Inferring the Equation of State from Neutron Star Observables via Machine Learning

Based on *Phys.Lett.B 865 (2025) 130*

Presented by

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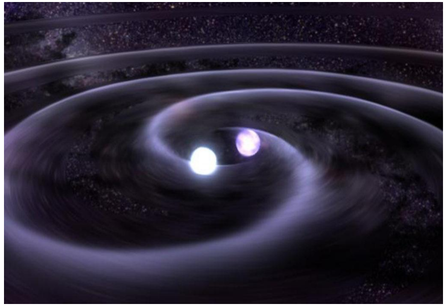
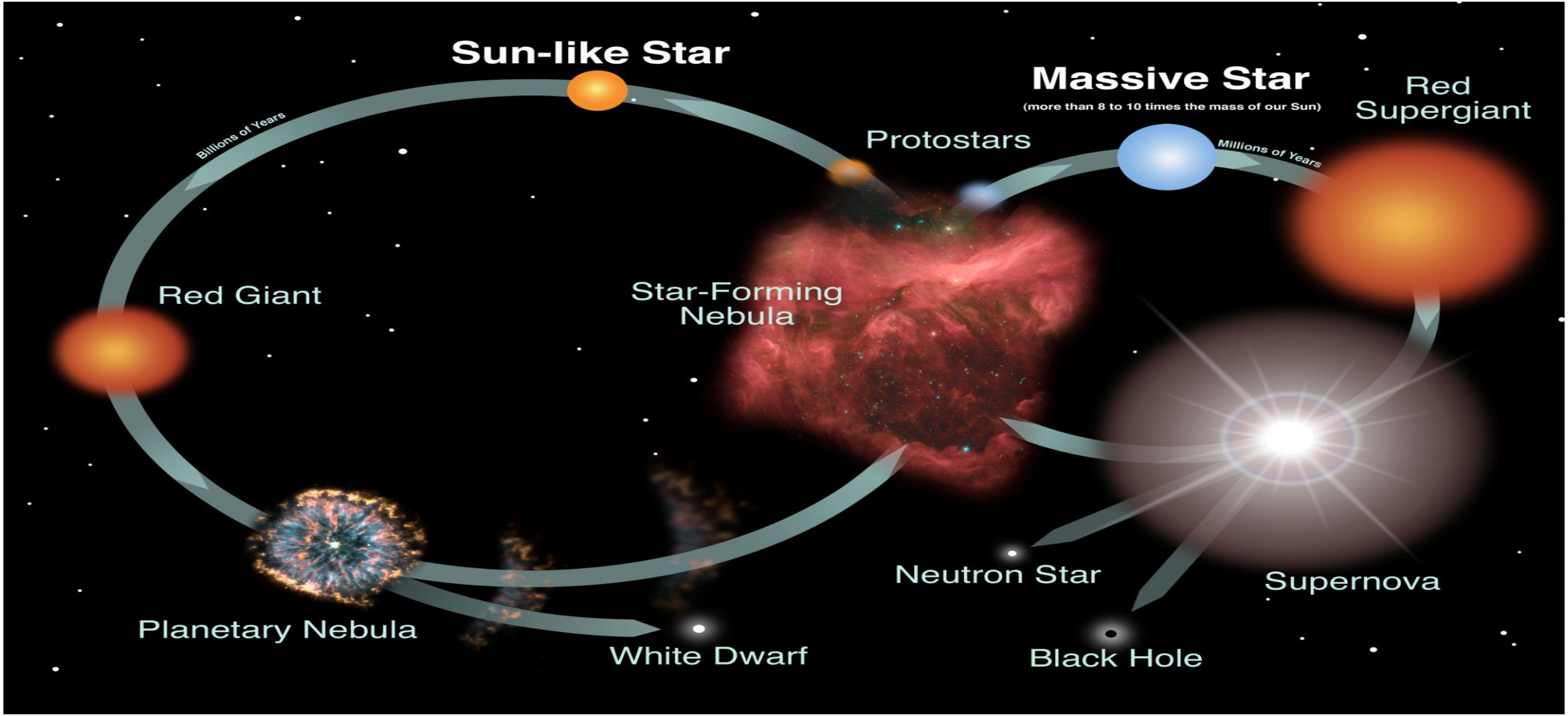


Outlines

- Introduction
  - Life cycle of stars
  - Schwarzschild radius
  - Phase diagram of dense matter
  - Equation of State
  - TOV Equations
- Methodology
- Results
- Summary

Life Cycle of Stars



Credit: NASA and the Night Sky Network.

Schwarzschild radius

The Schwarzschild radius is given as,

$$R_s = \frac{2GM}{c^2} \tag{1}$$

The simplified form is $\frac{2GM}{Rc^2}$.

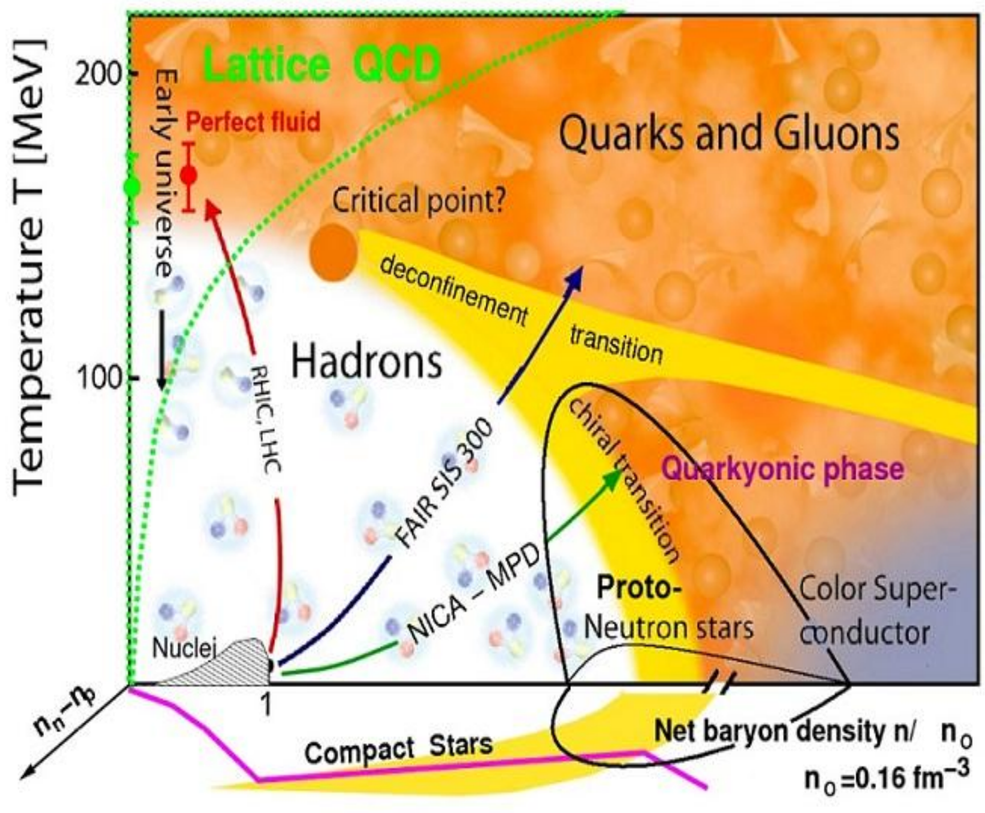
- **If** $\frac{2GM}{Rc^2} \geq \mathbf{1}$ then
  The remnant object is said to be a black hole. (R $\approx$ 9km. and M$\geq 3M_\odot$)
- **If** $\frac{2GM}{Rc^2} < \mathbf{1}$ then
  The remnant object is said to be a neutron star. (R $=$10-20km. and M$< 3M_\odot$)

