Resolving the PREX-CREX puzzle in nuclear density functional theory

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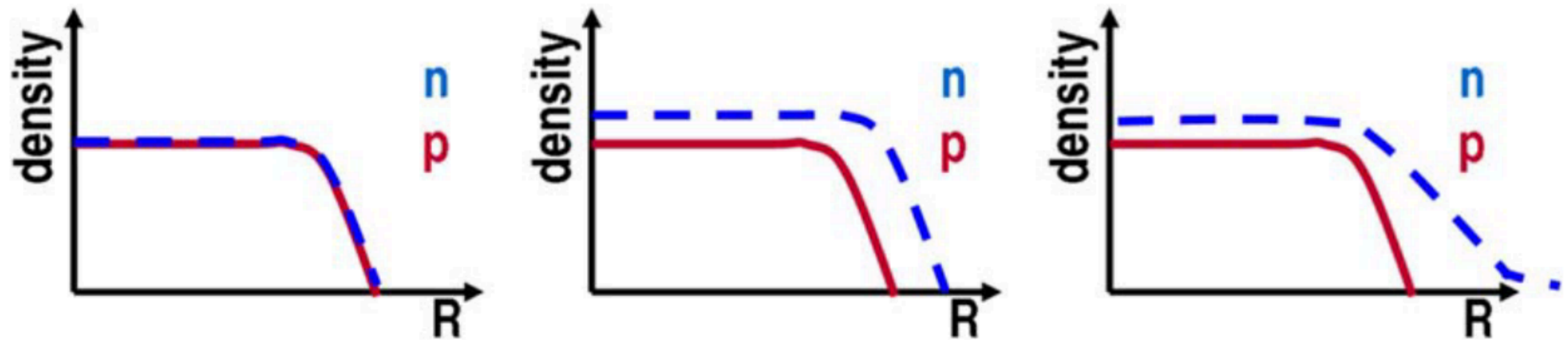
"Nuclear physics across energy scales", C3NT, Wuhan,
9.19-21, 2025

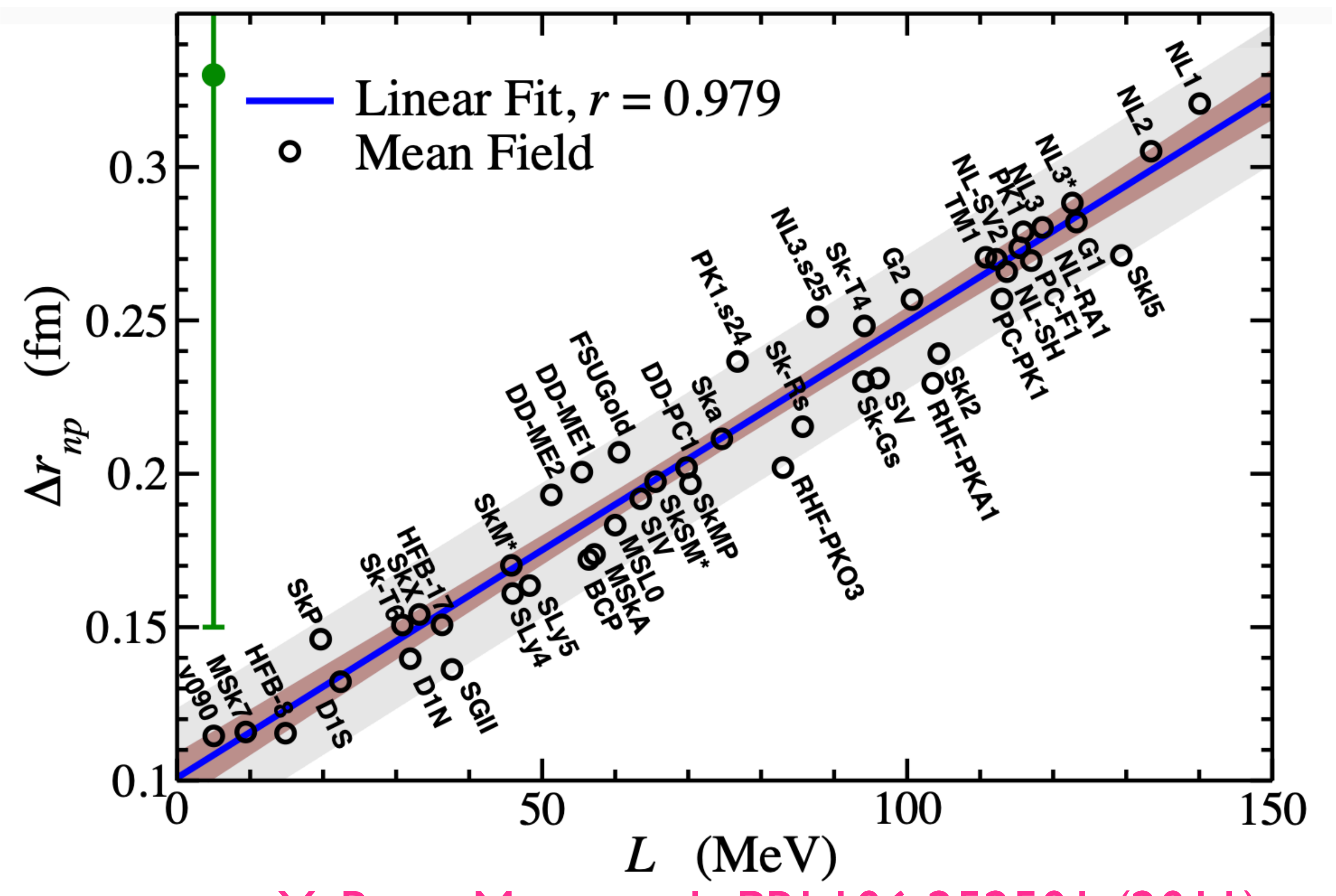
Outline

- ✦ Introduction: PREX-CREX puzzle
- ✦ Resolving the PREX-CREX puzzle via a strong isovector spin-orbit interaction
  - ✓ Nonrelativistic extended Skyrme energy density functional (EDF)
    T.G. Yue, **ZZ,** L.W. Chen, arXiv: 2406.03844
  - ✓ Relativistic density-dependent point-coupling EDF
    M. Qiu, **ZZ,** T.G. Yue, L.W. Chen, in preparation
- ✦ Summary

Neutron rms radius and neutron skin



M. Thiel et al. JPG 46 (2019) 093003



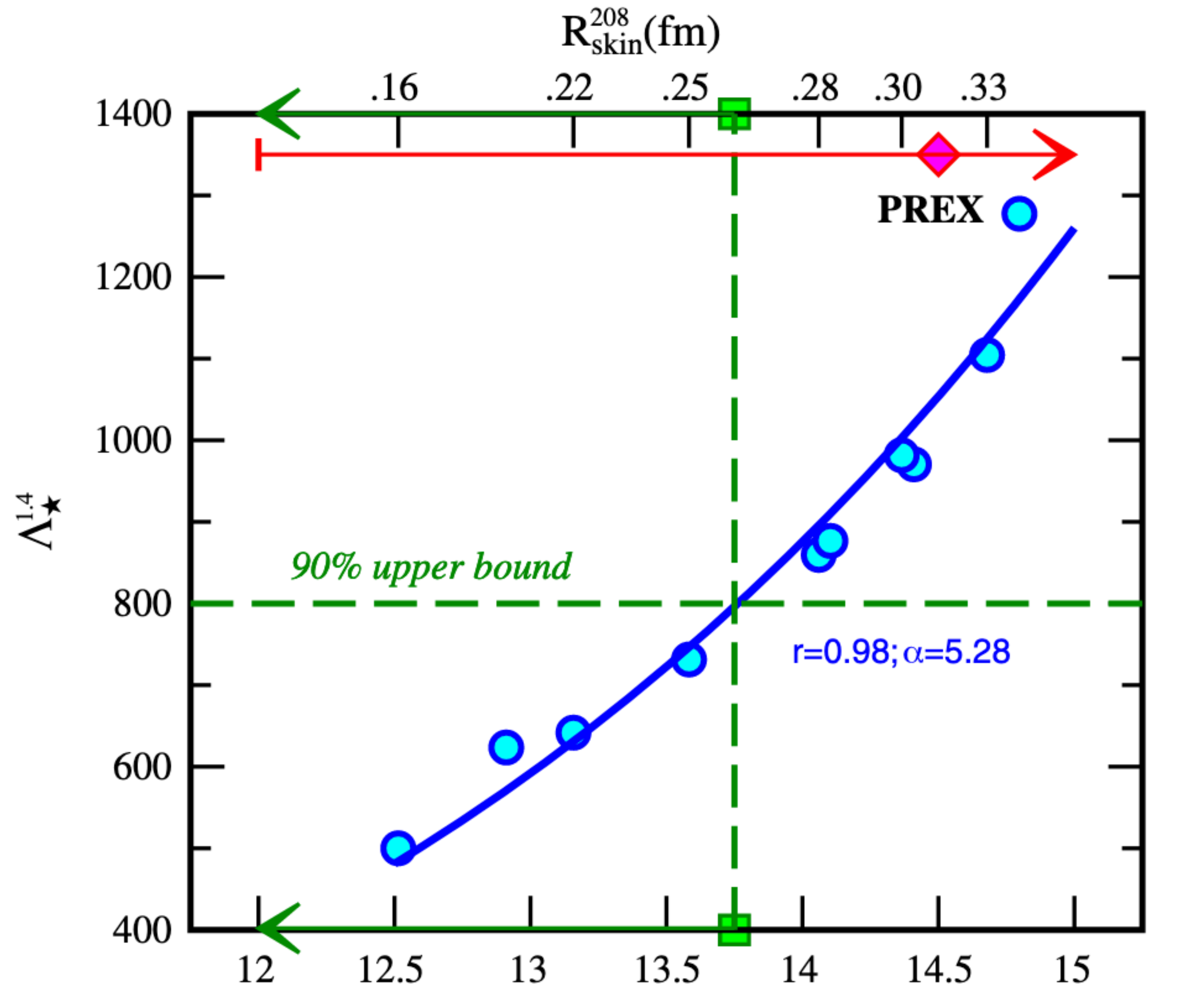
X. Roca-Maza et al., PRL106 252501 (2011)

Neutron skin thickness

- $\Delta r_{np} = R_n - R_p$
  - R_n: neutron rms radius
  - R_n: proton rms radius

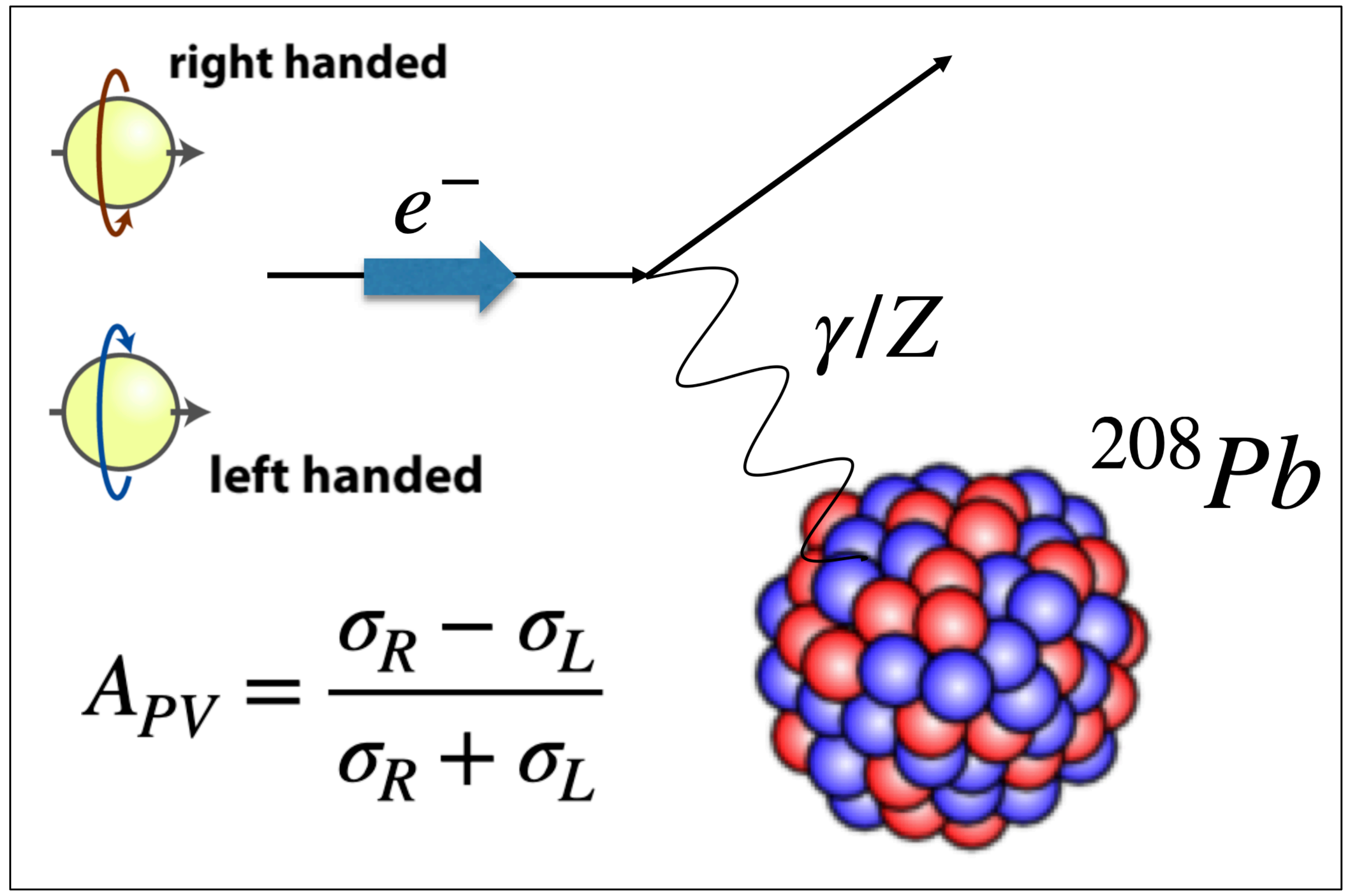
B. A. Brown, PRL (2000)
R. J. Furnstahl, NPA (2002)
L..-W. Chen, C. M. Ko & B.A. Li, PRC (2005)
M. Centelles et al., PRL102, 122502 (2009)
L.-W. Chen et al., PRC82, 024321 (2010)
ZZ & L.-W. Chen, PLB 726, 234 (2013)

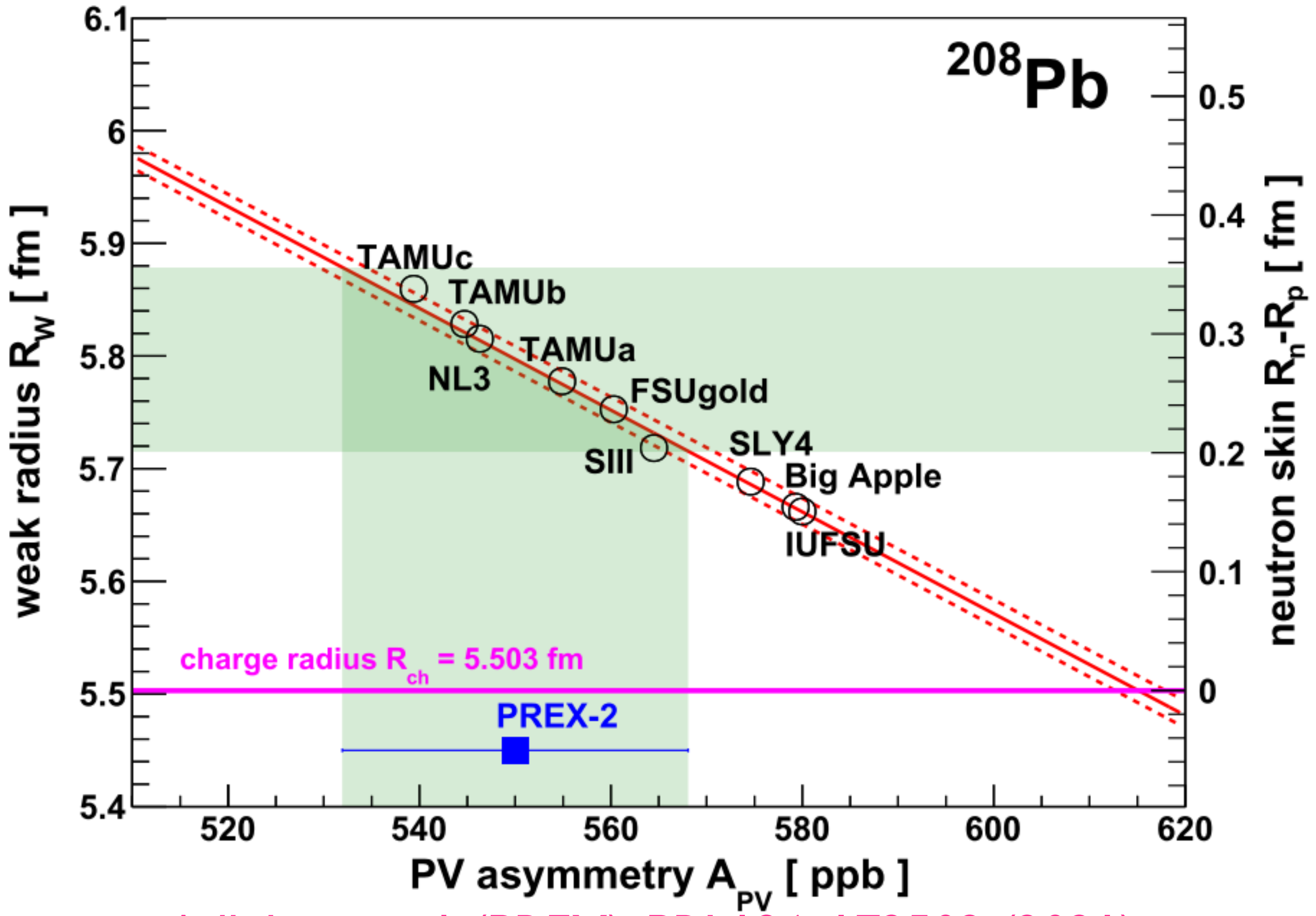
- an ideal probe of nuclear symmetry energy
- important for both nuclear physics and astrophysics.



F. J. Fattoyev, et al., PRL120, 172702 (2018)

PREX: Lead (^{208}Pb) Radius Experiment





Adhikari et al. (PREX), PRL126, 172502 (2021)

- Parity-violating asymmetry in longitudinally polarized elastic electron scattering :

$$A_{PV} = \frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L} \approx \frac{G_F Q^2 |Q_W|}{4\sqrt{2}\pi\alpha Z} \frac{F_W(Q^2)}{F_{ch}(Q^2)}$$

Donnelly et al., NPA503, 589 (1989);
Horowitz et al., PRC63, 025501 (2001).

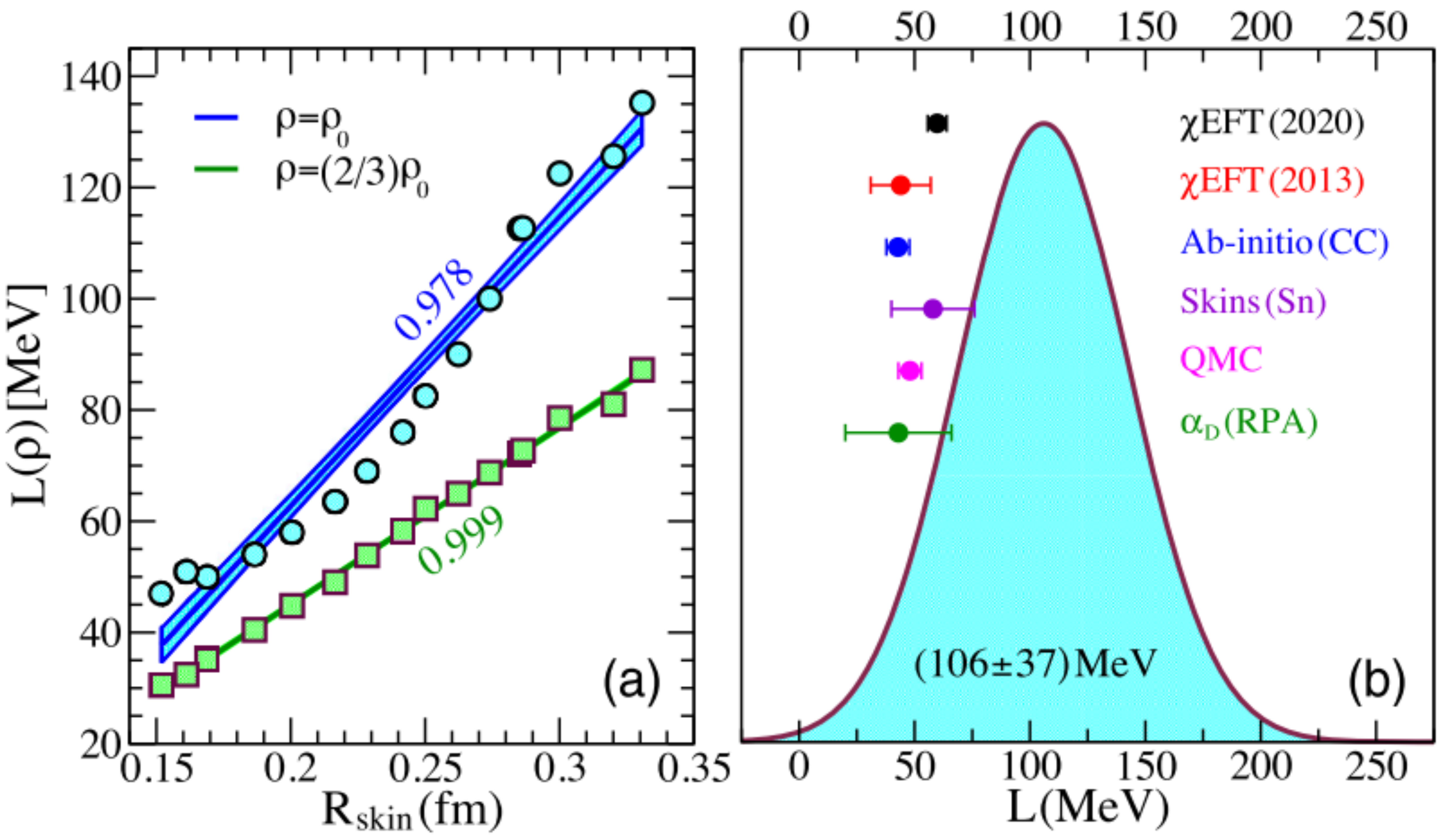
- Free from most strong interaction uncertainties.

