Toward constraining QCD phase transitions in neutron star interiors: Bayesian inference with a TOV linear response analysis

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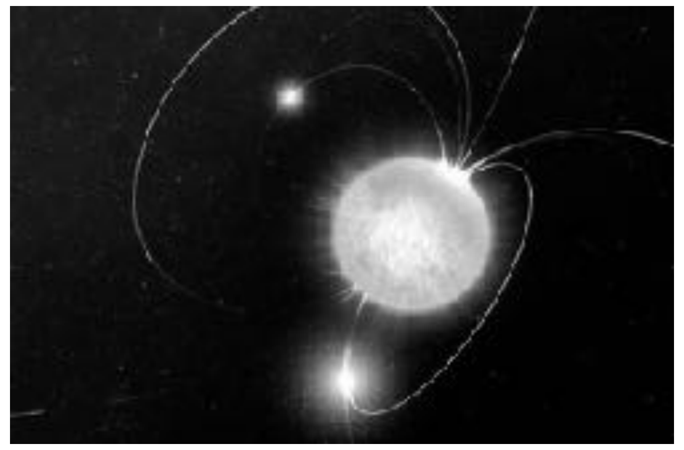
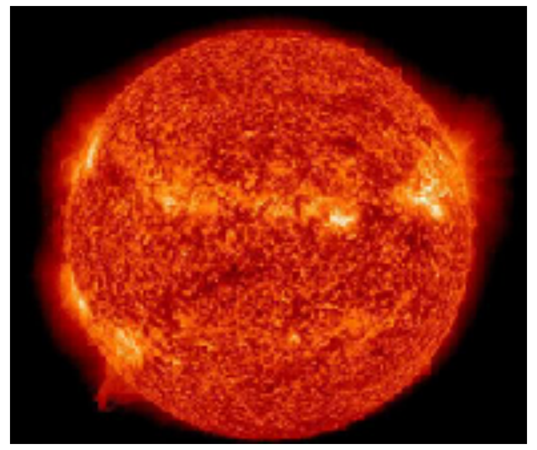
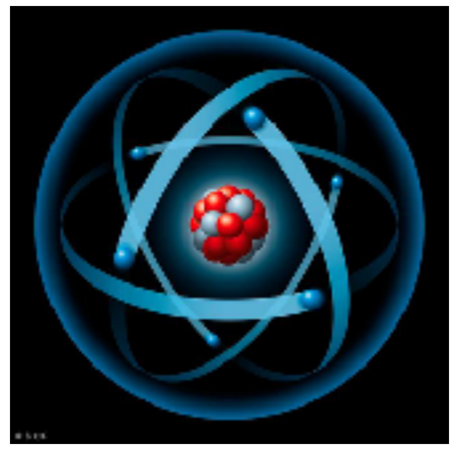
Ronghao Li, Sophia Han, Zidu Lin, Lingxiao Wang, Kai Zhou, ***SS***,
Phys. Rev. D, 111, 074026 + work in progress

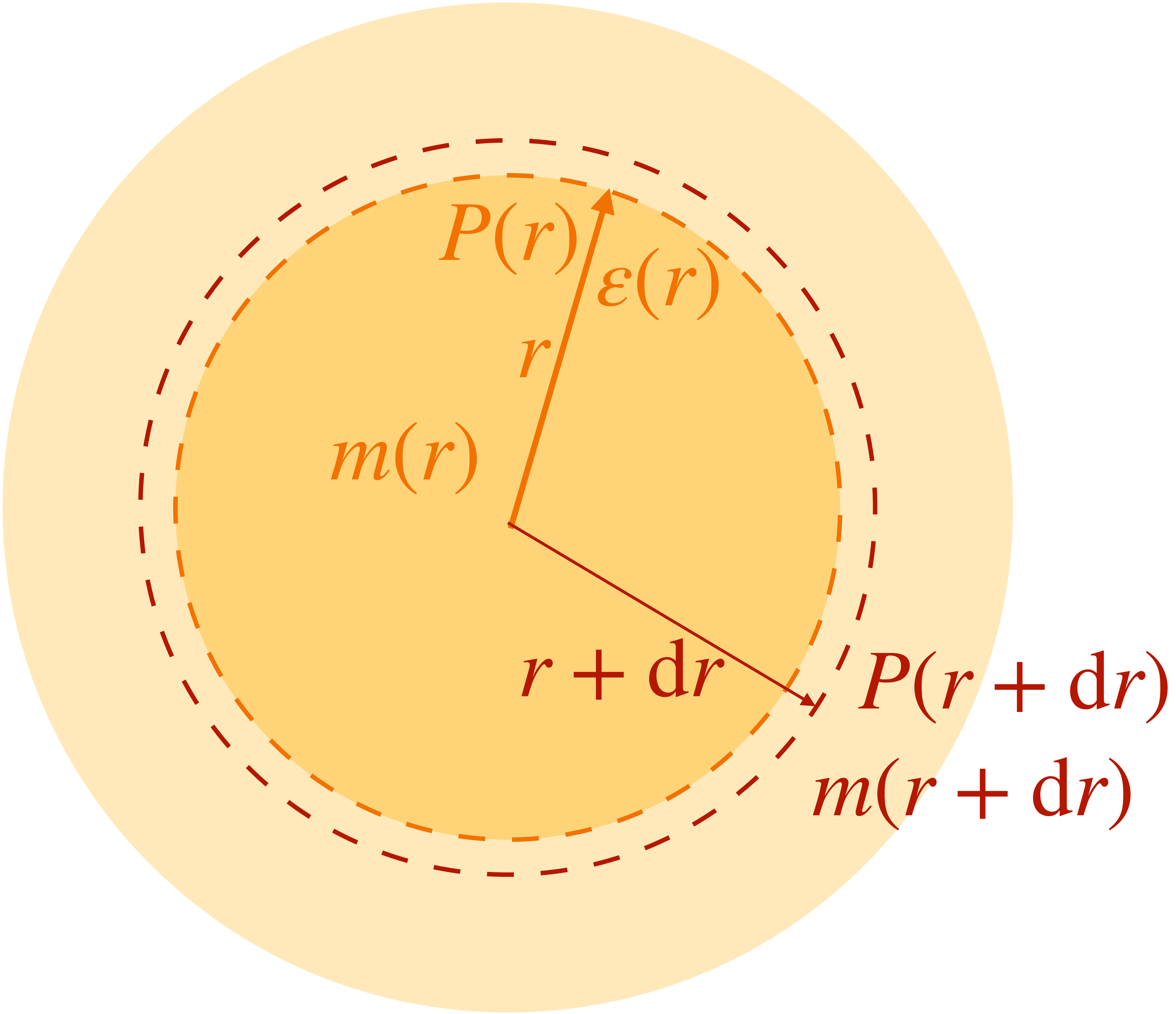
see also:

Soma, Wang, ***SS***, Stoecker, Zhou, JCAP08 (2022) 071; Phys. Rev. D 107, 083028

Neutron Star

			
	Neutron Star	Sun	Nucleus
Mass ($M_\odot$)	$0.5 \sim 2$	1	$\sim 10^{0\sim 2} \times 10^{-57}$
Radius (km)	~ 10	7×10^5	$\sim 10^{-18}$
Density ($M_\odot/\text{km}^3$)	$\sim 5 \times 10^{-3}$	$\sim 7 \times 10^{-19}$	$\sim 3 \times 10^{-3}$

Tolman-Oppenheimer-Volkov equations

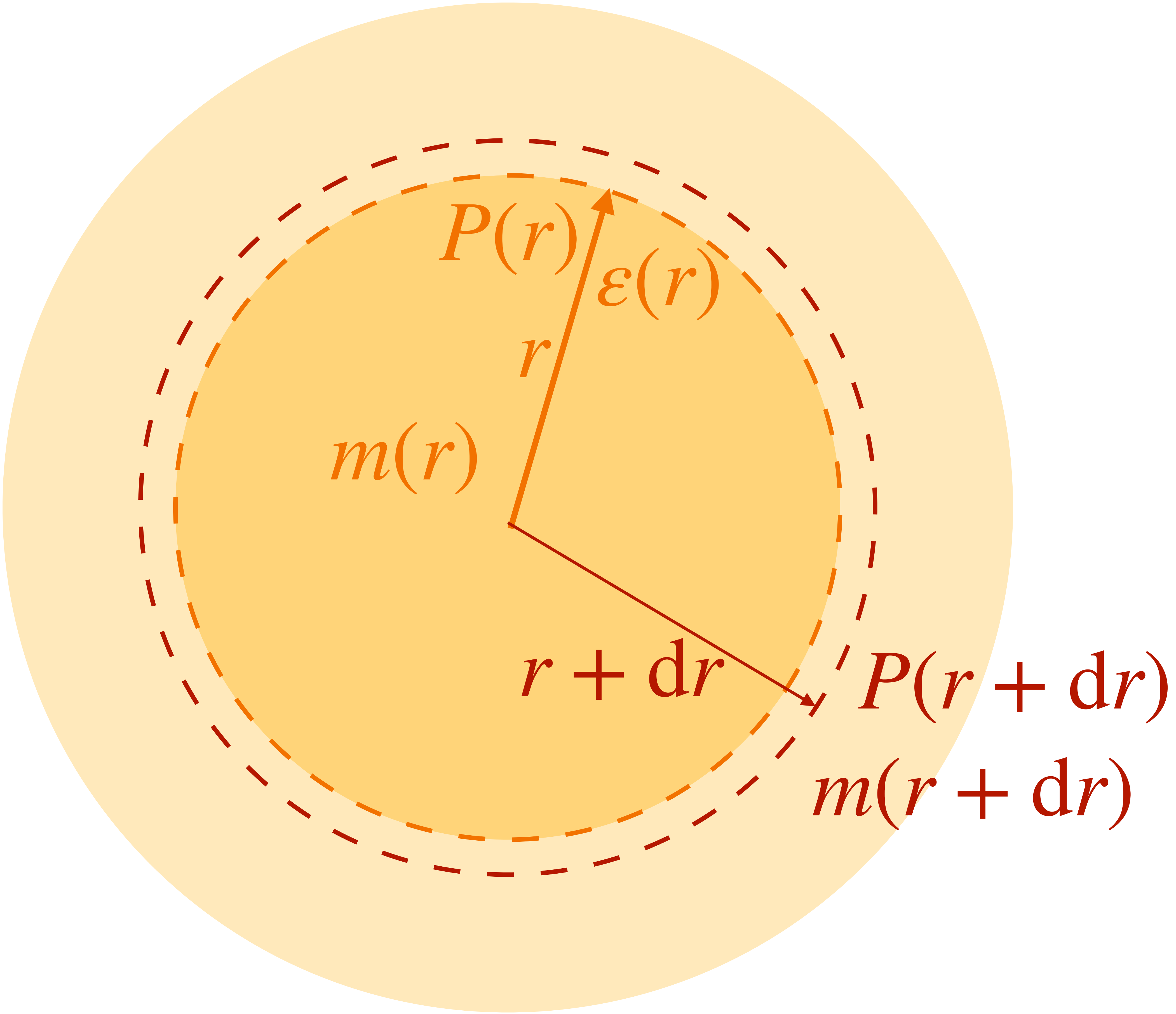


Pressure at radius r: $P(r)$

Mass density at r: $\varepsilon(r)$

Mass enclosed within shell: $m(r)$

Tolman-Oppenheimer-Volkov equations



Pressure at radius r: $P(r)$

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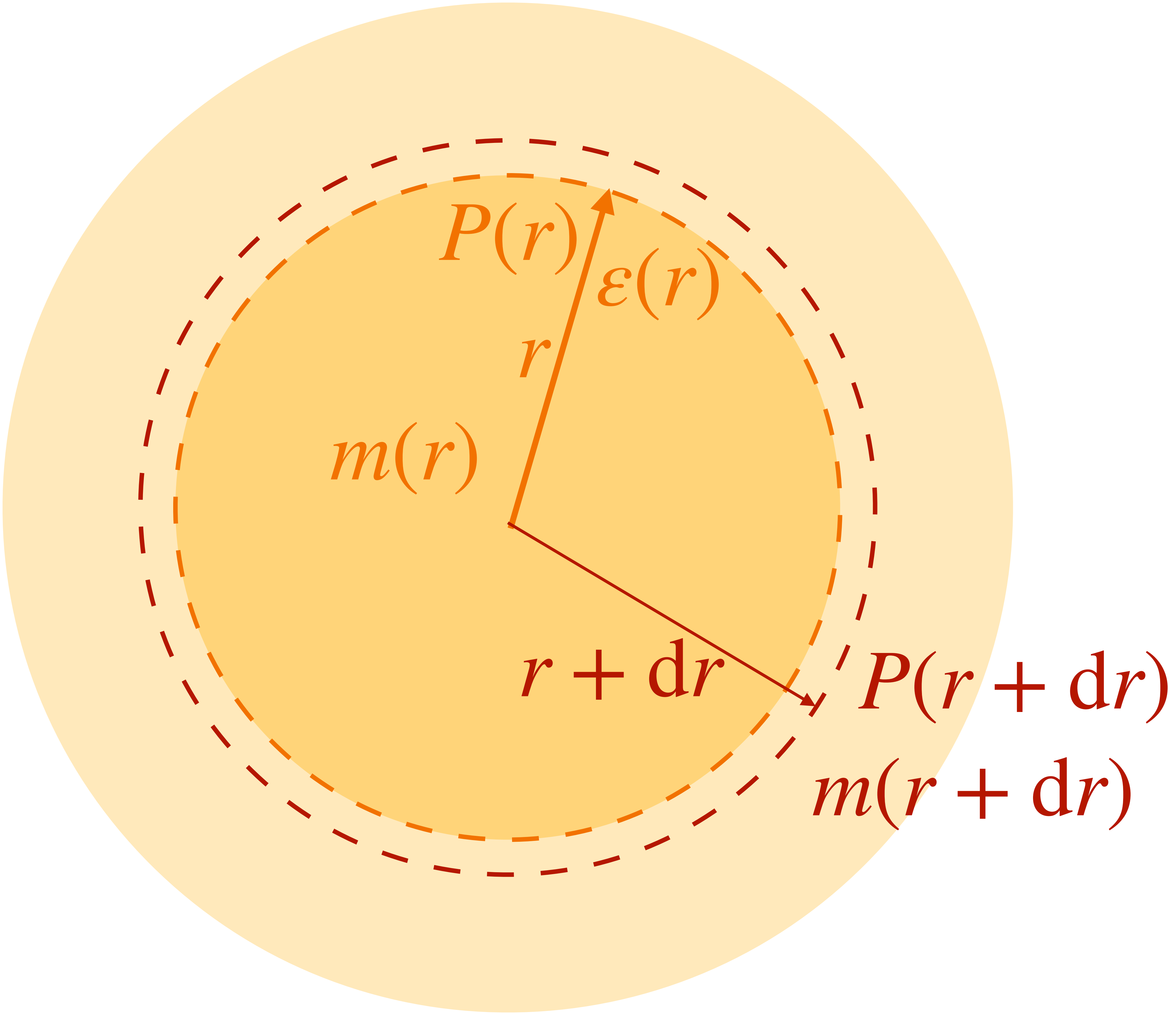
Mass enclosed within shell: $m(r)$

Continuation equation:

$$m(r + \mathrm{d}r) = m(r) + \underbrace{4\pi r^2}_{\text{area}} \overbrace{\varepsilon(r)}^{\text{density}} \underbrace{\mathrm{d}r}_{\text{thickness}},$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \varepsilon \,,$$

Tolman-Oppenheimer-Volkov equations



Pressure at radius r: $P(r)$

Mass density at r: $\varepsilon(r)$

Mass enclosed within shell: $m(r)$

Balance of force:

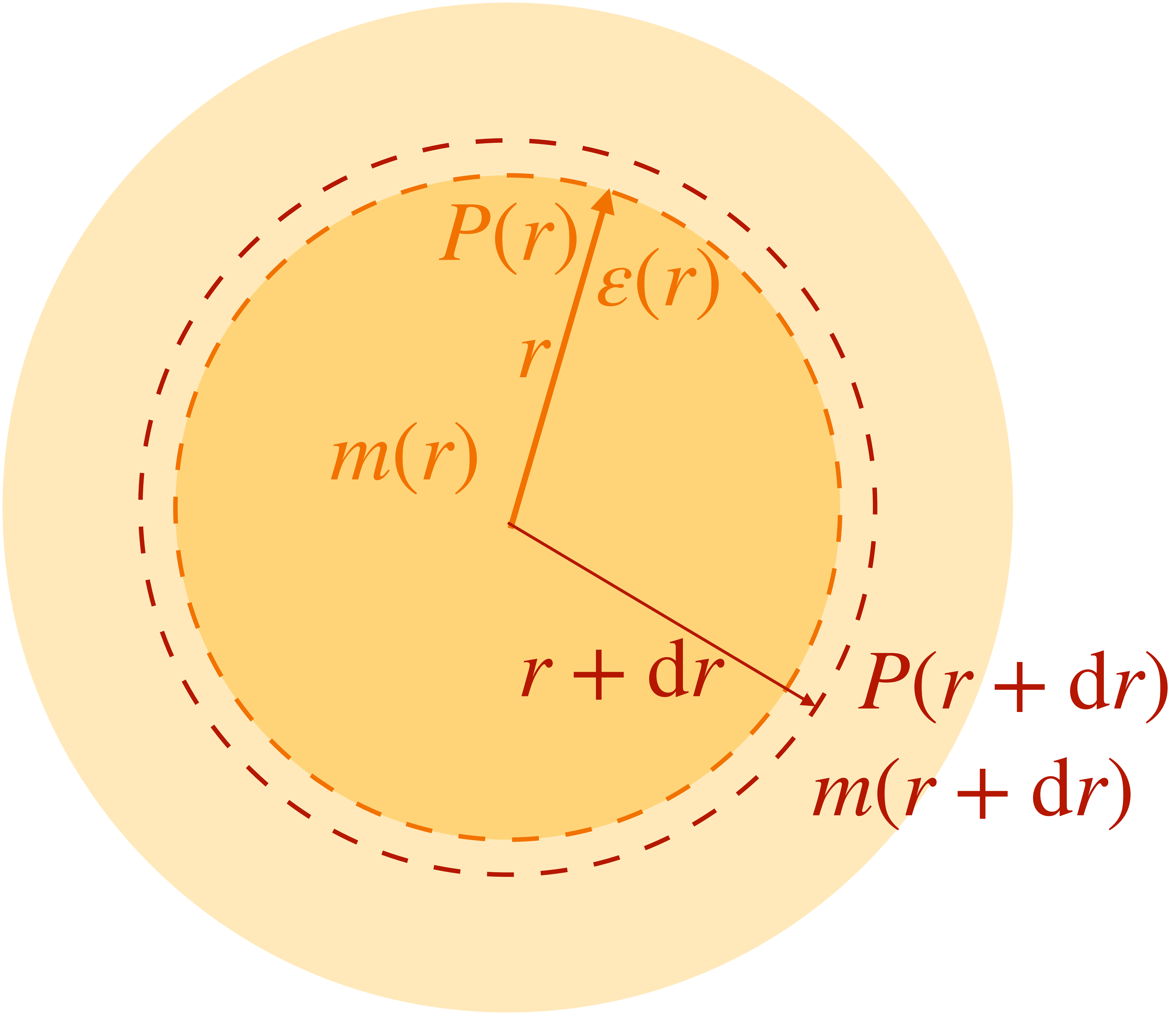
$$4\pi r^2 P(r) = 4\pi(r + \mathrm{d}r)^2 P(r + \mathrm{d}r) + F_G(r)\,,$$

$$F_G = \frac{G\, \underbrace{m(r)}_{m_1}\, \underbrace{4\pi r^2 \varepsilon \mathrm{d}r}_{m_2}}{r^2}$$

distance-squared

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \varepsilon\,,$$

Tolman-Oppenheimer-Volkov equations



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Mass enclosed within shell: $m(r)$