Ref:- T. X. Zhang "The principles and laws of black hole universe." JMP(9.9)(2018)

Note:-Gravity plays an important role in distinguishing between these two remnant objects.

Ref:- T.J. Maccarone (Black Holes and Neutron Stars)

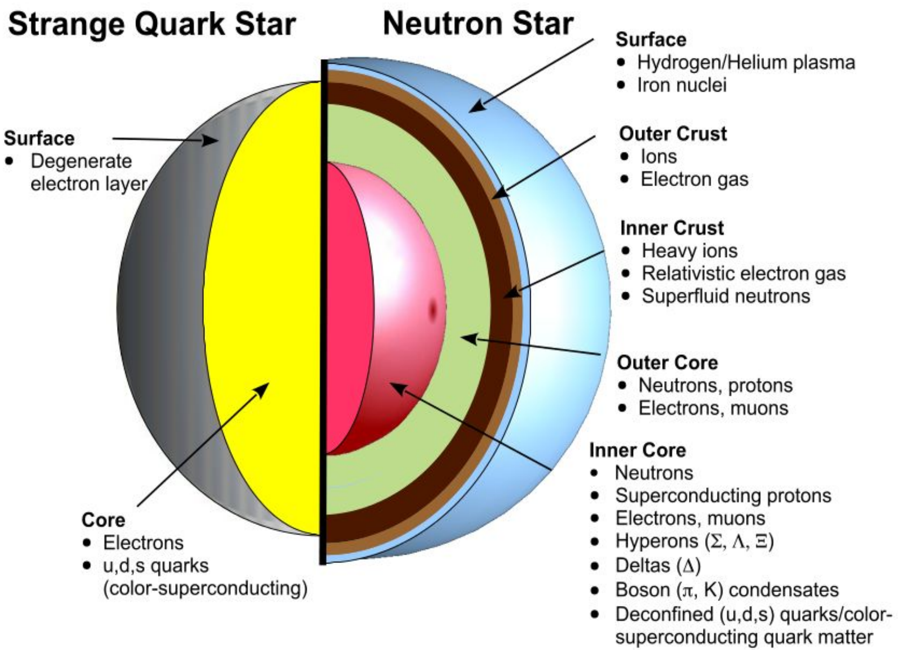
Phase Diagram of Dense Matter

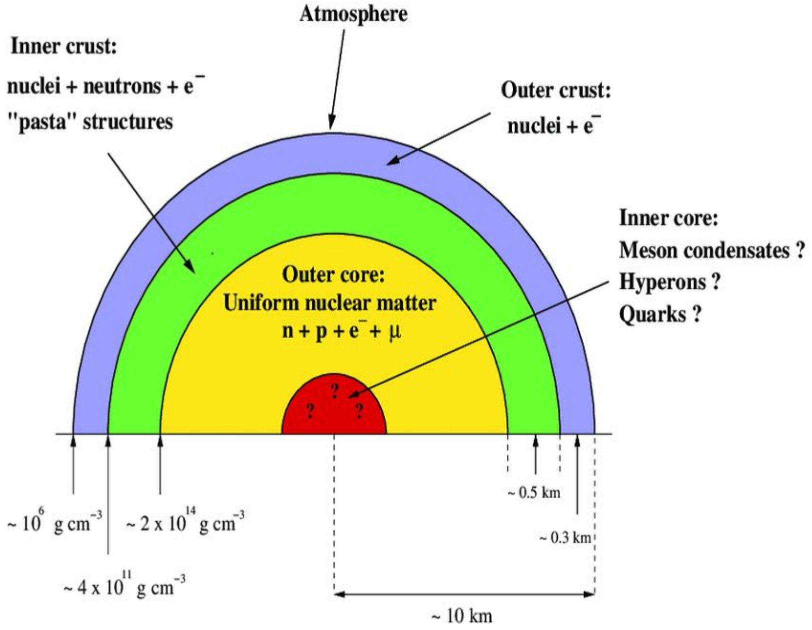


Credit-Larry McLerran, Warsaw University of Technology

- **Why NSs are so interesting in the universe?**
- **What possible structure, composition and dynamics could be of these stars?**

Structure and Composition of Neutron Stars





Credit-Burgio et al. 2020,Universe (119)

Equation of State of Dense Matter

- **Microscopic Models:** (upto 2.8×10^{14} $g.cm^{-3}$ density)
  - Brueckner-Hartree-Fock (BHF) and Dirac-BHF theories (R.Brockmann et al.(PRC 42,(1990))).
  - Chiral perturbation theory (K. Hebeler, PRL 105 (2010))

- **Phenomenological Models:**
  - Non-Relativistic mean field (Skyrme interactions) (Miller et al., PRC (2003))
  - Relativistic mean field (RMF) models (J. D. Walecka, Adv. Nucl. Phys. 16,(1986))

- **Meta Models:**
  - Taylor expansion
  - n/3 expansion
  - polytropic expansion
  - Speed of Sound Model

EOS from Phenomenological Models

Non-Relativistic Skyrme Model

- It is constructed for finite nuclei and nuclear matter at saturation density.
- The Hamiltonian density can be written as

$$\mathcal{H} = \mathcal{K} + \mathcal{H}_0 + \mathcal{H}_3 + \mathcal{H}_{\text{eff}}$$

where, $\mathcal{K}$: kinetic energy term

$\mathcal{H}_0$: zero-range term

$\mathcal{H}_3$: Density dependent term

$\mathcal{H}_{\text{eff}}$: Effective-mass term.