- PREX-2 results ($\langle Q^2 \rangle = 0.00616\ \mathrm{GeV}^2$) :

$$A_{\mathrm{PV}}^{\mathrm{meas}} = 550 \pm 16(\text{ stat }) \pm 8(\text{ syst })\mathrm{ppb}$$
$$F_W(\langle Q^2 \rangle) = 0.368 \pm 0.013(\exp) \pm 0.001(\text{ theo })$$

^{208}Pb Parameter	Value
Weak radius (R_W)	5.800 ± 0.075 fm
Interior weak density (ρ_W^0)	$-0.0796 \pm 0.0038\ \mathrm{fm}^{-3}$
Interior baryon density (ρ_b^0)	$0.1480 \pm 0.0038\ \mathrm{fm}^{-3}$
Neutron skin ($R_n - R_p$)	0.283 ± 0.071 fm

Super stiff symmetry energy from PREX-2

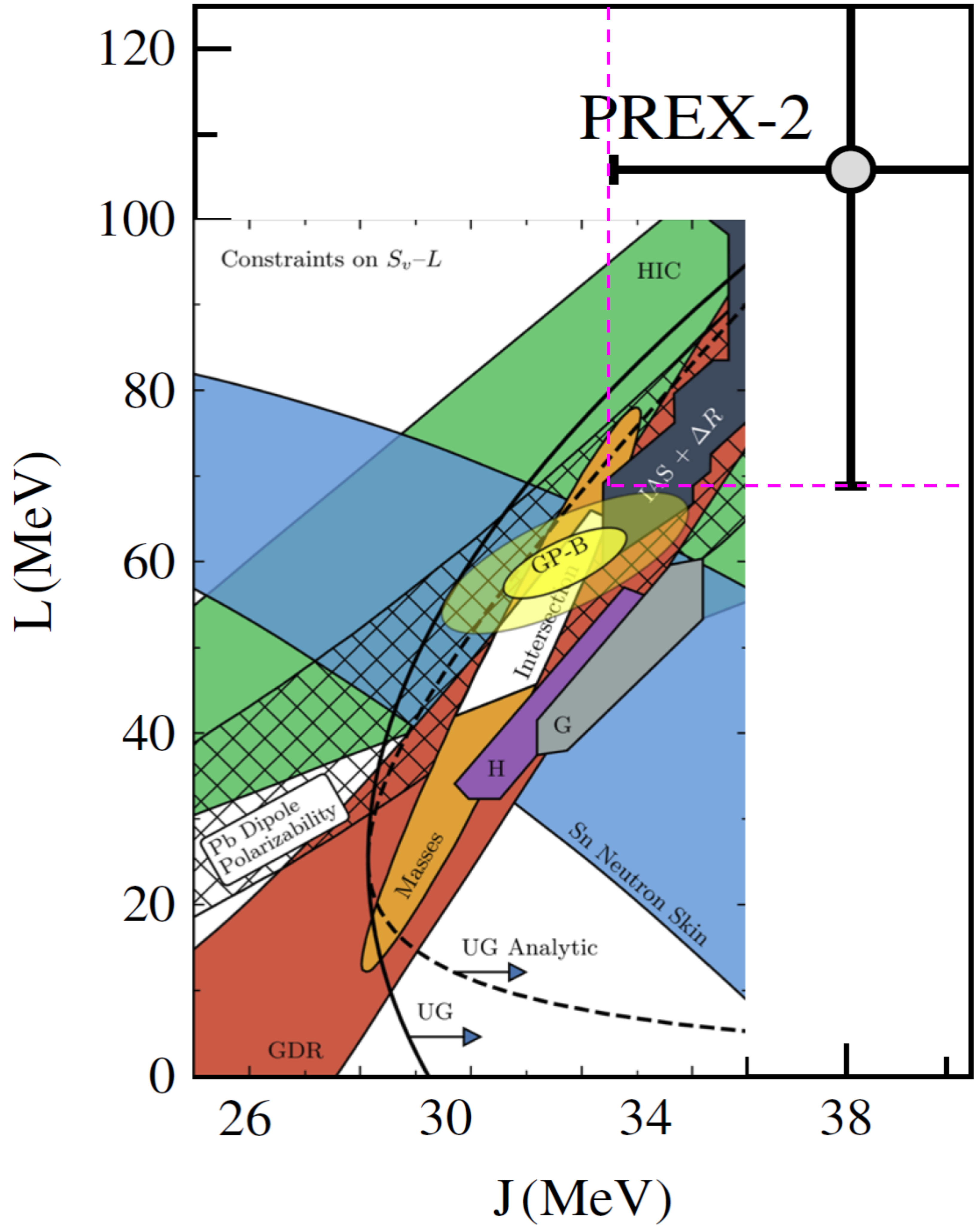


- **Super stiff symmetry energy from relativistic EDF analysis:**

$$J = (38.1 \pm 4.7)\ \text{MeV},$$

$$L = (106 \pm 37)\ \text{MeV},$$

- **Challenge our understanding of the symmetry energy.**



Reed et al., PRL126, 172503 (2021)

CREX: Calcium (^{48}Ca) Radius Experiment

- Model-independent determination of **charge-weak form factor difference**:

$$\Delta F^{48}_{\mathrm{CW}}(q) = 0.0277 \pm 0.0055, \quad q = 0.8733\mathrm{fm}^{-1}, \mathit{CREX}$$

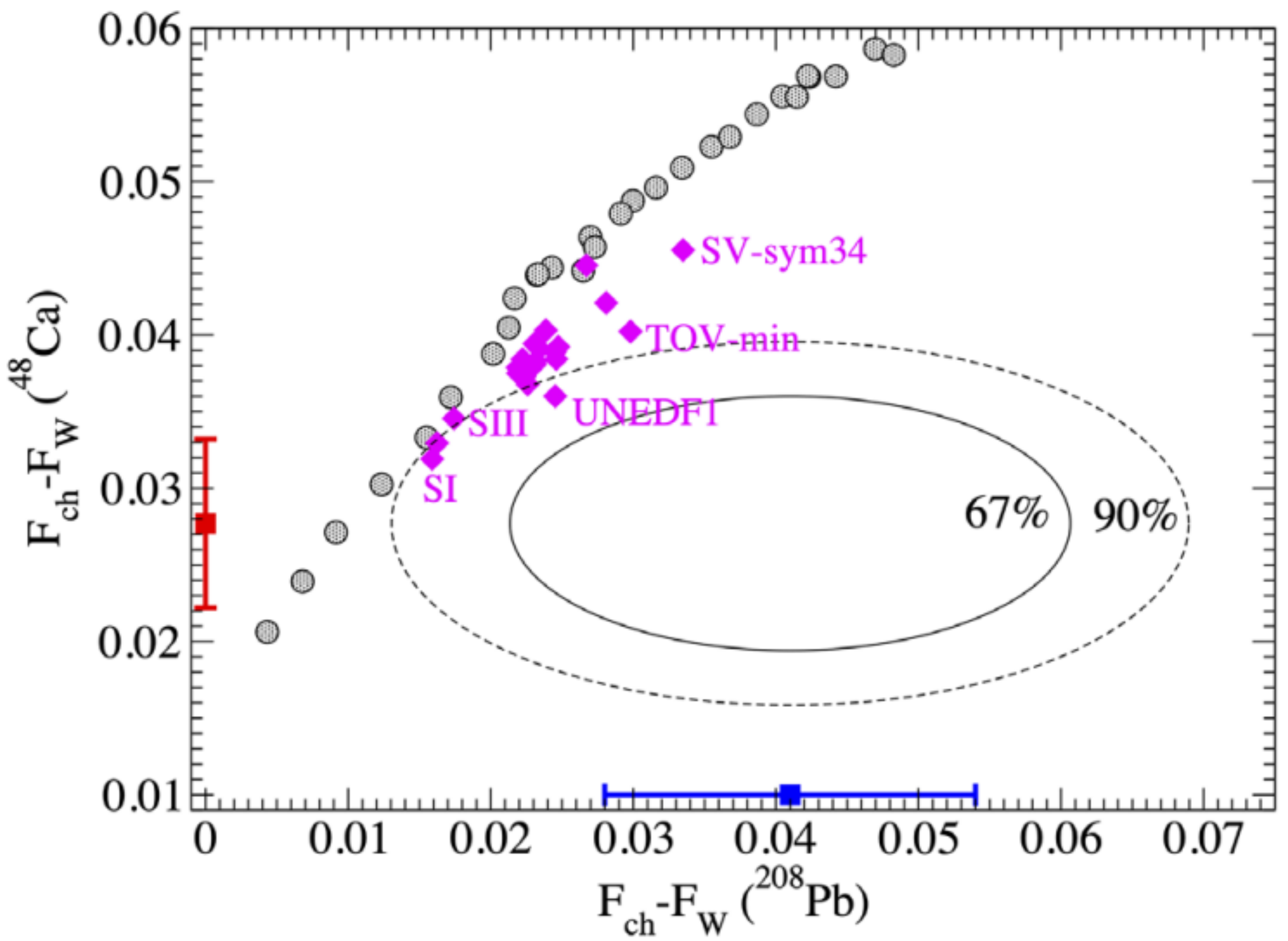
$$\Delta F^{208}_{\mathrm{CW}}(q) = 0.041 \pm 0.013, \quad q = 0.3977\mathrm{fm}^{-1}, \mathit{PREX}$$

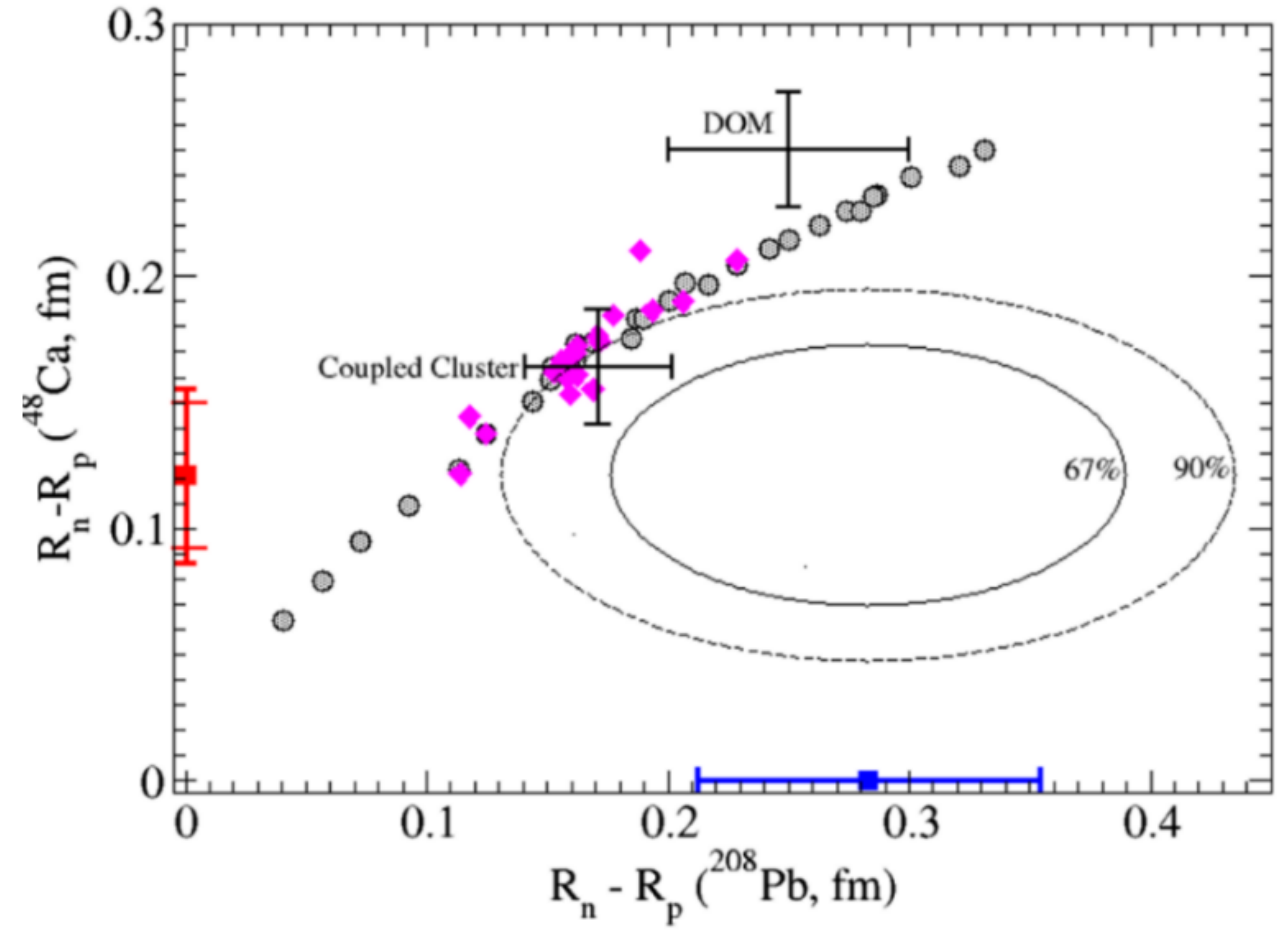
Adhikari et al. (CREX), PRL129, 042501 (2022)

- Extracted neutron skin of Ca48

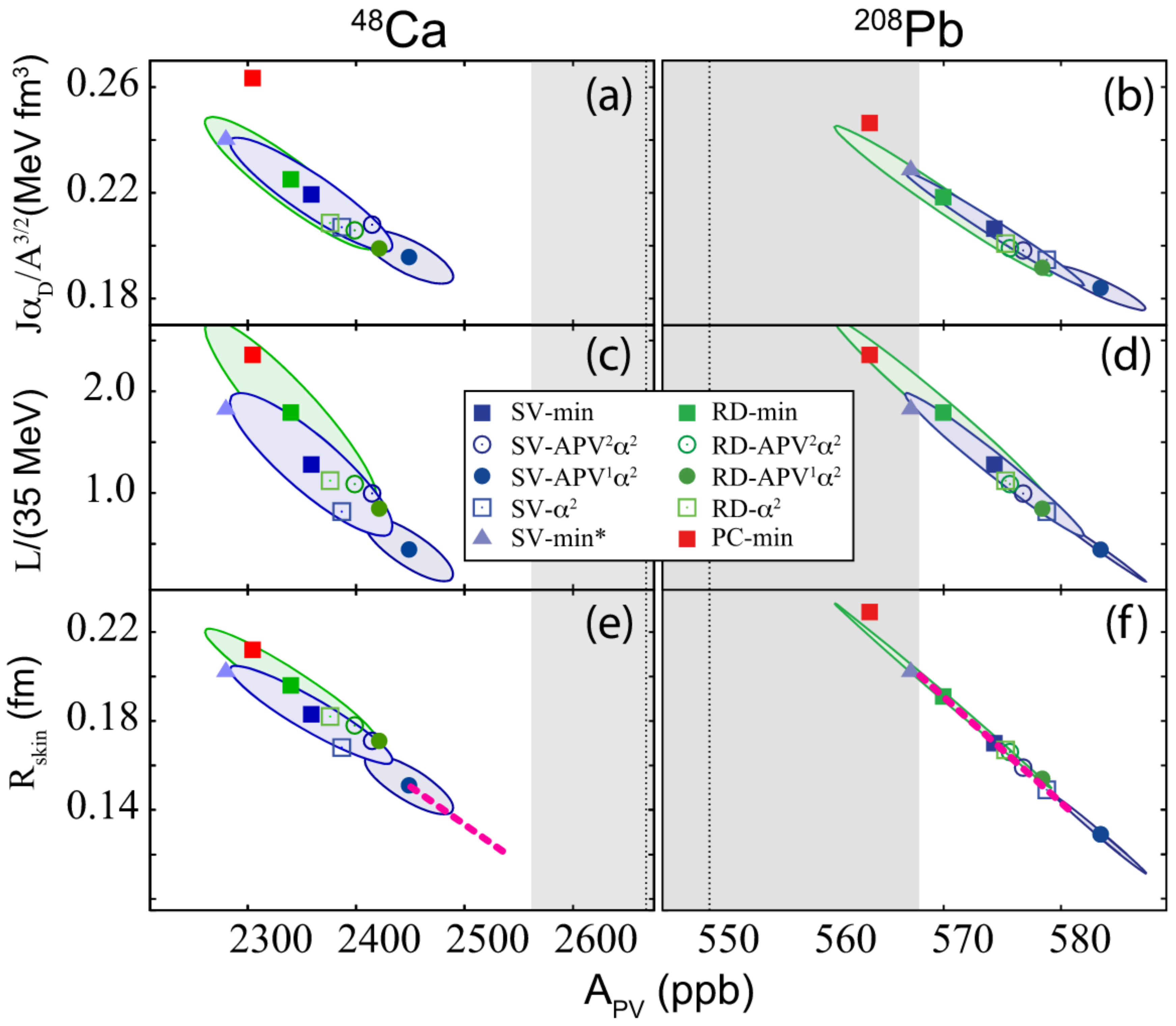
Quantity	Value ± (exp) ± (model) (fm)
$R_W - R_{\mathrm{ch}}$	0.159 ± 0.026 ± 0.023
$R_n - R_p$	0.121 ± 0.026 ± 0.024

- Too thin neutron skin of ^{48}Ca or too thick neutron skin of ^{208}Pb?

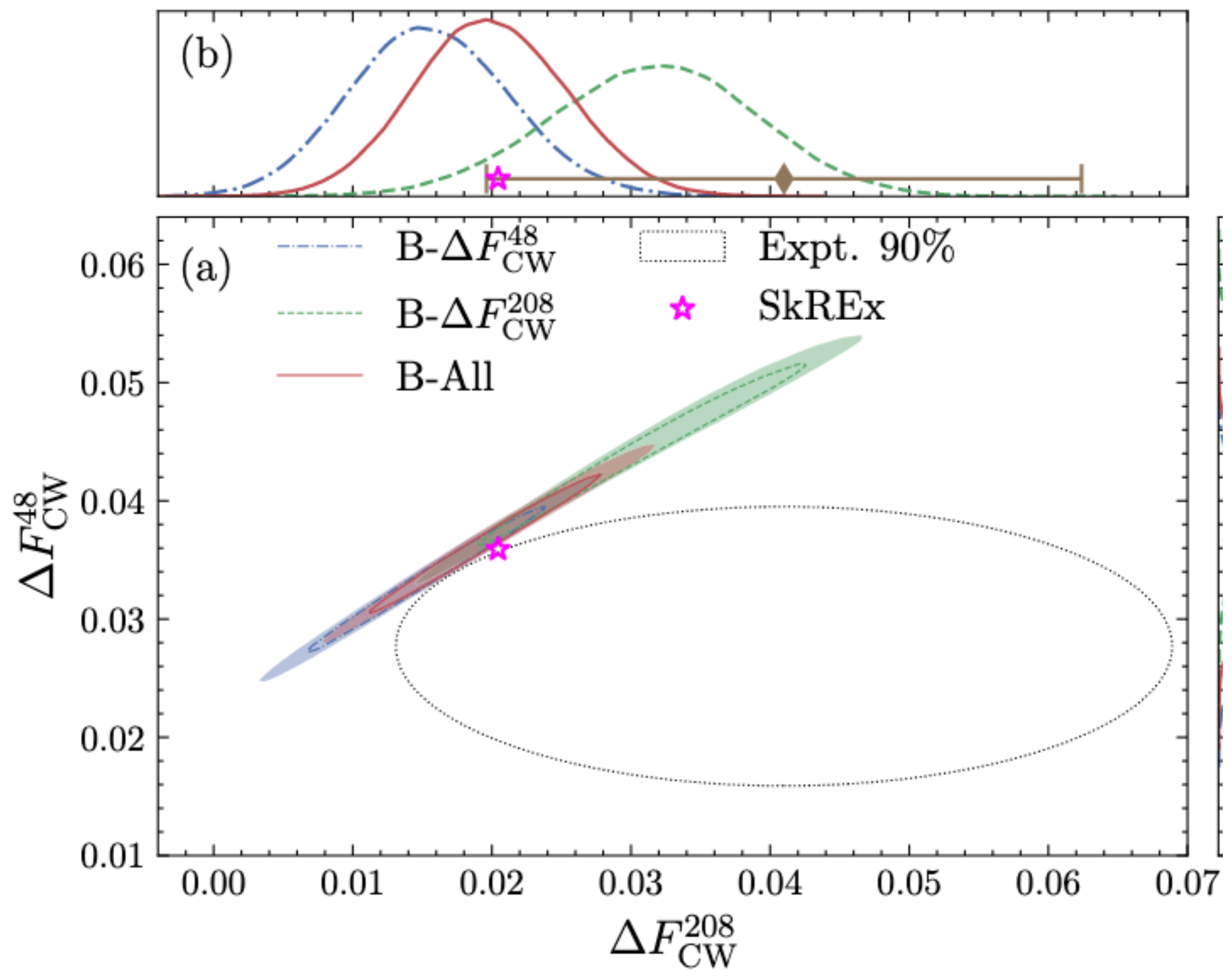


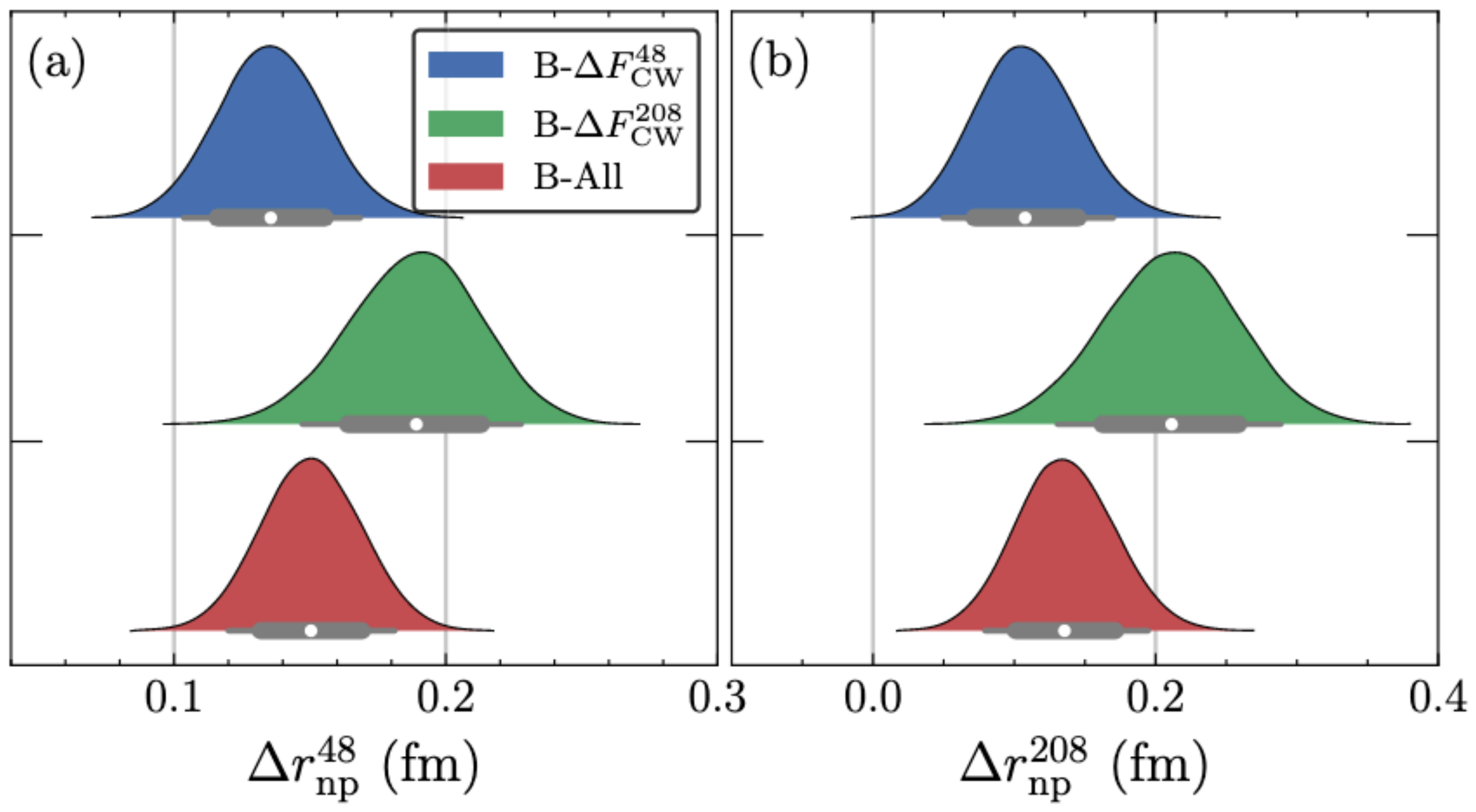


PREX-CREX puzzle

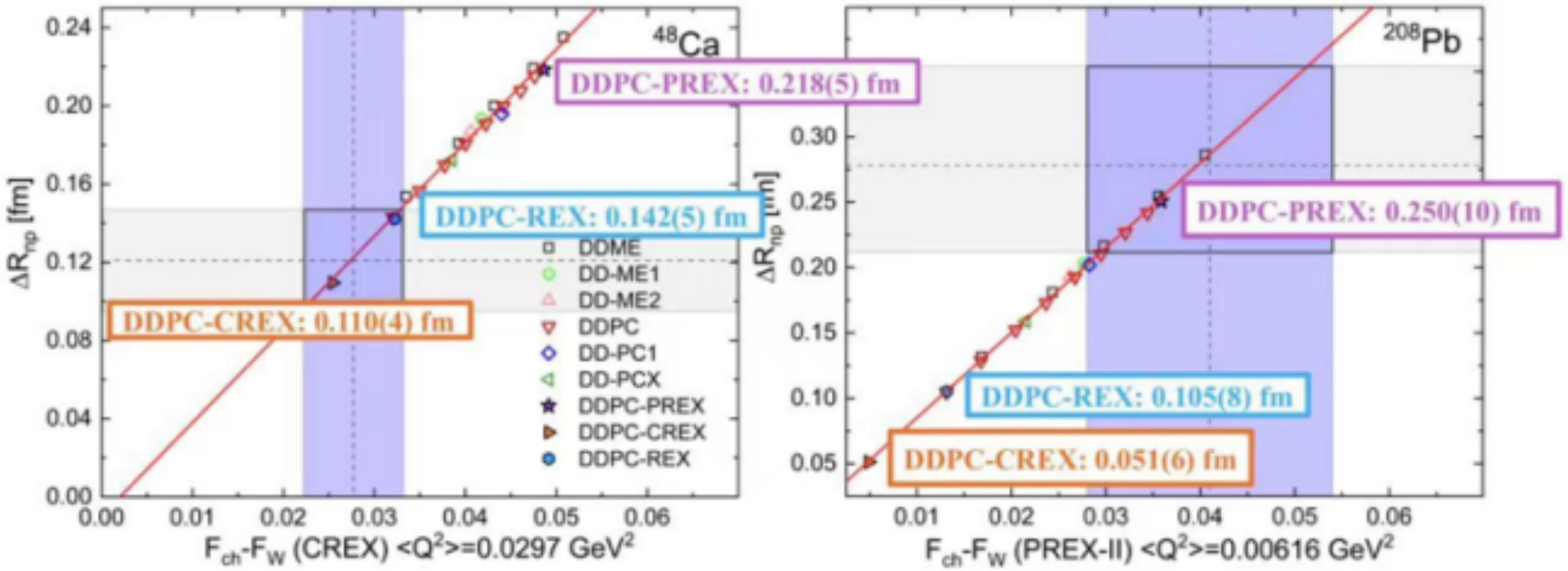


Reinhard, Roca-Maza & Nazarewicz PRL 129, 232501 (2022)





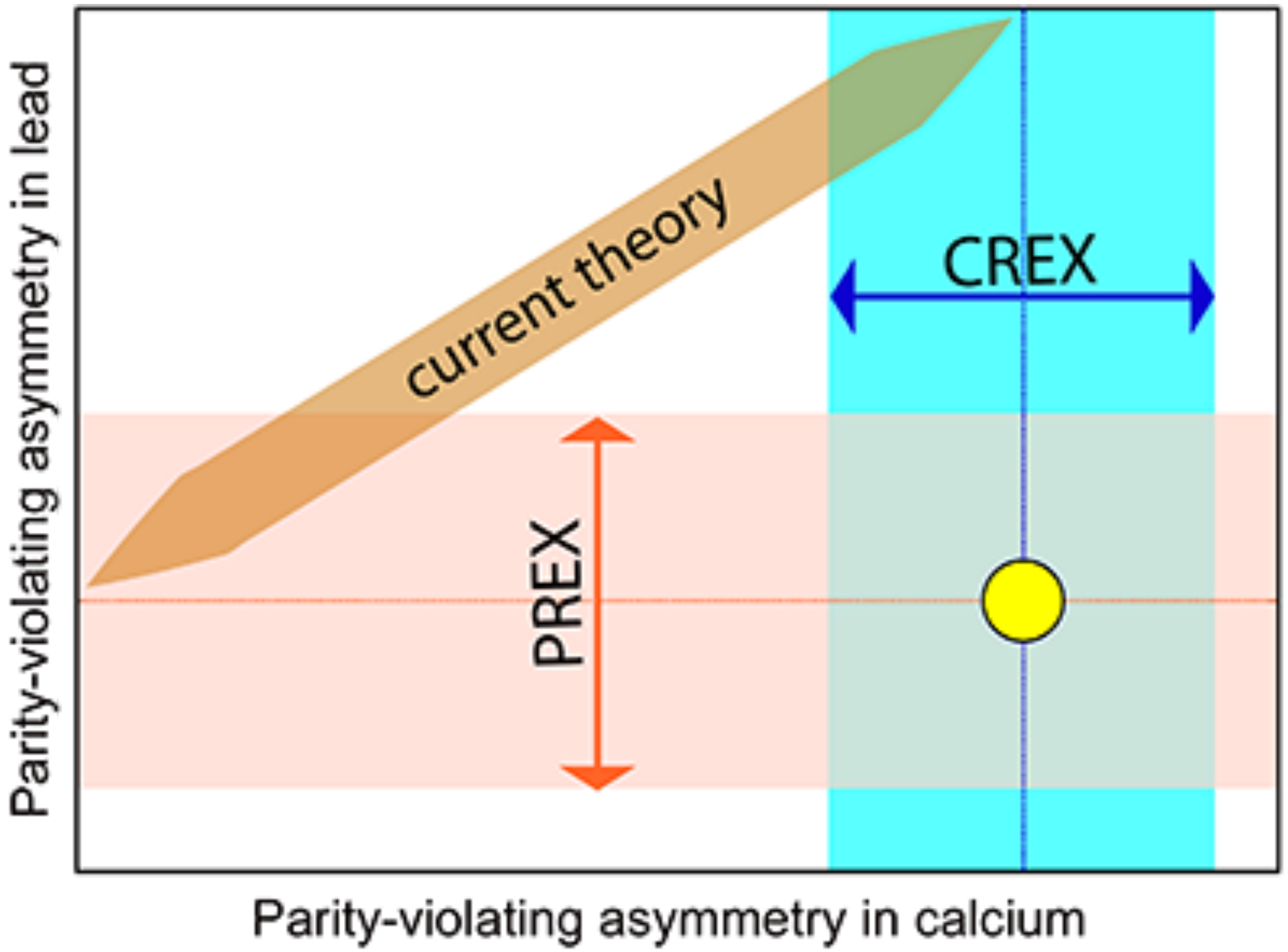
ZZ & L.W. Chen, PRC 108, 024317 (2023)



Yüksel & Paar, PLB 836, 137622 (2023)

✦Tension between CREX and PREX-2 results!

