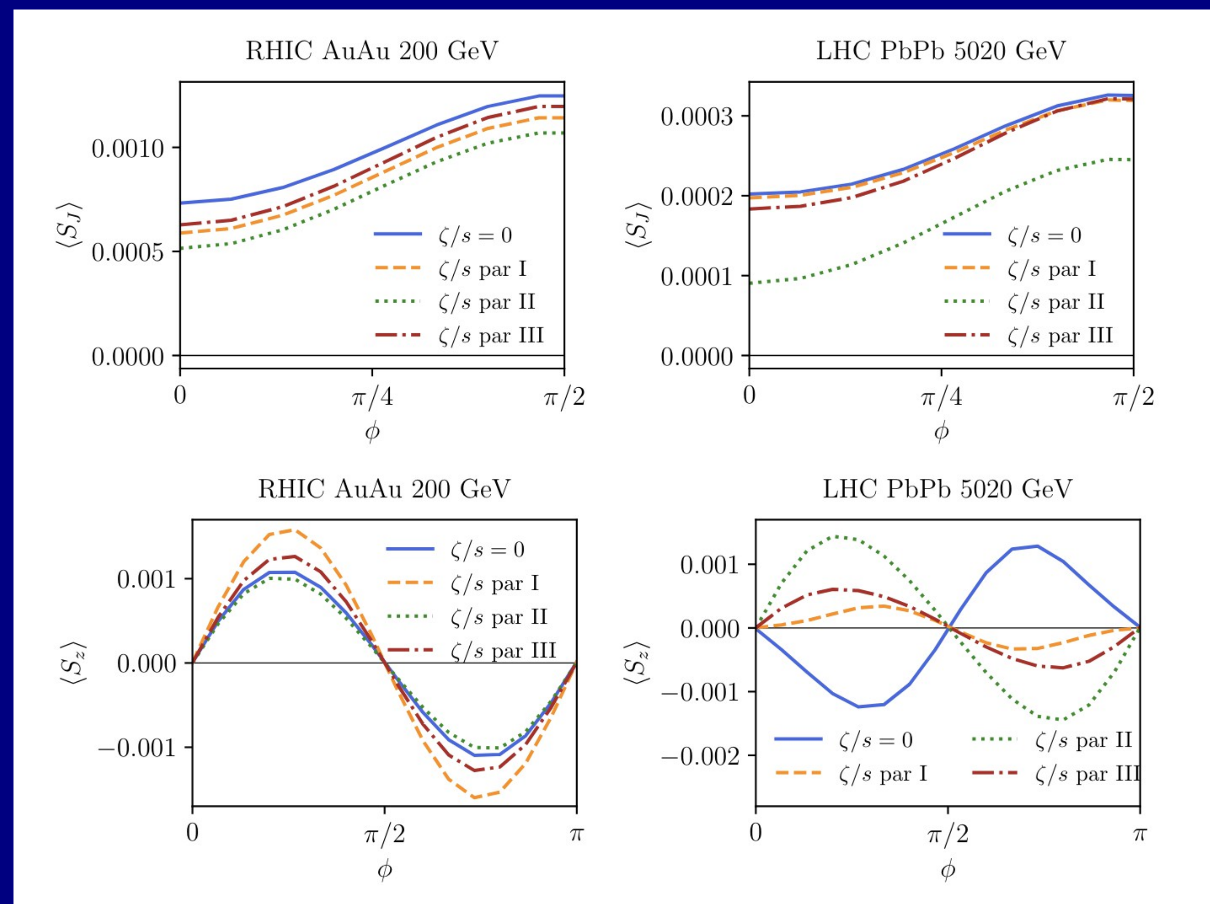


Polarization can be detected by studying parity-violating Λ decays

Why does polarization matter?

EXAMPLE: it turns out to be very sensitive to bulk viscosity of the QGP at the highest LHC energy





A. Palermo, E. Grossi, I. Karpenko and F.B., Eur. Phys. J. C 84 (2024) 920

Theoretical formulae of polarization

Spin polarization at local thermodynamic equilibrium, leading order:

< 2021

$$S^\mu(p) = -\frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int_\Sigma \mathrm{d}\Sigma_\tau p^\tau n_F(1-n_F)\varpi_{\nu\rho}}{\int_\Sigma \mathrm{d}\Sigma_\tau p^\tau \; n_F}$$

$$\varpi_{\rho\lambda} = -\frac{1}{2}\left(\partial_\rho\beta_\lambda - \partial_\lambda\beta_\rho\right)$$
$$\xi_{\rho\lambda} = \frac{1}{2}\left(\partial_\rho\beta_\lambda + \partial_\lambda\beta_\rho\right)$$

$$\beta^\mu = \frac{1}{T}u^\mu$$

Spin-thermal shear coupling or Shear-induced polarization, leading order:

F. B., M. Buzzegoli, A. Palermo, Phys. Lett. B 820 (2021) 136519

> 2021

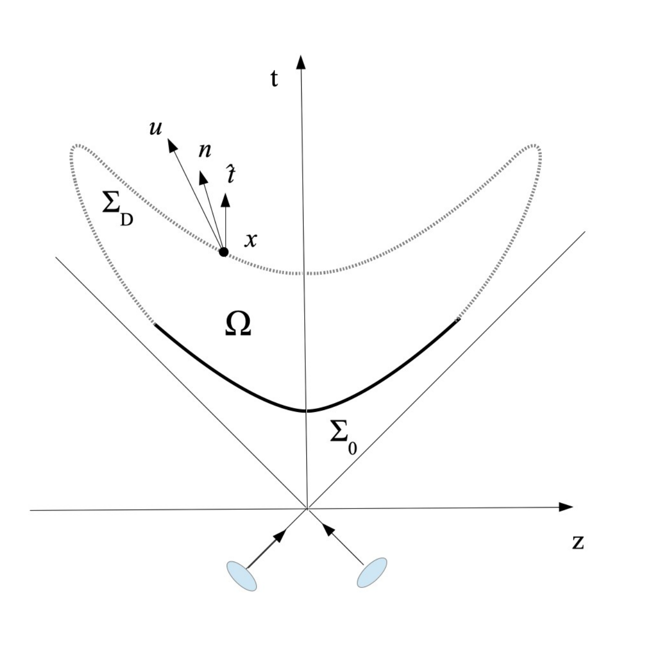
t

$$S^\mu_\xi(p) = -\frac{1}{4m}\epsilon^{\mu\nu\sigma\tau}\frac{p_\tau p^\rho}{\varepsilon}\frac{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F(1-n_F)\hat{t}_\nu\xi_{\sigma\rho}}{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F},$$

u

$$S^\mu_\xi = -\frac{1}{4m}\epsilon^{\mu\nu\sigma\tau}\frac{p_\tau p^\rho}{p\cdot u}\frac{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F(1-n_F)u_\nu\xi_{\sigma\rho}}{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F}$$

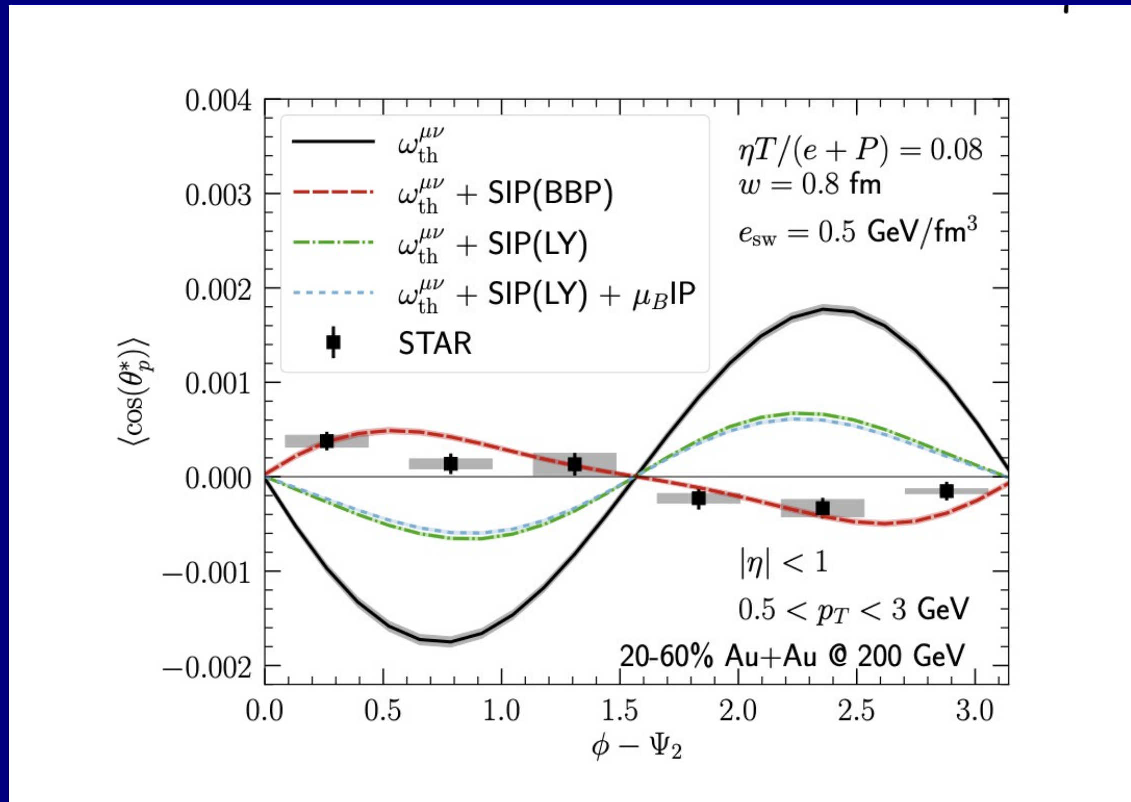
S. Liu, Y. Yin, JHEP 07 (2021) 188



Confirmed in:

C. Yi, S. Pu, D. L. Yang, Phys.Rev.C 104 (2021) 6, 064901
Y. C. Liu, X. G. Huang, arXiv 2109.15301, Sci. China Phys.Mech.Astron. 65 (2022) 7, 272011

Different formulae, different predictions



S. Alzhrani, S. Ryu and C. Shen, Phys. Rev. C 106 (2022) 014905

$$S^\mu_\xi(p) = -\frac{1}{4m}\epsilon^{\mu\nu\sigma\tau}\frac{p_\tau p^\rho}{\varepsilon}\frac{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F(1-n_F)\hat{t}_\nu\xi_{\sigma\rho}}{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F},$$

t

$$S^\mu_\xi = -\frac{1}{4m}\epsilon^{\mu\nu\sigma\tau}\frac{p_\tau p^\rho}{p\cdot u}\frac{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F(1-n_F)u_\nu\xi_{\sigma\rho}}{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F}$$

u

Why this difference?
Geometric assumptions

The Wigner operator/function

Wigner operator is an essential tool to calculate spectra and polarization of particles in a quantum-relativistic framework. For a free Dirac field

$$\Psi(x) = \frac{1}{\sqrt{(2\pi)^3}} \sum_s \int \frac{\mathrm{d}^3\mathrm{p}}{2\varepsilon_{\mathbf{p}}} \left(u_s(p)\widehat{a}_s(p)\mathrm{e}^{-ip\cdot x} + v_s(p)\widehat{b}_s v_s(p)\mathrm{e}^{ip\cdot x} \right) ,$$

$$\widehat{W}_{ab}(x,p) \equiv \int \frac{\mathrm{d}^4 y}{(2\pi)^4} \mathrm{e}^{-ip\cdot y} \overline{\Psi}_b \left(x + \frac{y}{2} \right) \Psi_a \left(x - \frac{y}{2} \right)$$

It can be decomposed into particle, antiparticle an mixed components:

$$\widehat{W}(x,k) \equiv \widehat{W}^+(x,k) + \widehat{W}^-(x,k) + \widehat{W}_S(x,k) ,$$

$$\widehat{W}^\pm(x,k) \equiv \widehat{W}(x,k)\,\theta\left(k^2\right)\theta\left(\pm k^0\right), \quad \widehat{W}_S \equiv \widehat{W}(x,k)\,\theta\left(-k^2\right)$$

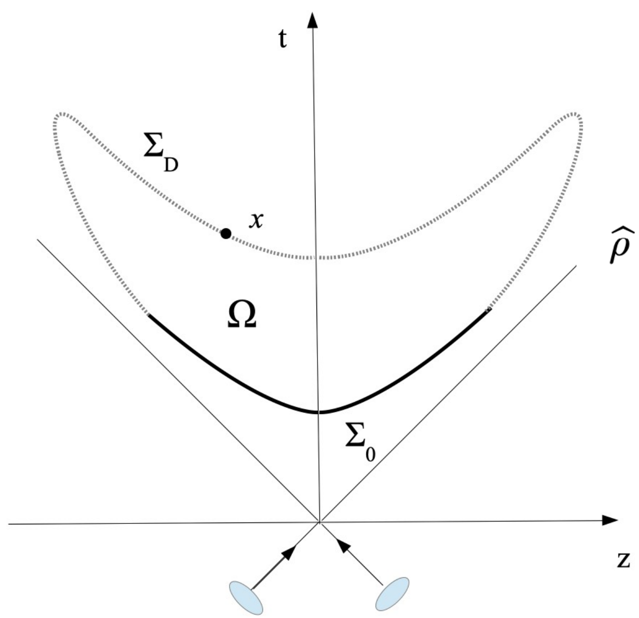
The covariant Wigner function is the mean value of the Wigner operator

$$W(x,k) \equiv \mathrm{Tr}\left[\widehat{\rho}\,\widehat{W}(x,k)\right] ,$$

Density operator

The density operator to describe the collision after local equilibrium is achieved:

$$\widehat{\rho} = \widehat{\rho}_{\rm LE}(\tau_0) = \frac{1}{Z} \exp\left[- \int_{\Sigma_0} {\rm d}\Sigma_\mu(y) \left(\widehat{T}^{\mu\nu}(y)\beta_\nu(y) - \widehat{j}^\mu(y)\zeta(y) \right) \right] ,$$



The density operator can be rewritten using the Gauss theorem:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma_D} {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta\widehat{j}^\mu \right) + \int_\Omega {\rm d}^4 y \left(\widehat{T}^{\mu\nu}(y)\partial_\mu\beta_\nu(y) - \partial_\mu\zeta(y)\widehat{j}^\mu(y) \right) \right]$$

local equilibrium *dissipation*

and the mean value of an operator at the point *x* on the decoupling hypersurface:

$$\langle \widehat{O}(x) \rangle = {\rm Tr}(\widehat{\rho}\,\widehat{O}(x)) = {\rm Tr}(\widehat{\rho}_{\rm LE}\widehat{O}(x)) + \Delta O(x) = \langle \widehat{O}(x) \rangle_{\rm LE} + \Delta O(x)$$