Non-Relativistic Skyrme Model

Zero range term :

$$\mathcal{H}_0 = \frac{1}{4} t_0 \left[(2 + x_0)\rho^2 - (2x_0 + 1)(\rho_p^2 + \rho_n^2) \right]$$

Density dependent term :

$$\mathcal{H}_3 = \frac{1}{24} t_3 \rho^\sigma \left[(2 + x_3)\rho^2 - (2x_3 + 1)(\rho_p^2 + \rho_n^2) \right]$$

$\sigma = 1$: Incompressibility, K_0 >300 MeV but $\frac{1}{3} \leq \sigma \leq \frac{2}{3}$: K_0 = 240 MeV (∼Experimental value)

Effective mass term :

$$\begin{aligned} \mathcal{H}_{\text{eff}} &= \frac{1}{8} \left[t_1(2 + x_1) + t_2(2 + x_2) \right] \tau\rho \\ &+ \frac{1}{8} \left[t_2(2x_2 + 1) - t_1(2x_1 + 1) \right] (\tau_p\rho_p + \tau_n\rho_n) \end{aligned}$$

Relativistic Mean Field(RMF)(σωρ) Model

- The interaction between nucleon happen through the exchange of mesons.

- Isoscalar meson: σ (long range attractive), m_σ=550MeV
- Isovector meson: ω (short range repulsive), m_ω=783MeV
- Isovector meson: ρ (strong repulsive), m_ρ=769 MeV

- The effective lagrangian density is,

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_M + \mathcal{L}_{NL} + \mathcal{L}_{leptons}$$

Relativistic Mean Field

- The all terms are as,

$$\mathcal{L}_N = \bar{\Psi}\Big[\gamma^\mu\left(i\partial_\mu - g_\omega\omega_\mu - \frac{1}{2}g_\varrho \boldsymbol{t}\cdot\boldsymbol{\varrho}_\mu\right) - (m_N - g_\sigma\sigma)\Big]\Psi$$

$$\begin{aligned}\mathcal{L}_M &= \frac{1}{2}\left[\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2\right] - \frac{1}{4}F^{(\omega)}_{\mu\nu}F^{(\omega)\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu \\ &- \frac{1}{4}\boldsymbol{F}^{(\varrho)}_{\mu\nu}\cdot\boldsymbol{F}^{(\varrho)\mu\nu} + \frac{1}{2}m_\varrho^2\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\varrho}^\mu\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{NL} &= -\frac{1}{3}b\, m_N\, g_\sigma^3(\sigma)^3 - \frac{1}{4}c(g_\sigma\sigma)^4 + \frac{\xi}{4!}g_\omega^4(\omega_\mu\omega^\mu)^2 \\ &+ \Lambda_\omega g_\varrho^2\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\varrho}^\mu g_\omega^2\omega_\mu\omega^\mu\end{aligned}$$

$$\mathcal{L}_{leptons} = \bar{\Psi}_l\Big[\gamma^\mu\left(i\partial_\mu - m_l\right)\Psi_l\Big], \text{ where } \Psi_l\ (l = e^-, \mu^-)$$

EOS from Meta Models

Taylor expansion:-

- The total energy density per nucleon is expressed as,

$$E(\rho, \delta) = \sum_n \frac{1}{n!}(a_n + b_n\delta^2)\left(\frac{\rho - \rho_0}{3\rho_0}\right)^n ,$$

- The coefficients are expressed in terms nuclear matter parameters.

n/3 expansion:-

$$E(\rho, \delta) = \sum_{n=2}^{6}(a'_{n-2} + b'_{n-2}\delta^2)\left(\frac{\rho}{\rho_0}\right)^{\frac{n}{3}} .$$

- The coefficients are expressed in the linear combinations of nuclear matter parameters.

Ref. Patra et. al. PRD (2022)

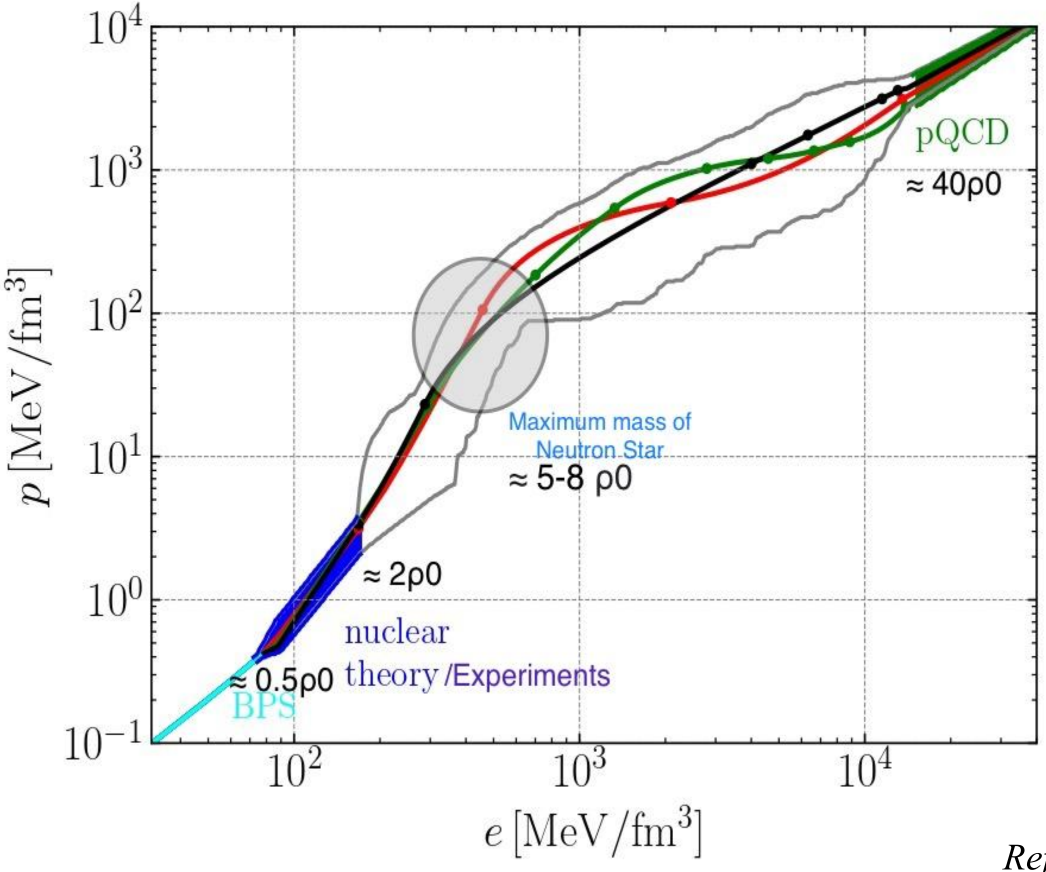
Polytropic expansion:-

$$p(\varepsilon) = K_i \varepsilon^{\gamma_i}, \quad \varepsilon_{i-1} < \varepsilon < \varepsilon_i .$$

- Depending upon the different segments, one can calculate the EOS.