PREX-CREX puzzle



https://frib.msu.edu/news/2022/prl-paper.html

What is missing in the theory?

α-clustering effects?

S. Yang, R.J. Li, and C. Xu, PRC108, L021303 (2023)

Symmetry energy?

Reed et al., PRC109, 035804 (2024)

Salinas & Piekarewicz PRC109, 045807 (2024)

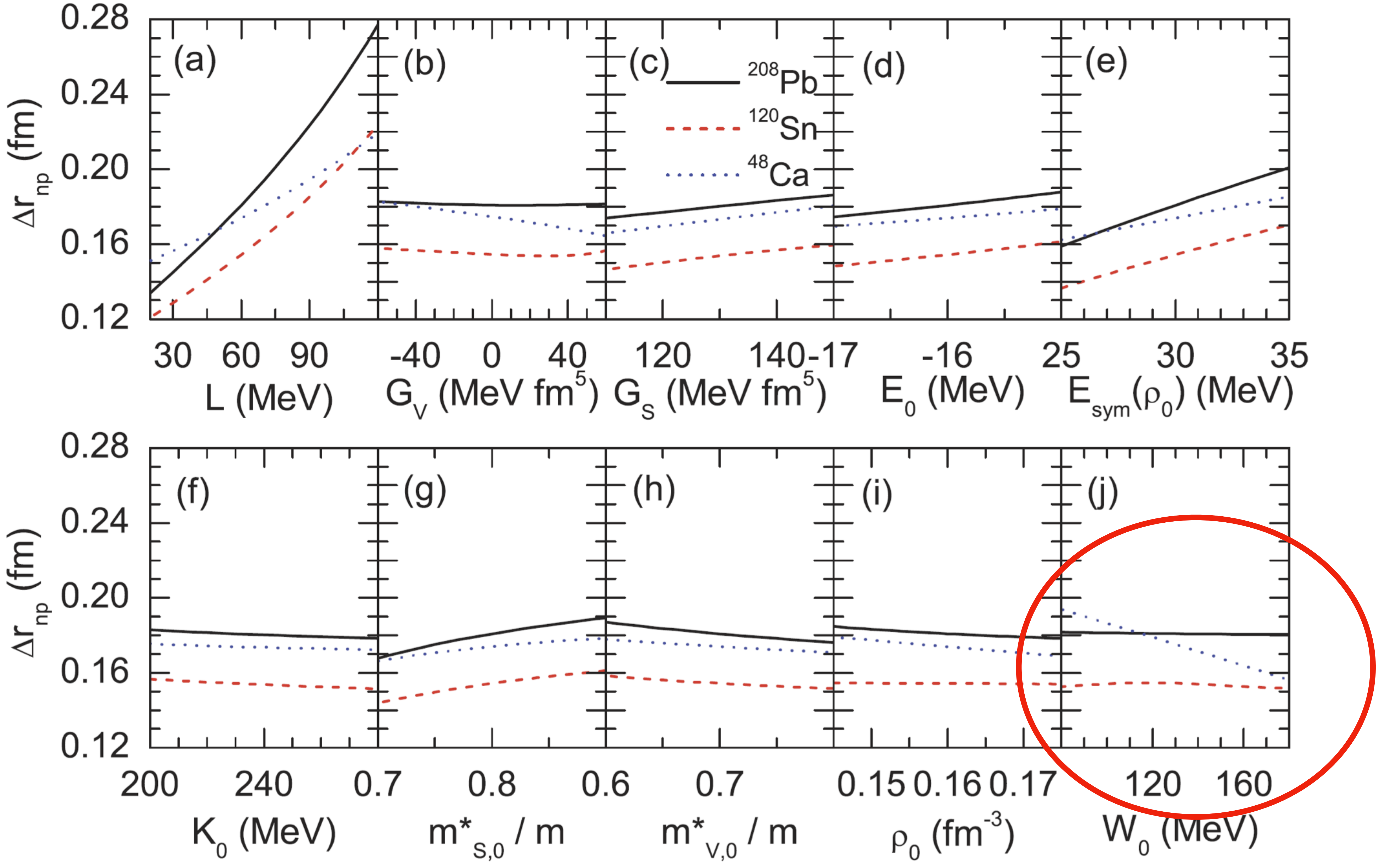
Resolving the PREX-CREX puzzle via a strong isovector spin-orbit interaction

— extended Skyrme energy density functional

T.G. Yue, **ZZ,** L.W. Chen, arXiv: 2406.03844

Spin-orbit interaction and neutron/electroweak skin

Chen, Ko, Li, &Xu, PRC82, 024321 (2010)



Horowitz & Piekarewitz, PRC86, 045503 (2012)

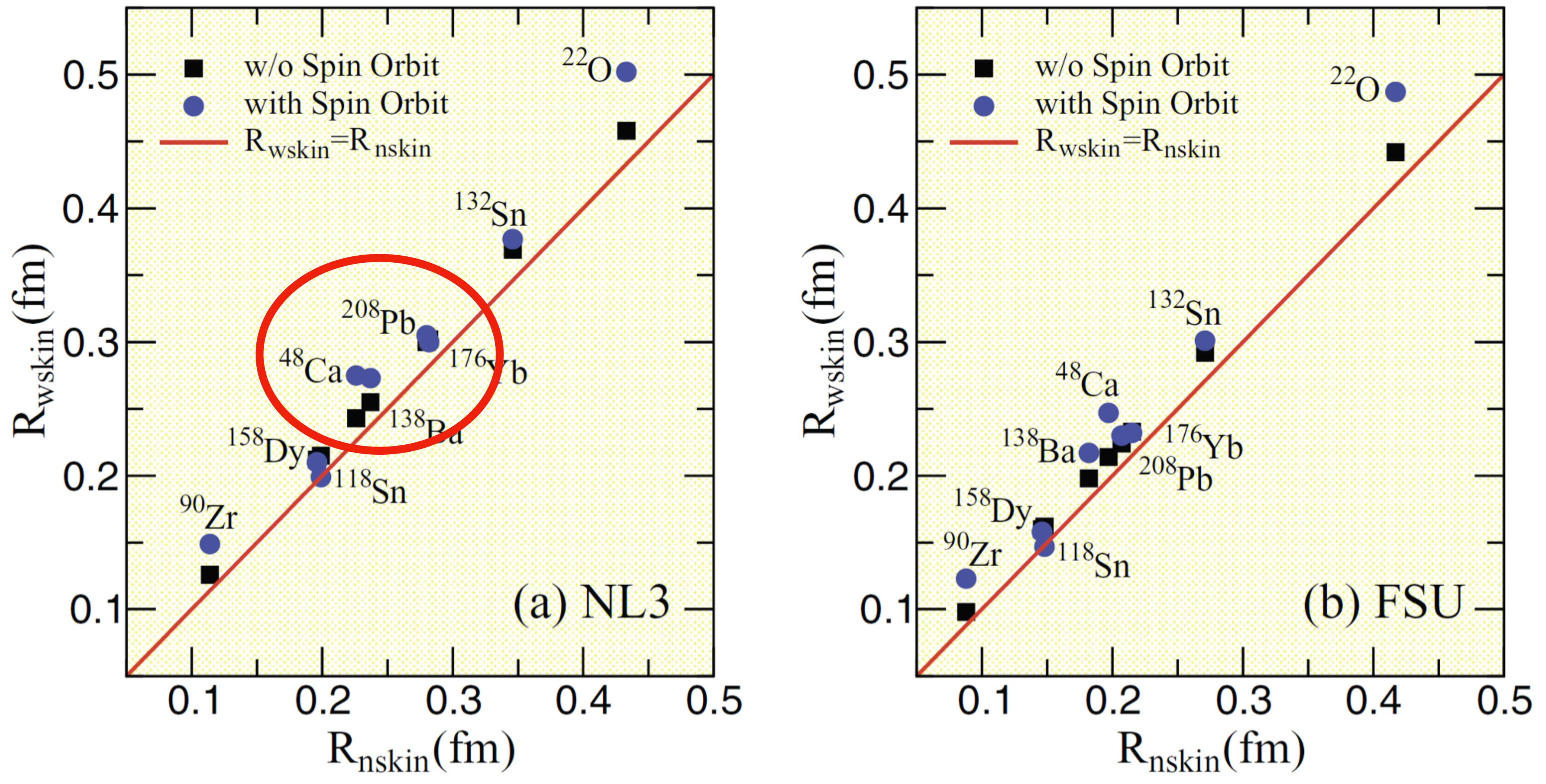
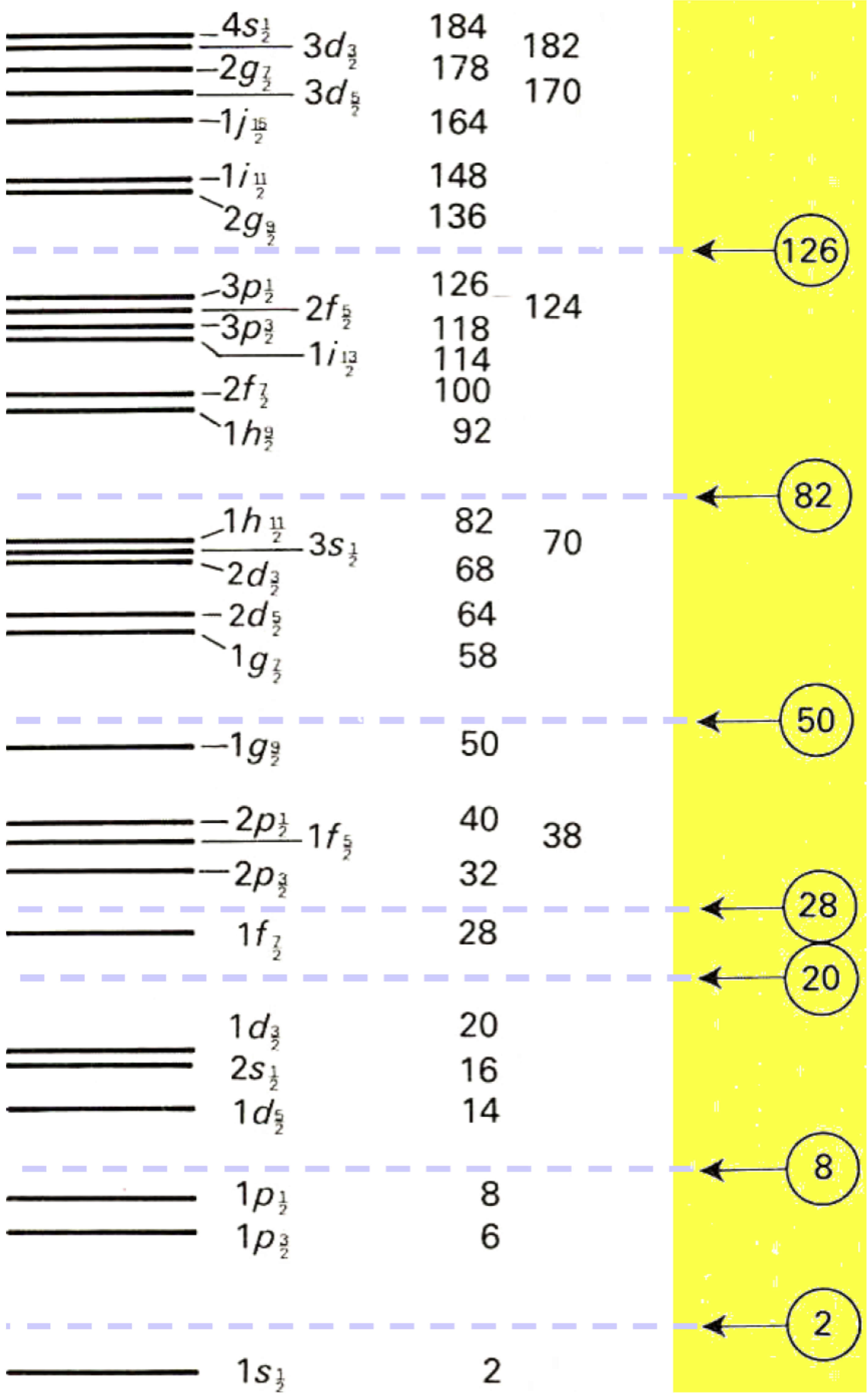


FIG. 2. (Color online) Electroweak skin (R_{wk}-R_{ch}) with and without spin-orbit corrections as a function of neutron skin (R_n-R_p) for the various neutron-rich nuclei considered in this work. Predictions are made using both the (a) NL3 and (b) FSU interactions.

- ✦ The Nskin of Ca48 is sensitive to spin-orbit coupling W0 in the standard SHF!
- ✦ Spin-orbit coupling makes significant contribution to the electroweak skin of Ca48.
- ✦ Ca48 and Pb208 have different shell and surface structures – Both are related to Spin-Orbit interaction.

Nuclear spin-orbit interaction



✦ Strong spin-orbit interaction ⟶ magic numbers

$$V(r) \to V(r) + W(r) L \bullet S$$

$$W(r) = -|V_{LS}| \left(\frac{\hbar}{m_\pi c} \right)^2 \frac{1}{r} \frac{dV(r)}{dr}$$

Relativistic effects
(Duerr, PR103), 469(1956)

Mayer and Jensen (1949)
Nobel Prize, 1963 (Also Wigner)

✦ Nonrelativistic energy density functionals (Skyrme):

- Spin-orbit interaction: $iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma})_2 \cdot \left[\boldsymbol{P}' \times \delta(\boldsymbol{r}) \boldsymbol{P} \right]$
- Spin-orbit energy:

Chabanat, et al., NPA 627, 710 (1997)

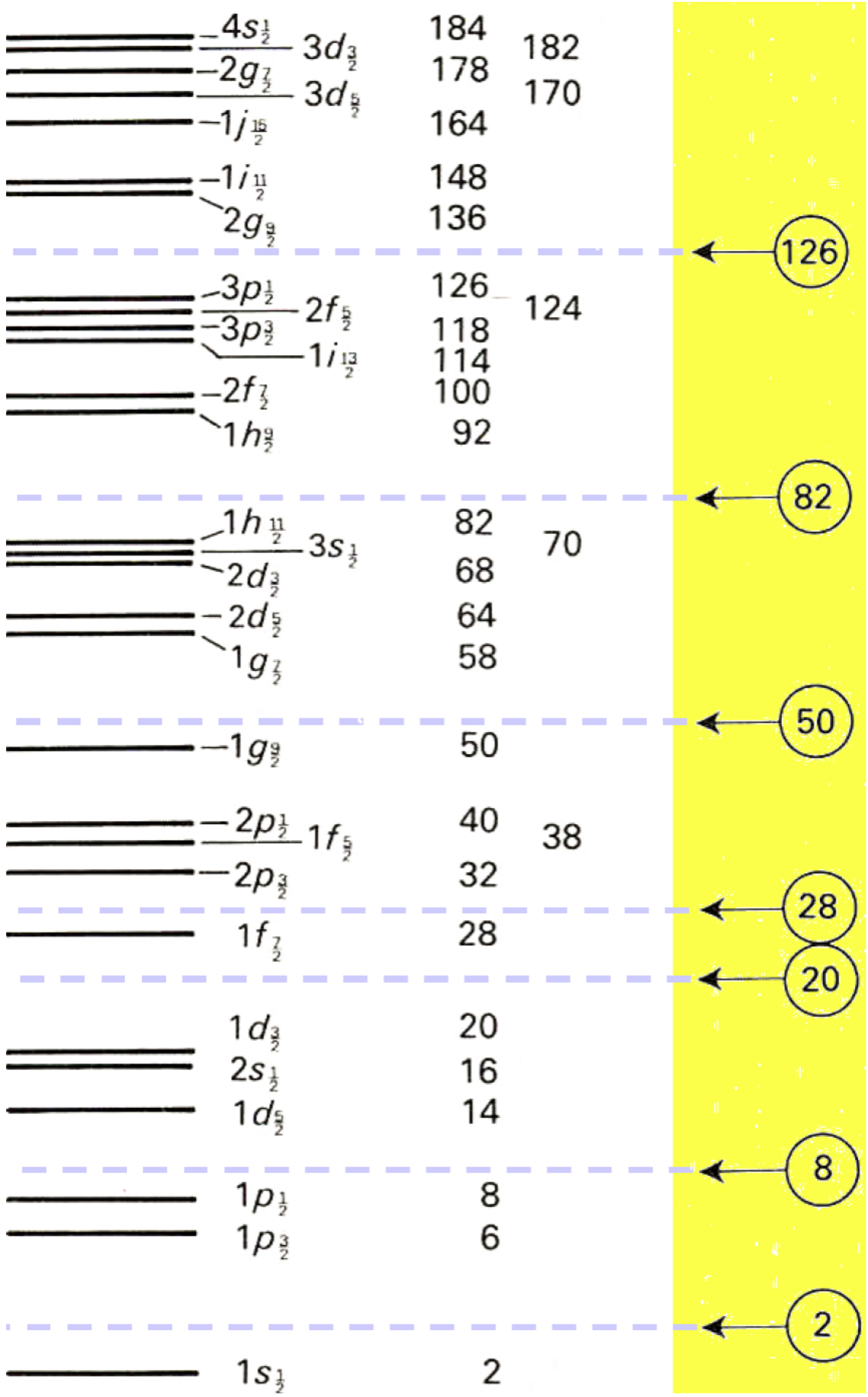
$$E_{\rm so} = \int dr^3 \frac{1}{2} W_0 \left[J \cdot \nabla \rho + J_p \nabla \rho_p + J_n \nabla \rho_n \right]$$

$$= \int dr^3 \left[\frac{b_{\rm IS}}{2} J \cdot \nabla \rho + \frac{b_{\rm IV}}{2} (J_n - J_p) \nabla (\rho_n - \rho_p) \right]$$

$$\mathbf{J}_q(\mathbf{r}) = -\mathrm{i} \sum v_i^2 \varphi_i^+(\mathbf{r}) \nabla \times \hat{\sigma} \varphi_i(\mathbf{r}).$$

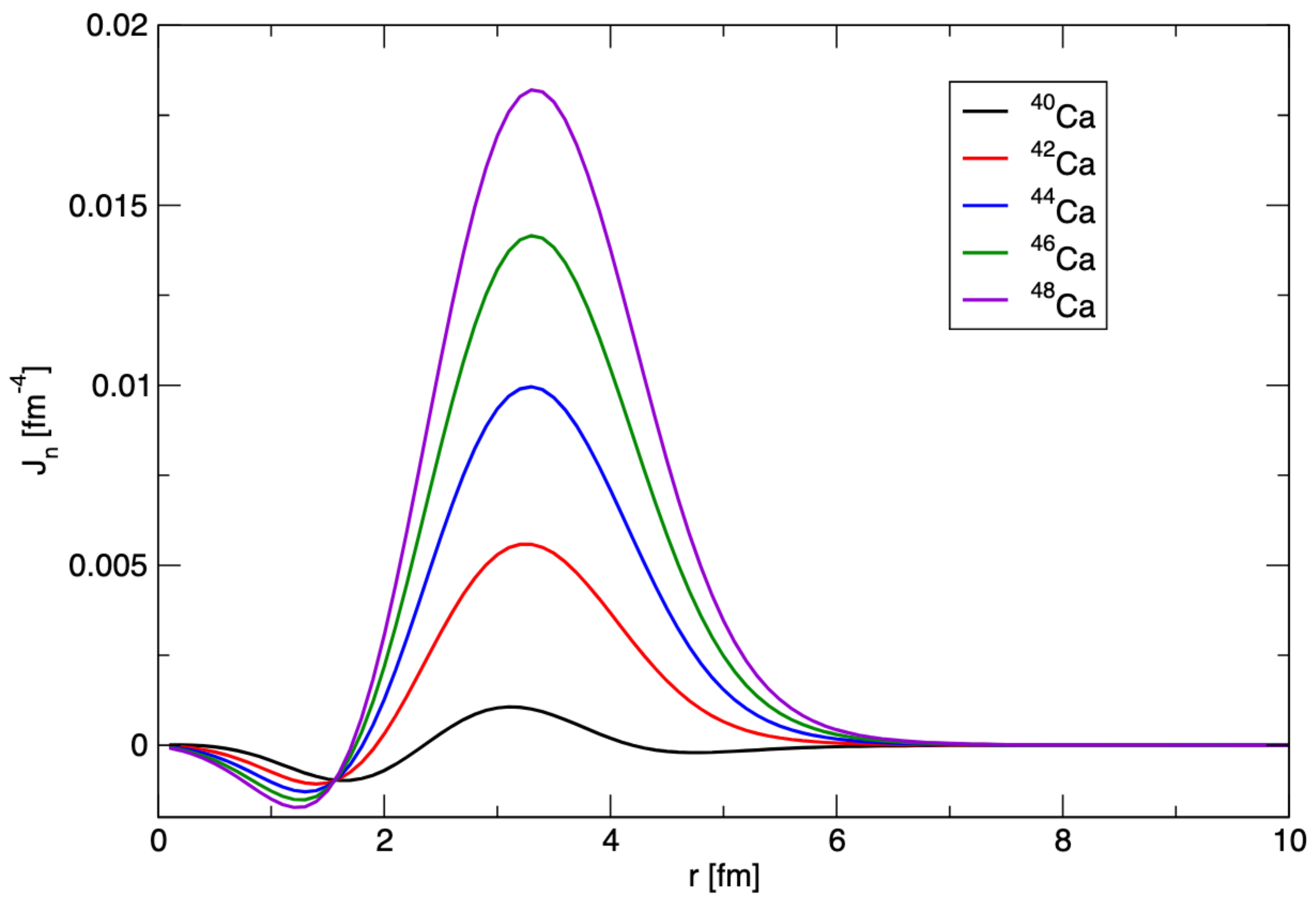
Reinhard and Flocard, NPA 584, 467488 (1995)
Bender, Heenen, and Reinhard, Rev. Mod. Phys. 75, 121 (2003).
Ebran, Mutschler, Khan, and Vretenar, PRC 94, 024304 (2016).

Spin-orbit density in ^{48}Ca and ^{208}Pb



✦ Spin-Orbit density (spherical nuclei):

$$J_q(r) = \frac{1}{4\pi r^3}\sum_i v_i^2\,(2j_i+1) \times \left[j_i\,(j_i+1) - l_i\,(l_i+1) - \frac{3}{4}\right] R_i^2(r)$$

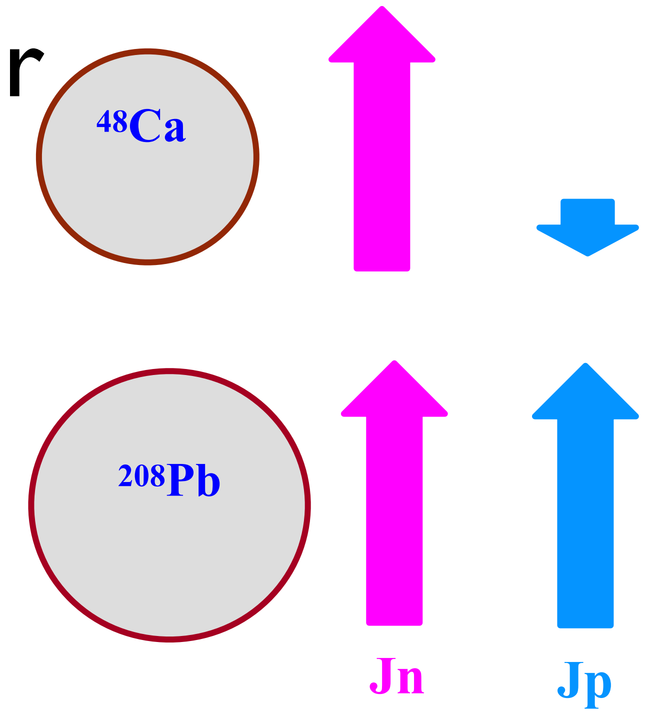


R_i: radial wave function; j & l: total & orbital angular momentum

Sagawa & Colo, PPNP 76 (2014) 76

✦ Contributions from $j_>$ and $j_<$ largely cancel with each other

- $j_> = l + 1/2$: positive contribution
- $j_< = l - 1/2$: negative contribution



✦ $|J_n - J_p|$ is large in ^{48}Ca, but relatively small in ^{208}Pb

- Ca40: $J_p \approx 0,\ J_n \approx 0$
- Ca48: $J_p \approx 0,\ J_n >> 0$ due to the 8 $1f_{\frac{7}{2}}$ neutrons of unpaired l•s partner
- Pb208: $J_p \approx J_n >> 0$ due to 14 $1i_{\frac{13}{2}}$ neutrons and 12 $1h_{\frac{11}{2}}$ protons

✦ The isovector spin-orbit coupling b_{IV} is expected to have significant effect on Ca48 while essentially no influence on Pb208!