Balance of force:

$$4\pi r^2 P(r) = 4\pi(r + \mathrm{d}r)^2 P(r + \mathrm{d}r) + F_G(r)\,,$$

m1   m2

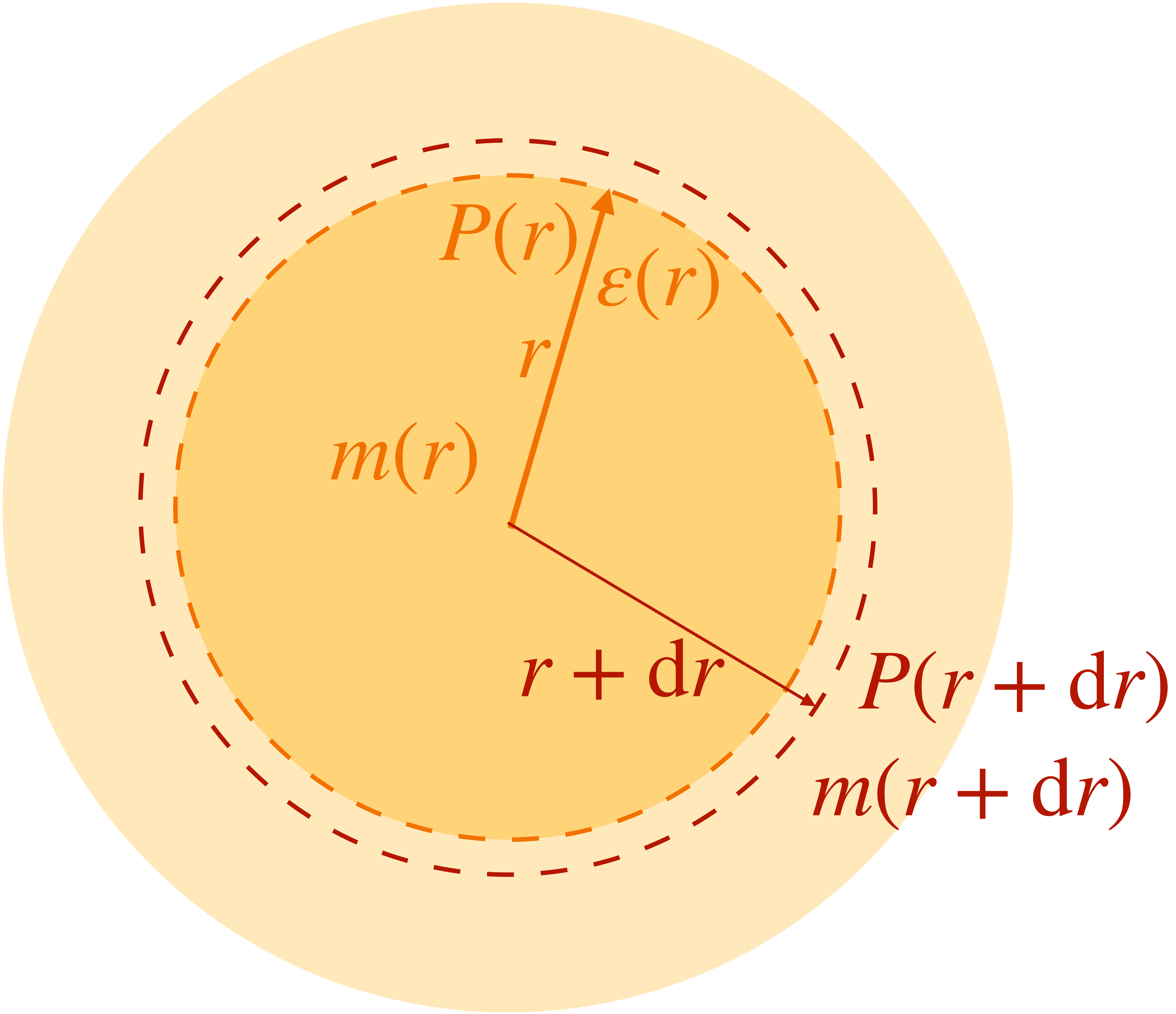
$$F_G = \frac{G\, m(r)\, 4\pi r^2 \varepsilon \mathrm{d}r}{r^2}$$

distance-squared

+GR corrections

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \varepsilon\,, \qquad \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2\,m\,r}\,,$$

Tolman-Oppenheimer-Volkov equations



Pressure at radius r: $P(r)$

Mass density at r: $\varepsilon(r)$

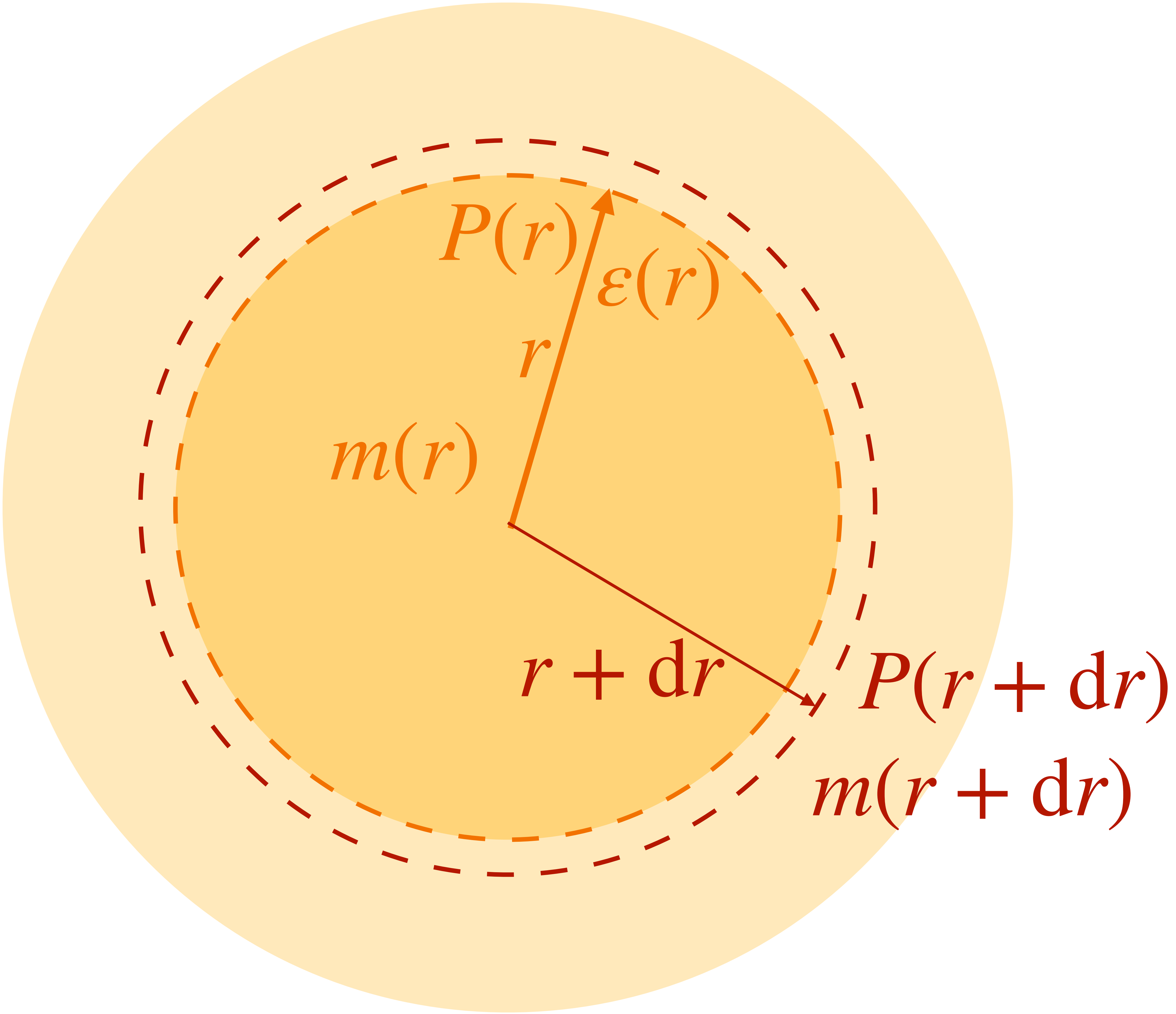
Mass enclosed within shell: $m(r)$

Equation of state:

$$\varepsilon = \varepsilon_{\mathrm{EoS}}(P)\,,$$
$$\varepsilon(r) = \varepsilon_{\mathrm{EoS}}(P(r))\,,$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \varepsilon\,, \qquad \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2\,m\,r}\,, \qquad \varepsilon = \varepsilon(P)\,,$$

Tolman-Oppenheimer-Volkov equations



Pressure at radius r: $P(r)$

Mass density at r: $\varepsilon(r)$

Mass enclosed within shell: $m(r)$

boundary conditions:

$$m(r=0) = 0, \qquad m(r=R) = M,$$
$$P(r=0) = P_{\rm cen}, \qquad P(r=R) = 0.$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \varepsilon\,, \qquad \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{(m + 4\pi r^3 P)(P+\varepsilon)}{r^2 - 2\,m\,r}\,, \qquad \varepsilon = \varepsilon(P)\,,$$

Tolman-Oppenheimer-Volkov equations

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Tolman, Phys.Rev. 55 (1939) 364-373

Oppenheimer, Volkov, Phys.Rev. 55 (1939) 374-381

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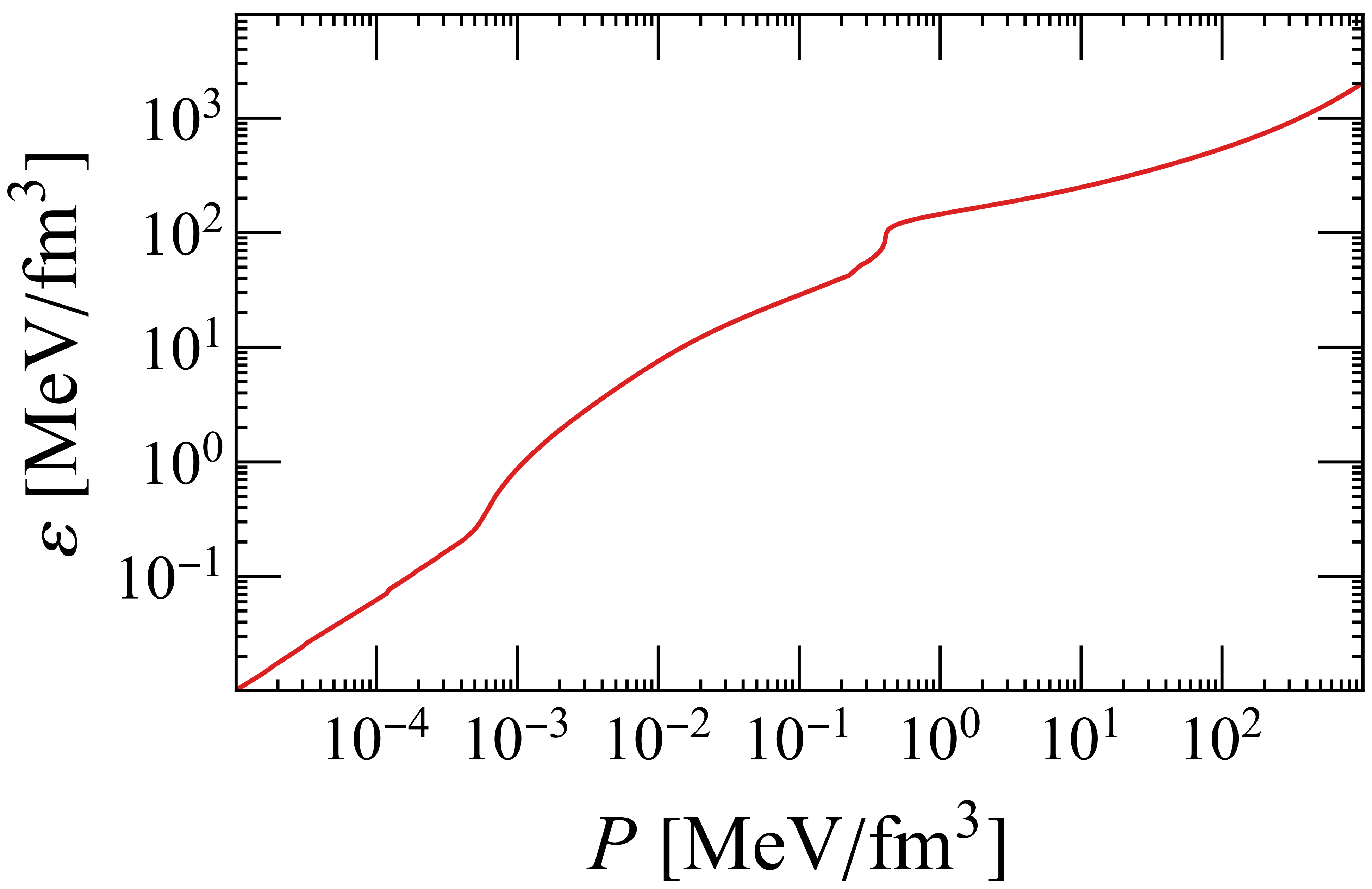