Ref. Huth et al. Nature (2022)

Validation of Equation of State

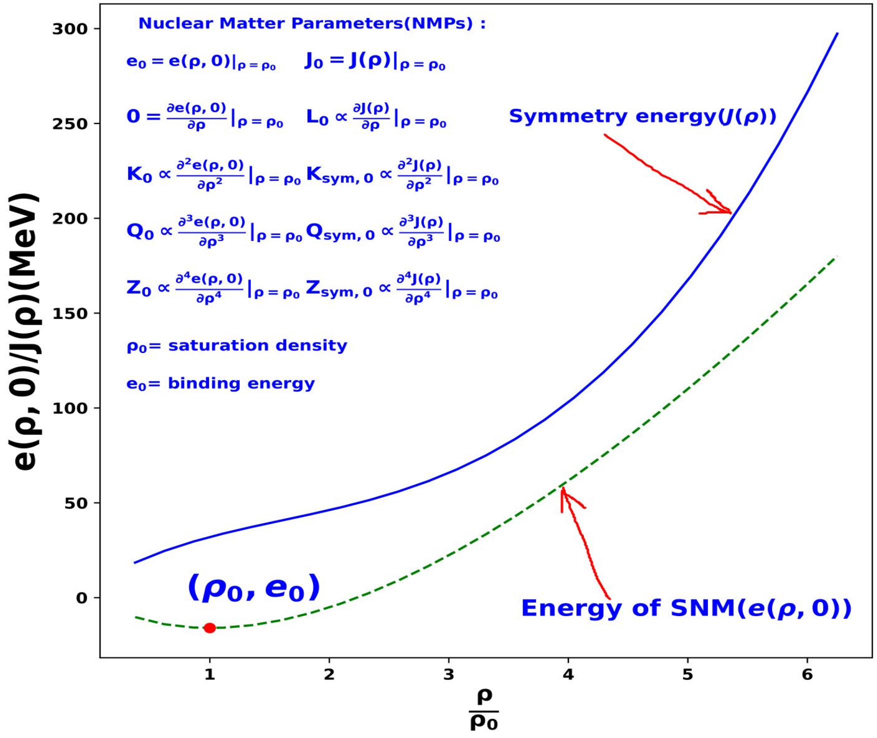


Ref. Altiparmak et. al. APJ (2022)

Relation of NMPs with Equation of State

$$e(\rho, \delta) \;=\; e_{snm}(\rho) + e_{sym}(\rho)\delta^2$$

$$
\begin{aligned}
e_0 &= e_{snm}(\rho_0) \\
K_0 &= 9(\rho_0)^2 e''_{snm}(\rho_0) \\
Q_0 &= 27(\rho_0)^3 e'''_{snm}(\rho_0) \\
J_0 &= e_{sym}(\rho_0) \\
L_0 &= 3(\rho_0) e'_{sym}(\rho_0) \\
K_{sym,0} &= 9(\rho_0)^2 e''_{sym}(\rho_0) \\
Q_{sym,0} &= 27(\rho_0)^3 e'''_{sym}(\rho_0)
\end{aligned}
$$



Nuclear Theory & Experiments Constraints

- e_0 = -16.0 ± 0.3 MeV and ρ_0=0.16± 0.005 (few percentage uncertainties)
- J_0 = 32.5 ± 2.5MeV & K_0 = 240 ± 20 MeV (about 10% uncertainties)
- L_0 = 20-130 MeV (more than 20% uncertainties)
- K_{sym0},Q_0, Z_0, Q_{sym0} and Z_{sym0} are not well known.

Symmetric matter			Ref.
Constraints	n (fm^{-3})	P_{SNM} (MeV/fm^3)	Ref.
HIC(DLL)	0.32	10.1 ± 3.0	[77]
HIC(FOPI)	0.32	10.3 ± 2.8	[50]

Asymmetric matter				
Constraints	n (fm^{-3})	S(n) (MeV)	P_{sym} (MeV/fm^3)	Ref.
Nuclear structure				
α_D	0.05	15.9 ± 1.0		[69]
PREX-II	0.11		2.38 ± 0.75	[66, 70, 71]
Nuclear masses				
Mass(Skyrme)	0.101	24.7 ± 0.8		[65, 66]
Mass(DFT)	0.115	25.4 ± 1.1		[66, 67]
IAS	0.106	25.5 ± 1.1		[66, 68]
Heavy-ion collisions				
HIC(Isodiff)	0.035	10.3 ± 1.0		[66, 72]
HIC(n/p ratio)	0.069	16.8 ± 1.2		[66, 73]
HIC(π)	0.232	52 ± 13	10.9 ± 8.7	[66, 74]
HIC(n/p flow)	0.240		12.1 ± 8.4	[51, 66, 75, 76]

Ref: Tsang et al.Nature Astron. 8 (2024)

Tolman–Oppenheimer–Volkoff Equations

- Consider the general static, spherically symmetric metric as

$$d\tau^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2d\theta^2 - r^2 sin^2\theta d\phi^2$$