Extended Skyrme EDF

✦ Energy density functional:

$$\begin{aligned}\mathcal{E}_{\text{Skyrme}} &= \frac{B_0+B_3\rho^\alpha}{2}\rho^2 - \frac{B_0'+B_3'\rho^\alpha}{2}\tilde{\rho}^2 + (B_1+B_4\rho^\beta+B_5\rho^\gamma)\rho\tau - (B_1'+B_4'\rho^\beta+B_5'\rho^\gamma)\tilde{\rho}\tilde{\tau}\\ &+\frac{2B_2+(2\beta+3)B_4\rho^\beta-B_5\rho^\gamma}{4}(\nabla\rho)^2 - \frac{2B_2'+3B_4'\rho^\beta-B_5'\rho^\gamma}{4}(\nabla\tilde{\rho})^2 - \frac{\beta B_4'}{2}\rho^{\beta-1}\tilde{\rho}\nabla\rho\cdot\nabla\tilde{\rho}\\ &+\frac{C_1+C_2\rho^\beta+C_3\rho^\gamma}{2}\boldsymbol{J}^2 + \frac{C_1'+C_2'\rho^\beta+C_3'\rho^\gamma}{2}\tilde{\boldsymbol{J}}^2\\ &+\frac{b_{\text{IS}}}{2}\nabla\rho\cdot\mathbf{J} + \frac{b_{\text{IV}}}{2}\nabla\tilde{\rho}\cdot\tilde{\mathbf{J}} + \frac{\alpha_T+\beta_T}{4}\boldsymbol{J}^2 + \frac{\alpha_T-\beta_T}{4}\tilde{\boldsymbol{J}}^2.\end{aligned}$$

$$\begin{aligned}\rho_q(\mathbf{r}) &= \sum_i v_i^2\,|\varphi_i(\mathbf{r})|^2,\\ \tau_q(\mathbf{r}) &= \sum_i v_i^2\,|\nabla\varphi_i\mathbf{r})|^2,\\ \mathbf{J}_q(\mathbf{r}) &= -\mathrm{i}\sum v_i^2\varphi_i^+(\mathbf{r})\nabla\times\hat{\sigma}\varphi_i(\mathbf{r}).\end{aligned}$$

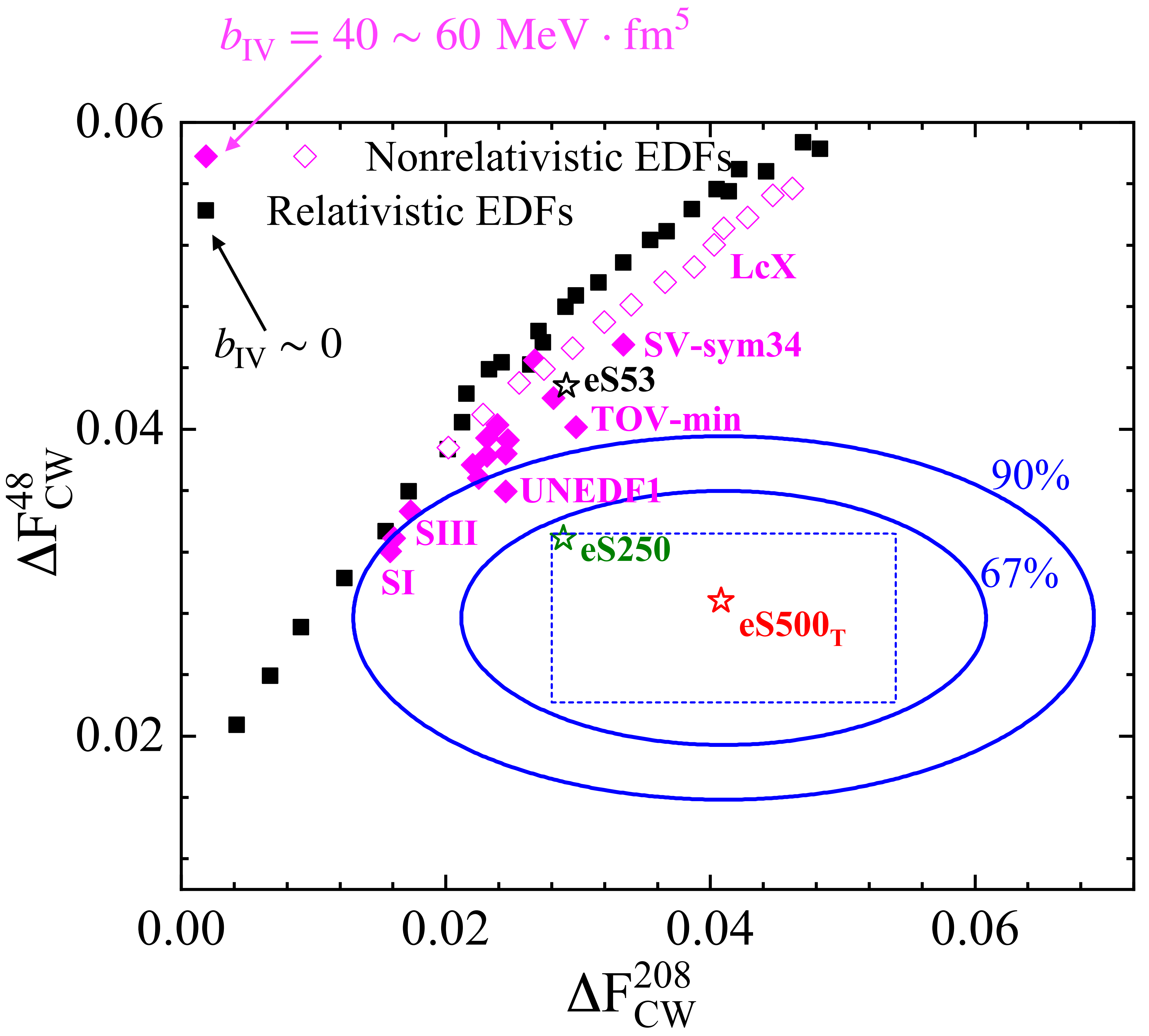
Tong-Gang Yue, **ZZ**, Lie-Wen Chen, arXiv:2406.03844

$$\rho=\rho_n+\rho_p,\quad \tau=\tau_n+\tau_p,\quad \boldsymbol{J}=\boldsymbol{J}_n+\boldsymbol{J}_p,$$

$$\tilde{\rho}=\rho_n-\rho_p,\quad \tilde{\tau}=\tau_n-\tau_p,\quad \tilde{\boldsymbol{J}}=\boldsymbol{J}_n-\boldsymbol{J}_p,$$

✦ Construct 3 new EDFs to simultaneously fit CREX and PREX results, ground- and excited-state of a number of typical (semi-)closed-shell nuclei, and constraints on EOS of nuclear matter. **eS250, eS500T, and eS53.** [e: extended, T: tensor force, number: the value of b_{IV}]

New EDFs with strong IVSO interaction

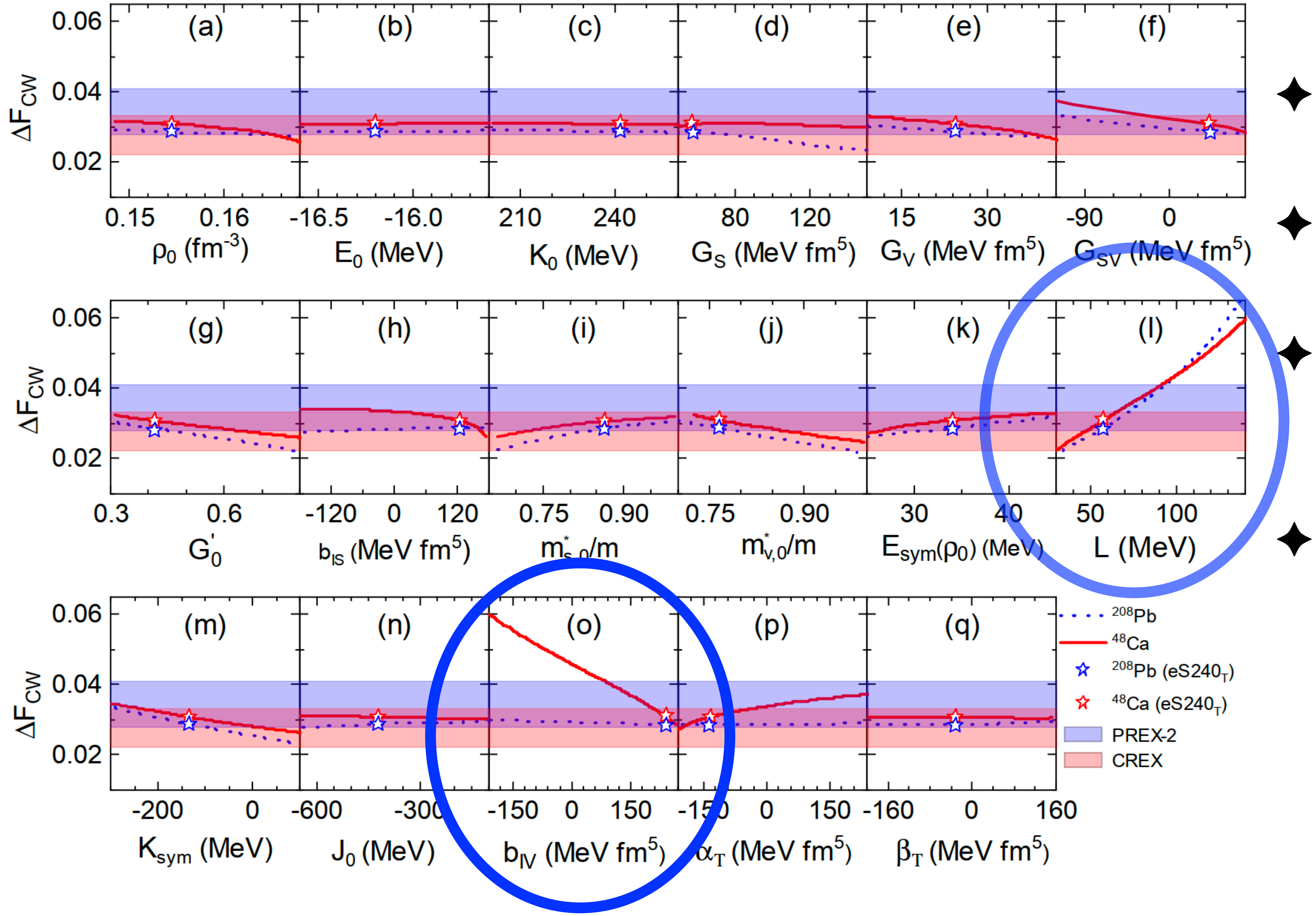


Tong-Gang Yue, **ZZ**, Lie-Wen Chen, arXiv:2406.03844

quantity	eS250	eS53	eS500$_T$	quantity	eS250	eS53	eS500$_T$
t_0 (MeV · fm^5)	-1803.9	-1803.9	-1632.7	b_{IS} (MeV · fm^5)	160.461	160.461	118.87
t_1 (MeV · fm^5)	450.79	450.79	545.566	b_{IV} (MeV · fm^5)	250	53.4870	500
t_2 (MeV · fm^5)	287.09	287.09	-946.38	ρ_0 (fm^{-3})	0.15881	0.15881	0.15089
t_3 (MeV · fm$^{3+3\alpha}$)	12514.	12514.	12807.8	E_0 (MeV)	-16.1209	-16.1209	-15.957
t_4 (MeV · fm$^{3+3\beta}$)	-922.51	-922.51	-1815.3	K_0 (MeV)	238.125	238.125	236.22
t_5 (MeV · fm$^{3+3\gamma}$)	-1064.2	-1064.2	7734.2	J_0 (MeV)	-393.323	-393.323	-463.64
x_0	0.25037	0.25037	-0.1312	$E_{sym}(\rho_0)$ (MeV)	33.9279	33.9279	36.96
x_1	-0.95171	-0.95171	0.8692	L (MeV)	58.9409	58.9409	80.6
x_2	-2.2637	-2.2637	-0.9003	K_{sym} (MeV)	-111.833	-111.833	-189.5
x_3	0.06042	0.06042	-0.9644	$m_{s,0}/m$	0.92297	0.92297	0.921
x_4	-1.5503	-1.5503	1.6947	$m_{v,0}/m$	0.77078	0.77078	0.662
x_5	-1.8795	-1.8795	-1.1009	G'_0	0.75906	0.75906	0.951
α	0.33162	0.33162	0.43201	G_S (MeV · fm^5)	81.1891	81.1891	44.66
β	1	1	1	G_V (MeV · fm^5)	-26.4039	-26.4039	7.19
γ	1	1	1	G_{SV} (MeV · fm^5)	19.2338	19.2338	-75.15
B_0 (MeV · fm^3)	-1352.90	-1352.90	-1224.51	B'_0 (MeV · fm^3)	-676.78	-676.78	-301.036
B_1 (MeV · fm^5)	-30.4933	-30.4933	19.5659	B'_1 (MeV · fm^5)	37.8390	37.8390	40.1165
B_2 (MeV · fm^5)	163.162	163.162	194.804	B'_2 (MeV · fm^5)	-69.8261	-69.8261	166.682
B_3 (MeV · fm$^{3+3\alpha}$)	1564.21	1564.21	1600.97	B'_3 (MeV · fm$^{3+3\alpha}$)	584.414	584.414	-495.702
B_4 (MeV · fm$^{5+3\beta}$)	-172.971	-172.971	-340.371	B'_4 (MeV · fm$^{5+3\beta}$)	121.112	121.112	-498.017
B_5 (MeV · fm$^{5+3\beta}$)	167.480	167.480	288.309	B'_5 (MeV · fm$^{5+3\beta}$)	-183.508	-183.508	580.922
C_1 (MeV · fm^5)	0	0	-78.4280	C'_1 (MeV · fm^5)	0	0	93.2463
C_2 (MeV · fm$^{5+3\beta}$)	0	0	271.103	C'_2 (MeV · fm$^{5+3\beta}$)	0	0	-113.457
C_3 (MeV · fm$^{5+3\beta}$)	0	0	580.922	C'_3 (MeV · fm$^{5+3\beta}$)	0	0	483.384

The isovector spin-orbit coupling b_{IV} should be larger than ~ 250 MeV fm^5 to fit CREX/PREX data

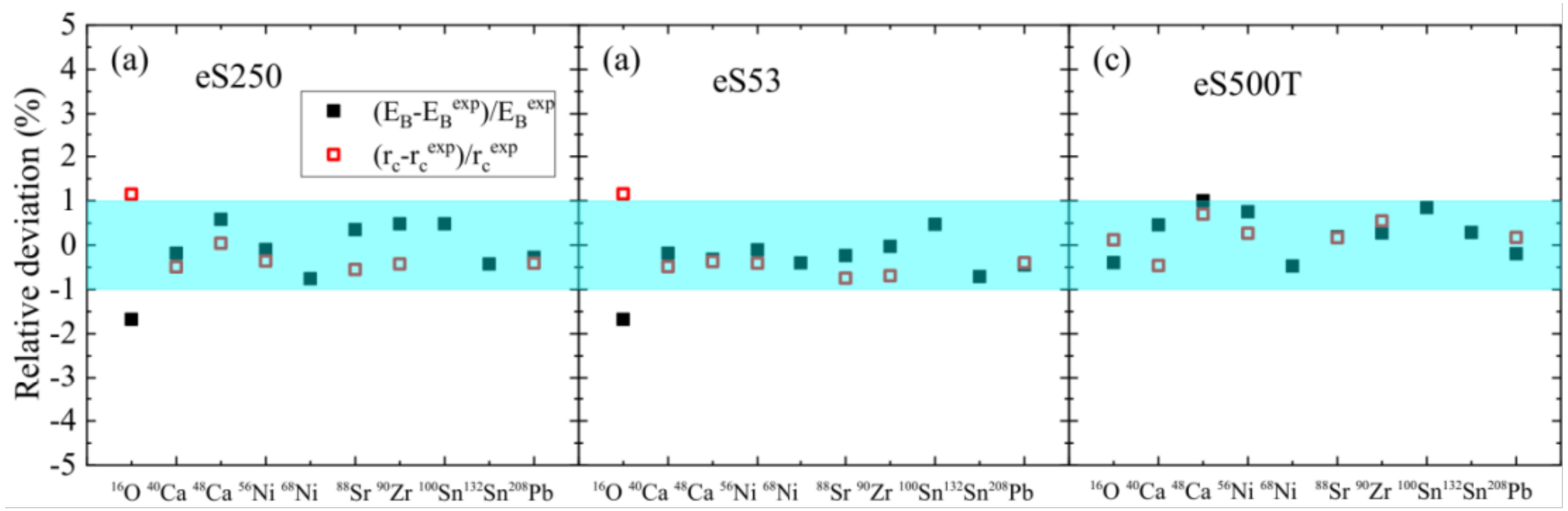
Correlation analysis for ΔF_{CW} in ^{48}Ca and ^{208}Pb

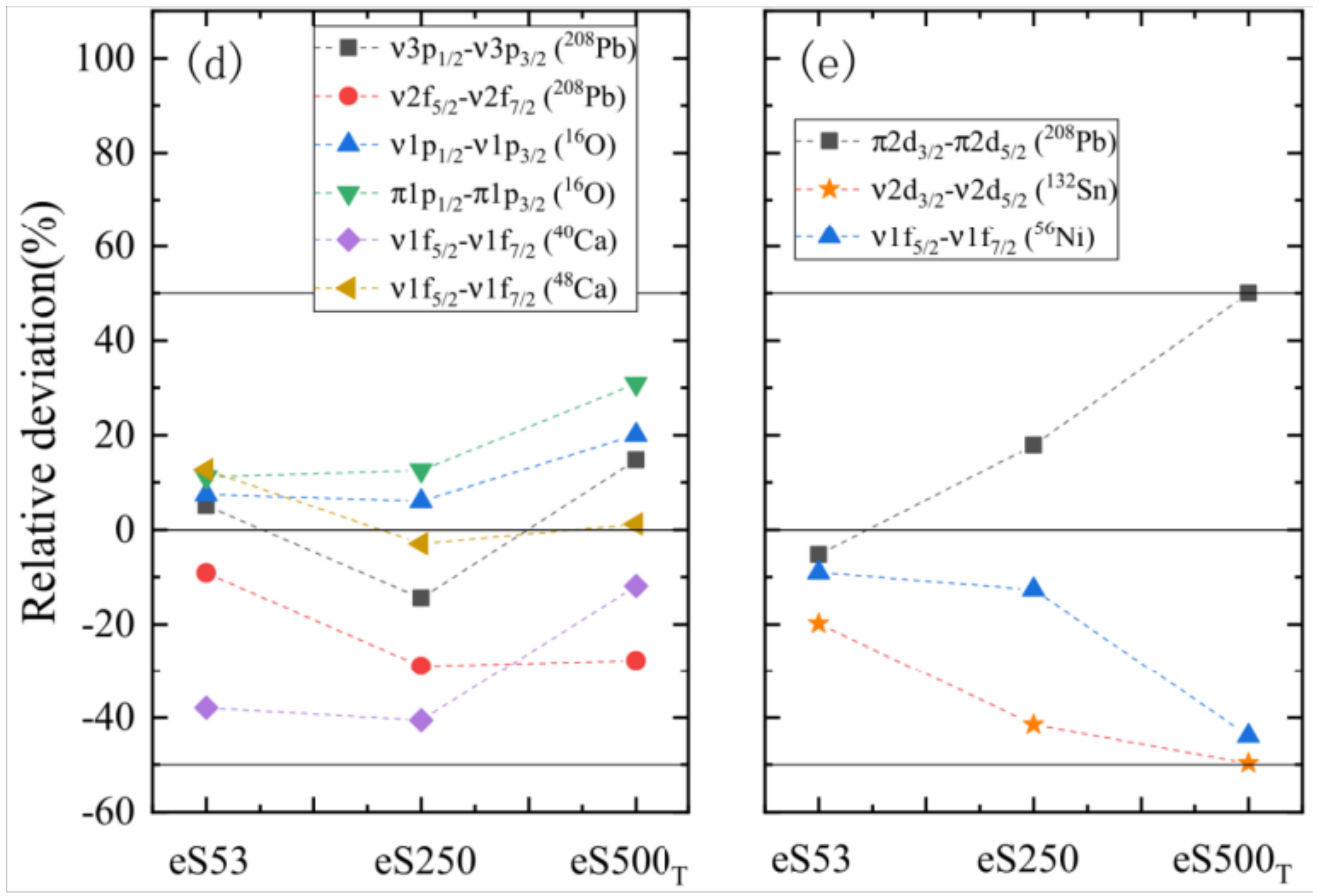


- Based on eS250 EDF
- ΔF_{CW} of ^{208}Pb is only sensitive to L.
- ΔF_{CW} of ^{48}Ca is positively(negatively) correlated to $L(b_{IV})$
- A large L can still reproduce the CREX result with a large b_{IV}

Tong-Gang Yue, **ZZ**, Lie-Wen Chen, arXiv:2406.03844

Ground-state properties: mass, radius, spin-orbit splitting

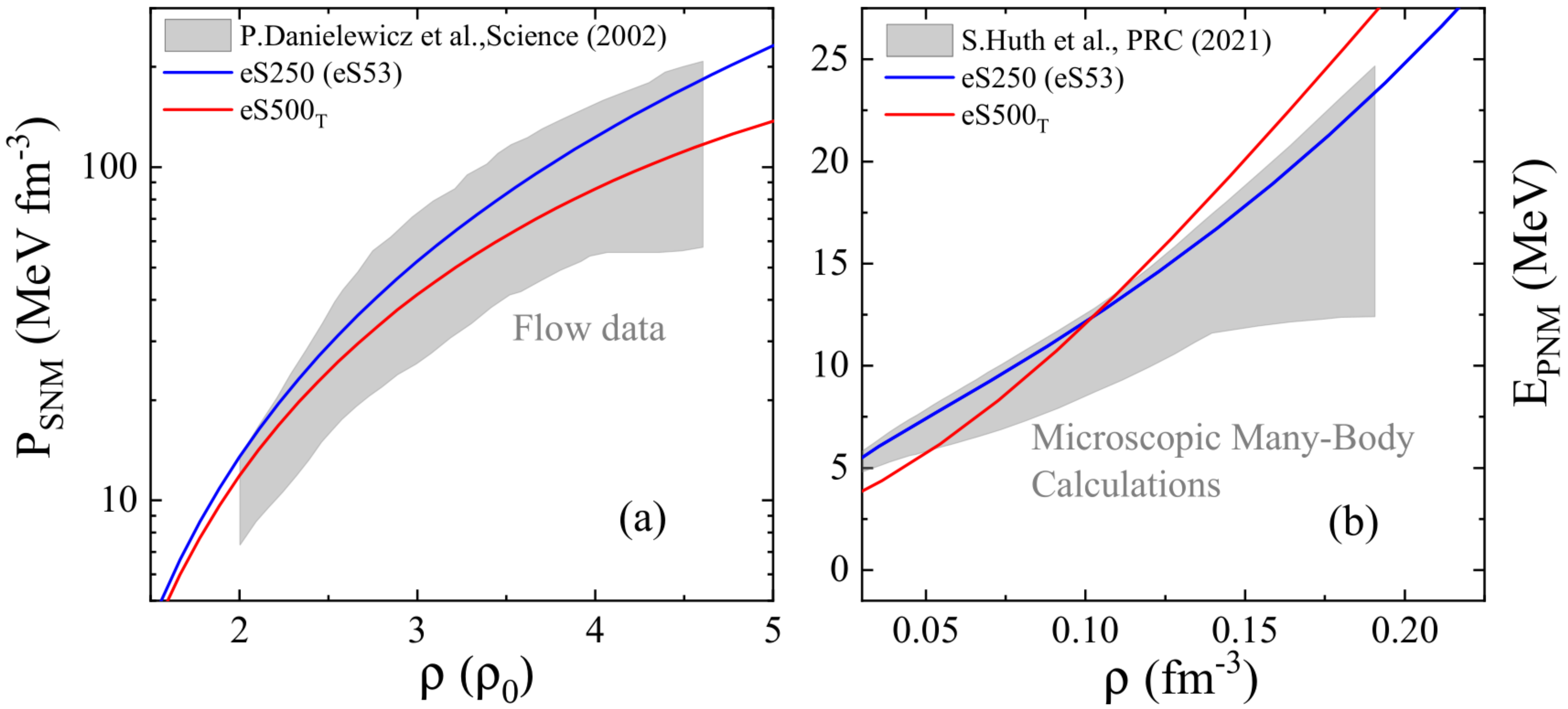




Tong-Gang Yue, **ZZ**, Lie-Wen Chen, arXiv:2406.03844

✦ The new EDFs with strong isovector spin-orbit interaction can well describe the nuclear ground-state properties!