Local equilibrium part *Dissipative part*

A new method to calculate the LTE term

Instead – as we did in previous studies – of Taylor-expanding the fields, we rewrite:

$$\beta_\nu(y) = \beta_\nu(x) + (\beta_\nu(y) - \beta_\nu(x)) = \beta_\nu(x) + \Delta\beta_\nu(y,x)$$

$$\zeta(y) = \zeta(x) + (\zeta(y) - \zeta(x)) = \zeta(x) + \Delta\zeta(y,x),$$

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-\beta(x)\cdot P + \zeta(x)\widehat{Q} - \int_{\Sigma_D} \mathrm{d}\Sigma_\mu(y)\left(\widehat{T}^{\mu\nu}(y)\Delta\beta_\nu(y,x) - \widehat{j}^\mu(y)\Delta\zeta(y,x)\right)\right] ,$$

Main term: global equilibrium at β(x), ζ(x)

Perturbation: Δβ, Δζ, small in the hydrodynamic limit

So, for a general local operator, truncating at the linear order (linear response), we have:

$$O_{\rm LE}(x) = O_{\rm GE}(x) + \Delta O_{\rm LE}(x)\,,$$

With

$$\langle \widehat{O}(x)\rangle_{\rm GE} = \frac{{\rm Tr}({\rm e}^{\widehat{A}}\widehat{O}(x))}{{\rm Tr}({\rm e}^{\widehat{A}})} = \frac{{\rm Tr}({\rm e}^{-\beta(x)\cdot\widehat{P}+\zeta(x)\widehat{Q}}\widehat{O}(x))}{{\rm Tr}({\rm e}^{-\beta(x)\cdot\widehat{P}+\zeta(x)\widehat{Q}})}\,,$$

$$\Delta O_{\rm LE}(x) = -\int_{\Sigma_{\rm D}} {\rm d}\Sigma_\mu(y)\int_0^1 {\rm d}z\;\Delta\beta_\nu(y,x) \times\left\langle \widehat{O}(x),\,{\rm e}^{-z\beta(x)\cdot\widehat{P}}\widehat{T}^{\mu\nu}(y){\rm e}^{z\beta(x)\cdot\widehat{P}}\right\rangle_{c,{\rm GE}}$$

$$\widehat{A} = -\beta(x)\cdot\widehat{P}$$

Expanding in $\Delta\beta$ instead of its gradients is highly beneficial to evaluate the Wigner function, as we will see

Working out the Wigner function

$$\langle\widehat{W}^+(x,k)\rangle \simeq \langle\widehat{W}^+(x)\rangle_{\mathrm{GE}} + \Delta W^+(x,k) \ ,$$

Responsible for spin polarization

By using:

$$\widehat{T}^{\mu\nu}(y) = \mathrm{e}^{\mathrm{i}\widehat{P}\cdot y}\widehat{T}^{\mu\nu}(0)\mathrm{e}^{-\mathrm{i}\widehat{P}\cdot y} \ , \qquad \widehat{j}^{\mu}(y) = \mathrm{e}^{\mathrm{i}\widehat{P}\cdot y}\widehat{j}^{\mu}(0)\mathrm{e}^{-\mathrm{i}\widehat{P}\cdot y} \ ,$$

$$\mathrm{e}^{\mathrm{i}\widehat{P}\cdot y}\widehat{a}^\dagger(p)\mathrm{e}^{-\mathrm{i}\widehat{P}\cdot y} = \mathrm{e}^{\mathrm{i}p\cdot y}\widehat{a}^\dagger(p) \ ,$$
$$\mathrm{e}^{\mathrm{i}\widehat{P}\cdot y}\widehat{a}(p')\mathrm{e}^{-\mathrm{i}\widehat{P}\cdot y} = \mathrm{e}^{-\mathrm{i}p'\cdot y}\widehat{a}(p') \ ,$$

$$\Delta W^+_{\mathrm{LE}}(x,p) = \frac{1}{2(2\pi)^3}\sum_{r,s}\int_{\Sigma_\mathrm{D}}\mathrm{d}\Sigma_\mu(y)\int \mathrm{d}^4q\ \mathrm{e}^{iq\cdot(x-y)}$$
$$\times \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta(p\cdot q)\delta\left(p^2+\frac{q^2}{4}-m^2\right)\theta(p^0_+)\theta(p^0_-)$$
$$\times\, u_s(p_-)\bar{u}_r(p_+)\left\langle \widehat{a}^\dagger_r(p_+)\widehat{a}_s(p_-), \widehat{T}^{\mu\nu}(0)\right\rangle_{c,\mathrm{GE}}\Delta\beta_\nu(y,x) \ .$$

In previous derivations, the integration in dΣ was done (tacitly or not) first, with *strong geometric assumptions* on the shape of the hypersurfaces.