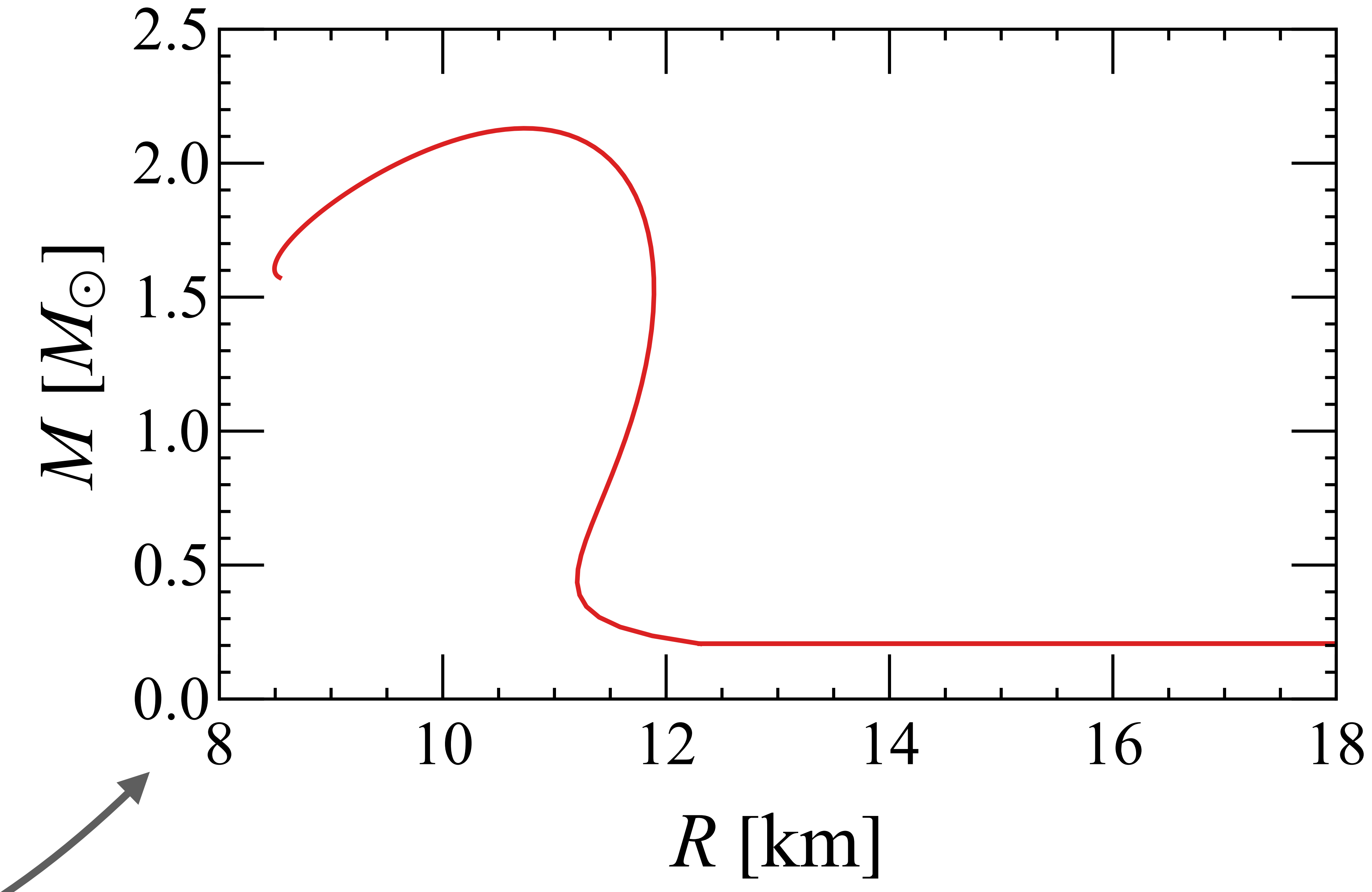

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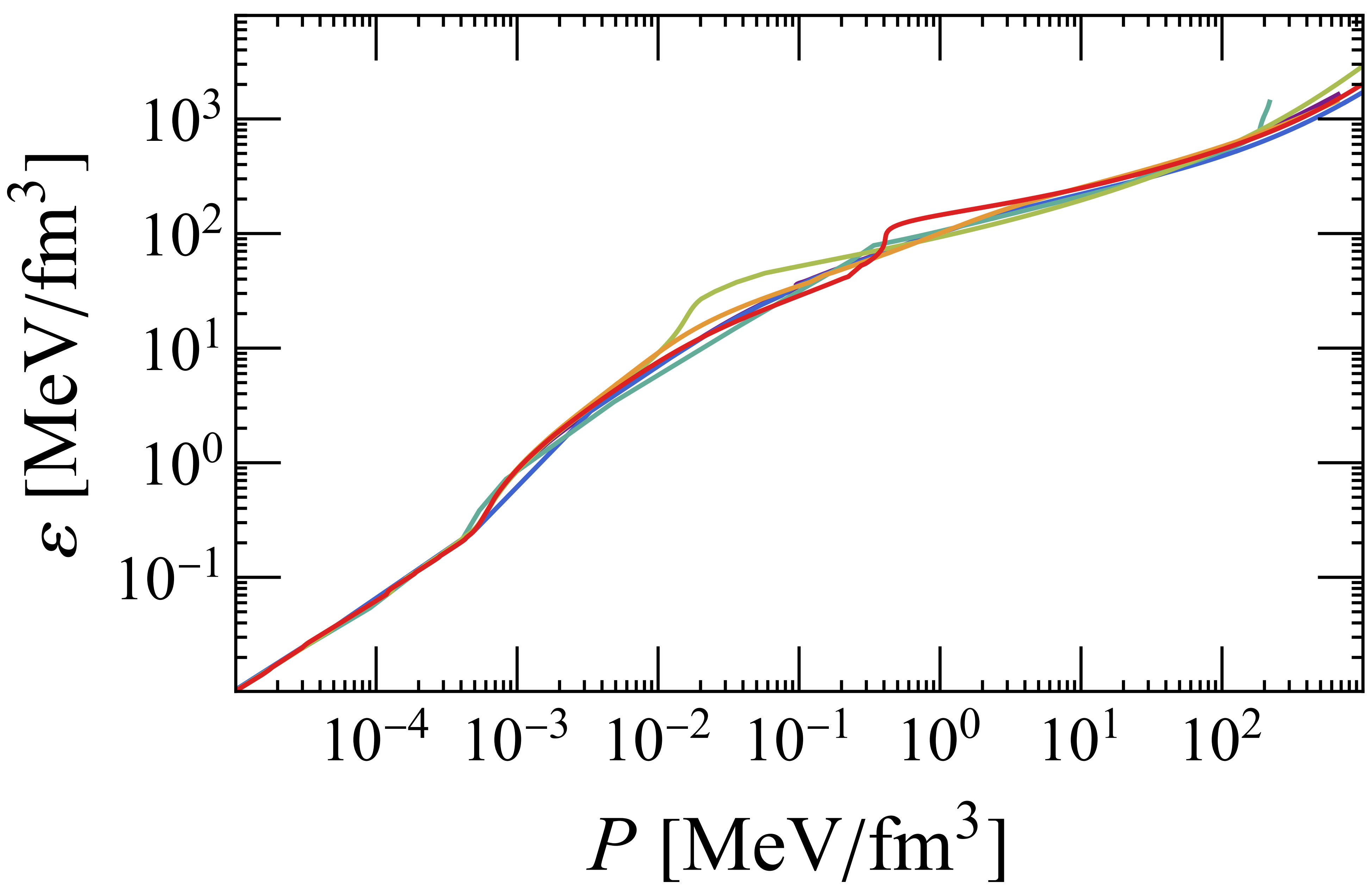

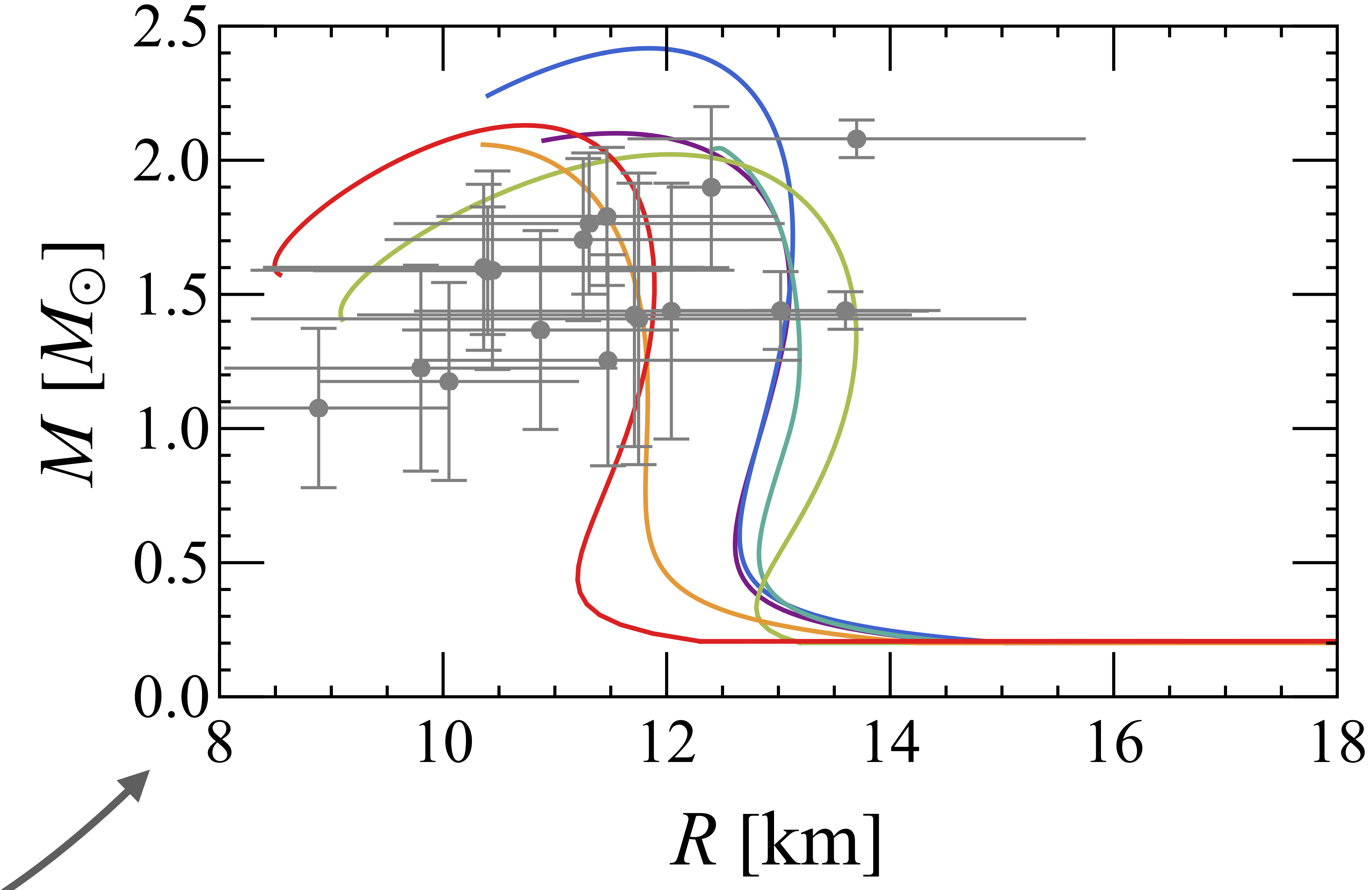

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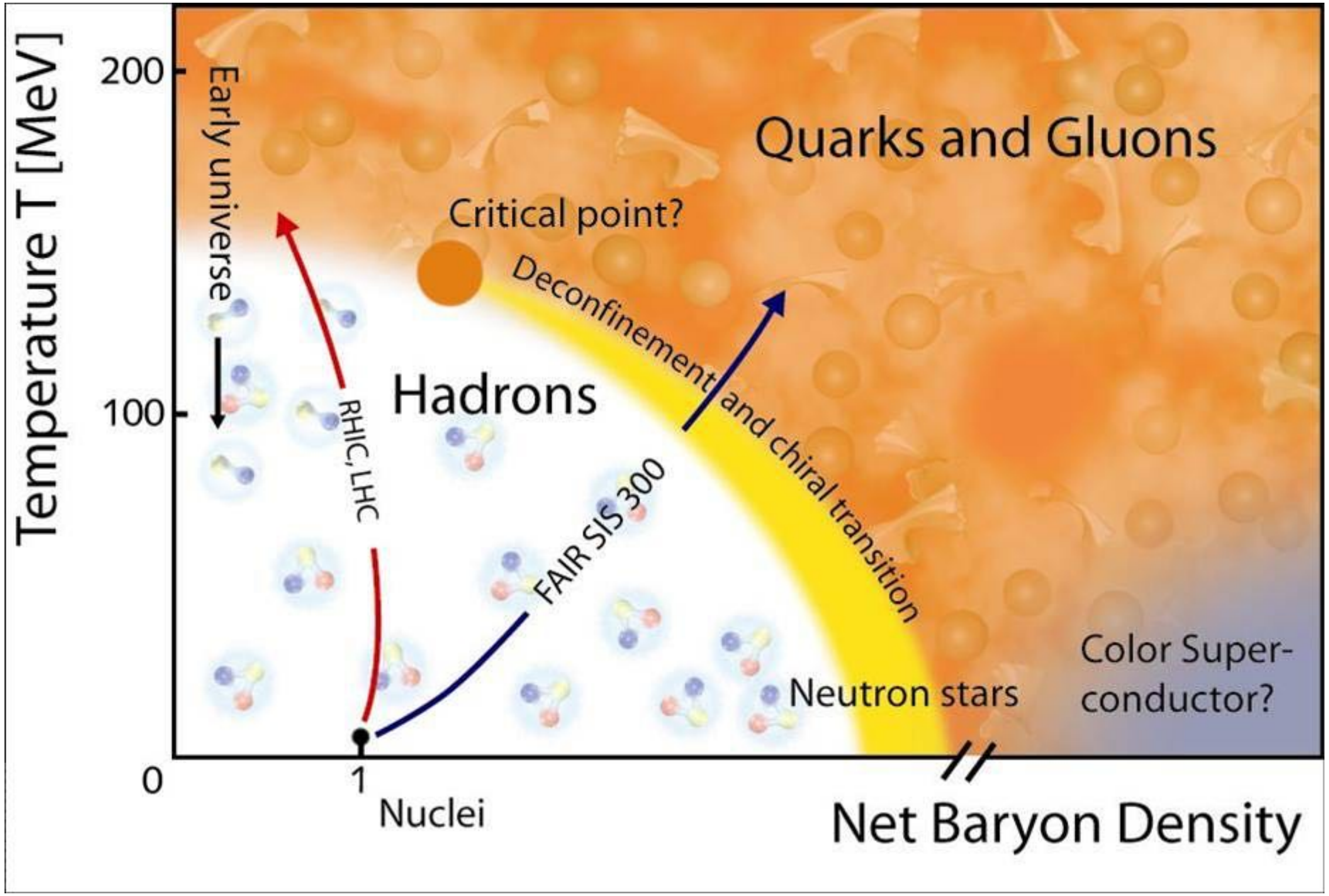
why do we still bother?

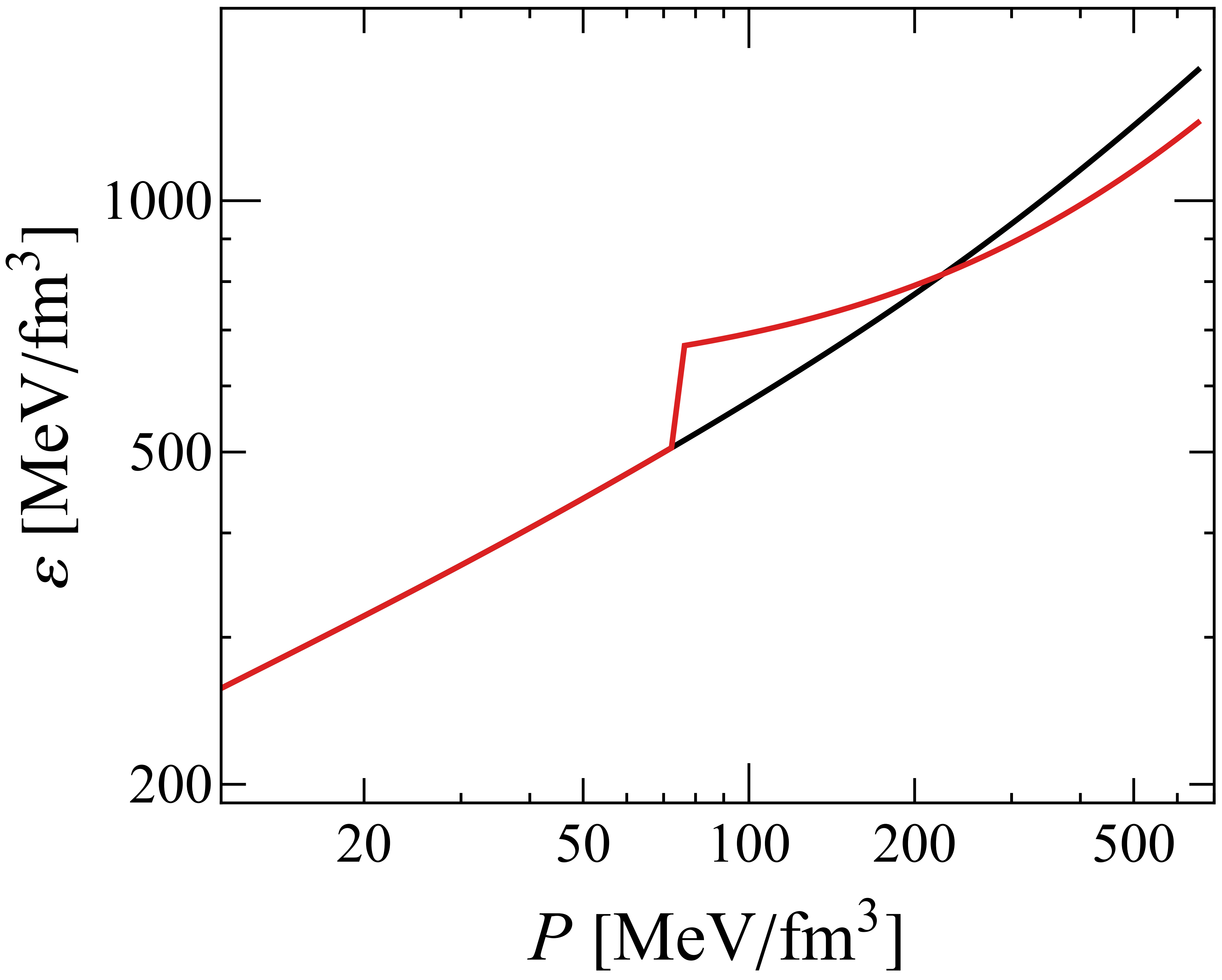
from heavy-ion collisions:

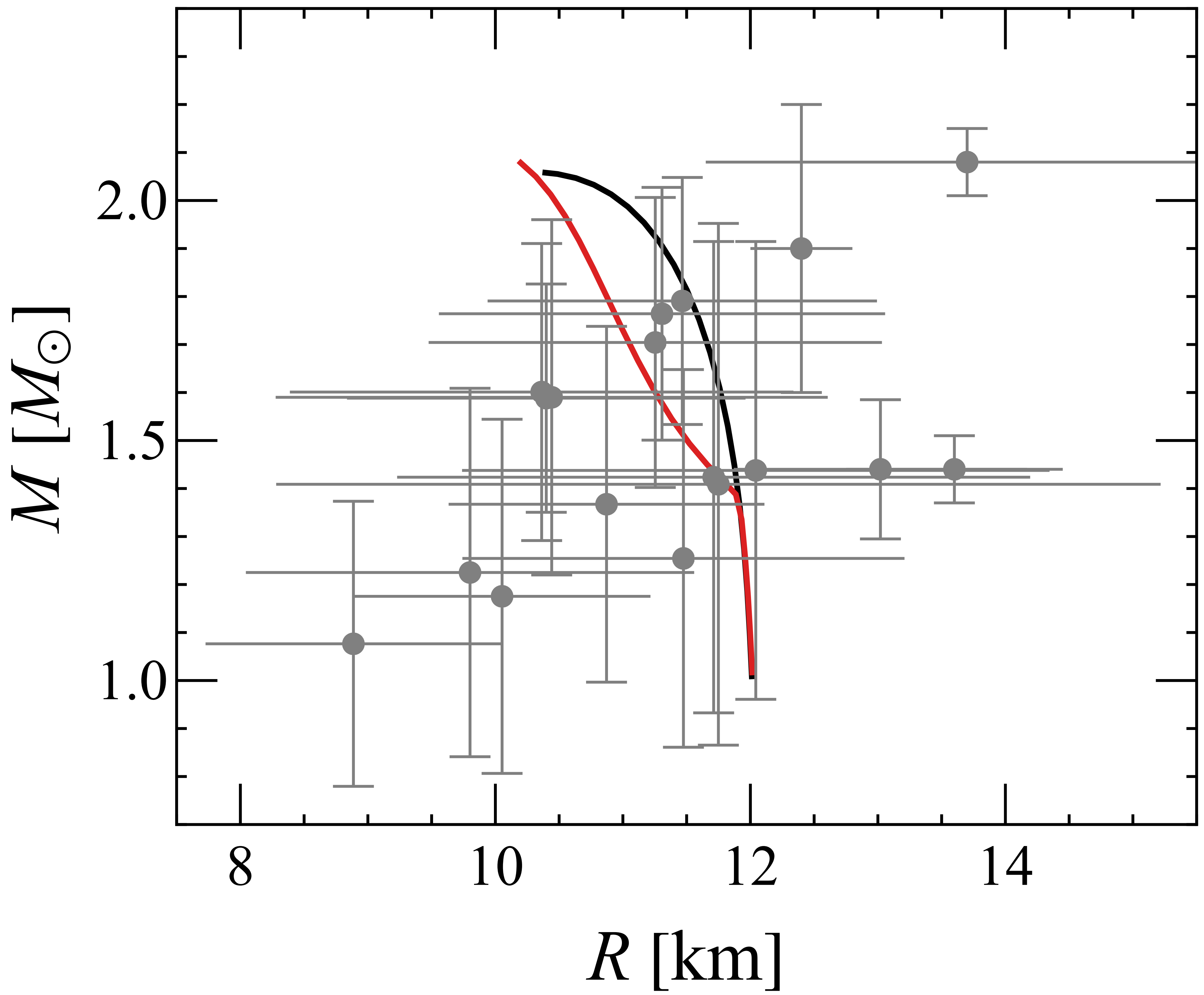
crossover @ zero μ_B



no experimental evidence for a first-order phase transition, yet

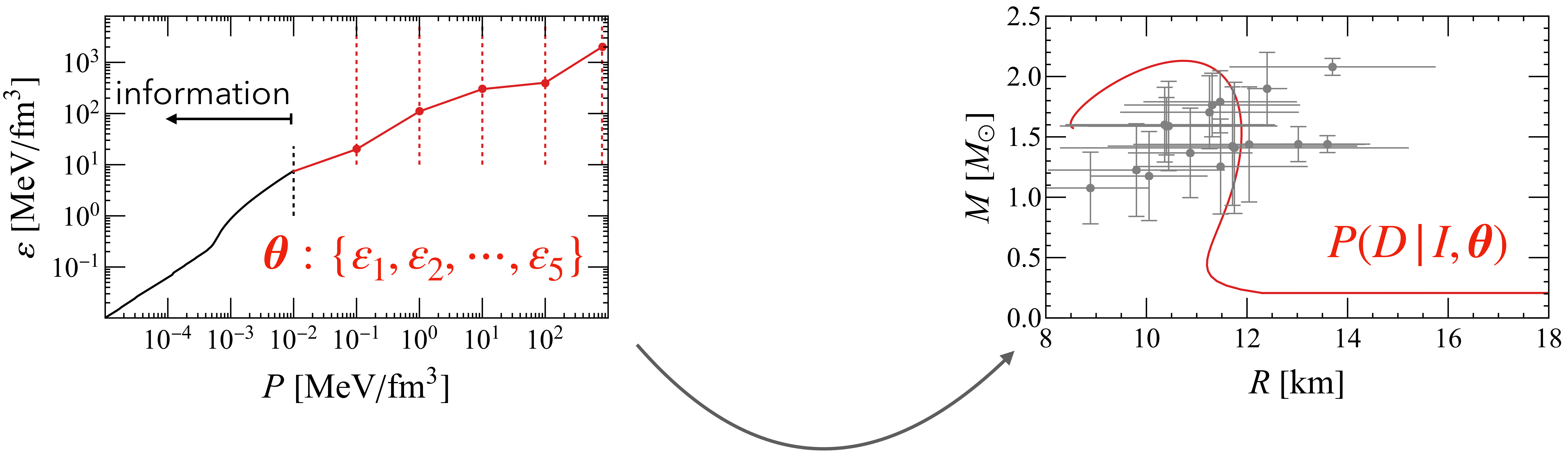
signature of first order phase transition?





reconstruction given finite number and accuracy?
Bayesian Analysis!

"conventional" Bayesian Analysis



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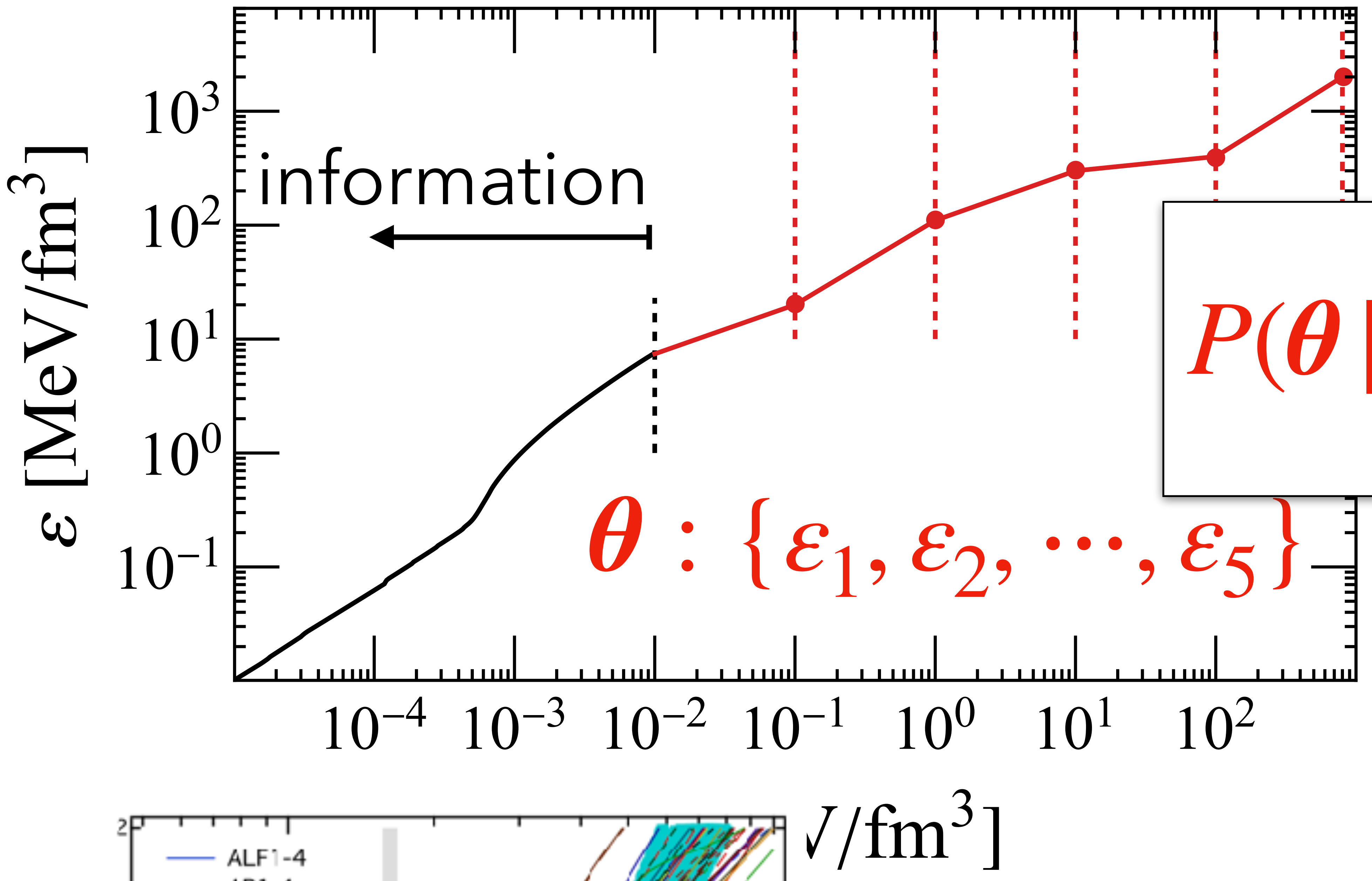
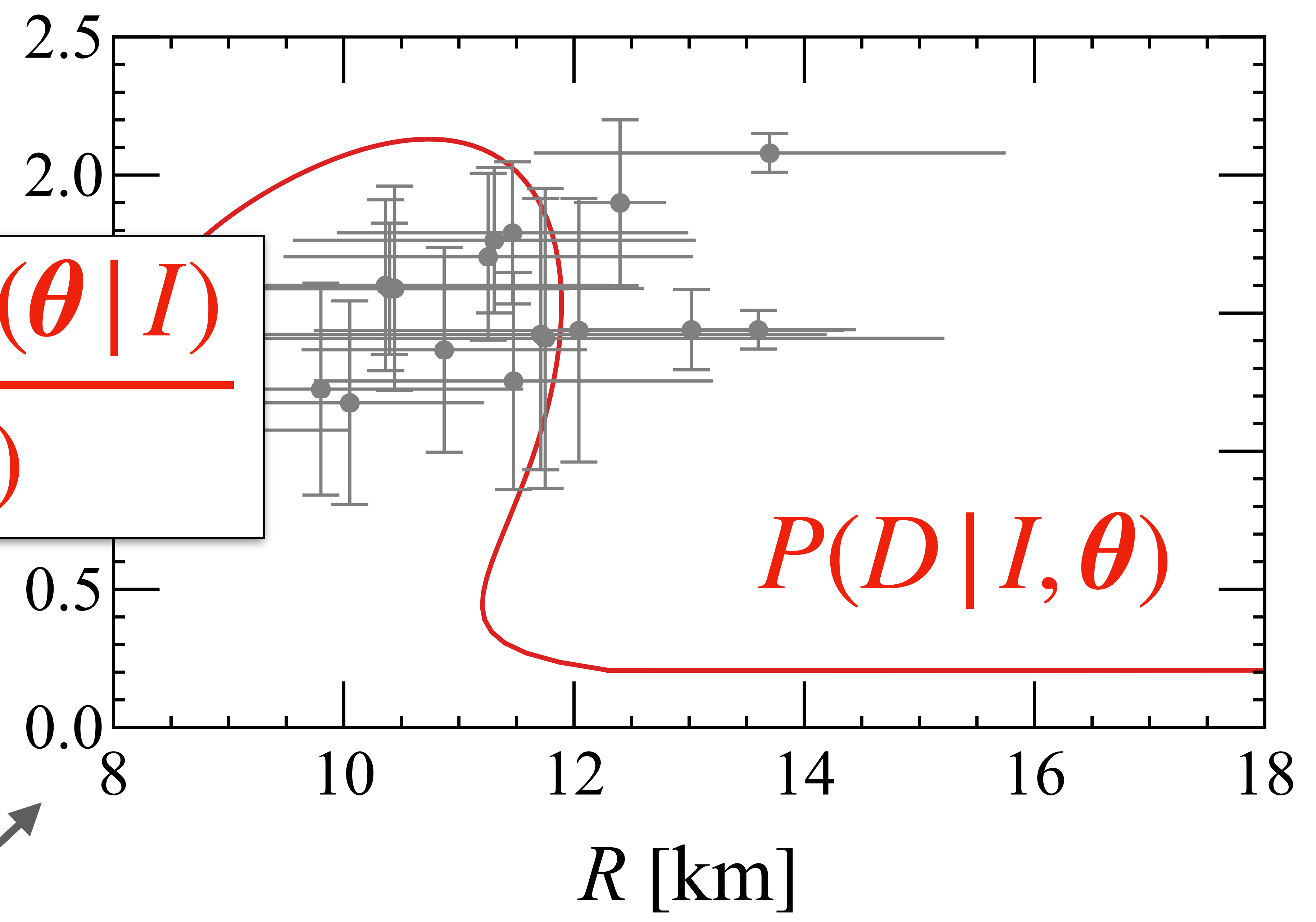
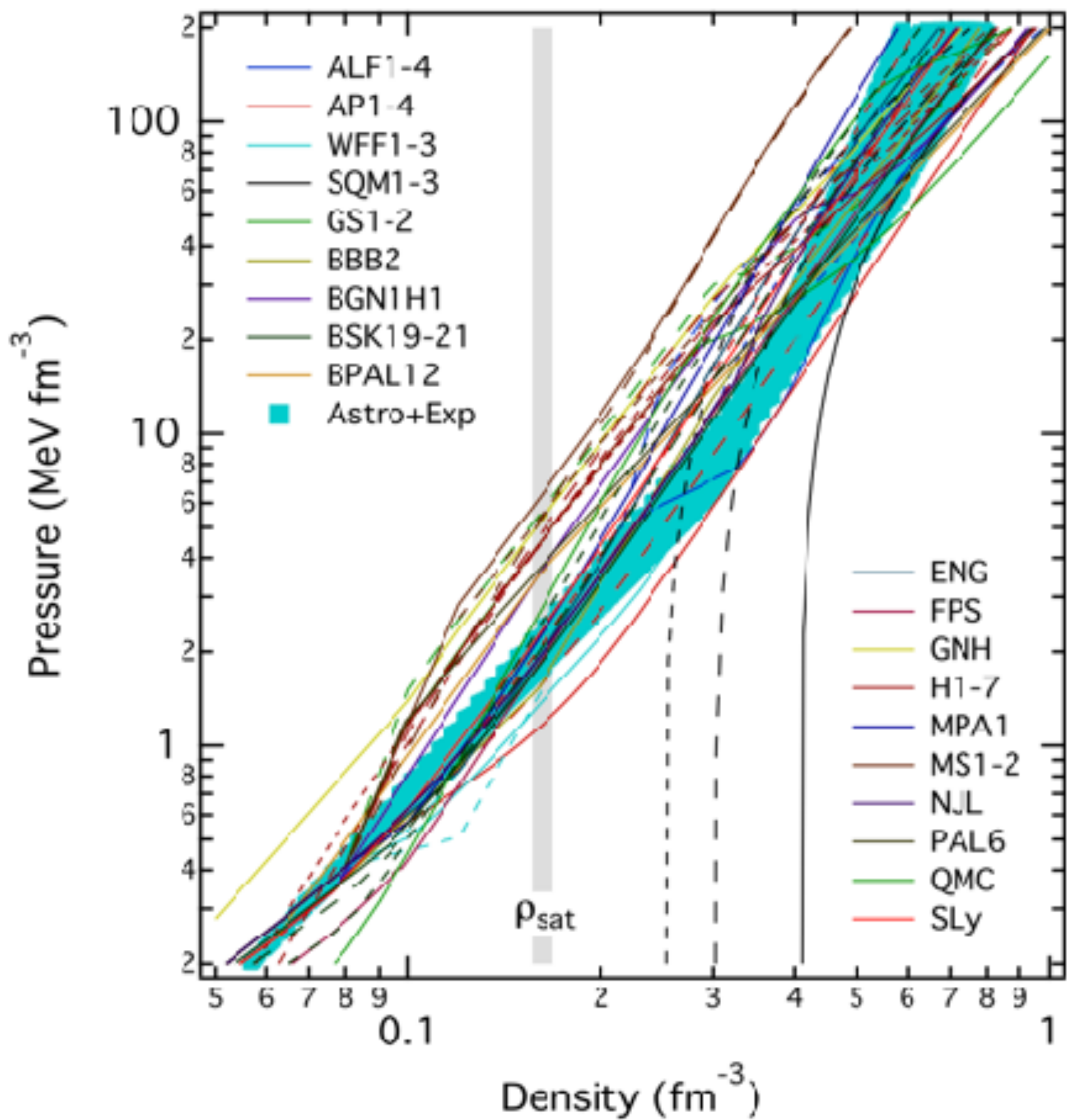


inverse TOV eq?

$$P(\boldsymbol{\theta} \mid D, I) = \frac{P(D \mid \boldsymbol{\theta}, I) P(\boldsymbol{\theta} \mid I)}{P(D \mid I)}$$





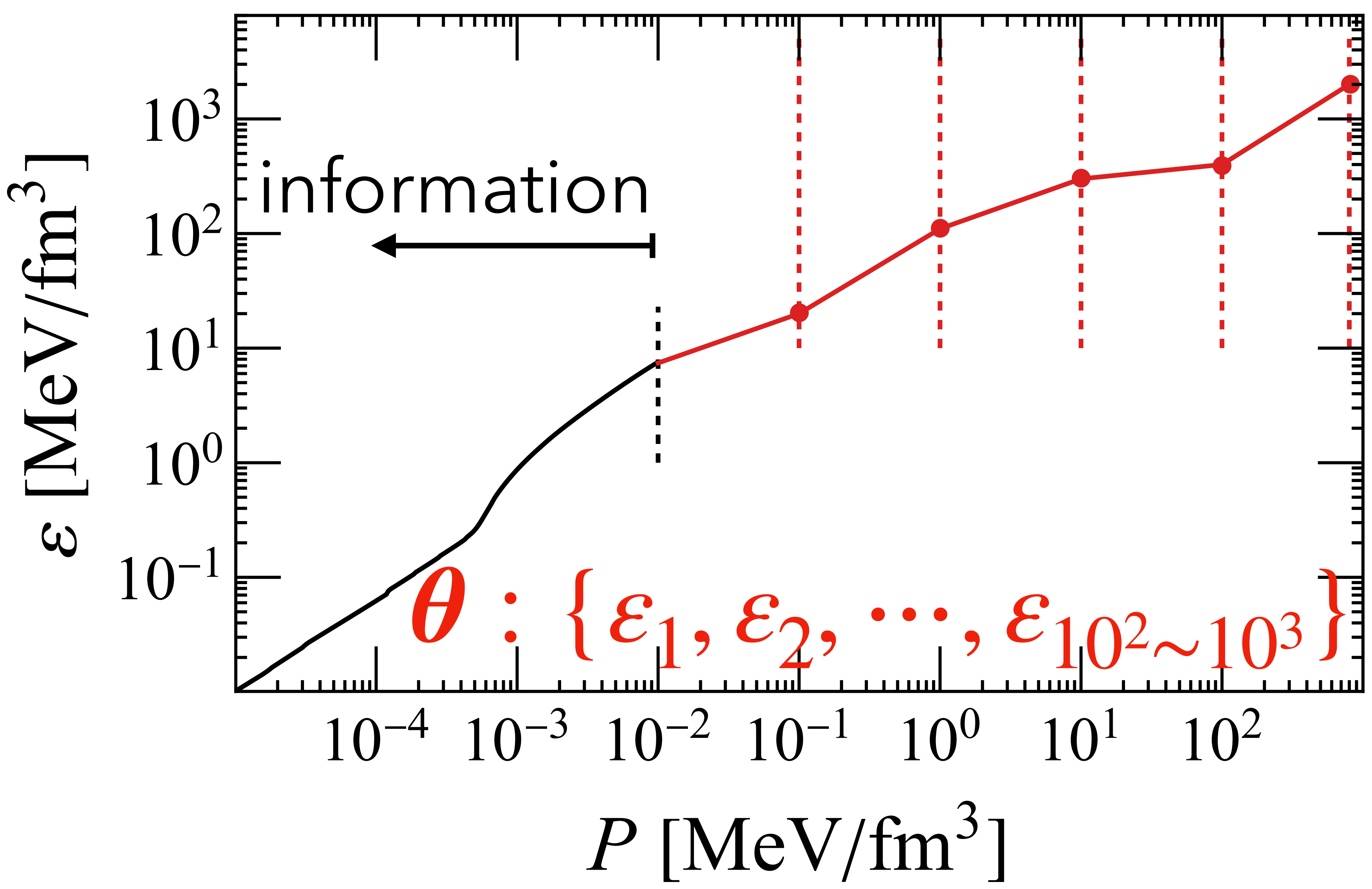
Refs:

Astrophys.J.Lett. 765 (2013) L5

Nature Astron. 4 (2020) 6, 625-632

Ann.Rev.Astron.Astrophys. 54 (2016) 401-440

“non-conventional” Bayesian Analysis



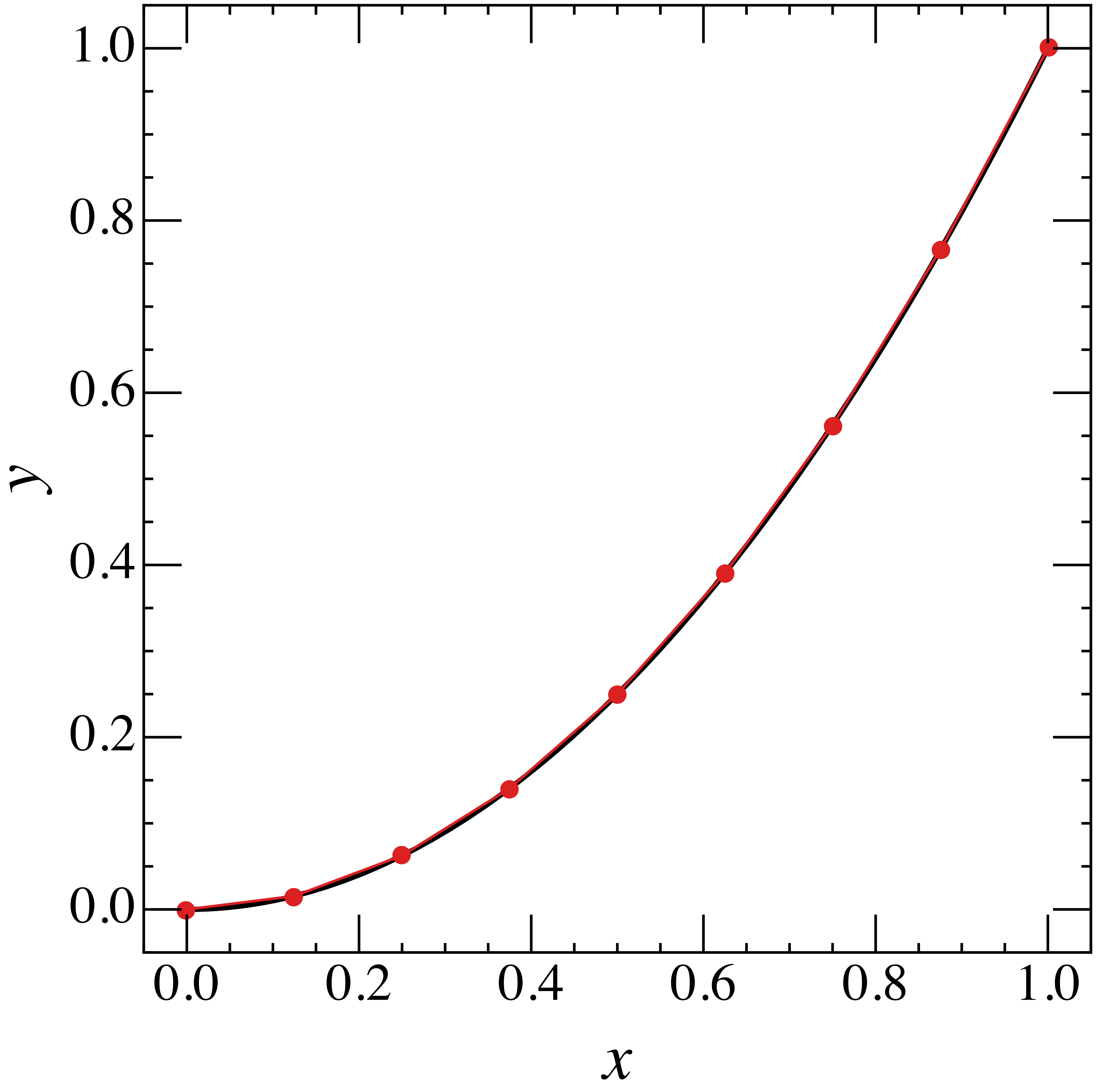
more differential information?

Deep Neural Networks!

What are Deep Neural Networks?

--- a general parameterization scheme to approximate continuous functions.

example: approximate $y = x^2$ for $x \in [0,1]$



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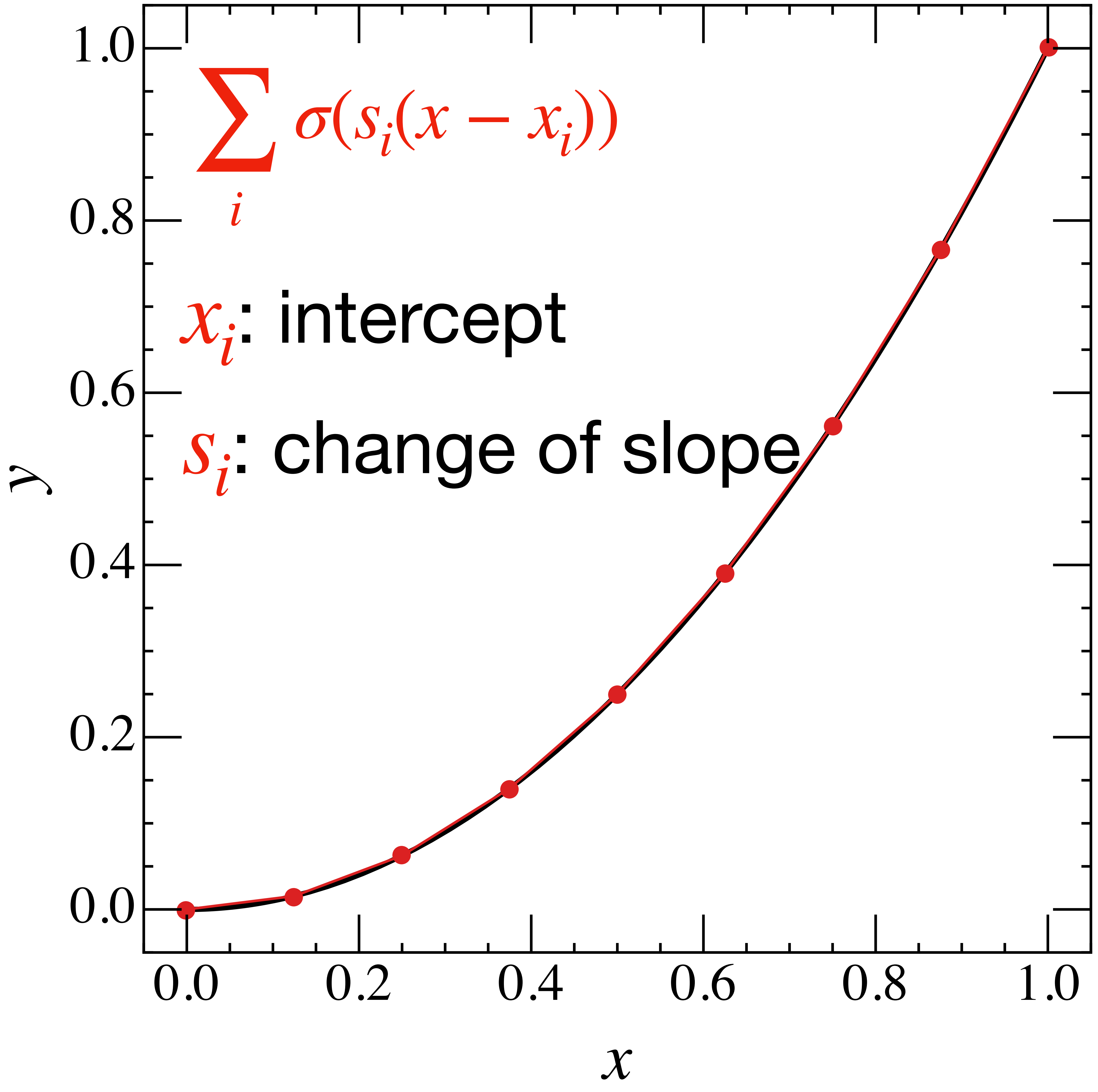
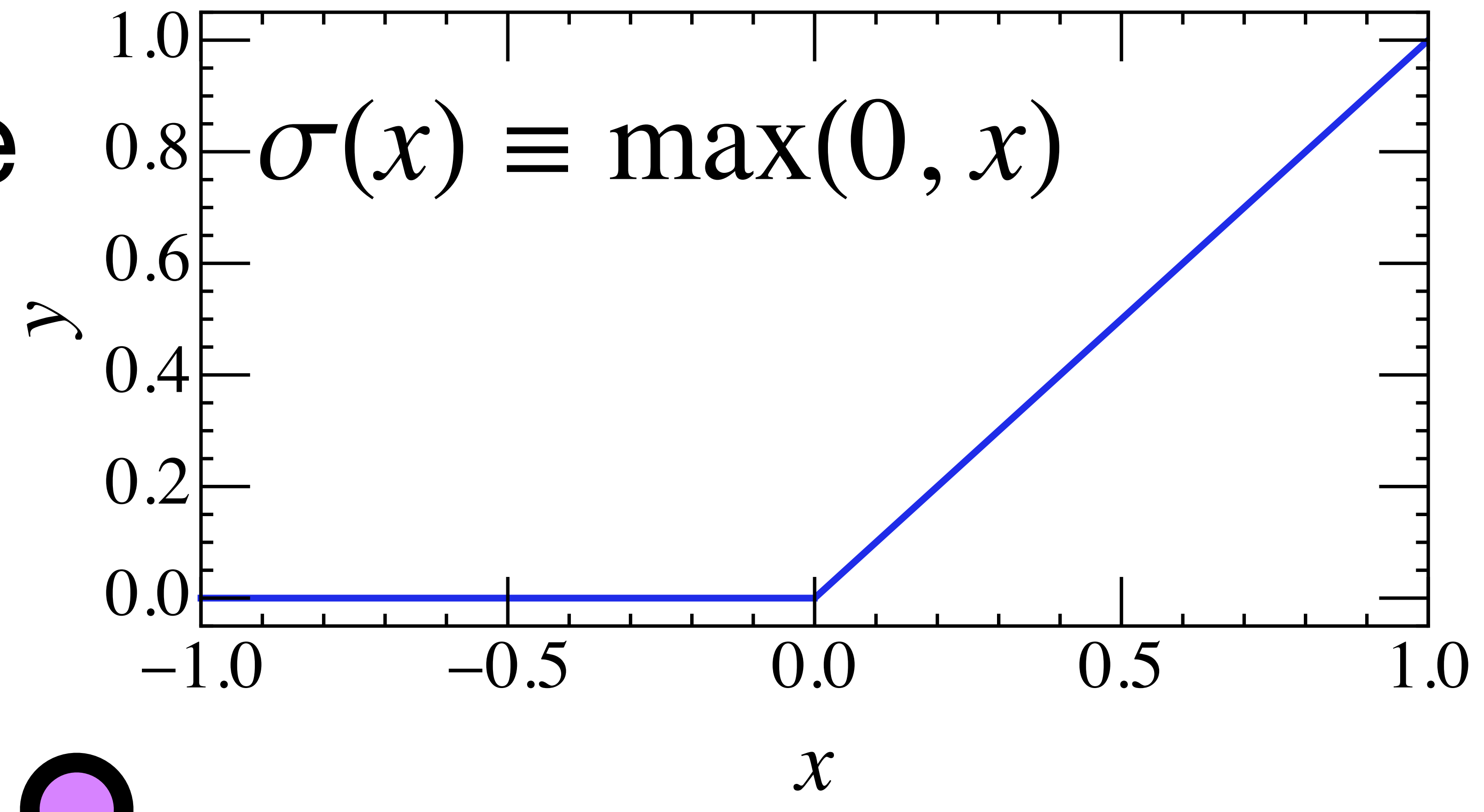
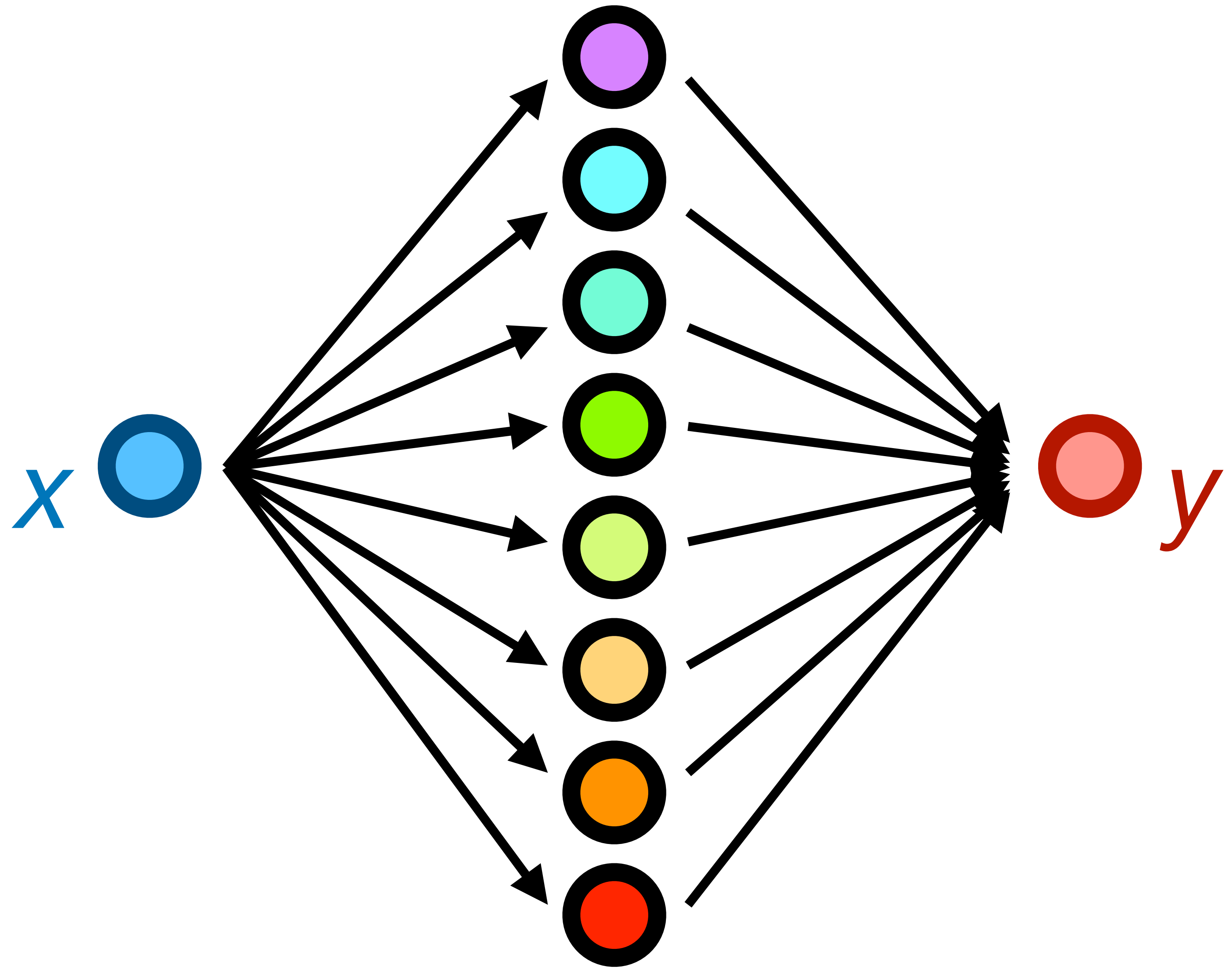


define





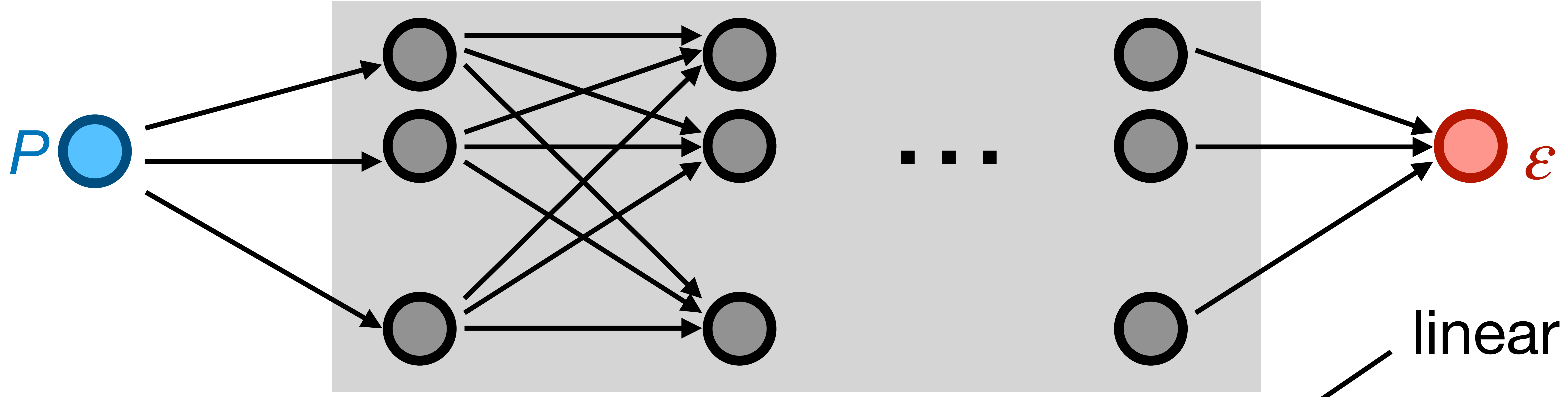
each ● represents one of $\sigma(s_i(x - x_i))$

What are Deep Neural Networks?