Equation of state



❖ The eS250 with strong isovector spin-orbit interaction can well describe the empirical EOSs of SNM and PNM! (but eS500T predict too stiff PNM EOS)

Tong-Gang Yue, **ZZ**, Lie-Wen Chen, arXiv:2406.03844

Resolving the PREX-CREX puzzle via a strong isovector spin-orbit interaction

—Density-dependent point-coupling RMF

M. Qiu, **ZZ**, T.G. Yue, L.W. Chen, in preparation

Strong IVSO potential in RMF model

Dirac equation

$$\hat{H} = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta(M - S(r)) + V(r) + i\,\boldsymbol{\gamma} \cdot \hat{\boldsymbol{r}}\, T(r),$$

$$\left(\frac{d}{dr} + \frac{\kappa^*(r)}{r}\right) g_{n\kappa}(r) - (E^*(r) + M^*(r)) f_{n\kappa}(r) = 0,$$

$$\left(\frac{d}{dr} - \frac{\kappa^*(r)}{r}\right) f_{n\kappa}(r) + (E^*(r) - M^*(r)) g_{n\kappa}(r) = 0.$$

Equivalent Schrödinger-like equation

$$\left[\frac{d^2}{dr^2} - p^2 - \frac{l(l+1)}{r^2} - U_{\mathrm{eff}}(r;\kappa,E)\right] u_{n\kappa}(r) = 0,$$

$$U_{\mathrm{eff}}(r;\kappa,E) = U_{\mathrm{c}}(r) - (1+\kappa) U_{\mathrm{so}}(r) + U_{\mathrm{D}}(r) + U_{\mathrm{t}}(r),$$

$$U_{\mathrm{so}}(r) \equiv U_{\mathrm{so}}^{(0)}(r) \pm U_{\mathrm{so}}^{(1)}(r) \rightarrow U_{\mathrm{so}}^{(0)}(r) \pm \beta U_{\mathrm{so}}^{(1)}(r)$$

EDITORS' SUGGESTION

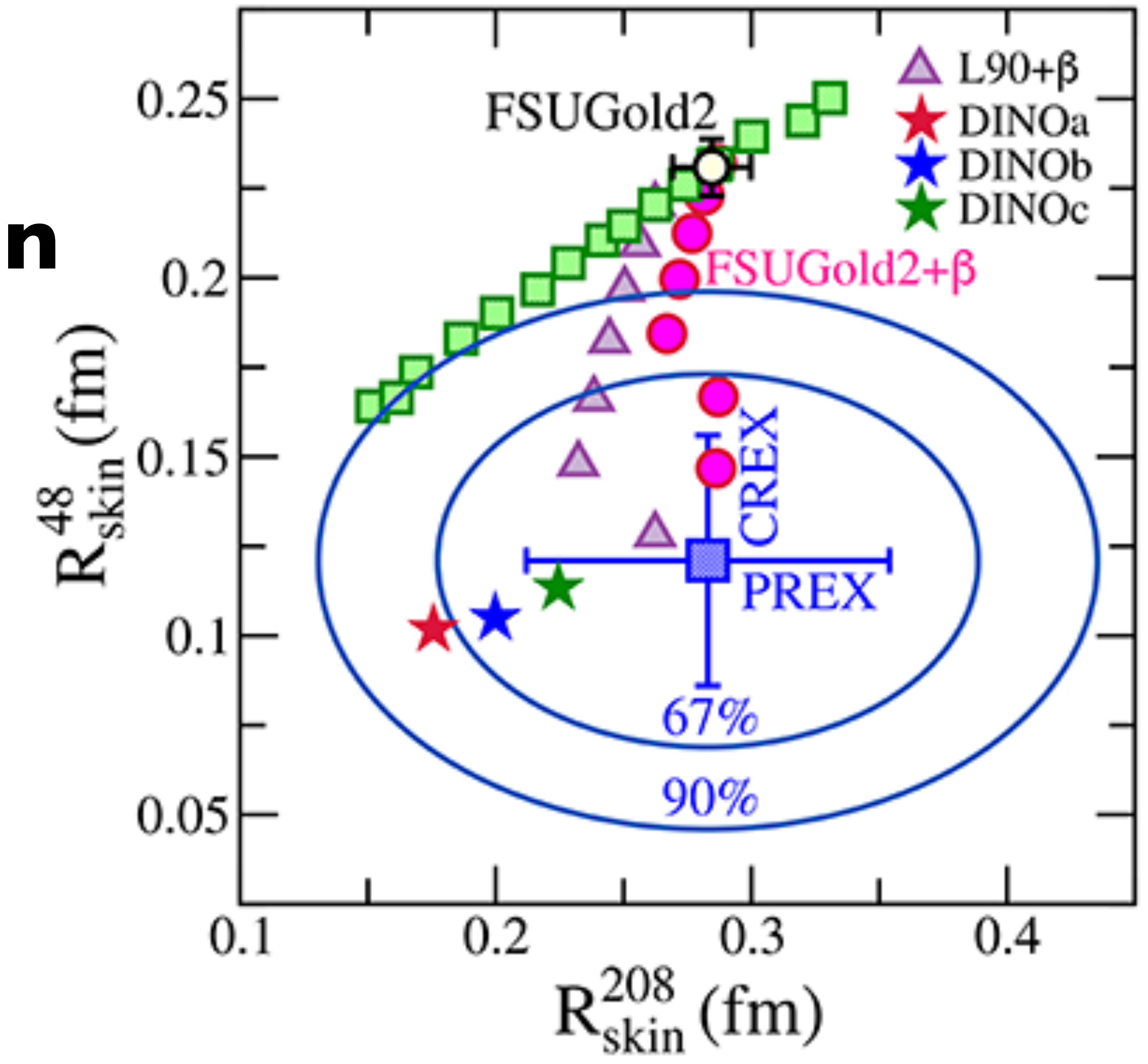
Role of the isovector spin-orbit potential in mitigating the CREX-PREX dilemma

Athul Kunjipurayil and J. Piekarewicz

Marc Salinas

Show more

Phys. Rev. C **112**, 014310 – **Published 8 July, 2025**



- Enhanced isovector potential by an enhancement factor β.
- β ≈-300 leads to perfect agreement with CREX and PREX
- Confirm the effects of IVSO interaction

IVSO coupling in RMF model

✦ Isovector-scalar δ meson exchange

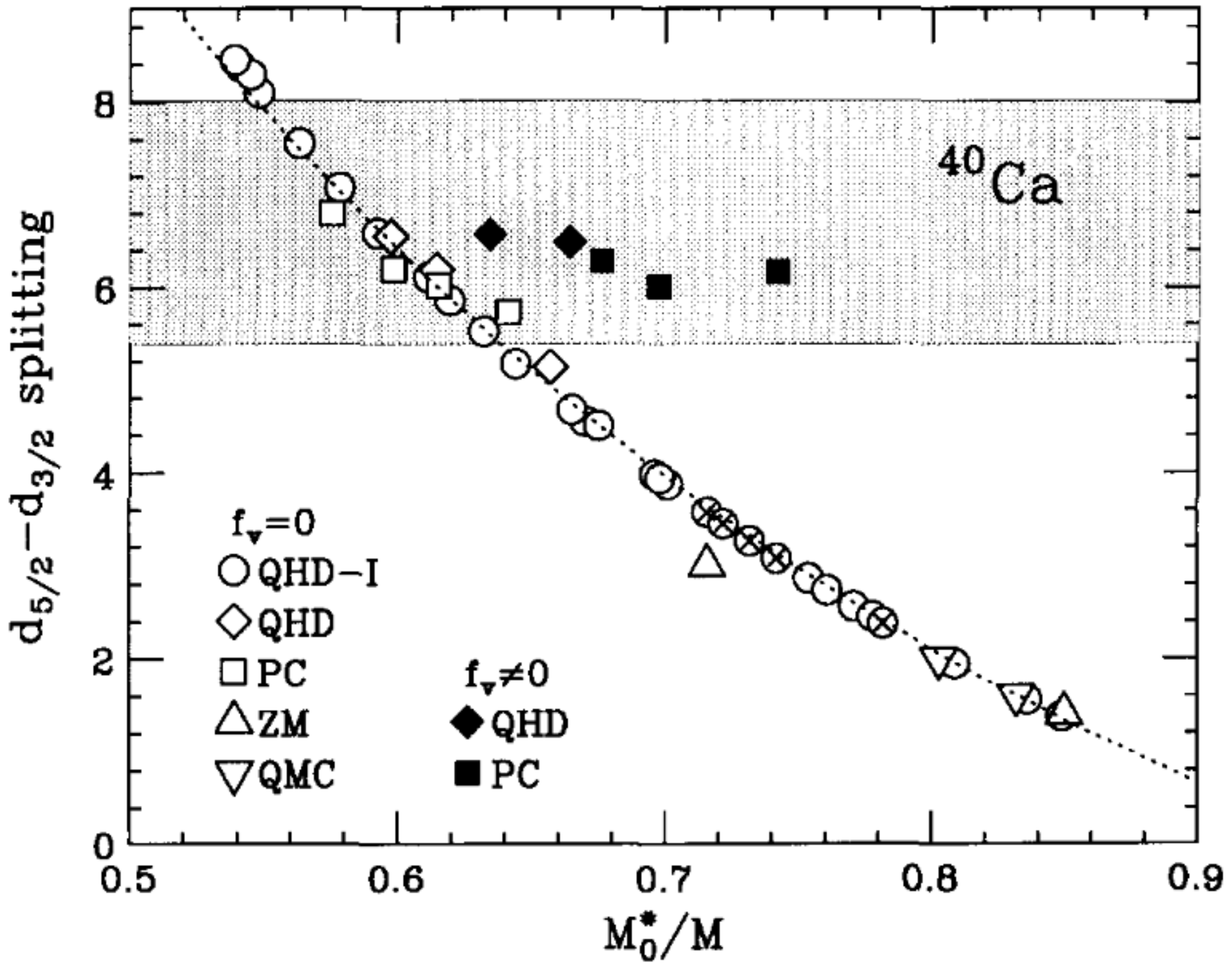
✦ Exchange (Fork) term

J.-P. Ebran, Mutschler, E. Khan, and D.Vretena, PRC 94, 024304 (2016)

✦ Tensor coupling

R.J. Furnstahl et al./Nuclear Physics A 632 (1998) 607-623

$$h_{\mathrm{LS,T}} = \left[\frac{1}{4\overline{M}^2}\frac{1}{r}\left(\frac{d\Phi}{dr}+\frac{dW}{dr}\right)+\frac{f_v}{2M\overline{M}}\frac{1}{r}\frac{dW}{dr}+O(v^4)\right]\boldsymbol{\sigma}\cdot\boldsymbol{L},$$



Possible key for PREX-CREX puzzle in covariant DFT：

✓Isovector tensor coupling

Density-dependent point-coupling RMF model

$$\mathscr{L} = \mathscr{L}^{\text{free}} + \mathscr{L}^{\text{4f}} + \mathscr{L}^{\text{der}} + \mathscr{L}^{\text{em}}$$

$$\begin{aligned}
\mathscr{L}^{\text{4f}} = & -\frac{1}{2}\alpha_S(\bar\psi\psi)(\bar\psi\psi) - \frac{1}{2}\alpha_{tS}(\bar\psi\vec\tau\psi)(\bar\psi\vec\tau\psi) \\
& -\frac{1}{2}\alpha_V\left(\bar\psi\gamma_\mu\psi\right)\left(\bar\psi\gamma^\mu\psi\right) - \frac{1}{2}\alpha_{tV}\left(\bar\psi\gamma_\mu\vec\tau\psi\right)\left(\bar\psi\gamma^\mu\vec\tau\psi\right) \\
& -\frac{1}{2}\alpha_T\left(\bar\psi\sigma_{\mu\nu}\psi\right)\left(\bar\psi\sigma^{\mu\nu}\psi\right) \\
& -\frac{1}{2}\alpha_{tT}\left(\bar\psi\sigma_{\mu\nu}\vec\tau\psi\right)\left(\bar\psi\sigma^{\mu\nu}\vec\tau\psi\right) \\
& -\frac{1}{2}\alpha_{PS}\left(\bar\psi\gamma_5\psi\right)\left(\bar\psi\gamma_5\psi\right) - \frac{1}{2}\alpha_{tPS}\left(\bar\psi\gamma_5\vec\tau\psi\right)\left(\bar\psi\gamma_5\vec\tau\psi\right) \\
& -\frac{1}{2}\alpha_{PV}\left(\bar\psi\gamma_5\gamma_\mu\psi\right)\left(\bar\psi\gamma_5\gamma^\mu\psi\right) \\
& -\frac{1}{2}\alpha_{tPV}\left(\bar\psi\gamma_5\gamma_\mu\vec\tau\psi\right)\left(\bar\psi\gamma_5\gamma^\mu\vec\tau\psi\right)
\end{aligned}$$

Q. Zhou, et al. PRC 106, 034315 (2022)

PHYSICAL REVIEW C **106**, 034315 (2022)

Editors' Suggestion

Covariant density functional theory with localized exchange terms

Qiang Zhao (赵强)[1,2] Zhengxue Ren (任政学)[1] Pengwei Zhao (赵鹏巍)[1,*] and Jie Meng (孟杰)[1,3,4,†]

$$\alpha_i(\rho) = \alpha_i\left(\rho_{\text{sat}}\right) f_i(x) \quad \text{for} \quad \text{i} = \text{S, V, tS, and tV},$$

$$f_i(x) = a_i\frac{1 + b_i\left(x + d_i\right)^2}{1 + c_i\left(x + d_i\right)^2}, \quad x = \rho/\rho_{\text{sat}}, \quad f_i(1) = 1, \quad f_i''(0) = 0$$

PCF-PK1:

$$\alpha_{tT} = \frac{1}{18}\left(-\alpha_S + 3\alpha_{tS} + 2\alpha_V - 6\alpha_{tV} + 6\alpha_T\right)$$

determined from Fierz transformation

✦ We treat α_{tT} as a free parameter

Non-relativistic reduction

Nuclear energy density (ground-state even-even nuclei)**:**

$$\epsilon = \sum_\alpha \bar{\varphi}_\alpha(-i\gamma\cdot\nabla + M)\varphi_\alpha + \frac{1}{2}\alpha_S\rho_S^2 + \frac{1}{2}\alpha_V\rho_V^2 + \frac{1}{2}\alpha_{\tau S}\rho_{\tau S}^2 + \frac{1}{2}\alpha_{\tau V}\rho_{\tau V}^2 - \alpha_{\tau T}\mathbf{j}_{\tau T}^{0^2} - \alpha_T\mathbf{j}_T^{0^2} - \frac{1}{2}\delta_S\left|\nabla\rho_S\right|^2$$

Expansion up to $\mathscr{B}_0^2$ ($\mathscr{B}_0 = \dfrac{1}{2M^*}$):

$$\begin{aligned}
\rho_V &= \rho,\\
\rho_S &= \sum \phi^{cl\dagger}\hat{I}^{-1/2}\left(1-\boldsymbol{\sigma}\cdot\overleftarrow{\Pi}\mathcal{B}_0^2\boldsymbol{\sigma}\cdot\overrightarrow{\Pi}\right)\hat{I}^{-1/2}\phi^{cl}\\
&= \rho - 2\mathcal{B}_0^2\left[\tau - \nabla\boldsymbol{J} - \nabla\rho\cdot\boldsymbol{T}^0 + 2\boldsymbol{T}^0\cdot\boldsymbol{J} + \rho\boldsymbol{T}^{0^2}\right],\\
\mathbf{j}_T^0 &= \sum\left[\phi^{cl\dagger}\hat{I}^{-1/2}\left(i\boldsymbol{\sigma}\mathcal{B}_0\boldsymbol{\sigma}\overrightarrow{\Pi}\right)\hat{I}^{-1/2}\phi^{cl} - c.c\right]\\
&= \mathcal{B}_0\nabla\rho - 2\mathcal{B}_0\boldsymbol{T}^0\rho - 2\mathcal{B}_0\boldsymbol{J}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}^{NR} = &\left(\frac{1}{2}\alpha_S + \frac{1}{2}\alpha_V\right)\rho^2 + \left(\frac{1}{2}\alpha_{\tau S} + \frac{1}{2}\alpha_{\tau V}\right)\tilde{\rho}^2 - \frac{1}{2}(\nabla\rho)^2\\
&- 2\mathcal{B}_0^2\alpha_S\rho\tau - 2\mathcal{B}_0^2\alpha_{\tau S}\tilde{\rho}\tilde{\tau} - \alpha_T\mathcal{B}_0^2(\nabla\rho)^2 - \alpha_{\tau T}\mathcal{B}_0^2(\nabla\tilde{\rho})^2\\
&- 2\mathcal{B}_0^2\alpha'_S\rho\nabla\rho\cdot\boldsymbol{J} - 2\mathcal{B}_0^2\alpha_S\nabla\rho\cdot\boldsymbol{J} + 4\mathcal{B}_0^2\alpha_T\nabla\rho\cdot\boldsymbol{J}\\
&- 2\mathcal{B}_0^2\alpha'_{\tau S}\tilde{\rho}\nabla\rho\cdot\tilde{\boldsymbol{J}} - 2\mathcal{B}_0^2\alpha_{\tau S}\nabla\tilde{\rho}\cdot\tilde{\boldsymbol{J}} + 4\mathcal{B}_0^2\alpha_{\tau T}\nabla\tilde{\rho}\cdot\tilde{\boldsymbol{J}}\\
&- 4\alpha_T\mathcal{B}_0^2\boldsymbol{J}^2 - 4\alpha_{\tau T}\mathcal{B}_0^2\tilde{\boldsymbol{J}}^2.
\end{aligned}$$

$$\begin{aligned}
b_{IS}^{NR} &= 8\mathcal{B}_0^2\alpha_T - 4\mathcal{B}_0^2\alpha_S - 4\mathcal{B}_0^2\alpha'_S\rho,\\
b_{IV}^{NR} &= 8\mathcal{B}_0^2\alpha_{\tau T} - 4\mathcal{B}_0^2\alpha_{\tau S}.
\end{aligned}$$

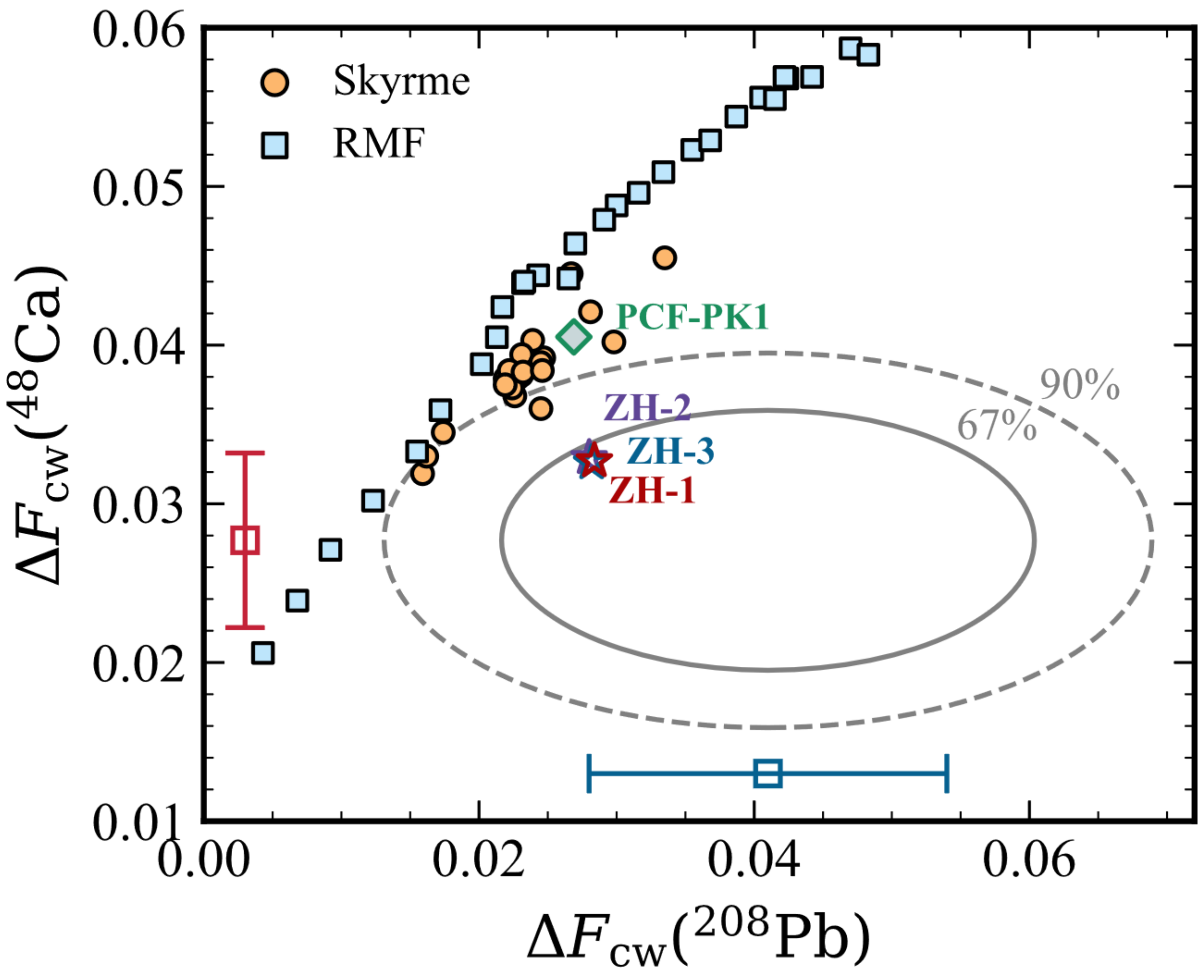
Fitting strategy

- ✦ Isoscalar sector $\alpha_S(\rho)$ and $\alpha_V(\rho)$ are taken from the PCF-PK1.

- ✦ 9 adjustable parameters:
  tS(3), tV(3), α_T, α_{tT}, δ

- ✦ 12 Data:
  - Binding energies and charge radii of ^{48}Ca and ^{208}Pb
  - Spin-orbit splitting in ^{48}Ca (1) and ^{208}Pb (3)
  - The symmetry energy at $2\rho_0/3$: 25.6 ± 1.35 MeV — Qiu et al. PLB 849, 138435 (2024)
  - Neutron matter EOS at $\rho = 0.04805\ \mathrm{fm}^{-3}$: $6.72 \pm 0.05\ \mathrm{MeV}$ — Machleidt &Sammarruca PPNP 137, 104117 (2024)
  - Form factor difference in ^{48}Ca and ^{208}Pb

Three new relativistic EDFs : ZH-1,2,3

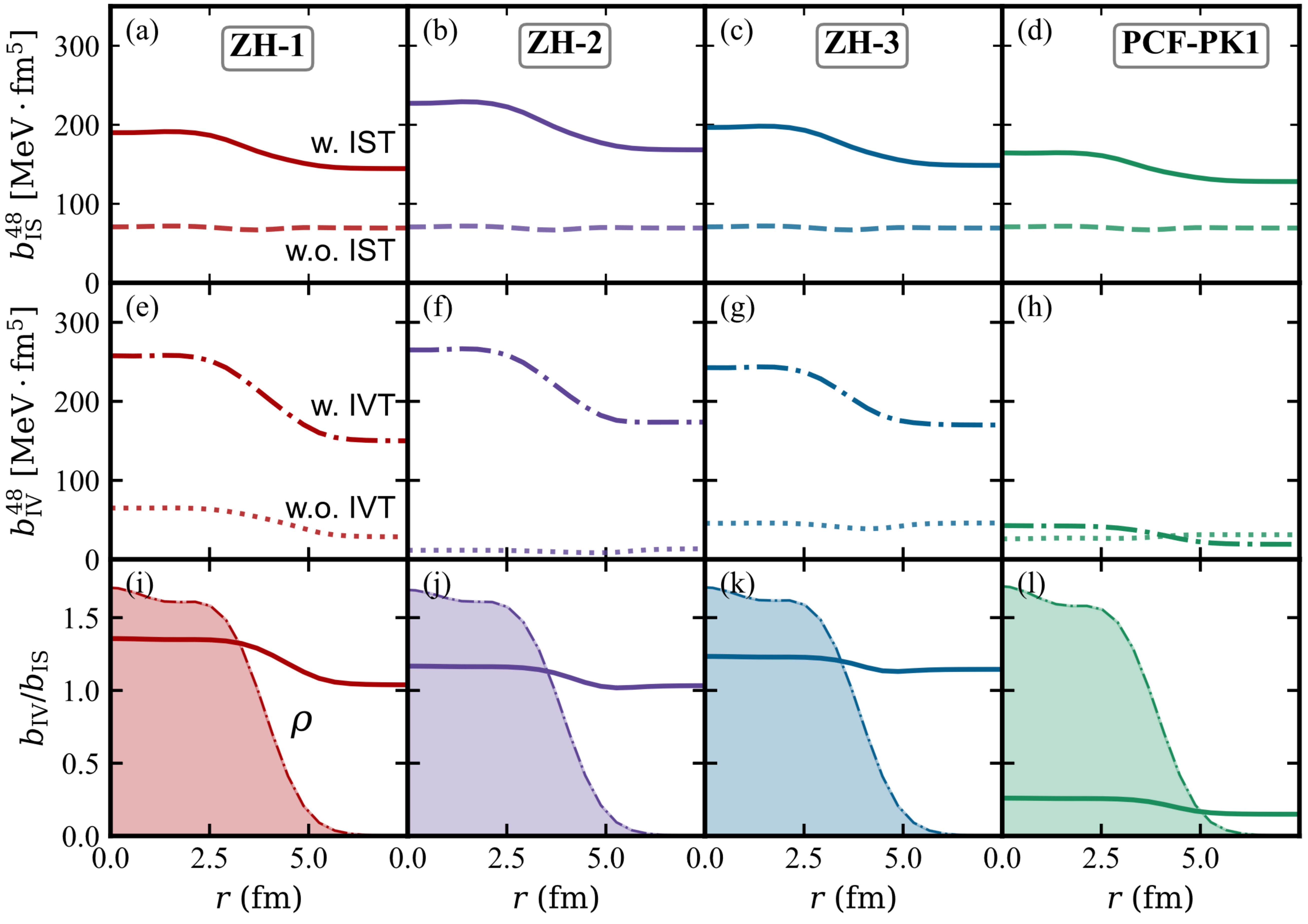


✦Consistent with both PREX-II and CREX results at 68.3% confidence level.