In the new method, we invert the integrations: first d*q*, then dΣ

Rewriting

$$\Delta W^{+}_{\rm LE}(x,p) = \frac{1}{(2\pi)^3} \int {\rm d}^4 q \; \delta(p\cdot q) G^{\mu\nu}(q) F_{\mu\nu}(q) \, ,$$

with:

$$G^{\mu\nu}(q) = \frac{1}{2}\delta\left(p^2 + \frac{q^2}{4} - m^2\right) \frac{1-{\rm e}^{\beta(x)\cdot q}}{\beta(x)\cdot q} \theta(p^0_+)\theta(p^0_-)$$
$$\times \sum_{r,s} u_s(p_-)\bar{u}_r(p_+) \left\langle \widehat{a}^\dagger_r(p_+)\widehat{a}_s(p_-), \widehat{T}^{\mu\nu}(0)\right\rangle_{c,{\rm GE}} \qquad (9)$$

$$F_{\mu\nu}(q) = \int_{\Sigma_{\rm D}} {\rm d}\Sigma_\mu(y)\; {\rm e}^{-iq\cdot(y-x)} \Delta\beta_\nu(y,x) \, .$$

In the hydrodynamic limit, the *F* is PEAKED around *q=0 :* we can then expand the *G(q)* around *q=0* and integrate term by term in *q*.

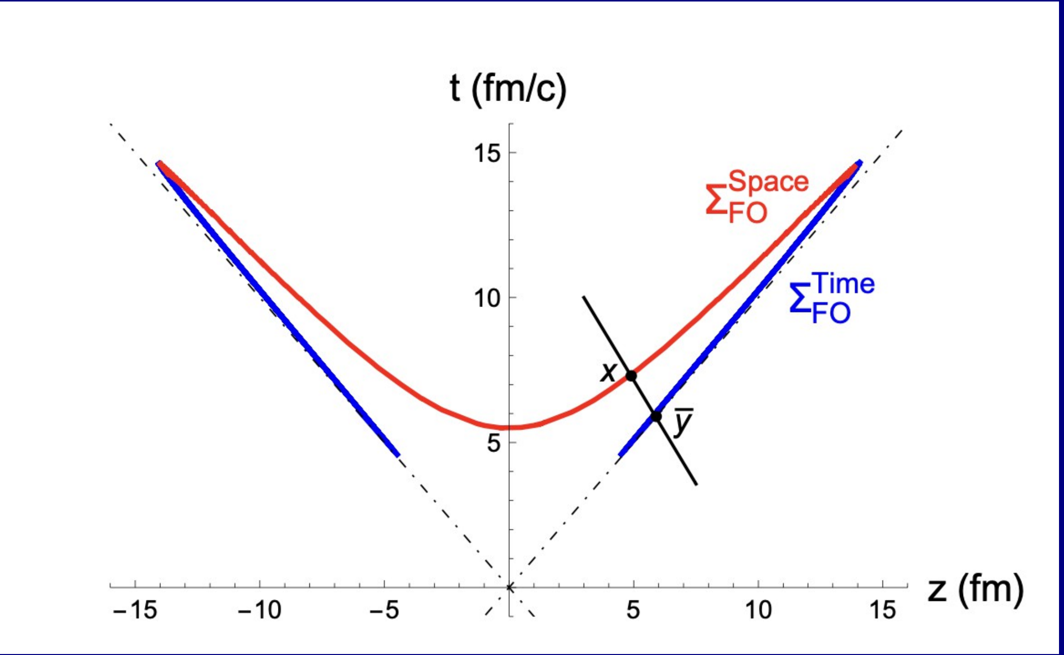
We obtain:

$$\Delta W^{+}_{\mathrm{LE}}(x,p)= \frac{1}{(2\pi)^3} \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma_\mu(y)\, I_n^{\nu_1\nu_2\cdots\nu_N}(y-x) \times \Delta\beta_\nu(y,x) \left[\partial^q_{\nu_1}\partial^q_{\nu_2}\cdots\partial^q_{\nu}G^{\mu\nu}(q)\right]\Big|_{q=0} \, , \quad (10)$$

$$I_N^{\nu_1\nu_2\cdots\nu_n}(y-x) \equiv \int \mathrm{d}^4 q\, \delta(p\cdot q) \mathrm{e}^{-iq\cdot(y-x)} q^{\nu_1}\cdots q^{\nu_N}$$
$$= (2\pi)^3 \frac{(-i)^N}{|p^0|} \partial_x^{\nu_1}\cdots\partial_x^{\nu_N}\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{p}}{p^0}(y^0-x^0)\right).$$

The δ(k·q) leads to the appearance of Dirac deltas and derivatives thereof, selecting a world-line

$$\mathbf{y} = \mathbf{x} - \frac{\mathbf{p}}{p^0}(y^0-x^0)$$



possibly having multiple intersections with the the decoupling hypersurface besides the trivial one *y*=*x*