--- a general parameterization scheme to approximate continuous functions.

$$\varepsilon(P) \approx \varepsilon_{\mathrm{DNN}}(P \,|\, \boldsymbol{\theta})$$

(iterative function substitution)

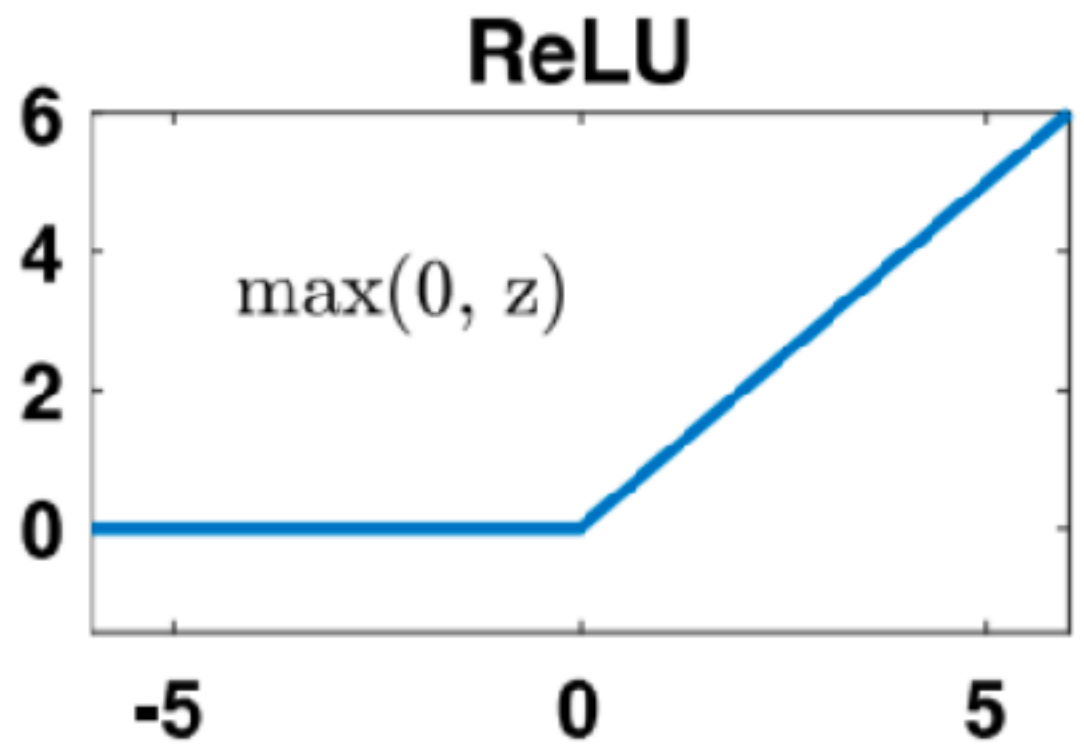


linear

nonlinear

Each ⬤ is an intermediate function ($a_i^{(l)}$):

- At the first layer: $z_i^{(1)} = b_i^{(1)} + W_{i,1}^{(1)}\, r, \qquad a_i^{(1)} = \sigma(z_i^{(1)})$
- At later layers: $z_i^{(l)} = b_i^{(l)} + \sum_j W_{i,j}^{(l)}\, a_j^{(l-1)}, \qquad a_i^{(l)} = \sigma(z_i^{(l)})$



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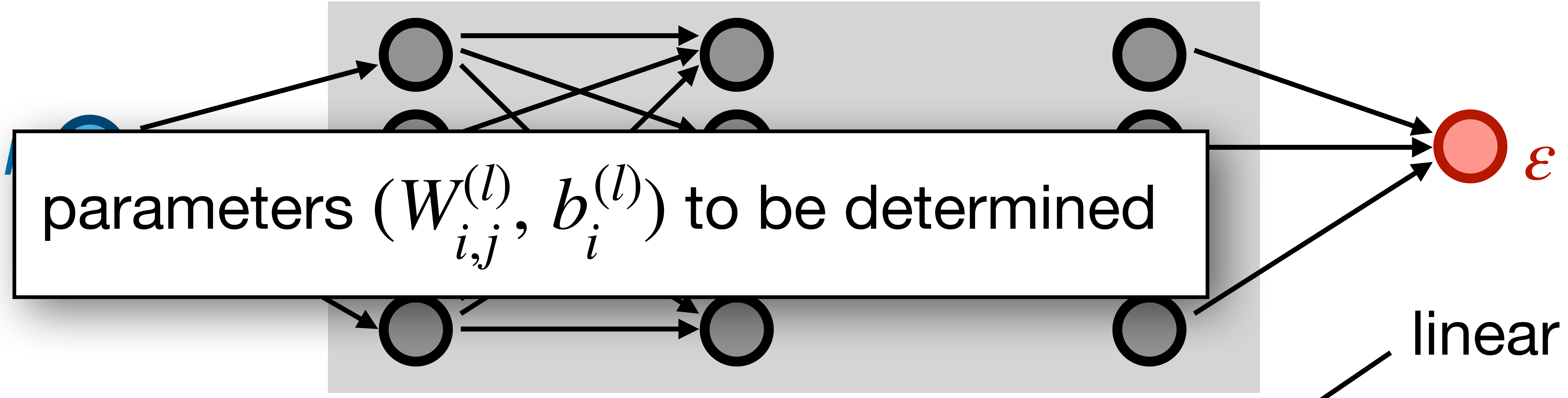

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linear

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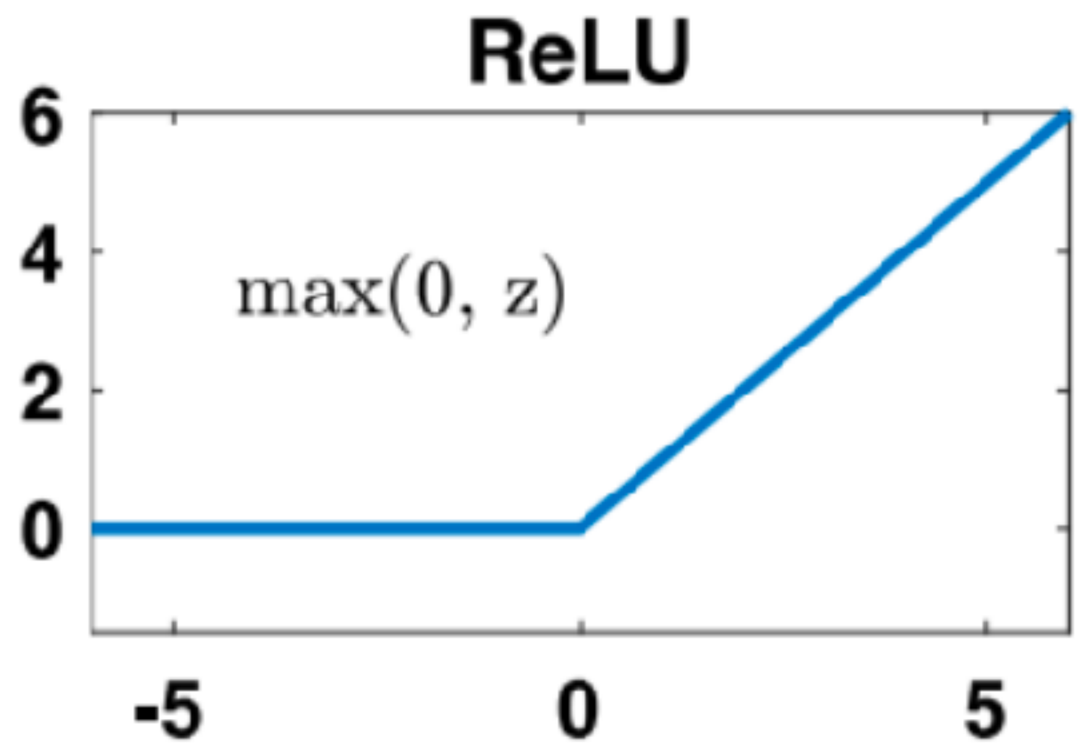
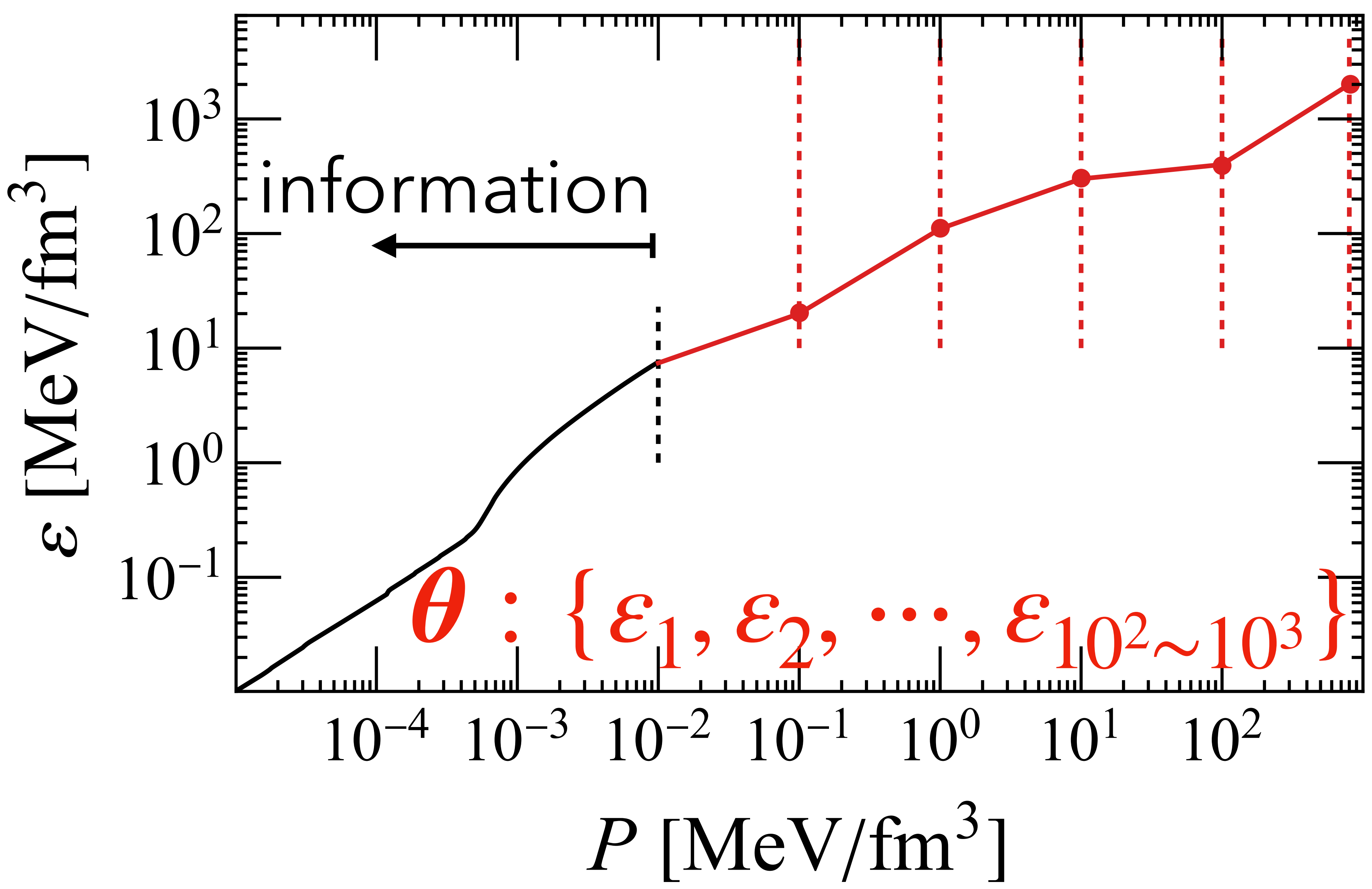


"non-conventional" Bayesian Analysis



more differential information?

Deep Neural Networks!

BA with $\sim 10^2 - 10^3$ parameters?

$\nabla_{\boldsymbol{\theta}} P(\boldsymbol{\theta})$?

gradient: linear response of TOV

input:
EoS

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2\,m\,r}\,,$$
$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \varepsilon\,,$$
$$\varepsilon = \varepsilon(P)\,,$$

output:
M-*R* curve

gradient: linear response of TOV

input:
EoS

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2\,m\,r}\,,$$

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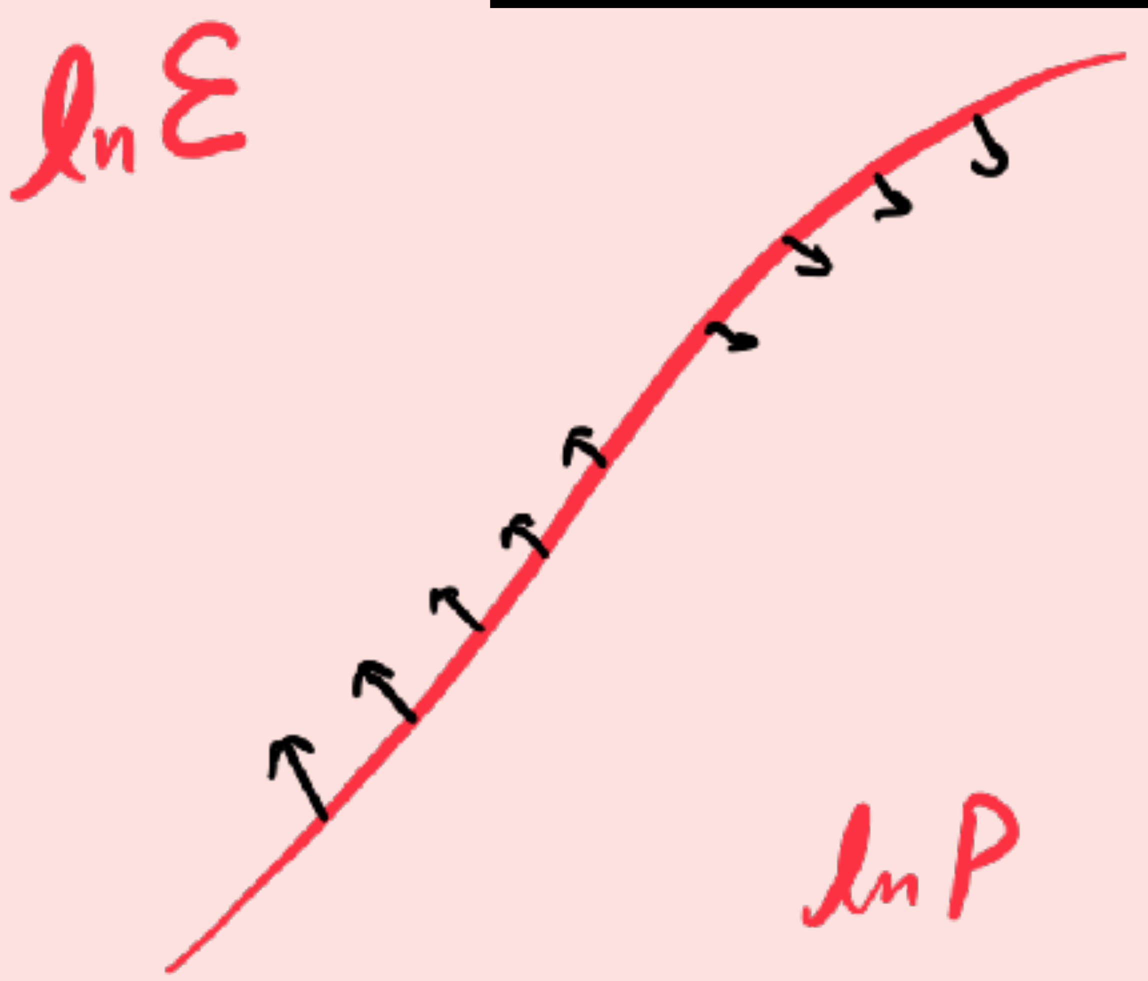
output:
M-R curve

TOV equation: functional mapping EoS to M-R;
Inverse TOV: functional derivatives!



change in EoS

$$\frac{\delta\,(M\text{-}R)}{\delta\,(\mathrm{EoS})}$$

$(M\text{-}R)_{\mathrm{desired}} - (M\text{-}R)_{\mathrm{curr.}}$

gradient: linear response of TOV

input:
EoS

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2\,m\,r},$$
$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \varepsilon\,,$$
$$\varepsilon = \varepsilon(P)\,,$$

output:
M-*R* curve

central pressure

radius

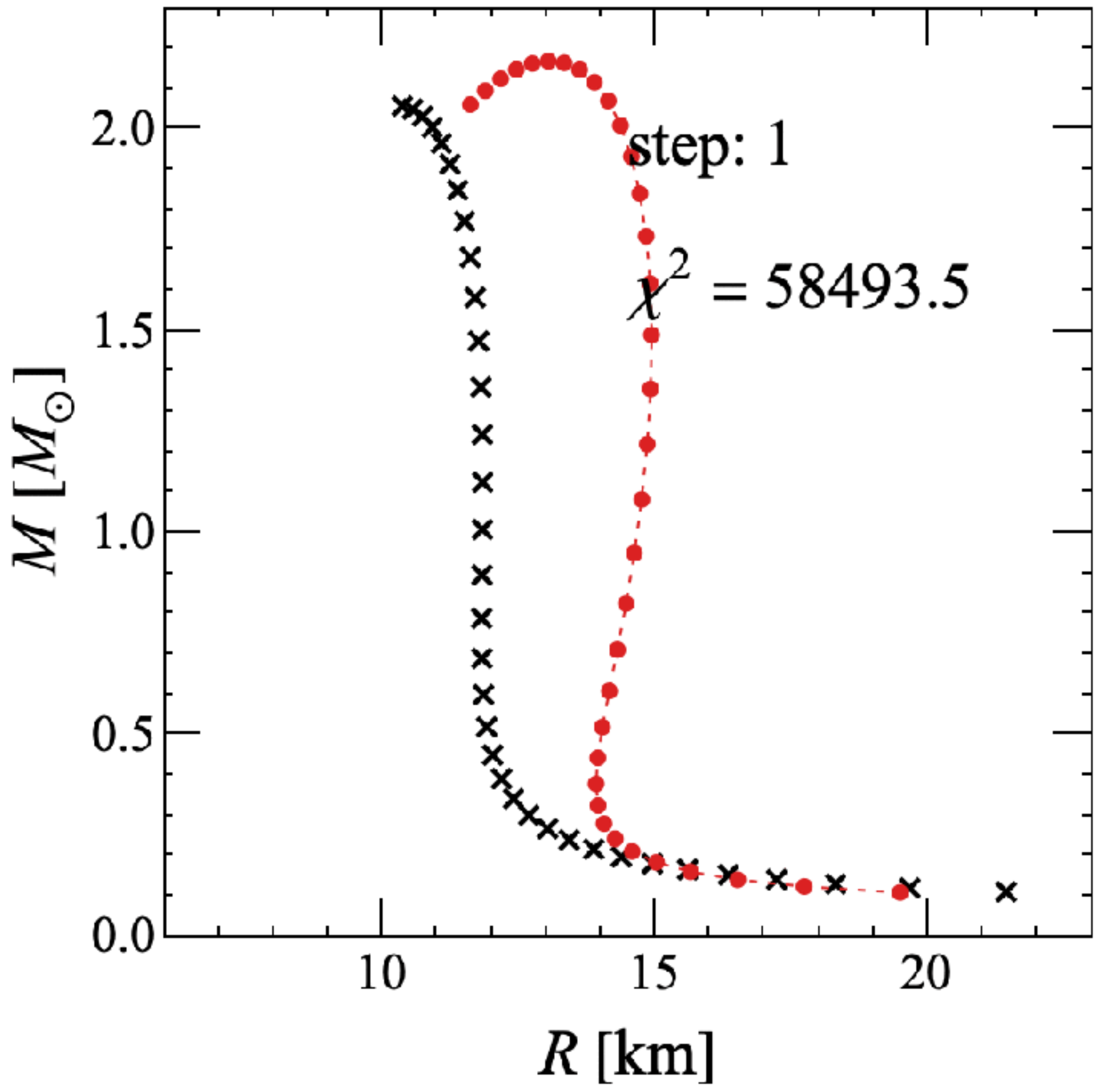
$$P_c = \int_0^R \frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2\,m\,r}\mathrm{d}r\,,$$