- Using Einstein's field equation, we will get,

$$\frac{dp}{dr} = -\frac{G.\epsilon(r).m(r)}{r^2}.\left[1+\frac{p(r)}{\epsilon(r)}\right].\left[1+\frac{4\pi.r^3.p(r)}{m(r)}\right].\left[1-\frac{2G.m(r)}{r}\right]^{-1}$$

$$\frac{dm}{dr} = 4\pi r^2\epsilon(r)$$

- Conditions to solve above the eqs.
  - At, r=0, M(r=0)=0 At, r=R, P(r=R)≈ 0

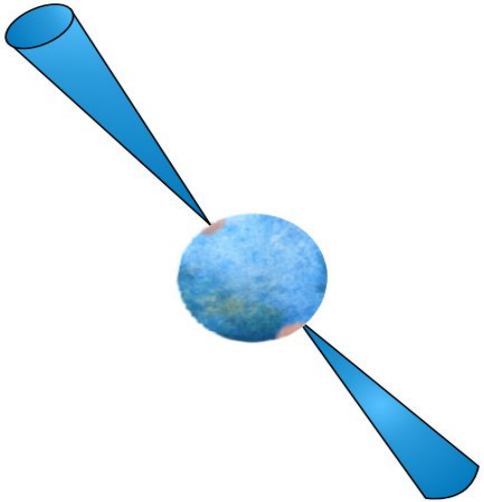
Ref. Tolman, Phys. Rev. 55, 364 (1939)

Multimessenger Observables

Radio timing

Mass measurement PSR J0740

Fonseca et al. (2021), [2104.00880](2104.00880)

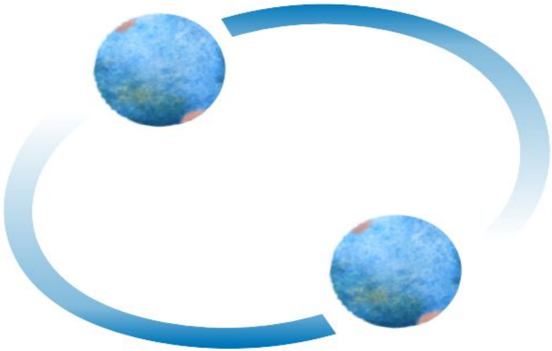


Radio pulsars

Gravitational wave signals

Tidal deformability measurement

GW170817 & GW190425



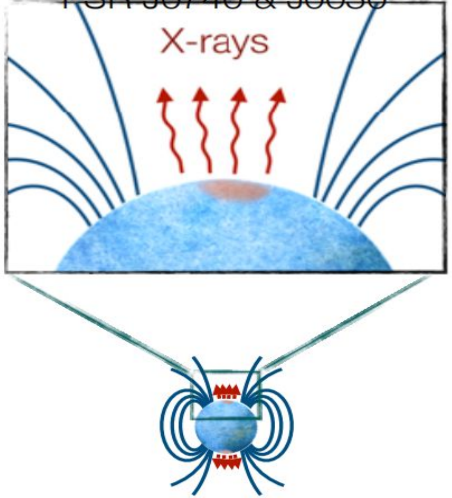
Binary neutron star &

black hole - neutron star mergers

X-ray pulse profile modeling

Mass-radius measurement

PSR J0740 & J0030

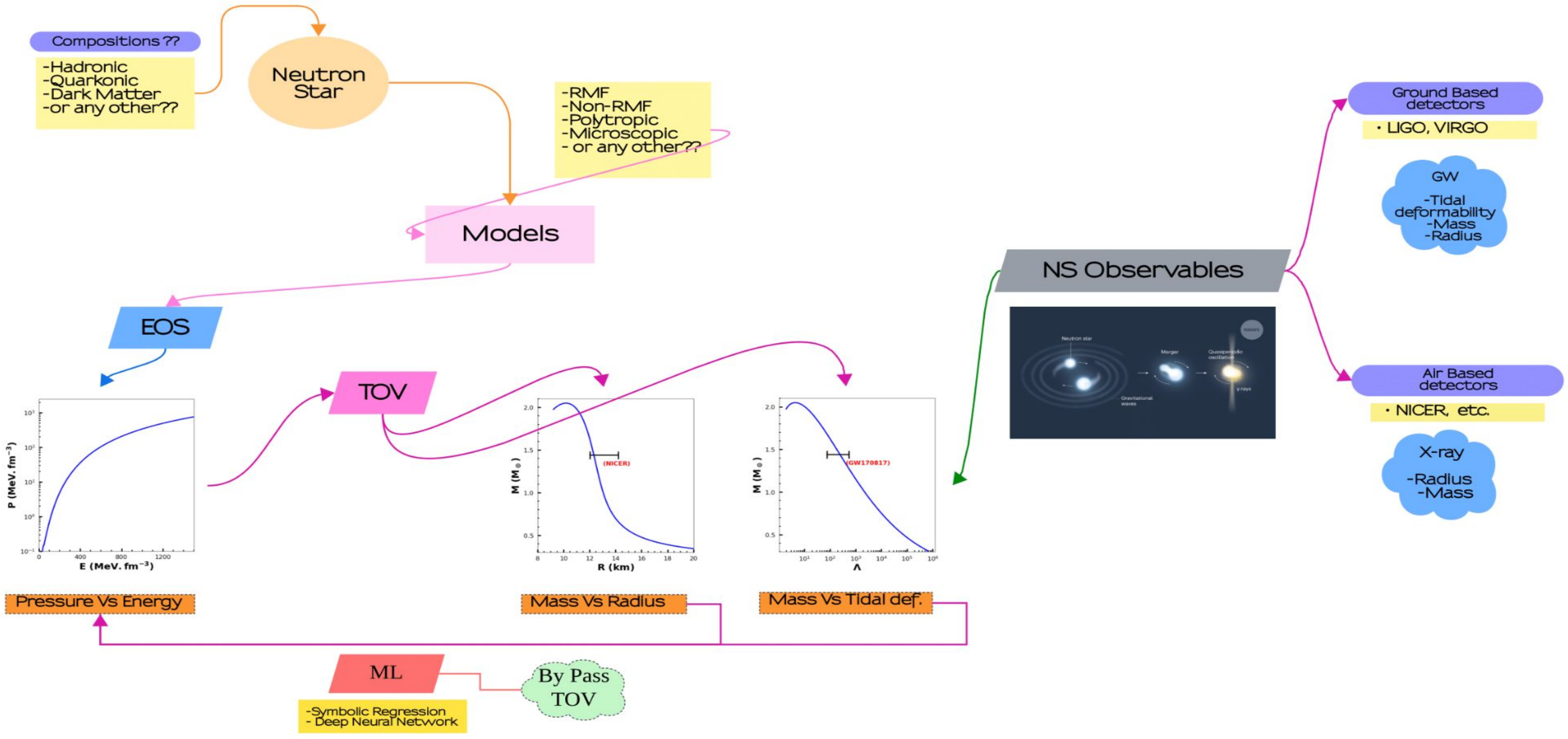


X-ray pulsars

Astrophysical Constraints

		$M(M_\odot)$	R (km)	Λ
NANO Grav	PSR J0740+6620	$2.14^{+0.10}_{-0.09}$	$12.39^{+1.30}_{-0.98}$	
LIGO/Virgo	BNS	1.4	$12.0^{+1.1}_{-1.1}$	190^{+390}_{-120}
NICER	PSR J0030+0451	$1.34^{+0.15}_{-0.16}$	$12.71^{+1.14}_{-1.19}$	
		$1.44^{+0.15}_{-0.14}$	$13.02^{+1.24}_{-1.06}$	
	PSR J0740+6620	$2.07^{+0.07}_{-0.07}$	$12.39^{+1.30}_{-0.98}$	
		$2.08^{+0.07}_{-0.07}$	$13.7^{+2.6}_{-1.5}$	