	PCF-PK1	ZH-1	ZH-2	ZH-3
$\Delta F_{\rm CW}^{48}$	0.0405	0.0327	0.0329	0.0326
$\Delta F_{\rm CW}^{208}$	0.0269	0.0284	0.0280	0.0282
$\Delta r_{\rm np}^{48}$ [fm]	0.172	0.116	0.128	0.127
$\Delta r_{\rm np}^{208}$ [fm]	0.183	0.188	0.190	0.191
$E_{\rm sym}(\rho_{\rm sat})$ [MeV]	33.0	33.6	32.4	32.1
$E_{\rm sym}(2\rho_{\rm sat}/3)$ [MeV]	24.5	25.4	25.8	26.9
L [MeV]	78.4	66.7	43.8	19.4
$L(2\rho_{\rm sat}/3)$ [MeV]	50.4	54.2	50.2	49.0
$K_{\rm sym}$ [MeV]	61.4	-114	-276	-469
$\alpha_{\rm T}$ [fm^2]	3.374	4.312	5.673	4.546
$\alpha_{\tau\rm T}$ [fm^2]	≈2	6.967	9.195	7.114
$\langle b_{\rm IS}^{(48)}\rangle$ [MeV · fm^5]	150	172	205	178
$\langle b_{\rm IS}^{(208)}\rangle$ [MeV · fm^5]	153	176	210	182
$\langle b_{\rm IV}^{(48)}\rangle$ [MeV · fm^5]	35.0	222	230	213
$\langle b_{\rm IV}^{(208)}\rangle$ [MeV · fm^5]	36.7	229	238	219

M. Qiu, **ZZ**, T.G. Yue, L.W. Chen, in preparation

ZH series: ISSO & IVSO

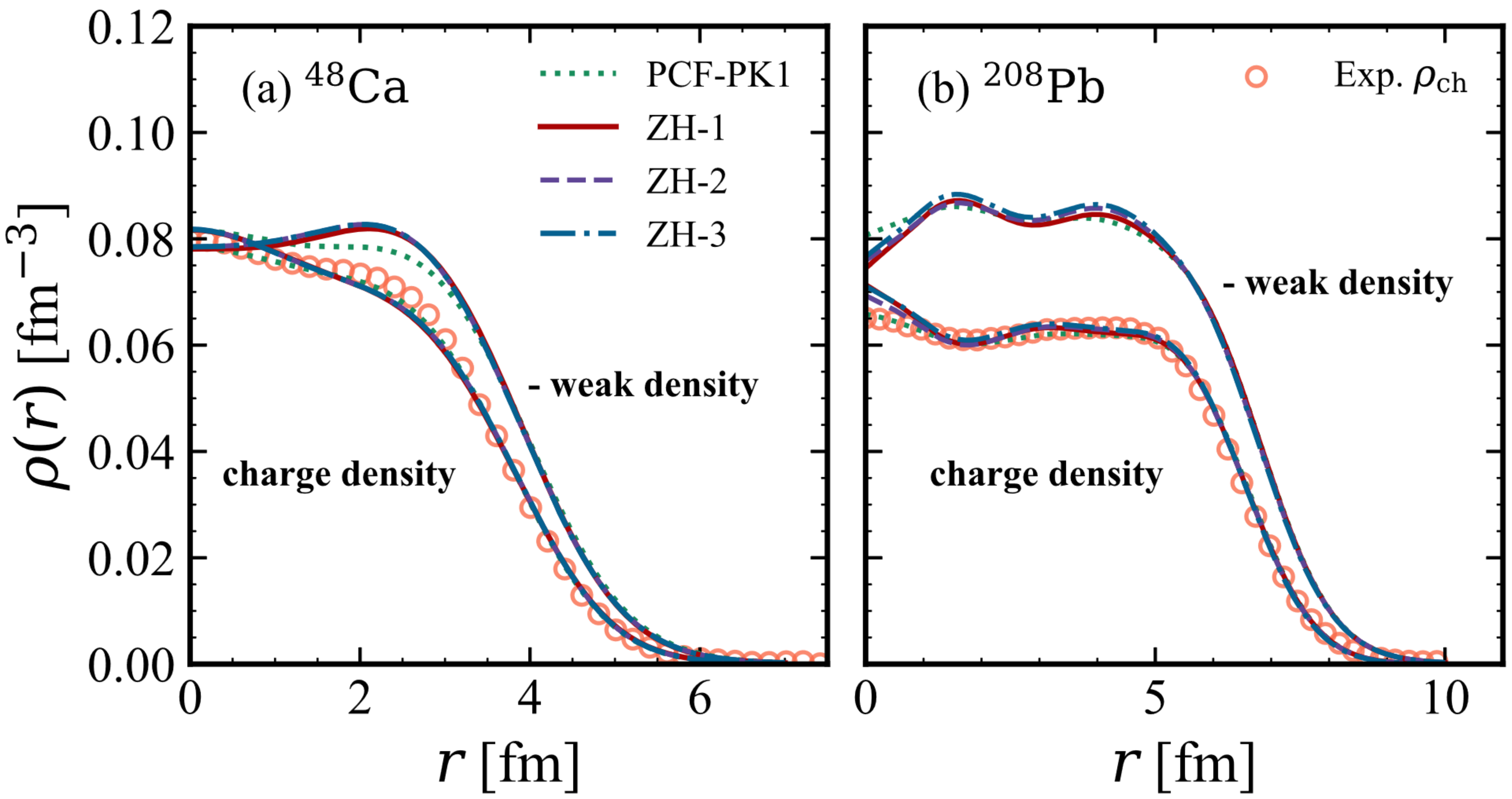


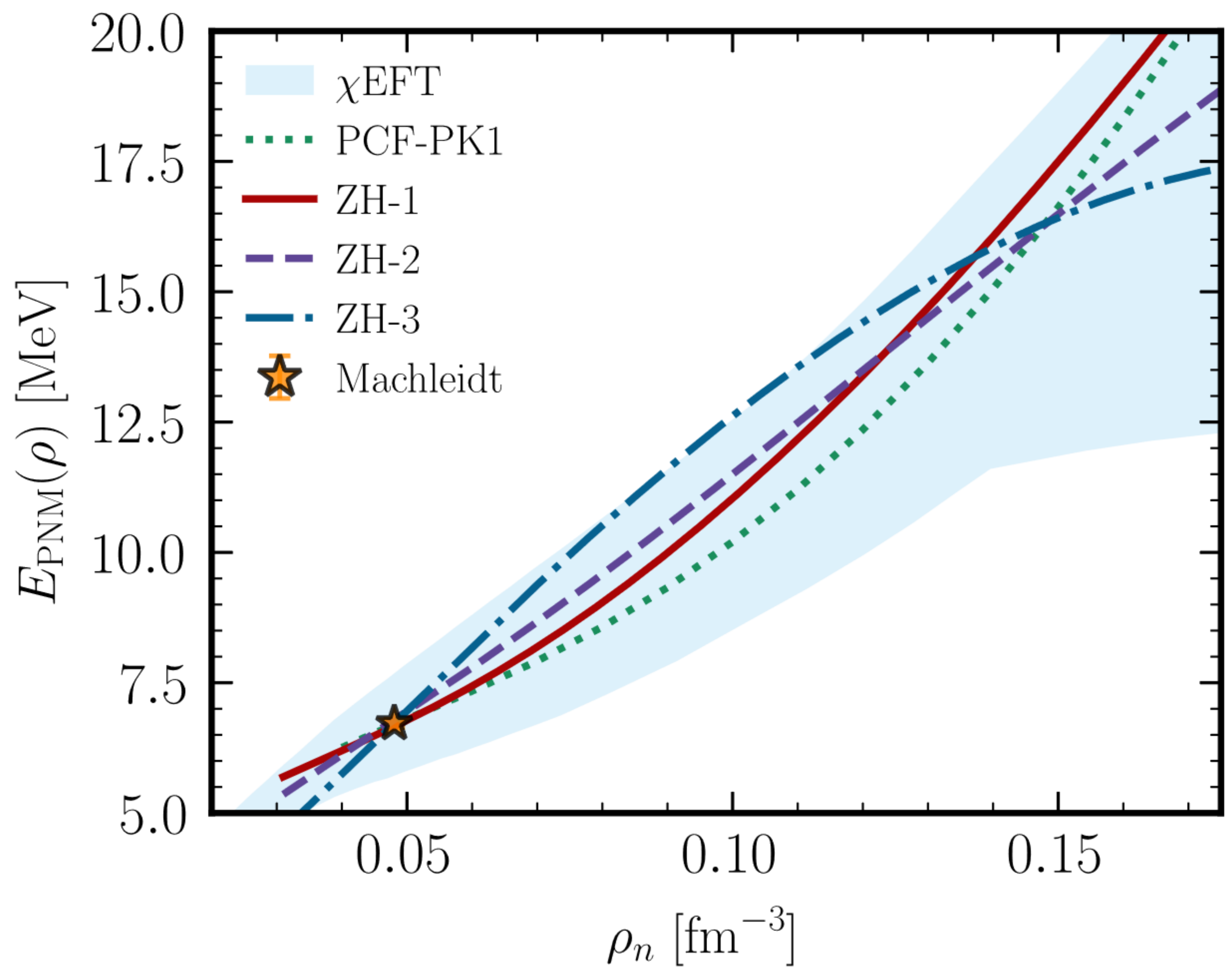
$$b_{\mathrm{IS}}^{\mathrm{NR}} = 8\mathcal{B}_0^2\alpha_{\mathrm{T}} - 4\mathcal{B}_0^2\alpha_{\mathrm{S}} - 4\mathcal{B}_0^2\alpha'_{\mathrm{S}}\rho,$$
$$b_{\mathrm{IV}}^{\mathrm{NR}} = 8\mathcal{B}_0^2\alpha_{\tau\mathrm{T}} - 4\mathcal{B}_0^2\alpha_{\tau\mathrm{S}}.$$

- ✦ Similar isoscalar SO coupling strength.
- ✦ Significantly larger isovector SO coupling strength in ZH family comparied with the PCF-PK1.
- ✦ Central values ≈ 250 $\mathrm{MeV} \cdot \mathrm{fm}^5$, consistent with the fining in nonrelativistic nuclear EDF.

M. Qiu, **ZZ**, T.G. Yue, L.W. Chen, in preparation

ZH series:density distribution and neutron matter EOS

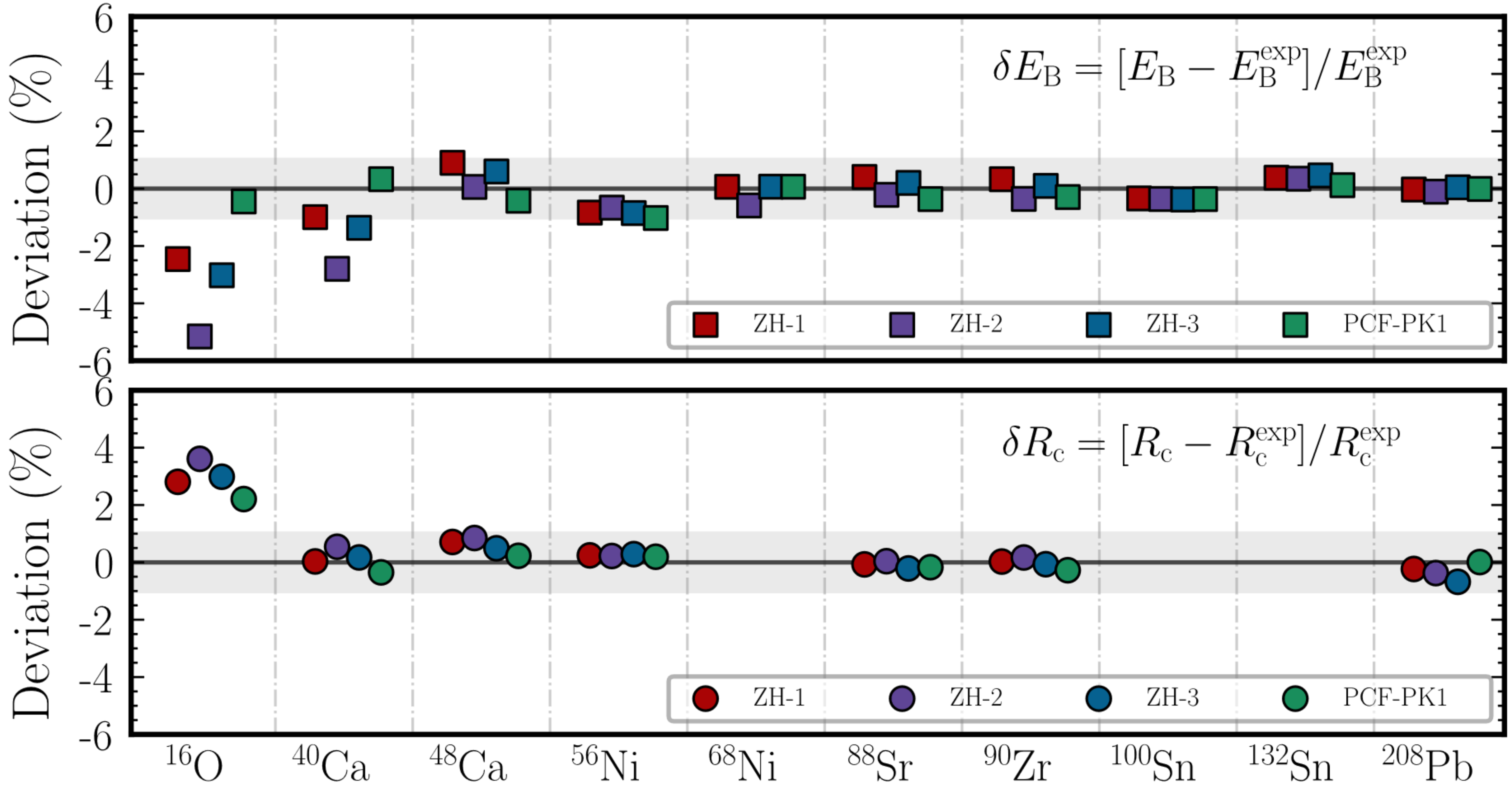




- ✦ No unphysical density fluctuations in the nuclear interior.
- ✦ Neutron matter EOSs are consistent with predictions of chiral EFT.

M. Qiu, **ZZ**, T.G. Yue, L.W. Chen, in preparation

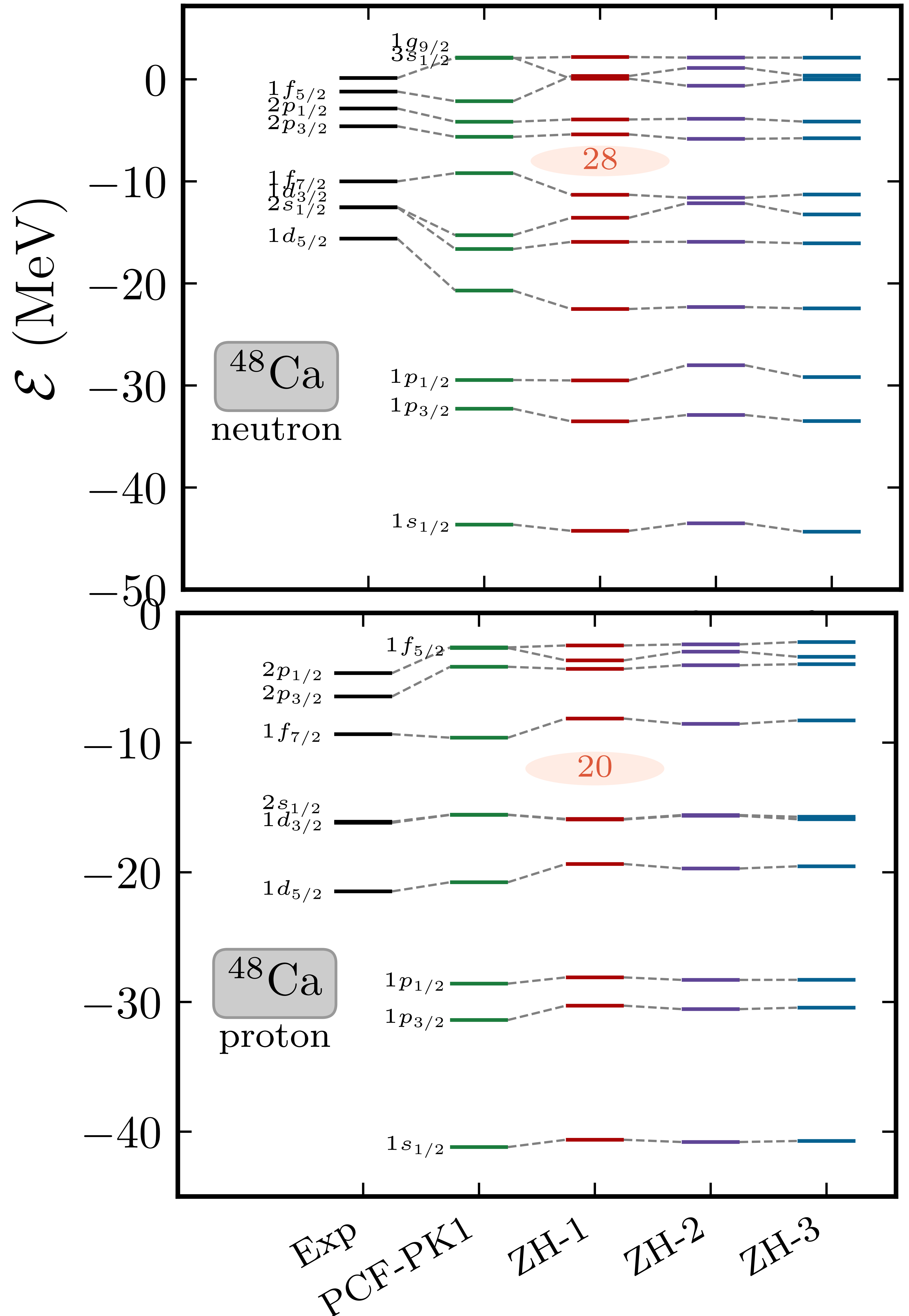
ZH series: binding energies and charge radii

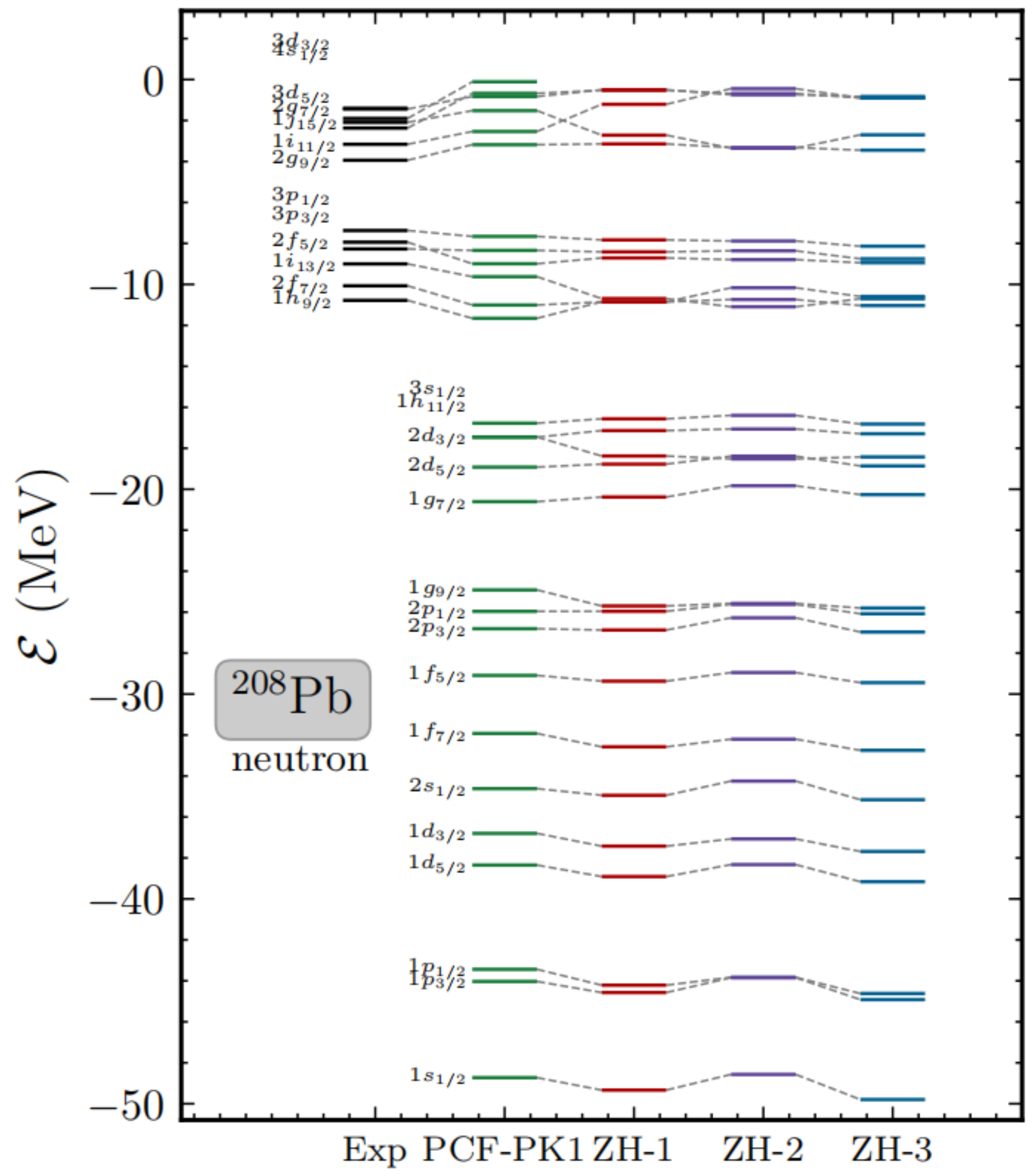


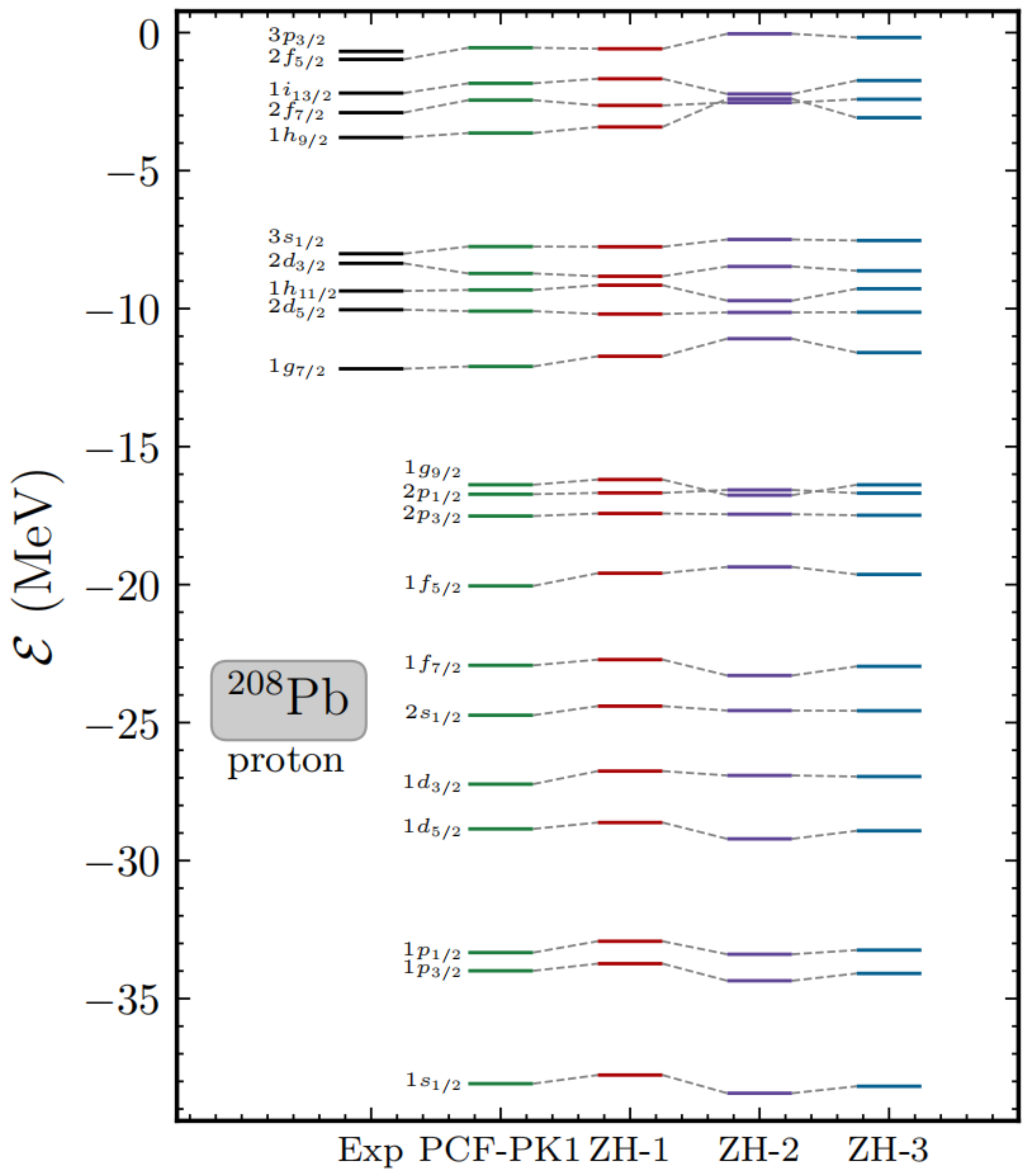
✦ Reasonable description for ground state properties of (semi-) doubly magic nuclei

M. Qiu, **ZZ**, T.G. Yue, L.W. Chen, in preparation

Spin-orbit partner ordering in Ca48 and Pb208







✓Correct ordering

✓Shell closure

M. Qiu, **ZZ**, T.G. Yue, L.W. Chen, in preparation

Summary

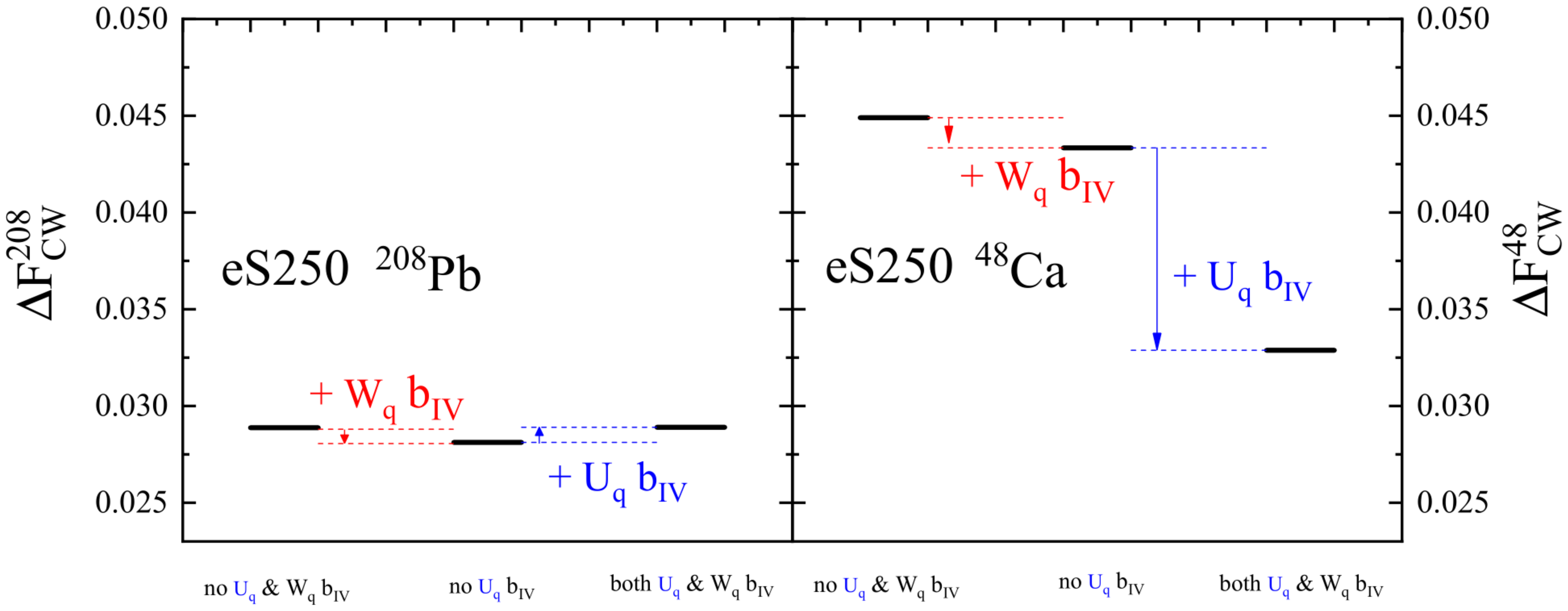
- ✦ The PREX-CREX puzzle can be resolved through a strong isovector spin-orbit interaction within both relativistic and nonrelativistic nuclear density functional theory.