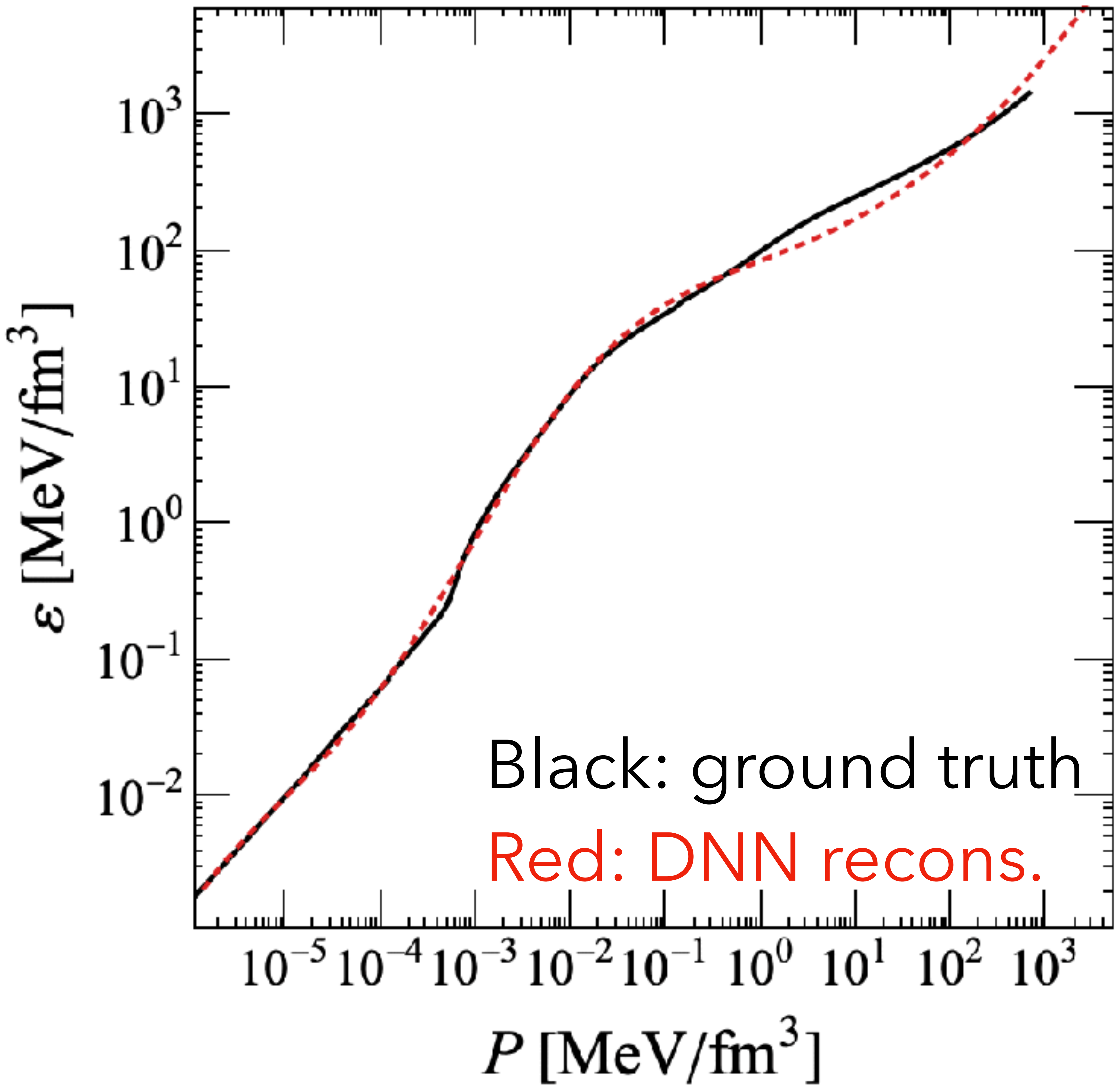
$$M = 4\pi \int_0^R r^2 \varepsilon\, \mathrm{d}r\,,$$

mass

$$\varepsilon(P) \to \varepsilon(P) + \delta\varepsilon\, \delta(P - P')$$
$$R \to R + \delta R, \quad M \to M + \delta M$$

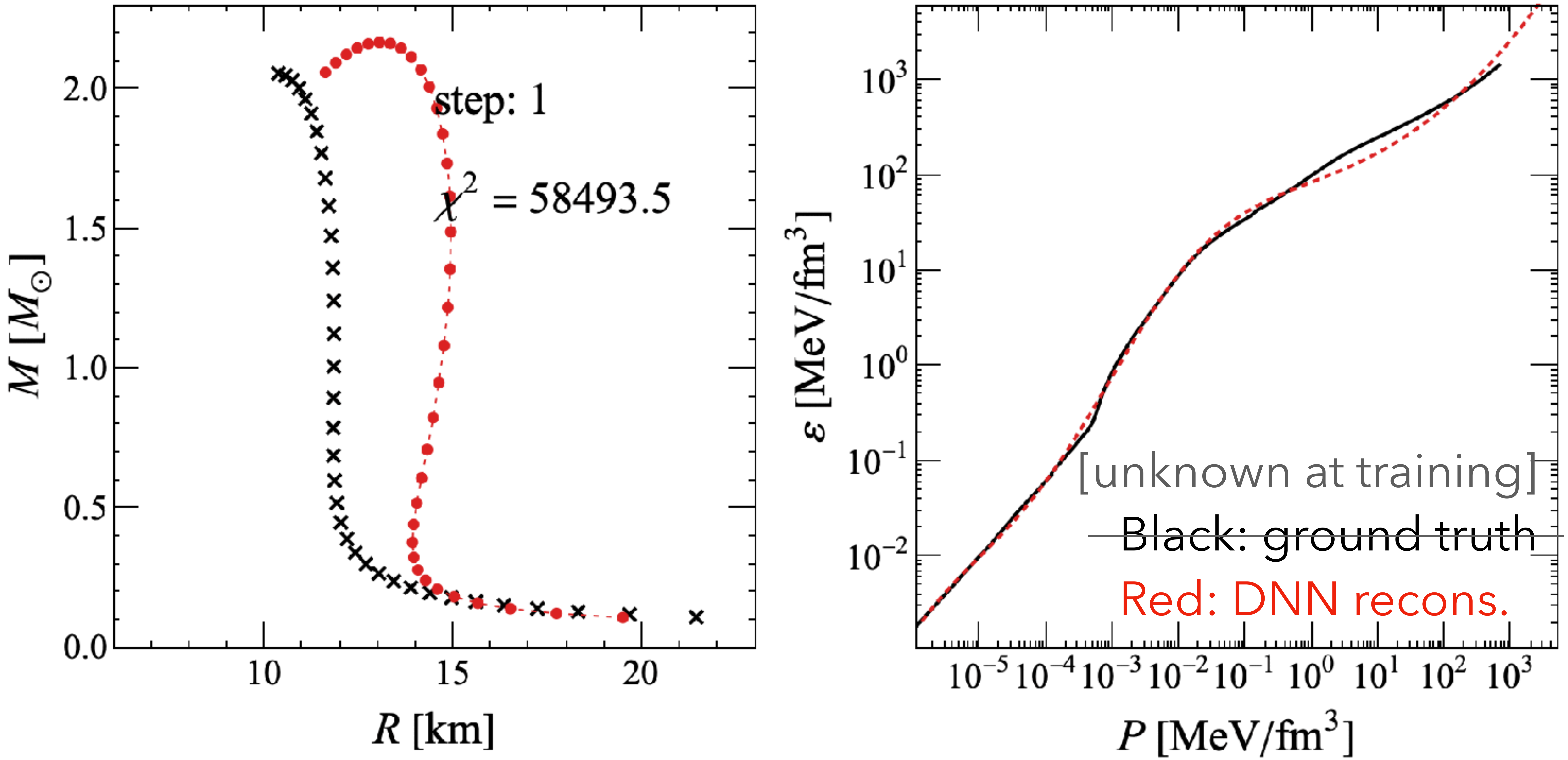
closure test





$$\chi^2 = \sum_i \left(\frac{m_i - m_i^{\rm obs}}{\Delta m_i}\right)^2 + \left(\frac{R_i - R_i^{\rm obs}}{\Delta R_i}\right)^2$$

closure test



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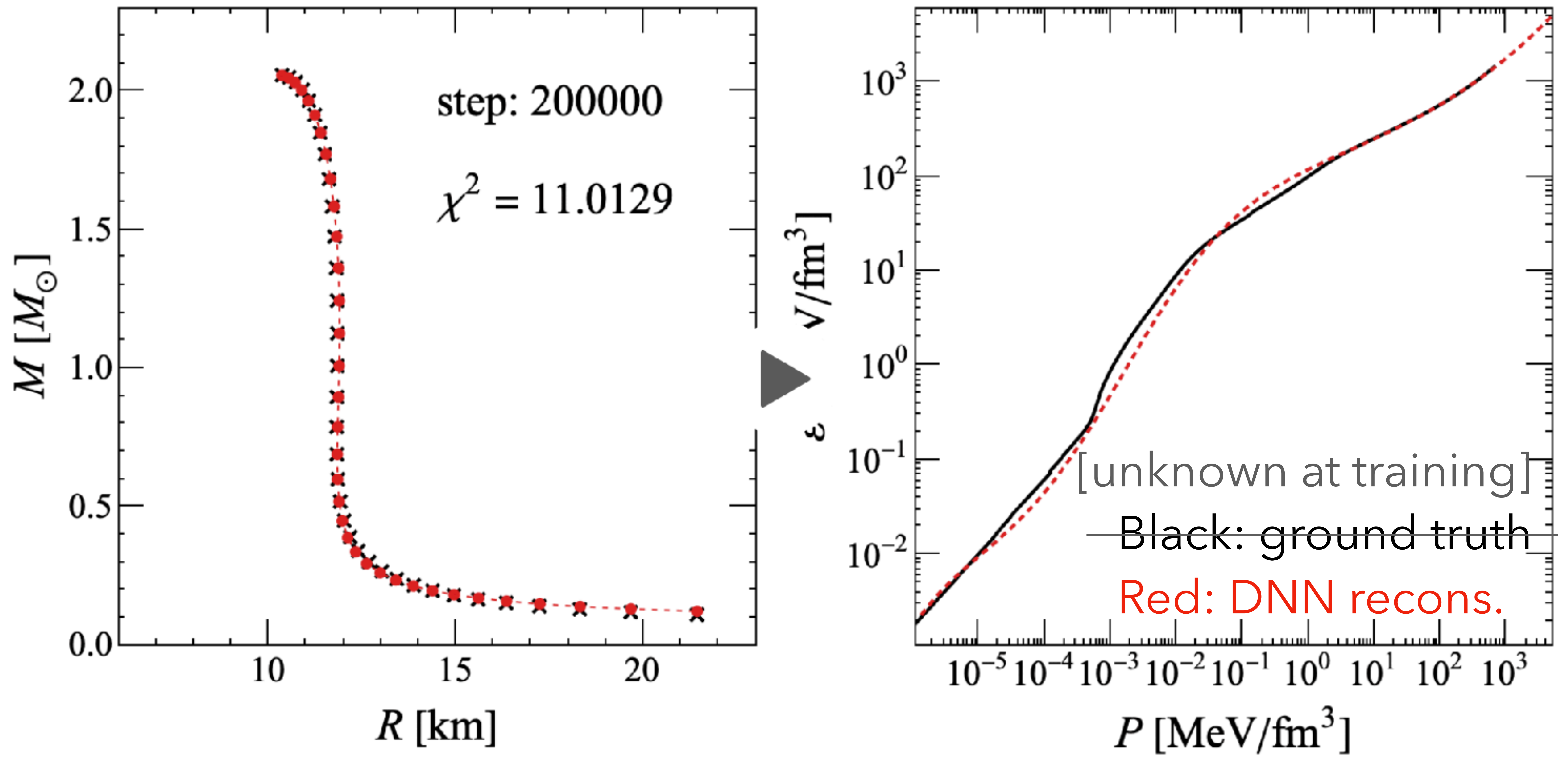

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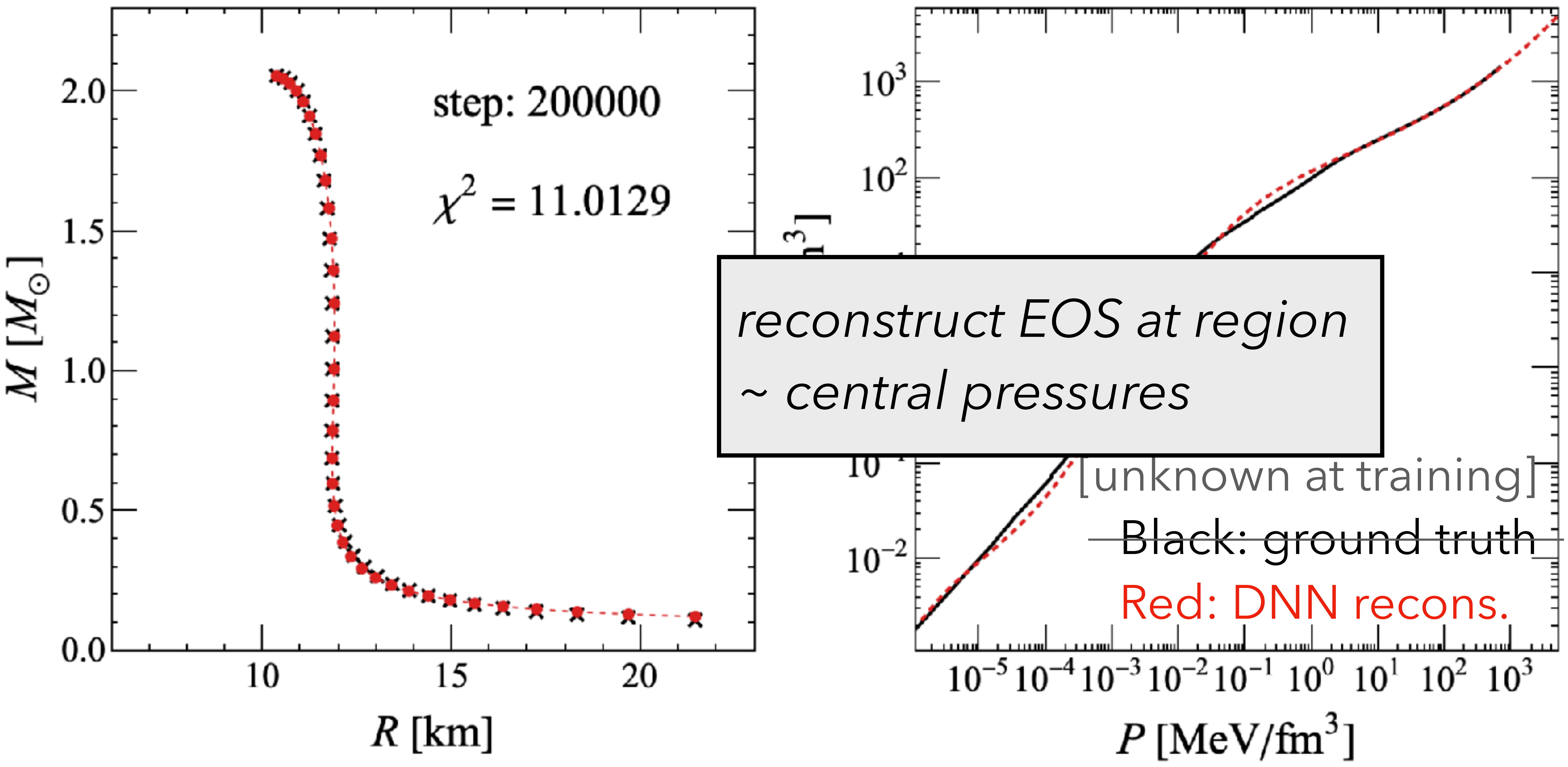
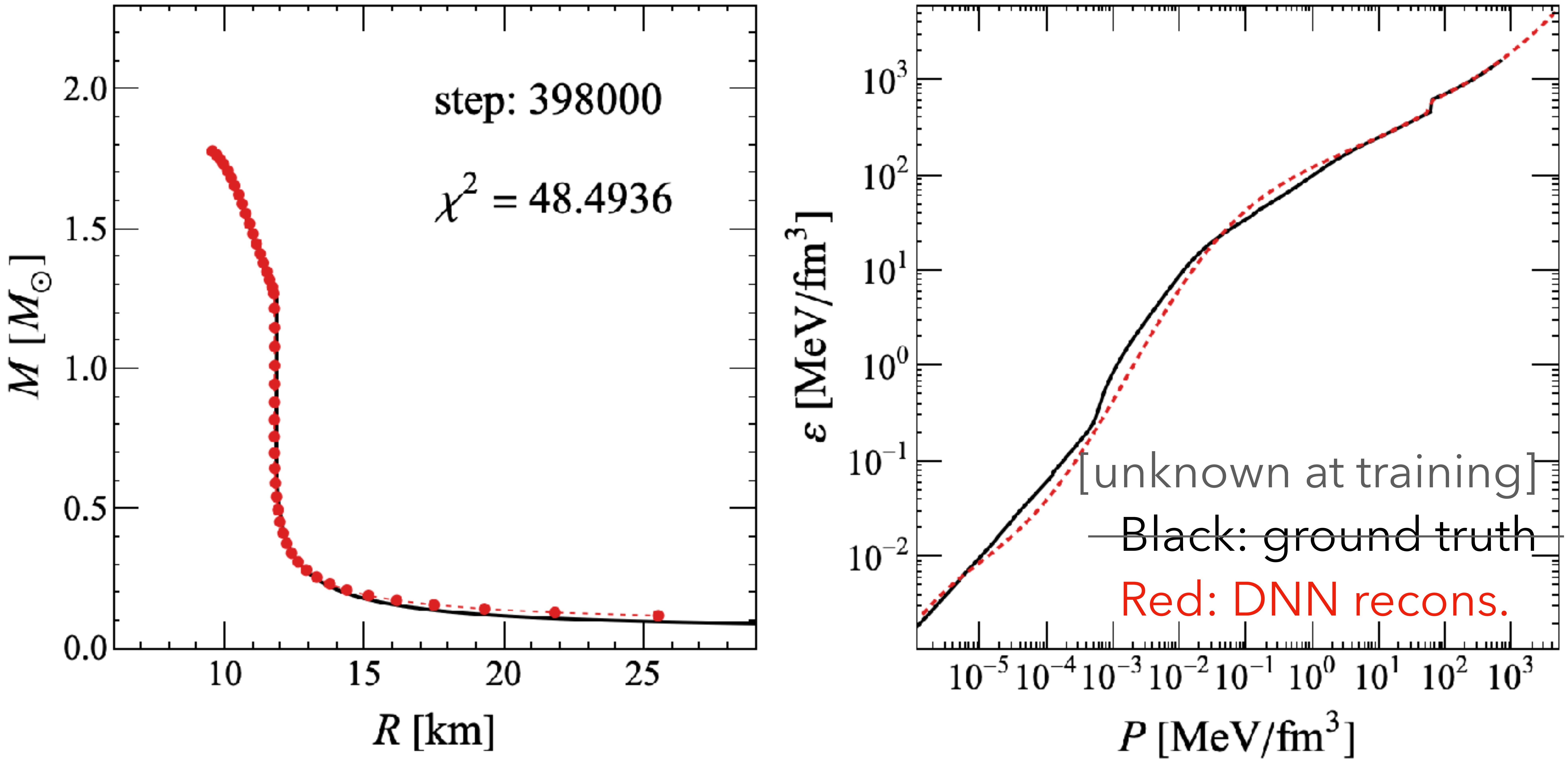