Motivation


Compositions ??
-Hadronic
-Quarkonic
-Dark Matter
-or any other??
Neutron Star
-RMF
-Non-RMF
-Polytropic
-Microscopic
- or any other??
Models
EOS
TOV
NS Observables
Ground Based detectors
• LIGO, VIRGO
GW
-Tidal deformability
-Mass
-Radius
Air Based detectors
• NICER, etc.
X-ray
-Radius
-Mass
P (MeV. fm^{-3})
E (MeV. fm^{-3})
M (M$_\odot$)
R (km)
(NICER)
(GW170817)
Λ
Pressure Vs Energy
Mass Vs Radius
Mass Vs Tidal def.
ML
-Symbolic Regression
- Deep Neural Network
By Pass TOV

Methodology

<u>Collection of EOSs</u>

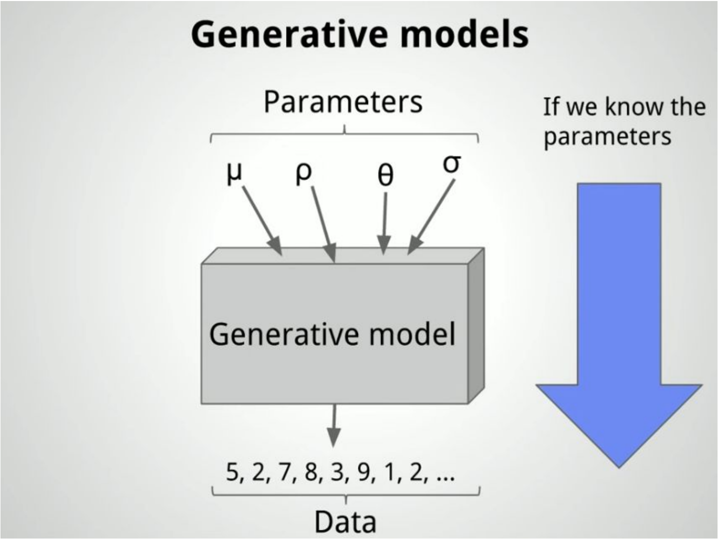
-Non-Linear RMF (NL) — *Malik et. al (2023)*
-Density Dependent RMF (DDB) — *Malik et. al (2022)*
-NL RMF including Hyperon (NL-Hyp) — *Providencia et. al (2022)*
-ComPOSE (ComP) — *Compose.obspm.fr*
-Speed of Sound Extraction (CSE) — *Huth et. al (2022)*
-Piecewise Polytrope (PWP) — *Huth et. al (2022)*
-Spectral Representation (SR) — *Lindblom et. al (2022)*

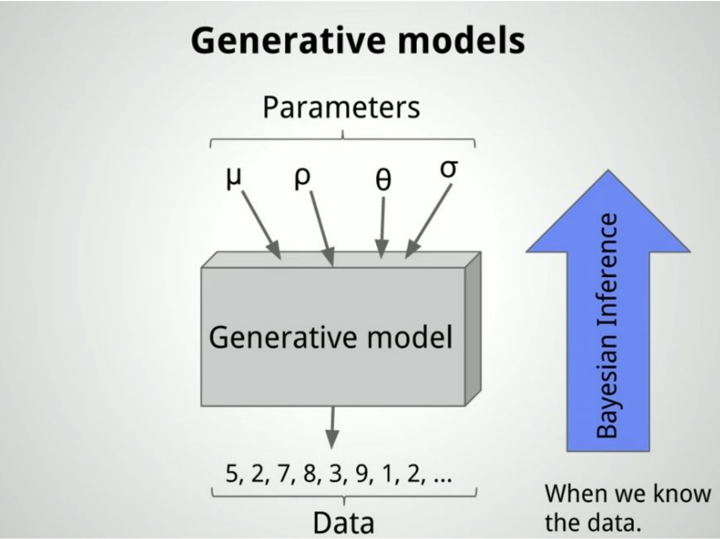
What is Bayesian Analysis?

This method requires basically three things such as,

- **Data**
- **A generative model**
- **Priors**

What is the Generative model?





Ref.- A. Gelman et al. (CRC Press)

Bayesian Analysis

The Bayes' theorem is defined by,

$$P(\theta|D) = \frac{\mathcal{L}(D|\theta)P(\theta)}{\mathcal{Z}}.$$

- θ and D denote the set of model parameters and the fit data.
- The $P(\theta|D)$ is the posterior distribution of the parameters.
- $\mathcal{L}(D|\theta)$ is the likelihood function and $P(\theta)$ is the prior for the model parameters.
- $\mathcal{Z}$ is called as evidence/normalization-constant.

Likelihoods

Gaussian Likelihood:-

$$\mathcal{L}(D|\theta) = \prod_j \frac{1}{\sqrt{2\pi\sigma_j^2}} e^{-\frac{1}{2}\left(\frac{d_j - m_j(\theta)}{\sigma_j}\right)^2}.$$

- The index j runs over all the data. d_j and m_j are the data and corresponding median values.
- The σ_j are the adopted uncertainties.