- ✦ Such a strong isovector spin-orbit interaction is expected to have significant impacts on essentially all properties of neutron-rich nuclei: The location of neutron-drip line, shell evolution in exotic nuclei, the new magic number, the properties of superheavy nuclei, …

- ✦ Future PVEs for some stable nuclei (MREX/MESA):
  Pb208, Ni60,…: Not sensitive to the isovector Spin-Orbit interactions (probe Esym);
  Ca48, Zr90,…: Sensitive to the isovector Spin-Orbit interactions (probe b_{IV})

Thanks for your attention

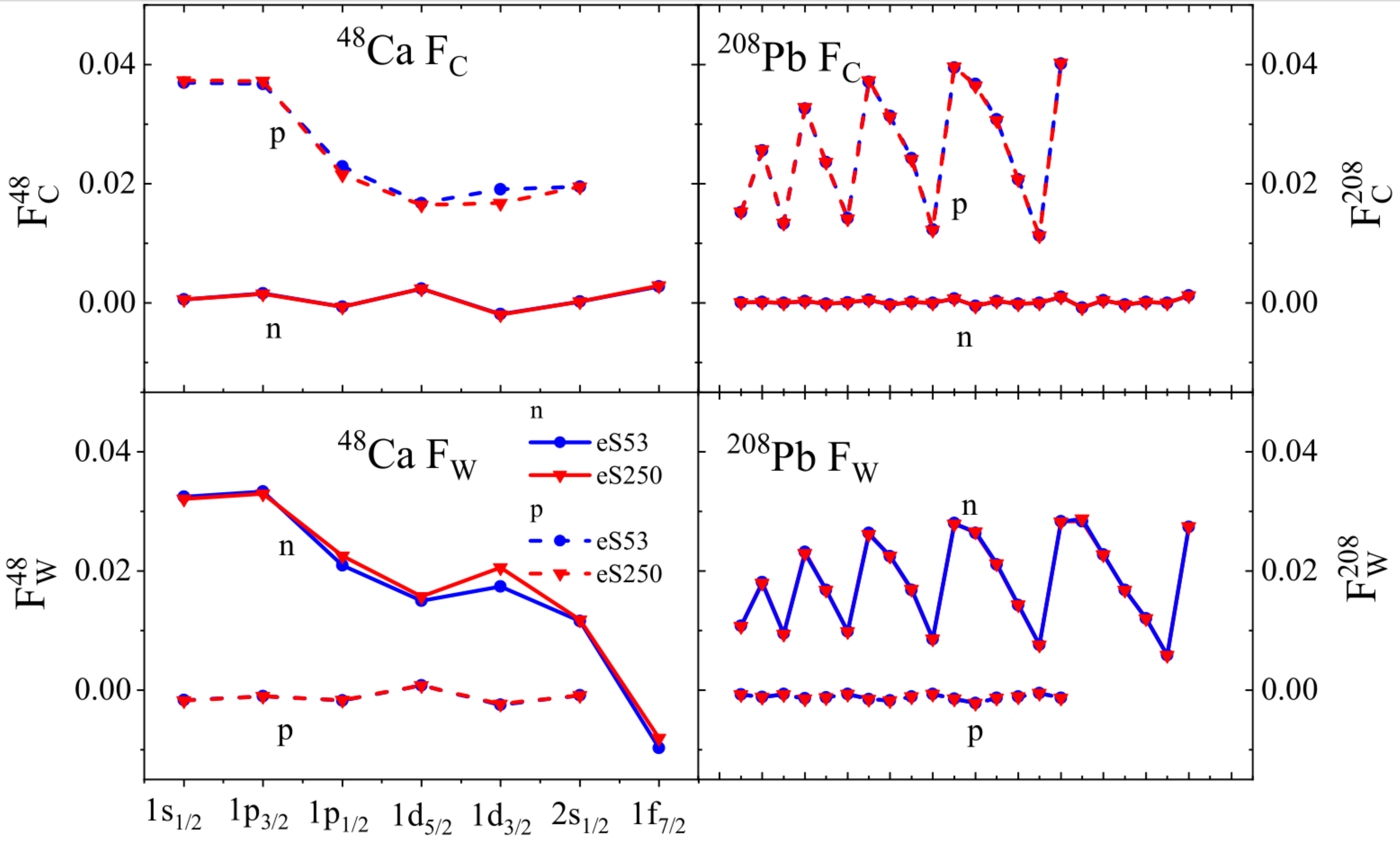
Central average mean field v.s. SO potential



$U_q(b_{IV} = 0)$ $W_q(b_{IV} = 0)$ $U_q(b_{IV} = 0)$ $W_q(250)$ $U_q(250)$ $W_q(250)$

$$\hat{h}_q = -\nabla \cdot \frac{\hbar^2}{2m_q^*}\nabla + U_q + i\boldsymbol{W}_q \cdot (\boldsymbol{\sigma} \times \nabla),$$

IVSO effects in U_q dominate.



IVSO interaction affects

- **Not only the 8 unpaired 1f7/2 neutrons in Ca48**
- But all orbitals via the U_q

Isovector spin-orbit interaction

★ Spin-orbit (SO) energy:

$$E_{\rm SO} = \int d^3r \left[\frac{b_{\rm IS}}{2} \boldsymbol{J} \cdot \nabla \rho + \frac{b_{\rm IV}}{2} (\boldsymbol{J}_n - \boldsymbol{J}_p) \cdot \nabla (\rho_n - \rho_p) \right]$$

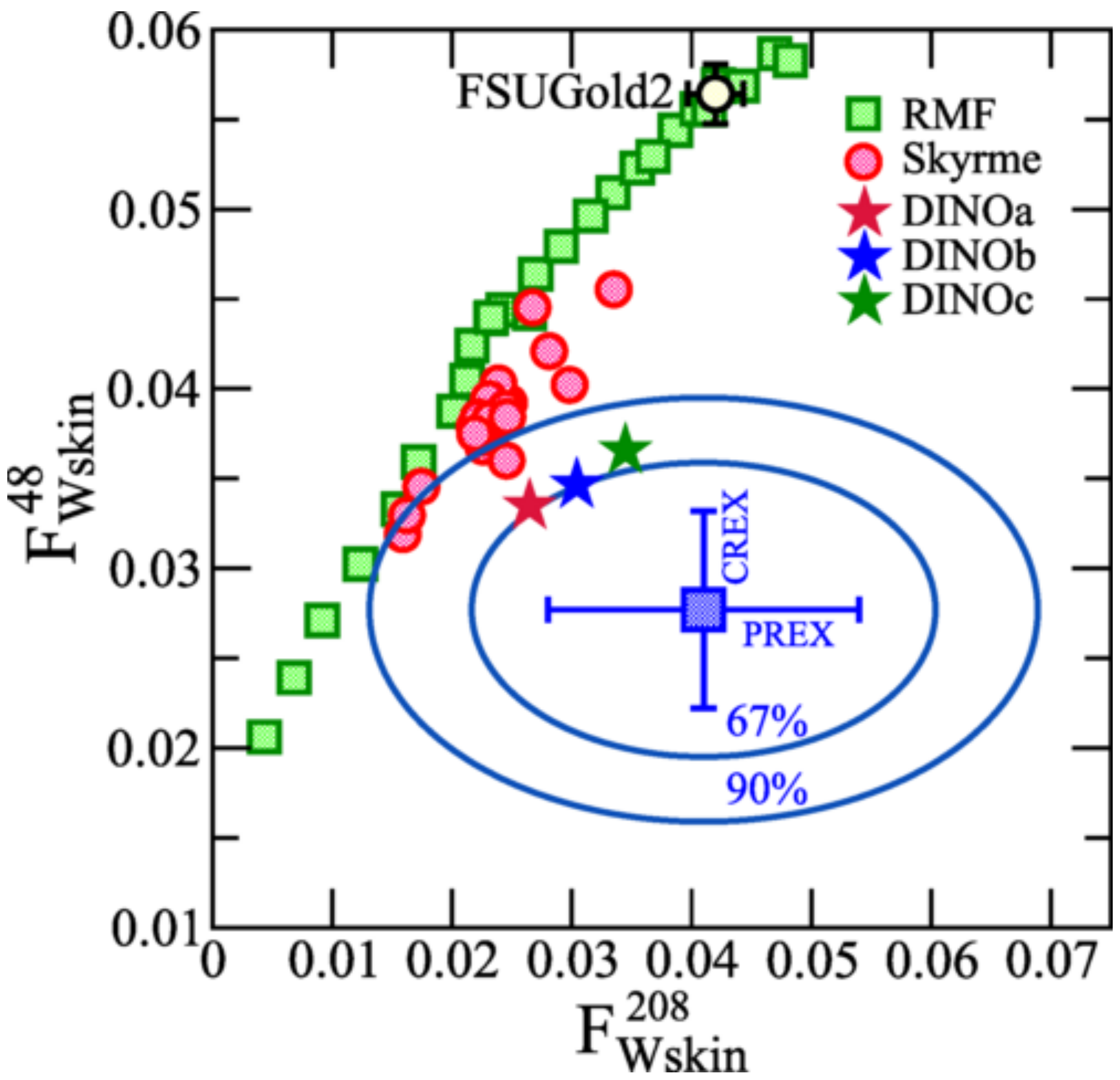
Reinhard and Flocard, NPA 584, 467488 (1995)
Bender, Heenen, and Reinhard, Rev. Mod. Phys. 75, 121 (2003).
Ebran, Mutschler, Khan, and Vretenar, PRC 94, 024304 (2016).

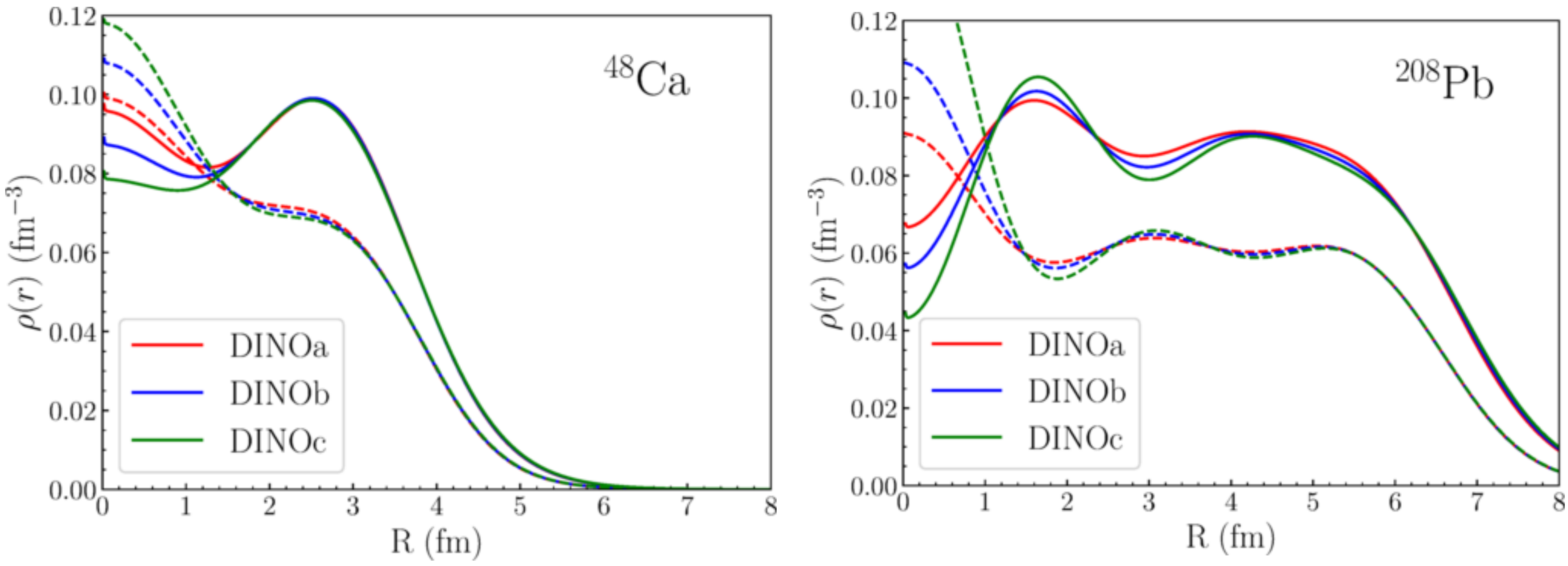
- Standard Skyrme energy density functional (EDF):
  $b_{\rm IV} = b_{\rm IS}/3 = W_0/2 \approx 60\ {\rm MeV} \cdot {\rm fm}^5$
- Relativistic EDF based on meson-exchange Lagrangian (nonrelativistic reduction):
  $b_{\rm IV} \approx 0$ (Hartree approximation, W./O. Isovector-scalar meson)

★ Lack experimental probes to constraint $b_{\rm IV}$

★ The isovector spin-orbit coupling $b_{\rm IV}$ is expected to have significant effects on light nuclei with larger $\boldsymbol{J}_n - \boldsymbol{J}_p$.

Efforts to reconcile CREX and PREX results





Reed, Fattoyev, Horowitz & Piekarewicz, PRC109, 035804 (2024)

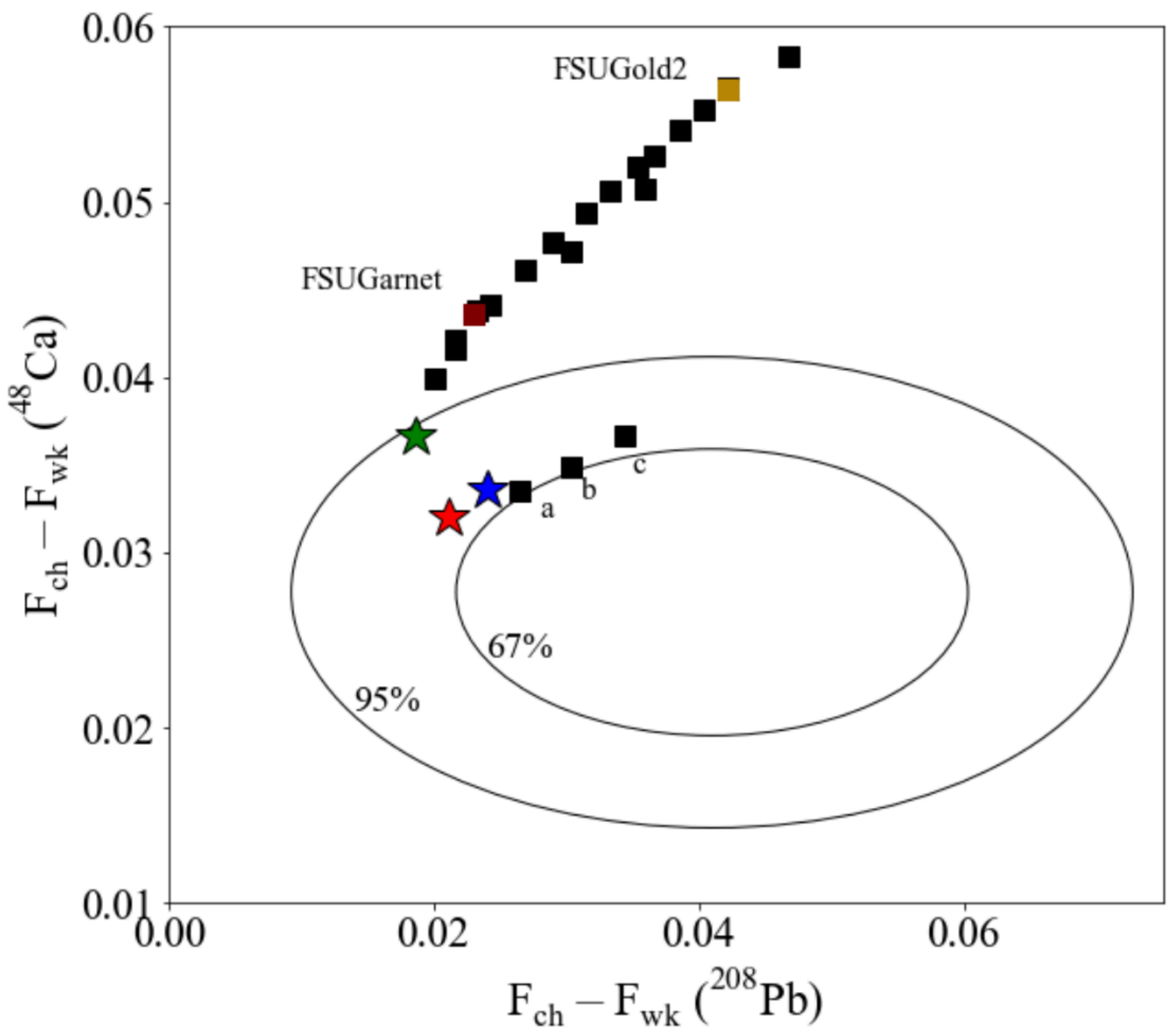
Meson-exchange relativistic mean field model:

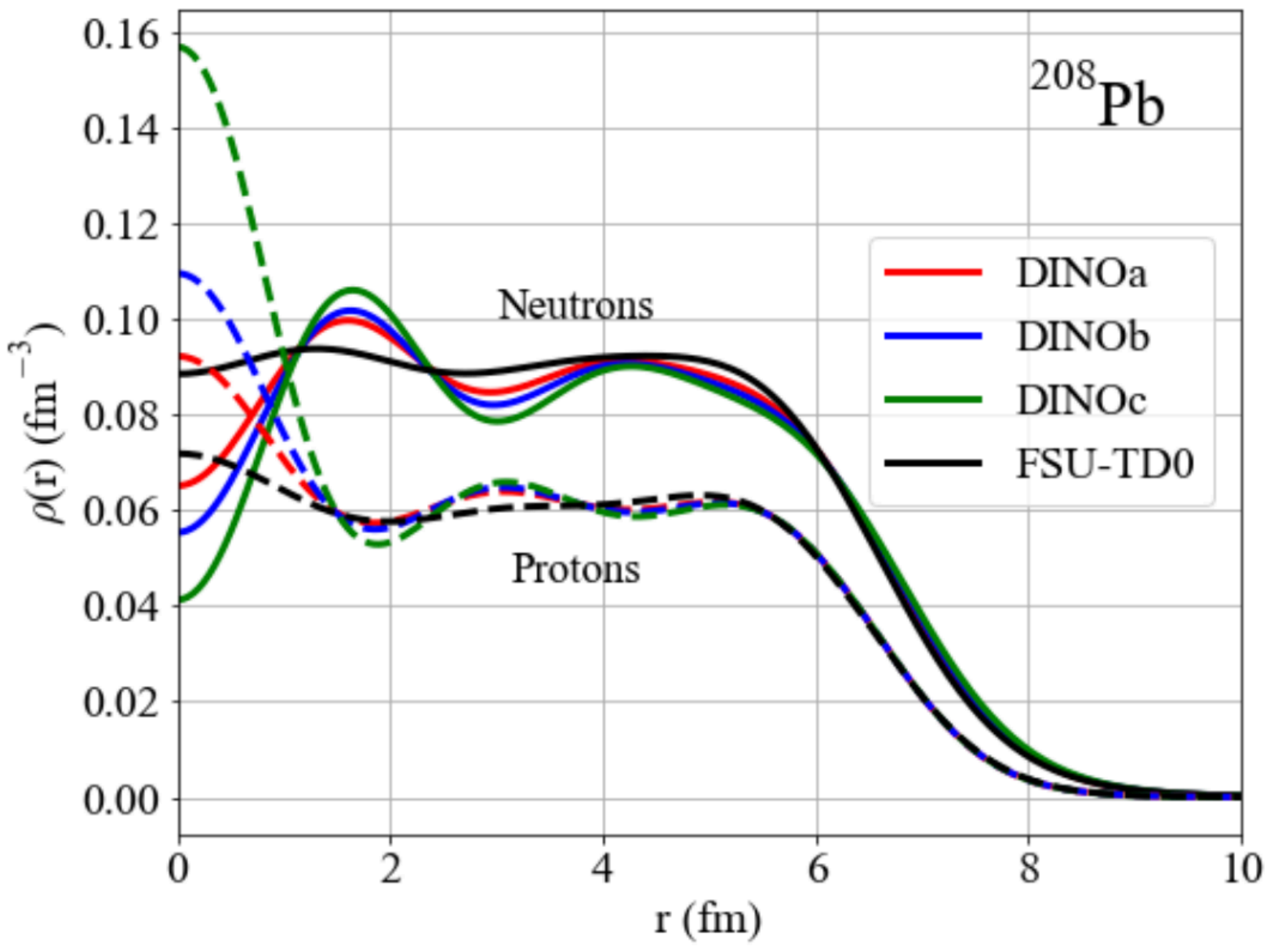
- ✦ Isovector-scalar δ meson
- ✦ Self-coupling of isovector-vector ρ meson
- ✓ Agree better with CREX&PREX