reconstruct EOS at region ~ central pressures

$$\chi^2 = \sum_i \left(\frac{m_i - m_i^{obs}}{\Delta m_i}\right)^2 + \left(\frac{R_i - R_i^{obs}}{\Delta R_i}\right)^2$$

closure test: with phase transition



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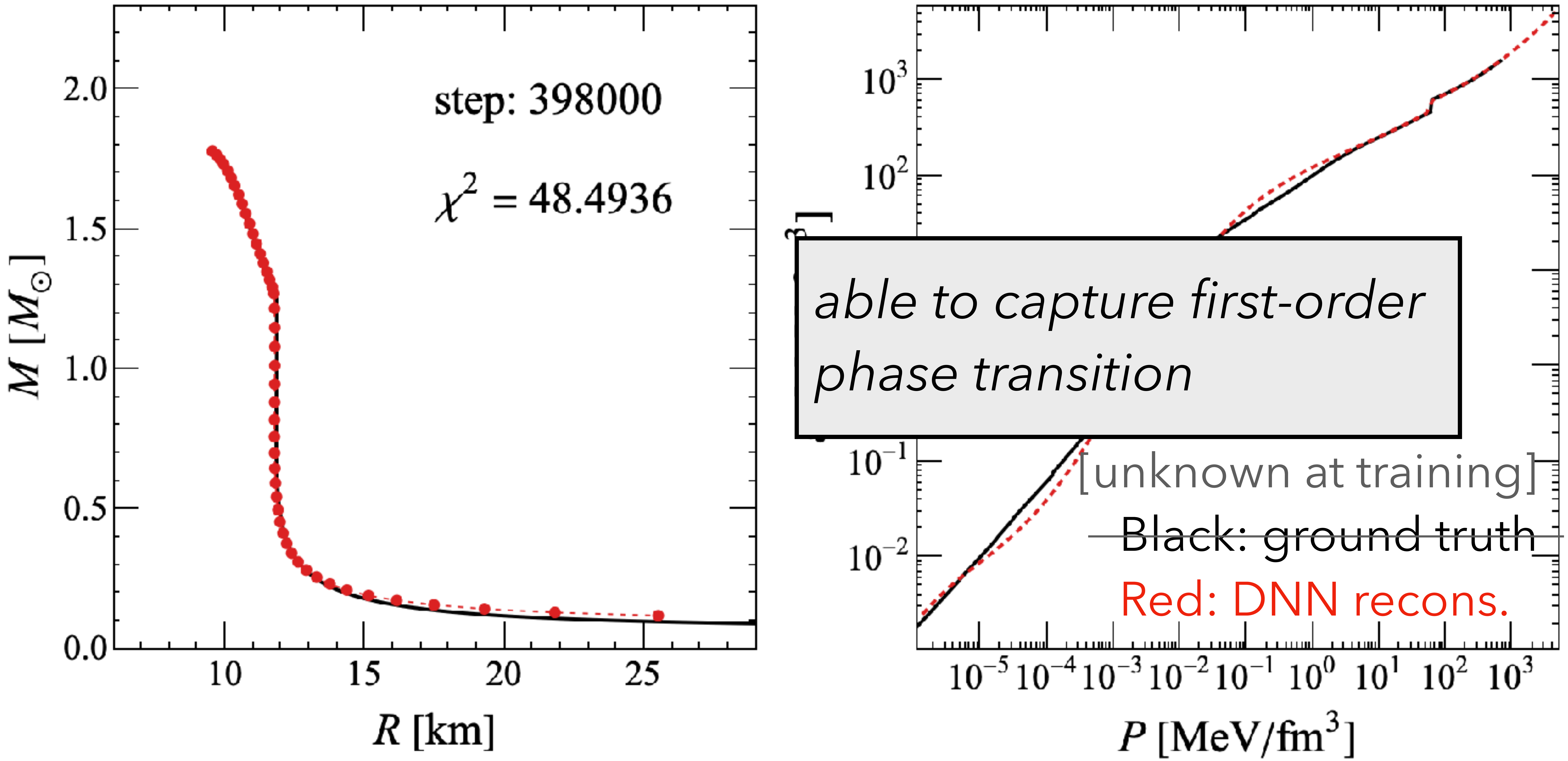
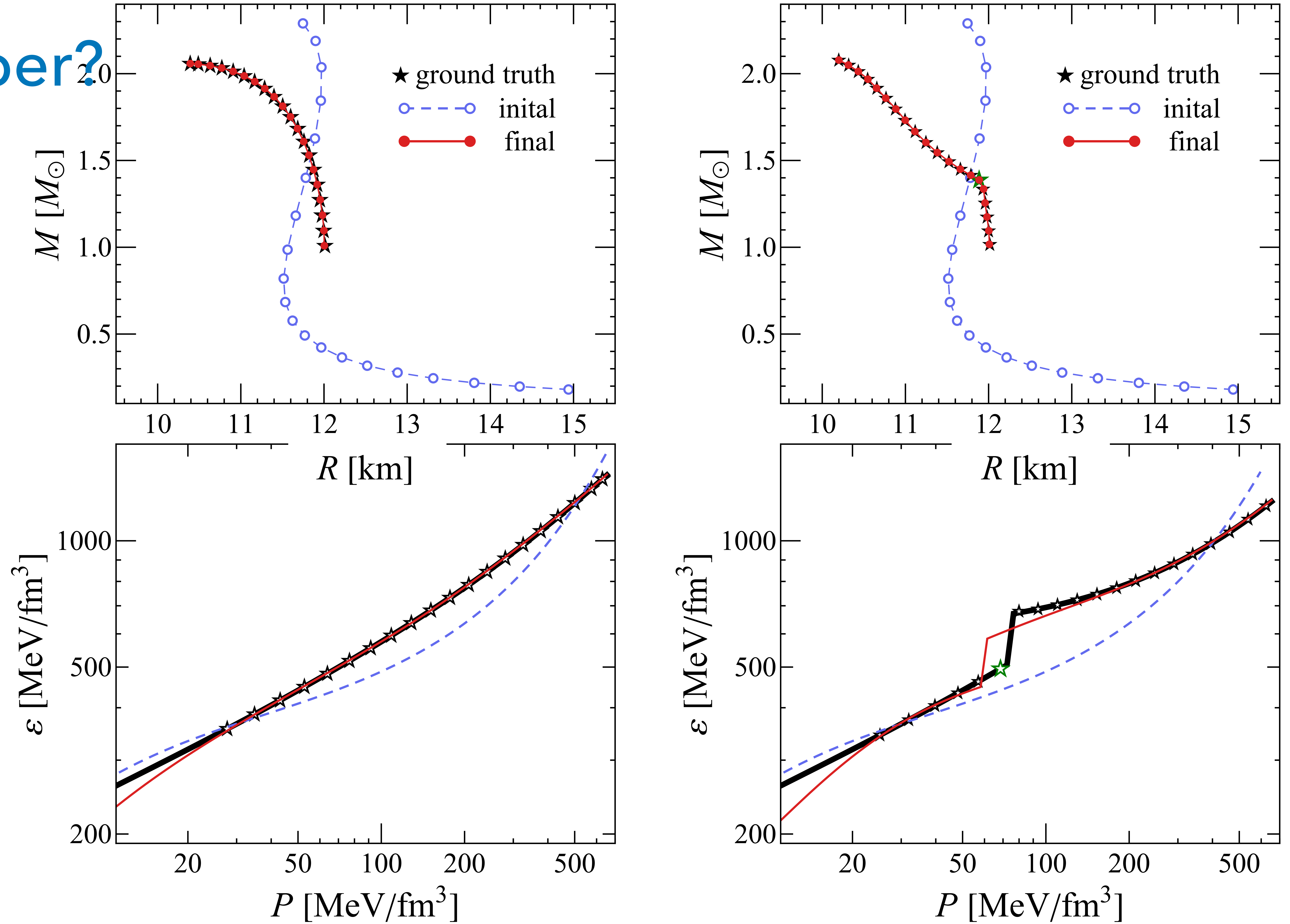


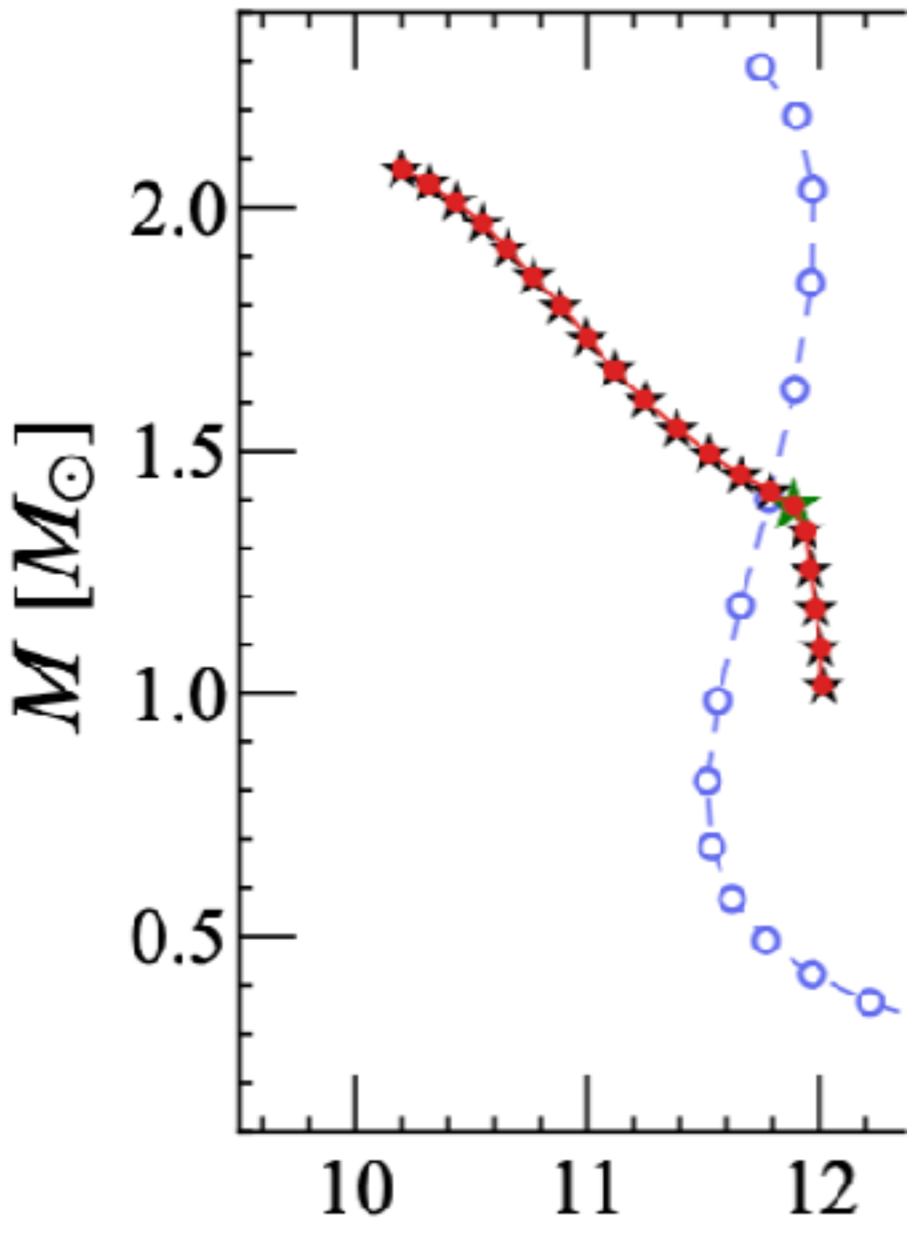
able to capture first-order phase transition

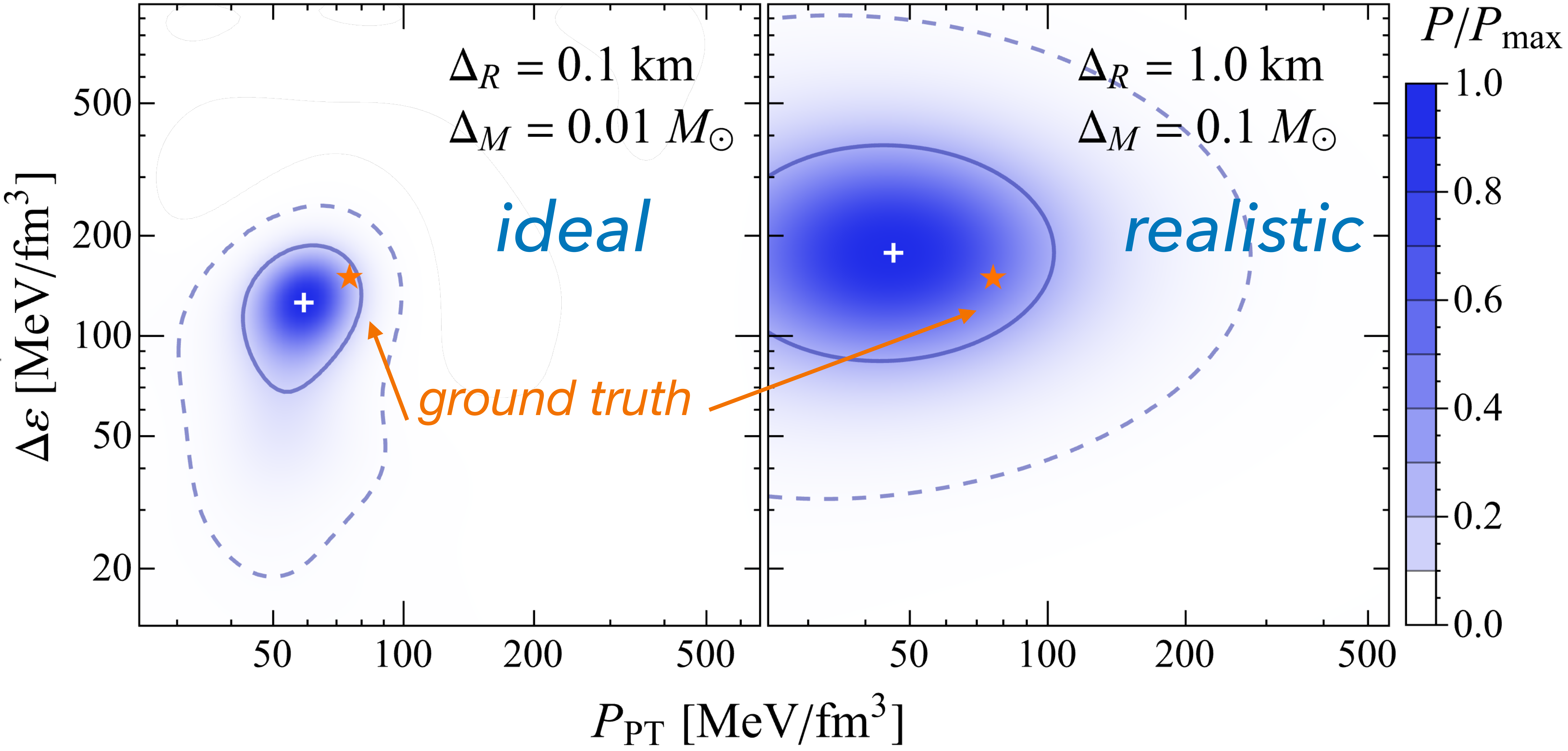
reconstruction w/ finite number?



reconstruction power for different uncertainties

marginal posterior distribution of phase transition pressure and latent heat





assign different level of uncertainties and perform BA.

summary and outlook

- Derived *analytical* differential equations for linear responses of the TOV eq.
  - enables gradient driven BA;
  - suitable for any parameterization of the EoS;

- Studied the *reconstruction power* of first-order phase transition based on NS observations with *finite number and accuracy*.

- On going: Apply to *tidal deformability* observables;

- On going: BA based on real NS observations