Correlation function

$$\Delta W^{+}_{\rm LE}(x,p) = \frac{1}{(2\pi)^3} \int_{\Sigma_{\rm D}} {\rm d}\Sigma_\mu(y) \Delta\beta_\nu(y,x) \times \left[\int {\rm d}^4 q \; \delta(p\cdot q) {\rm e}^{iq\cdot(x-y)} G^{\mu\nu}(q) \right]$$

$$C_{WT}(x-y) = \int {\rm d}^4 q \; \delta(k\cdot q) {\rm e}^{{\rm i}q\cdot(x-y)} G^{\mu\nu}(q) = -(2\pi)^3 \int_0^1 {\rm d}z \; \langle \widehat{W}^{+}(x,k), {\rm e}^{z\widehat{A}} \widehat{T}^{\mu\nu}(y) {\rm e}^{-z\widehat{A}} \rangle_{c,{\rm GE}} \; .$$

Due to the $\delta(k{\cdot}q)$, the correlation function is constant over the world-line

$$\mathbf{y} = \mathbf{x} - \frac{\mathbf{p}}{p^0}(y^0 - x^0)$$

and not only for $x=y$!

Result for the Wigner function in linear response

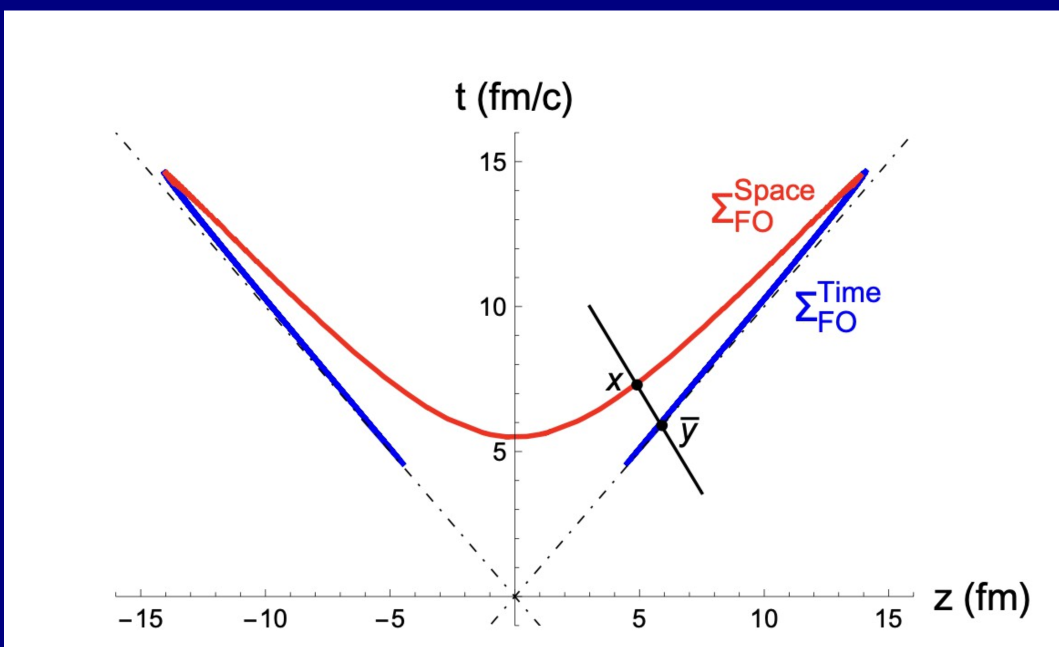
$$\Delta W^{+}_{\rm LE}(x,p) = \sum_{N=0}^{\infty} \sum_{\bar{y}(x,p)} \frac{1}{N!} D_y^N(\bar{y}) \times \left[G^{\mu\nu}(q) \frac{n_\mu(y)}{|p\cdot n(y)|} \Delta\beta_\nu(y,x) \right] \Bigg|_{q=0,y=\bar{y}(x,p)}$$

The q-expansion gave rise to a full gradient expansion

This formula includes gradients of n, that is of the geometry!

$$D_y(\bar{y}) \equiv -i\Delta^{\nu\rho}(\bar{y})\partial^y_\rho \partial^q_\nu \,,$$

$$\Delta^{\nu\rho}(\bar{y}) = g^{\nu\rho} - \frac{n^\nu(\bar{y})p^\rho}{p\cdot n(\bar{y})} \,.$$



It is a sum over the intersections of the worldline with the decoupling hypersurface

Spin polarization vector

$$S^\mu(p) = \frac{\int_{\Sigma_D} d\Sigma \cdot p \, \mathrm{tr}\left[\gamma^\mu \gamma^5 W^+(x,p)\right]}{2\int_{\Sigma_D} d\Sigma \cdot p \, \mathrm{tr}\left[W^+(x,p)\right]} \, ,$$