For GW:-

$$\begin{aligned} P(d_{\rm GW}|{\rm EoS}) = &\int_{m_2}^{M_u} dm_1 \int_{M_l}^{m_1} dm_2 P(m_1, m_2|{\rm EoS}) \\ &\times P(d_{\rm GW}|m_1, m_2, \Lambda_1(m_1, {\rm EoS}), \Lambda_2(m_2, {\rm EoS})) \\ &= \mathcal{L}^{\rm GW} \end{aligned}$$

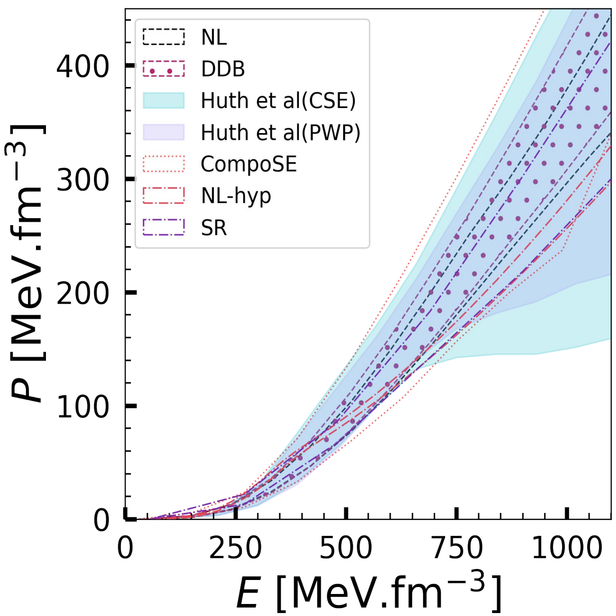
For NICER:-

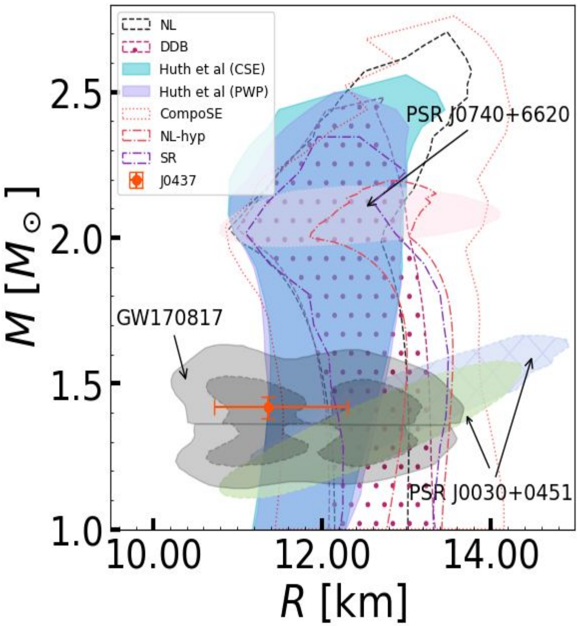
$$\begin{aligned} P(d_{\rm X-ray}|{\rm EoS}) = &\int_{M_l}^{M_u} dm P(m|{\rm EoS}) \\ &\times P(d_{\rm X-ray}|m, R(m, {\rm EoS})) \\ &= \mathcal{L}^{\rm NICER} \end{aligned}$$

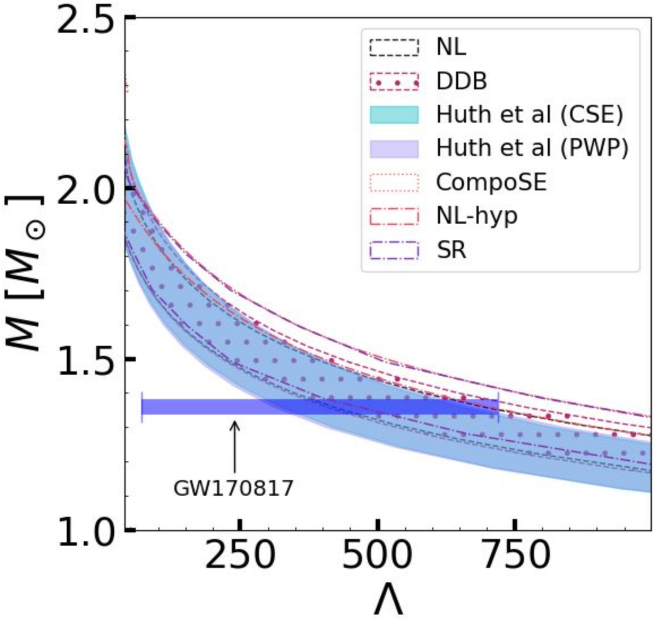
Aim

1. *Find Mmax in terms of EOS parameters*
2. *Find EOS parameters in terms of M-R & M-Λ*

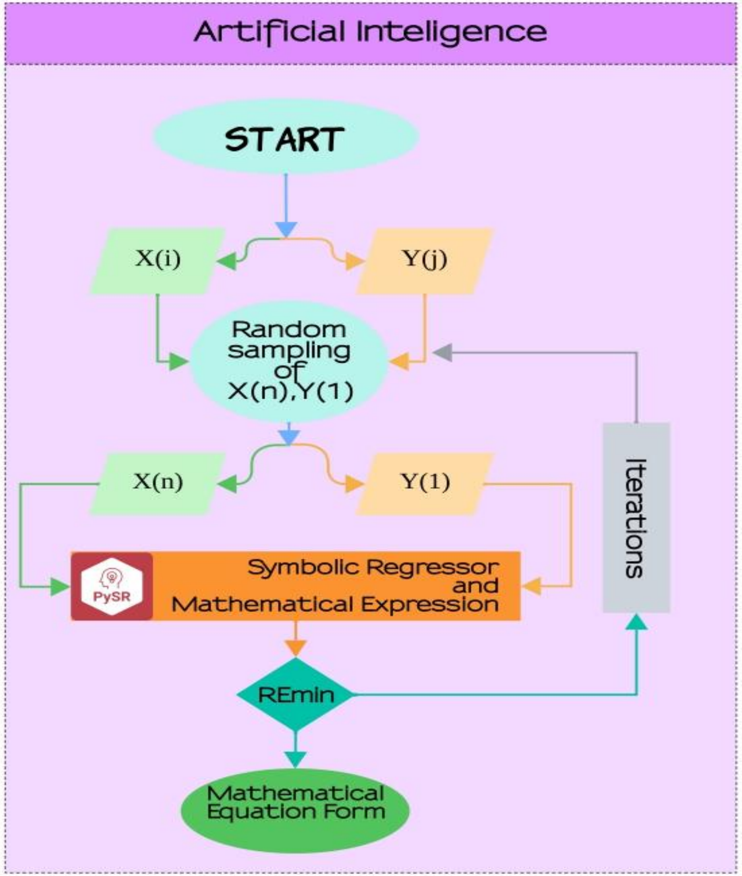
EOS and Mass-Radius-Tidal deformability Curves