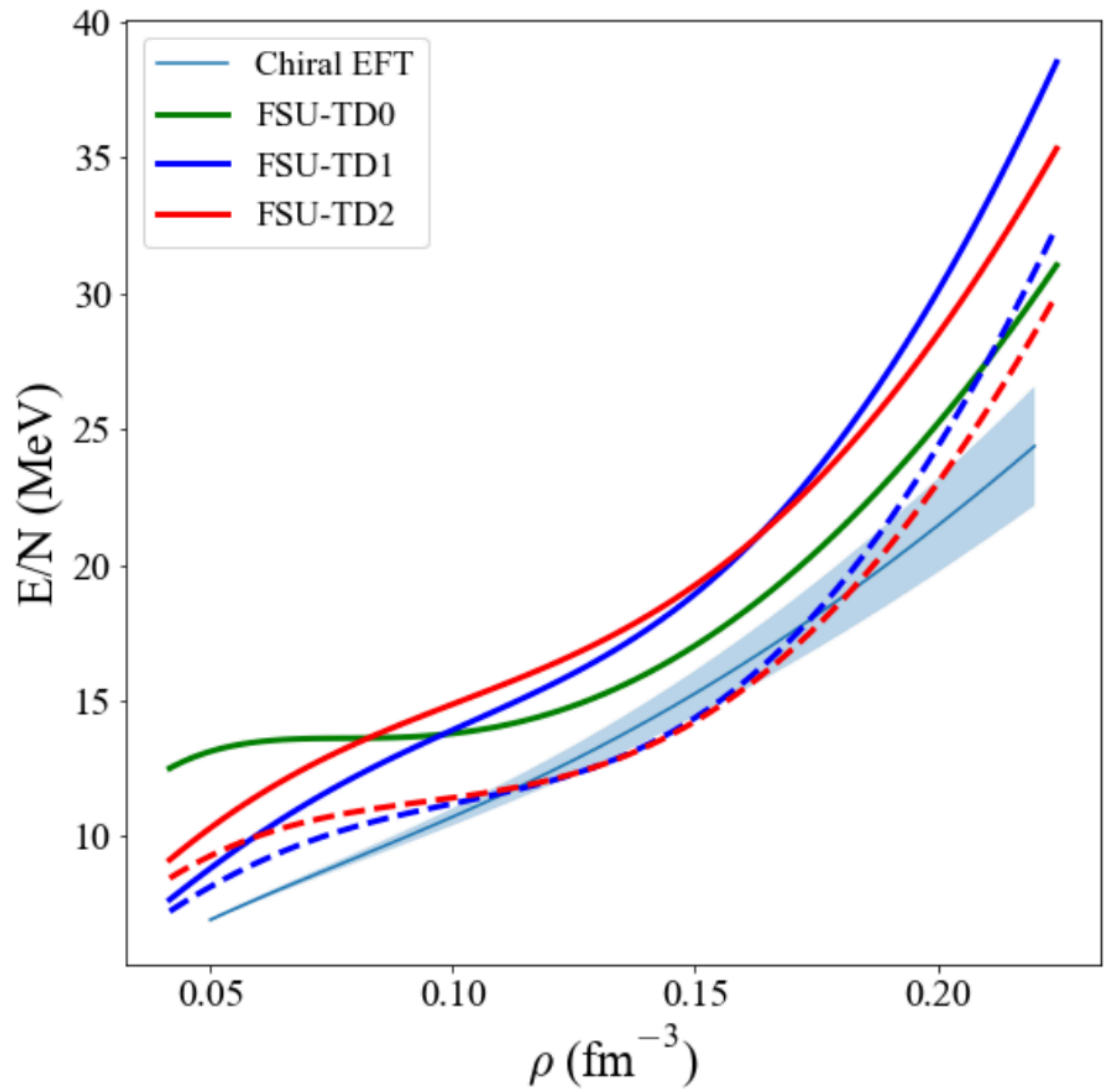
Price:

- ✦ large unphysical oscillations in the nuclear interior
- ✦ Too stiff symmetry energy with uncoventional large positive $K_{\rm sym}$

Efforts to reconcile CREX and PREX results







Salinas & Piekarewicz PRC109, 045807 (2024)

Three new RMF models including:

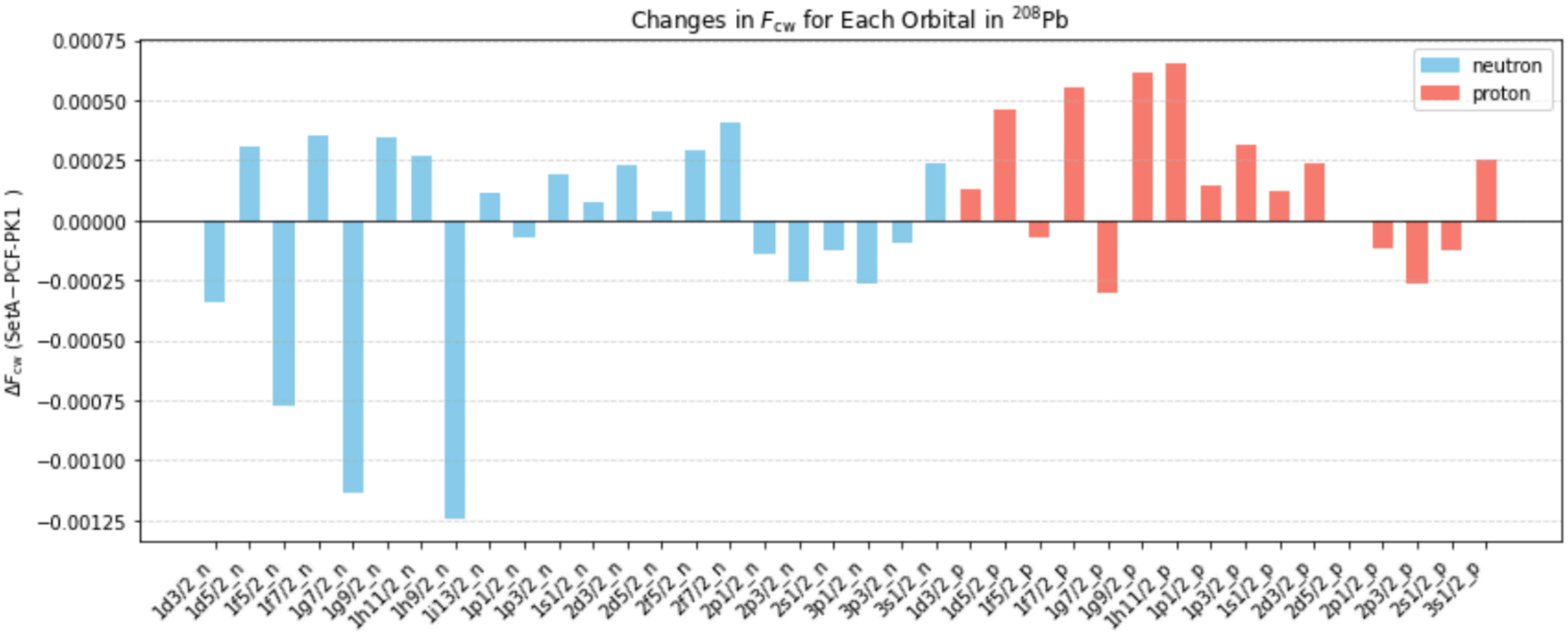
- ✦ Tensor coupling.
- ✦ isoscalar-isovector mixing term in the scalar sector.

- ✦ Remove unphysical density oscillations.
- ✦ In tension with Chiral EFT predictions for the neutron matter EOS.
- ✦ In slightly worse agreement with CREX and PREX

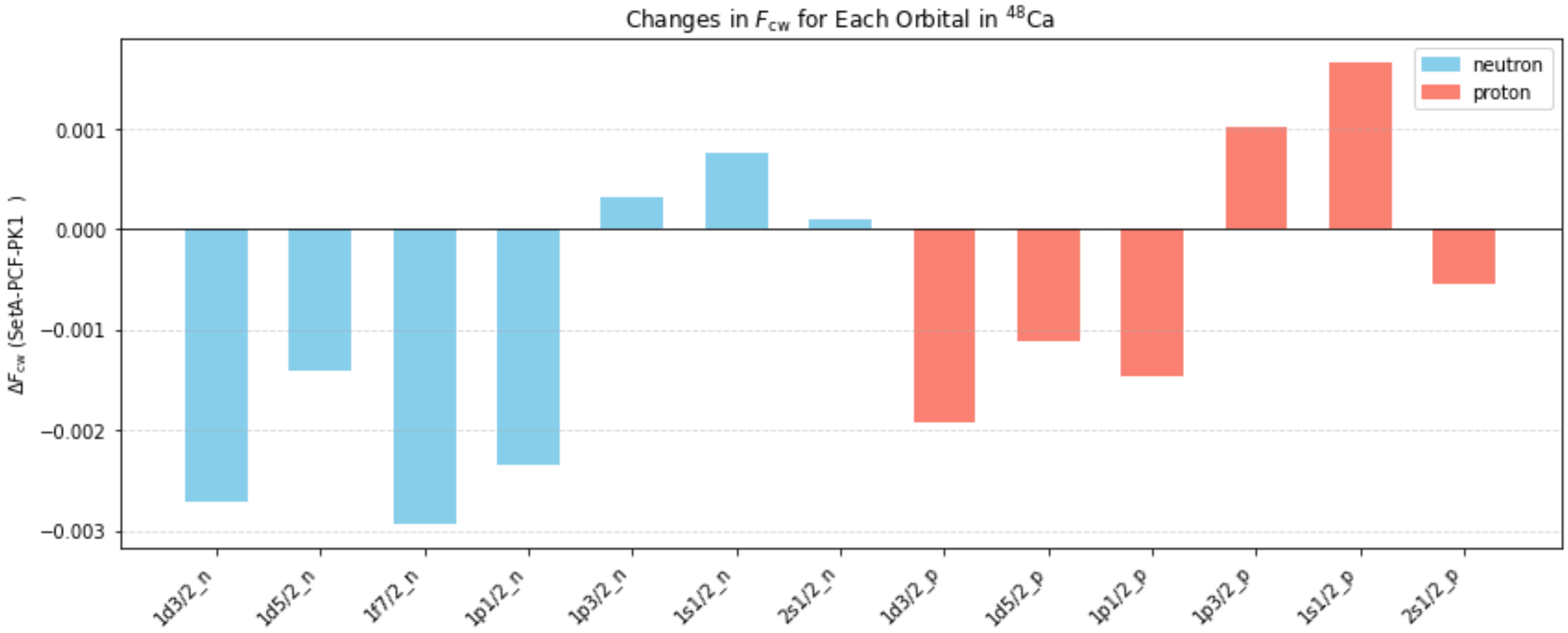
$\Delta F^{208}_{\rm CW}$

Central potential $$U_q = \frac{\partial \mathcal{E}}{\partial \rho_q} - \nabla \cdot \frac{\partial \mathcal{E}}{\partial [\nabla \rho_q]} \approx -\frac{b_{\rm IS}}{2} \nabla \cdot \boldsymbol{J} - \tau_3 \left(\frac{b_{\rm IV}}{2} + \frac{b_{\rm SV}}{2} \right) \nabla \cdot \tilde{\boldsymbol{J}},$$

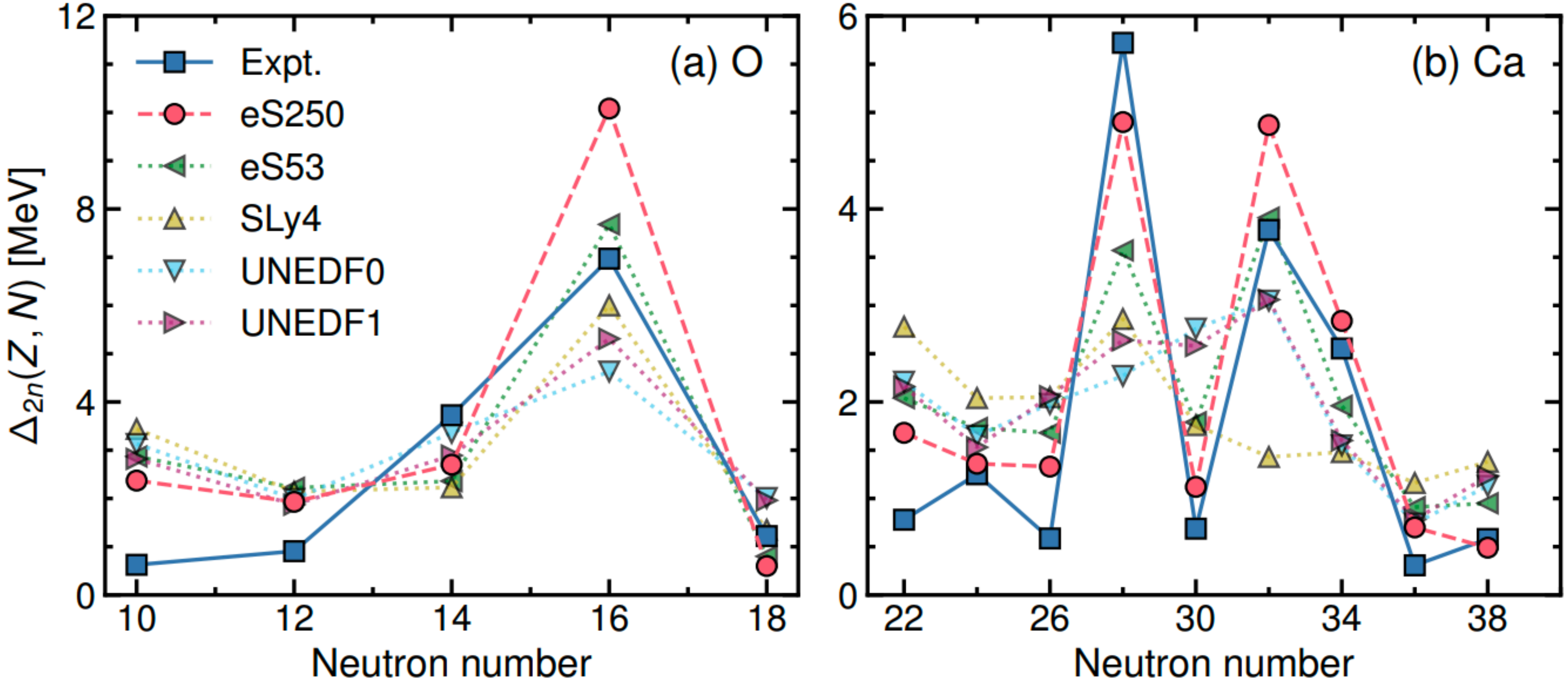
SO potential $$\boldsymbol{W}_q = \frac{\partial \mathcal{E}}{\partial \boldsymbol{J}_q} - \nabla \cdot \frac{\partial \mathcal{E}}{\partial [\nabla \boldsymbol{J}_q]} \approx \left(\frac{b_{\rm IS}}{2} + \tau_3 \frac{b_{\rm SV}}{2} \right) \nabla \rho + \tau_3 \frac{b_{\rm IV}}{2} \nabla \tilde{\rho}. \quad (4)$$



$\Delta F_{\mathrm{CW}}^{48}$



New magic number



1. 消去Dirac Spinor下分量

For a system with time-reversal invariance, the spatial components of the currents $\boldsymbol{j}_V$ and $\boldsymbol{j}_{\tau V}$ vanish, and $\boldsymbol{V}$ thus vanish. Rewriting Dirac equation in matrix form, we obtain

$$\begin{pmatrix} m_{\mathrm{B}} + S + V^0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} - i\boldsymbol{\sigma}\cdot\boldsymbol{T}^0 \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} + i\boldsymbol{\sigma}\cdot\boldsymbol{T}^0 & -m_{\mathrm{B}} - S + V^0 \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \epsilon \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{3}$$

Therefore component equation gives

$$\begin{cases} (m_{\mathrm{B}} + S + V^0)\varphi + (\boldsymbol{\sigma}\cdot\boldsymbol{p} - i\boldsymbol{\sigma}\cdot\boldsymbol{T}^0)\chi = \epsilon\varphi \\ (\boldsymbol{\sigma}\cdot\boldsymbol{p} + i\boldsymbol{\sigma}\cdot\boldsymbol{T}^0)\varphi + (-m_{\mathrm{B}} - S + V^0)\chi = \epsilon\chi \end{cases} \tag{4}$$

Using the second equation, we obtain

$$\chi = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p} + i\boldsymbol{\sigma}\cdot\boldsymbol{T}^0}{\epsilon + m_{\mathrm{B}} + S - V^0}\varphi = \mathcal{B}\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} + i\boldsymbol{T}^0\right)\varphi \tag{5}$$

where the inverse effective mass is definded as

$$\mathcal{B} = \frac{1}{\epsilon + m_{\mathrm{B}} + S} \tag{6}$$

$$S + V^0 \sim v^2, \quad \epsilon - m \sim v^2, \quad \boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - i\boldsymbol{T}^0\right)\mathcal{B}\boldsymbol{\sigma}\cdot\vec{\Pi} \sim v^2.$$

$$S_0 = \alpha_{\mathrm{S}}\rho_{\mathrm{S}} \text{ and } S_3 = \tau_3\alpha_{\tau\mathrm{S}}\rho_{\mathrm{S}}.$$

$$\mathcal{B} = \frac{1}{2(m+S) + (\epsilon - m) - (S + V^0)} \to \mathcal{B}^* = [2m+2S]^{-1}. \to \mathcal{B}_0 = [2m + 2S_0]^{-1}.$$

2. Spinor上分量和归一化条件

2.2 上分量和经典波函数的变换关系

The isoscalar vector density is not normalized and the norm is expressed as

$$\int d^3 r \bar{\psi} \gamma_0 \psi = 1 \rightarrow \text{ Norm } = \int \mathrm{d}^3 r \varphi^\dagger \left(1 + \boldsymbol{\sigma} \cdot \overleftarrow{\Pi} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \overrightarrow{\Pi}\right) \varphi = \int \mathrm{d}^3 r \varphi^\dagger \hat{I} \varphi \tag{10}$$

where the norm kernel is

$$\hat{I} = 1 + \boldsymbol{\sigma} \cdot \overleftarrow{\Pi} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \overrightarrow{\Pi} \tag{11}$$

The normalized ***classical*** wave function is expressed as

$$\phi^{\mathrm{cl}} = \hat{I}^{1/2} \varphi \tag{12}$$

and therefore

$$\varphi = \hat{I}^{-1/2} \phi^{\mathrm{cl}} \tag{13}$$

using Taylor expansion

$$\hat{I}^{-1} = \frac{1}{1 + \boldsymbol{\sigma} \cdot \overleftarrow{\Pi} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \overrightarrow{\Pi}} = 1 - \boldsymbol{\sigma} \cdot \overleftarrow{\Pi} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \Pi + \mathcal{O}\left[\left(\boldsymbol{\sigma} \cdot \Pi \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \overrightarrow{\Pi}\right)^2\right] \tag{14}$$

and

$$\hat{I}^{-1/2} = \frac{1}{\left(1 + \boldsymbol{\sigma} \cdot \overleftarrow{\Pi} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \overrightarrow{\Pi}\right)^{1/2}} = 1 - \frac{1}{2}\left(\boldsymbol{\sigma} \cdot \overleftarrow{\Pi} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \overrightarrow{\Pi}\right) + \mathcal{O}\left[\left(\boldsymbol{\sigma} \cdot \overleftarrow{\Pi} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \overrightarrow{\Pi}\right)^2\right] \tag{15}$$

3. 流和密度的非相对论极限

2.1 非相对论的流和密度

Non-relativistic density and current are

$$\text{density}\quad \rho=\sum_\alpha\left|\varphi_\alpha^{\mathrm{cl}}\right|^2,$$

$$\text{kinetic energy density}\quad \tau=\sum_\alpha\left|\nabla\varphi_\alpha^{\mathrm{cl}}\right|^2,$$

$$\text{S.O. current}\quad \mathbf{J}=-\frac{i}{2}\sum_\alpha\left[\varphi_\alpha^{\mathrm{cl}\dagger}(\nabla\times\boldsymbol{\sigma})\varphi^{\mathrm{cl}}-\left(\nabla\times\boldsymbol{\sigma}\varphi_\alpha^{\mathrm{cl}}\right)^\dagger\varphi_\alpha^{\mathrm{cl}}\right],$$

$$\mathbf{j}=-\frac{i}{2}\sum_\alpha\left[\varphi_\alpha^{\mathrm{cl}\dagger}\nabla\varphi_\alpha^{\mathrm{cl}}-\left(\nabla\varphi_\alpha^{\mathrm{cl}}\right)^\dagger\varphi_\alpha^{\mathrm{cl}}\right],$$

$$\text{spin density}\quad \boldsymbol{\rho}_\sigma=\sum_\alpha\varphi_\alpha^{\mathrm{cl}\dagger}\boldsymbol{\sigma}\varphi_\alpha^{\mathrm{cl}},$$

$$\rho_S=\sum_{\alpha>0}\bar{V}_\alpha V_\alpha,\quad \rho_{tS}=\sum_{\alpha>0}\bar{V}_\alpha\tau_3 V_\alpha,$$

$$\rho_V=\sum_{\alpha>0}\bar{V}_\alpha\gamma^0 V_\alpha,\quad \rho_{tV}=\sum_{\alpha>0}\bar{V}_\alpha\gamma^0\tau_3 V_\alpha,$$

$$\boldsymbol{j}_V=\sum_{\alpha>0}\bar{V}_\alpha\boldsymbol{\gamma}V_\alpha,\quad \boldsymbol{j}_{tV}=\sum_{\alpha>0}\bar{V}_\alpha\boldsymbol{\gamma}\tau_3 V_\alpha,$$

$$\boldsymbol{j}_T^0=\sum_{\alpha>0}\bar{V}_\alpha i\gamma^0\boldsymbol{\gamma}V_\alpha,\quad \boldsymbol{j}_{tT}^0=\sum_{\alpha>0}\bar{V}_\alpha i\gamma^0\boldsymbol{\gamma}\tau_3 V_\alpha.$$

$$\text{Scalar density}\quad \rho_S=\rho_0-2\mathcal{B}_0^2\left[\tau-\nabla\boldsymbol{J}-\boldsymbol{T}^0\nabla\rho+2\boldsymbol{T}^0\boldsymbol{J}+\rho_0\boldsymbol{T}^2\right]$$

$$\begin{aligned}\text{Tensor current}\quad \boldsymbol{j}_T^0=&\,i\mathcal{B}_0\sum_\alpha\left[-i\varphi^\dagger\nabla\varphi-i(\nabla\varphi^\dagger)\varphi\right]\\&+i\mathcal{B}_0\sum_\alpha\left[i\boldsymbol{T}^0\varphi^\dagger\varphi+i\boldsymbol{T}^0\varphi^\dagger\varphi\right]\\&+i\mathcal{B}_0\sum_\alpha\left[\varphi^\dagger\left(\nabla\times\boldsymbol{\sigma}\varphi\right)-\left(\nabla\times\boldsymbol{\sigma}\varphi^\dagger\right)\varphi\right]\\&=\mathcal{B}_0\nabla\rho_0-2\mathcal{B}_0\boldsymbol{T}^0\rho_0-2\mathcal{B}_0\boldsymbol{J}\end{aligned}$$

4. 能量密度的非相对论极限

首先，从拉式量出发，推导能量密度

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\psi\right)}\partial^{\nu}\psi + \partial^{\nu}\bar{\psi}\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\bar{\psi}\right)} - g^{\mu\nu}\mathcal{L} \tag{72}$$

又

$$\epsilon = T^{00} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{0}\psi\right)}\partial^{0}\psi + \partial^{0}\bar{\psi}\frac{\partial \mathcal{L}}{\partial\left(\partial_{0}\bar{\psi}\right)} - g^{00}\mathcal{L} \tag{73}$$

在静态原子核中，可写作

$$\epsilon = \psi^{\dagger}(-i\boldsymbol{\alpha}\cdot\nabla + \beta M)\psi + \frac{1}{2}\alpha_{\mathrm{S}}\rho_{\mathrm{S}}^{2} + \frac{1}{2}\alpha_{\mathrm{V}}\rho_{\mathrm{V}}^{2} + \frac{1}{2}\alpha_{\tau\mathrm{S}}\rho_{\tau\mathrm{S}}^{2} + \frac{1}{2}\alpha_{\tau\mathrm{V}}\rho_{\tau\mathrm{V}}^{2} - \alpha_{\tau\mathrm{T}}\mathbf{j}_{\tau\mathrm{T}}^{0\;2} - \alpha_{\mathrm{T}}\mathbf{j}_{\mathrm{T}}^{0\;2} - \frac{1}{2}\delta_{S}\left|\nabla\rho_{S}\right|^{2} \tag{74}$$

重新定义能量密度有

$$\epsilon = \epsilon^{\mathrm{kin}} + \epsilon^{\mathrm{4f}} + \epsilon^{\mathrm{em}} + \epsilon^{\mathrm{der}} \tag{75}$$

ϵ^{4f}保留到$\mathcal{B}_0^2$阶贡献为

$$\mathcal{E}^{\mathrm{NR}} = \left(\frac{1}{2}\alpha_{\mathrm{S}} + \frac{1}{2}\alpha_{\mathrm{V}}\right)\rho^{2} + \left(\frac{1}{2}\alpha_{\tau\mathrm{S}} + \frac{1}{2}\right.$$

(Extended) Skyrme EDF with tensor force

Standard Skyme interaction:

$$v(r_1, r_2) = t_0(1 + x_0 P_\sigma)\delta(\mathbf{r}) + \frac{1}{2}t_1(1 + x_1 P_\sigma)\left[\mathbf{k}'^2\delta(\mathbf{r}) + \delta(\mathbf{r})\mathbf{k}^2\right] + t_2(1 + x_2 P_\sigma)\mathbf{k}' \cdot \delta(\mathbf{r})\mathbf{k}$$
$$+ \frac{1}{6}t_3(1 + x_3 P_\sigma)[\rho(\mathbf{R})]^\alpha \delta(\mathbf{r}) + iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot [\mathbf{k}' \times \delta(\mathbf{r})\mathbf{k}],$$

Chabanat, et al., NPA 627, 710 (1997)

Momentum-dependent three-body interaction:

$$v' = v + \frac{1}{2}t_4(1 + x_4 P_\sigma)\left[\mathbf{k}'^2 \rho(R)^\beta \delta(\mathbf{r}) + \delta(\mathbf{r})\rho(R)^\beta \mathbf{k}^2\right] + t_5(1 + x_5 P_\sigma)\mathbf{k}' \cdot \rho(R)^\gamma \delta(\mathbf{r})\mathbf{k}.$$

Zhang & Chen, PRC 94, 064326 (2016)

Zero-range tensor force:

$$V_T = \frac{1}{2}T\left\{\left[(\sigma_1 \cdot \mathbf{k}')(\sigma_2 \cdot \mathbf{k}') - \frac{1}{3}\mathbf{k}'^2(\sigma_1 \cdot \sigma_2)\right]\delta(\mathbf{r}) + \delta(\mathbf{r})\left[(\sigma_1 \cdot \mathbf{k})(\sigma_2 \cdot \mathbf{k}) - \frac{1}{3}\mathbf{k}^2(\sigma_1 \cdot \sigma_2)\right]\right\}$$
$$+ U\left\{(\sigma_1 \cdot \mathbf{k}')\delta(\mathbf{r})(\sigma_2 \cdot \mathbf{k}) - \frac{1}{3}(\sigma_1 \cdot \sigma_2)[\mathbf{k}' \cdot \delta(\mathbf{r})\mathbf{k}]\right\},$$

Stancu, Brink, and Flocard, PLB 68, 108 (1977)