In the case of a free Dirac field, the exact solution for G reads

$$G^{\mu\nu}(q) = \frac{n_F(x,p_+) - n_F(x,p_-)}{(2\pi)^3(\beta \cdot q)} \delta\left(p^2 + \frac{q^2}{4} - m^2\right)$$
$$\times \theta(p_+^0)\theta(p_-^0)(\not{p}_- + m)(\gamma^\mu p^\nu + \gamma^\nu p^\mu)(\not{p}_+ + m).(2$$

$$\varpi_{\rho\lambda} = -\frac{1}{2}\left(\partial_\rho \beta_\lambda - \partial_\lambda \beta_\rho\right)$$
$$\xi_{\rho\lambda} = \frac{1}{2}\left(\partial_\rho \beta_\lambda + \partial_\lambda \beta_\rho\right)$$

$$S^\mu(p) \simeq -\frac{1}{8mN_p}\int_{\Sigma_D} d\Sigma(x) \cdot p \, n_F(1-n_F)\epsilon^{\mu\nu\rho\lambda} p_\nu$$
$$\times \sum_{\bar{y}(x,p)} \mathrm{sgn}[p \cdot n(\bar{y})] \left[\varpi_{\rho\lambda}(\bar{y}) + \frac{2n_\rho(\bar{y})\xi_{\lambda\alpha}(\bar{y})p^\alpha}{p \cdot n(\bar{y})}\right]$$

Comparison with previous expression

$$S^\mu(p) \simeq -\frac{1}{8mN_p}\int_{\Sigma_D} \mathrm{d}\Sigma(x)\cdot p\, n_F(1-n_F)\epsilon^{\mu\nu\rho\lambda}p_\nu \times \sum_{\bar{y}(x,p)} \mathrm{sgn}[p\cdot n(\bar{y})]\left[\varpi_{\rho\lambda}(\bar{y}) + \frac{2n_\rho(\bar{y})\xi_{\lambda\alpha}(\bar{y})p^\alpha}{p\cdot n(\bar{y})}\right]$$

F. B., M. Buzzegoli, A. Palermo, Phys. Lett. B 820 (2021) 136519

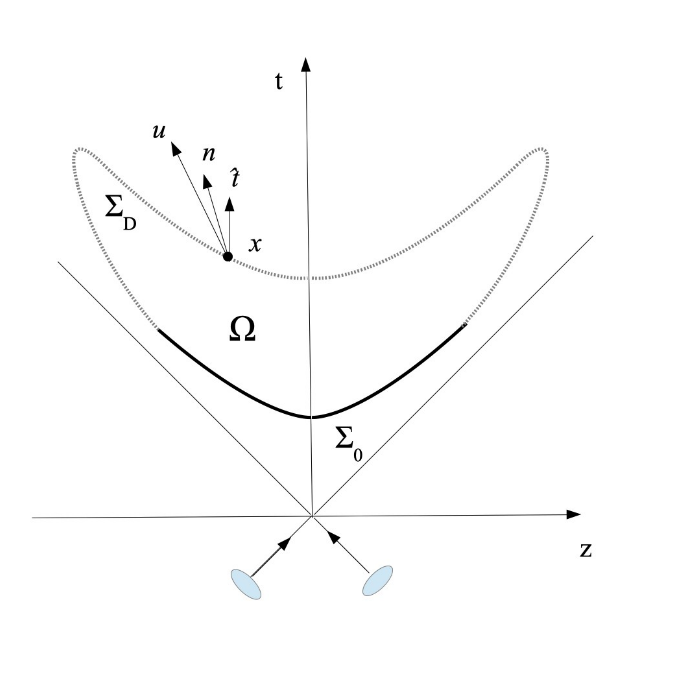
$$S^\mu_\xi(p) = -\frac{1}{4m}\epsilon^{\mu\nu\sigma\tau}\frac{p_\tau p^\rho}{\varepsilon}\frac{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F(1-n_F)\hat{t}_\nu\xi_{\sigma\rho}}{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F},$$

$\hat{t} \to n$

$$S^\mu_\xi = -\frac{1}{4m}\epsilon^{\mu\nu\sigma\tau}\frac{p_\tau p^\rho}{p\cdot u}\frac{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F(1-n_F)u_\nu\xi_{\sigma\rho}}{\int_\Sigma \mathrm{d}\Sigma\cdot p\, n_F}$$

$u \to n$

S. Liu, Y. Yin, JHEP 07 (2021) 188



Replacement of the *assumed* normal vector with the *actual* normal vector

Numerical consequence?

Discussion

• *Contribution from multiple intersections*

• A quantum interference effect or a spurious contribution due to the free-field approximation in W?

• *Contribution from the gradients of n are possible (curvature contributions)*

Yet, no contribution at the first order!

• *Isothermal hypersurface*

For a T=const hypersurface, the first order gradients in T do not contribute thanks to the combination $\left[\varpi_{\rho\lambda}(\bar{y}) + \frac{2n_\rho(\bar{y})\xi_{\lambda\alpha}(\bar{y})p^\alpha}{p\cdot n(\bar{y})}\right]$

In the new formula, only the gradients *along* the hypersurface matter.

• *Comparison with previous expressions*

A sign factor in the thermal vorticity expression; no difference if the hypersurface is space-like

Conclusions

• An improved formula for the spin polarization vector at local thermodynamic equilibrium properly including the geometry of the decoupling hypersurface

• Additional contribution from world-line intersections: phyisical or unphysical?

• Need of numerical test on relativistic hydrodynamic codes

The particle part of the Wigner operator can be written, by using the field expansion in plane waves:

$$\widehat{W}^{+}(x,p) = \frac{1}{2(2\pi)^3} \sum_{r,s} \int \mathrm{d}^4 q \; \mathrm{e}^{iq\cdot x} \theta(p_+^0)\theta(p_-^0)\delta(p\cdot q)$$
$$\times \delta\left(p^2 + \frac{q^2}{4} - m^2\right) \widehat{a}_r^{\dagger}(p_+)\widehat{a}_s(p_-) u_s(p_-)\bar{u}_r(p_+) \, , \, ($$

$$p_{\pm} = p \pm q/2$$

For instance, the momentum spectrum of the particles can be written as the integral of the Wigner function over an arbitrary hypersurface $\Longrightarrow$ *k* goes on-shell!

$$\frac{\mathrm{d}N_k}{\mathrm{d}^3\mathrm{k}} = \int \mathrm{d}k^0 \int_{\Sigma} \mathrm{d}\Sigma_{\mu} k^{\mu} \; W^{+}(x,k) = \frac{1}{2\varepsilon_{\mathbf{k}}} \langle \widehat{a}^{\dagger}(k)\widehat{a}(k) \rangle \; ,$$

NOTE: *in a scattering experiment, the creation-annihilation operators and the fields should be replaced – to be precise –* *with out-operators and fields*

$$\frac{\mathrm{d}N_k}{\mathrm{d}^3\mathrm{k}} = \lim_{t\to+\infty} \int \mathrm{d}k^0 \int_{\Sigma} \mathrm{d}\Sigma_{\mu} k^{\mu} \; W^{+}(x,k) = \frac{1}{2\varepsilon} \langle \widehat{a}_{\mathrm{out}}^{\dagger}(k)\widehat{a}_{\mathrm{out}}(k) \rangle$$

Discussion

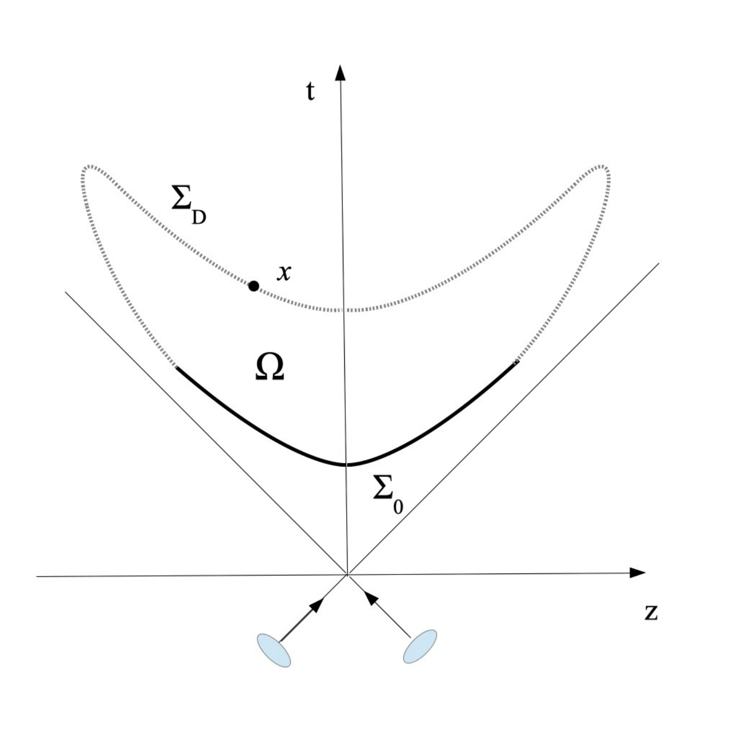
The point *x* should be chosen such as:

1) the thermo-hydrodynamics fields still exist
2) the approximation of out-fields is a good one

$$\widehat{W}(x,k) = \frac{2}{(2\pi)^4} \int \mathrm{d}^4 s \, \mathrm{e}^{-\mathrm{i}s\cdot k} : \widehat{\psi}^\dagger \left(x + \frac{s}{2}\right) \widehat{\psi} \left(x - \frac{s}{2}\right) :$$

$$\widehat{\psi}(x) = \widehat{\psi}_{\rm out}(x) - \int \mathrm{d}^4 y \, \theta(y^0 - x^0) \Delta(x-y) \widehat{J}(y)$$

$$(\Box + m^2)\widehat{\psi}(x) = \widehat{J}(x)$$



x on the *decoupling* (particlization) hypersurface

With this approximation post-decoupling interactions are neglected, i.e. the evolution of *W* for hadronic collisions is not taken into account