Sampling for Symbol Regression



Steps for SR

-Keep Features Vectors (X(i))

-Corresponding Target Vectors (Y(j))

-Choosing One Target (Y(1))

-Python Symbolic Regression (PySR)

-Finding Equation whose relative error (RE) has Minimum

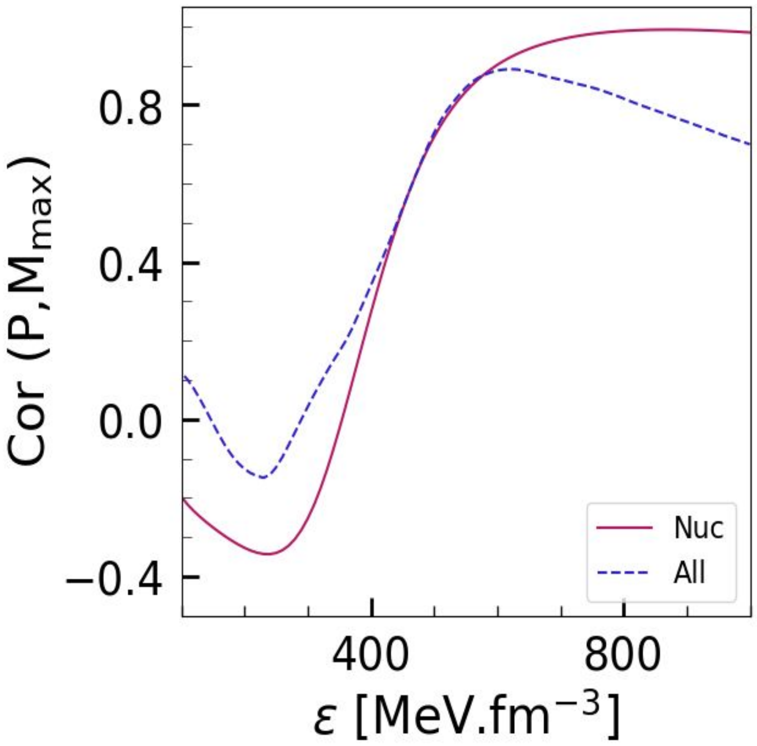
-Final Output expression

Find Mmax in terms of EOS parameters

- ❑ Inputs/Features : Density, Energy density & Pressure
- ❑ Outputs/Target : Maximum mass of NSs

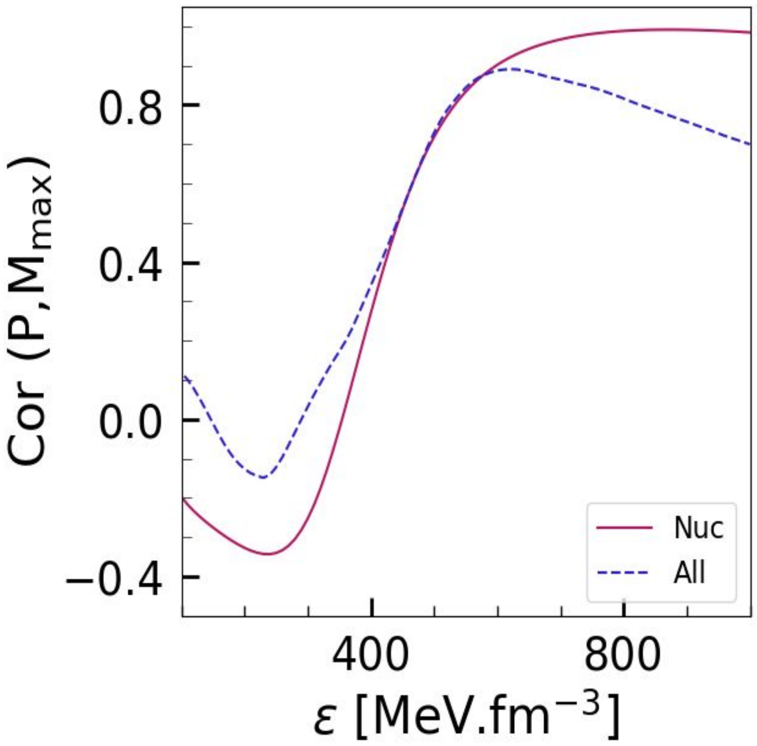
Note: All , We have 700 EOSs, obtained randomly selecting 100 EOSs from each sets.

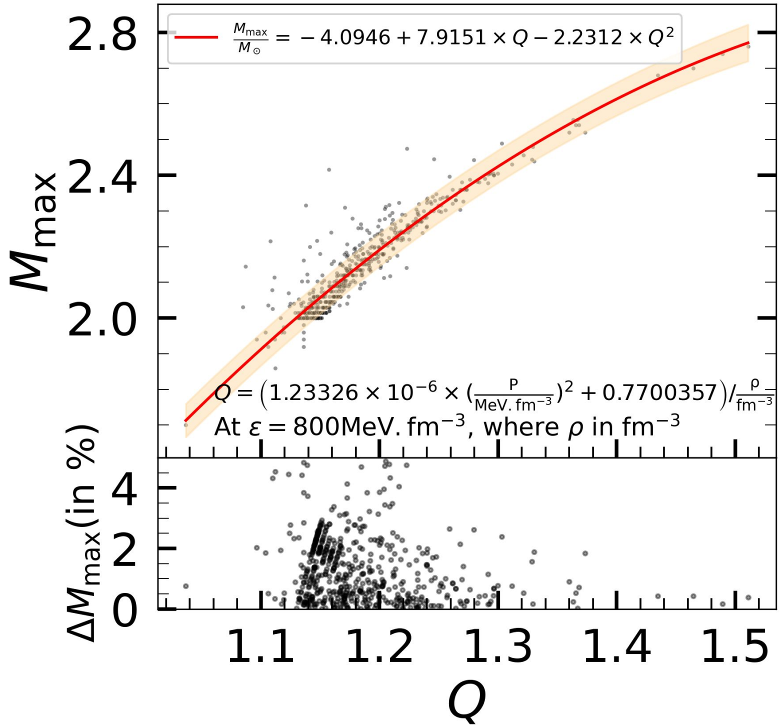
Relation of Maximum Mass of NS with EOS



Note: All (700 EOSs = 100 EOSs from each set)

Relation of Maximum Mass of NS with EOS





Note: All (700 EOSs = 100 EOSs from each set)

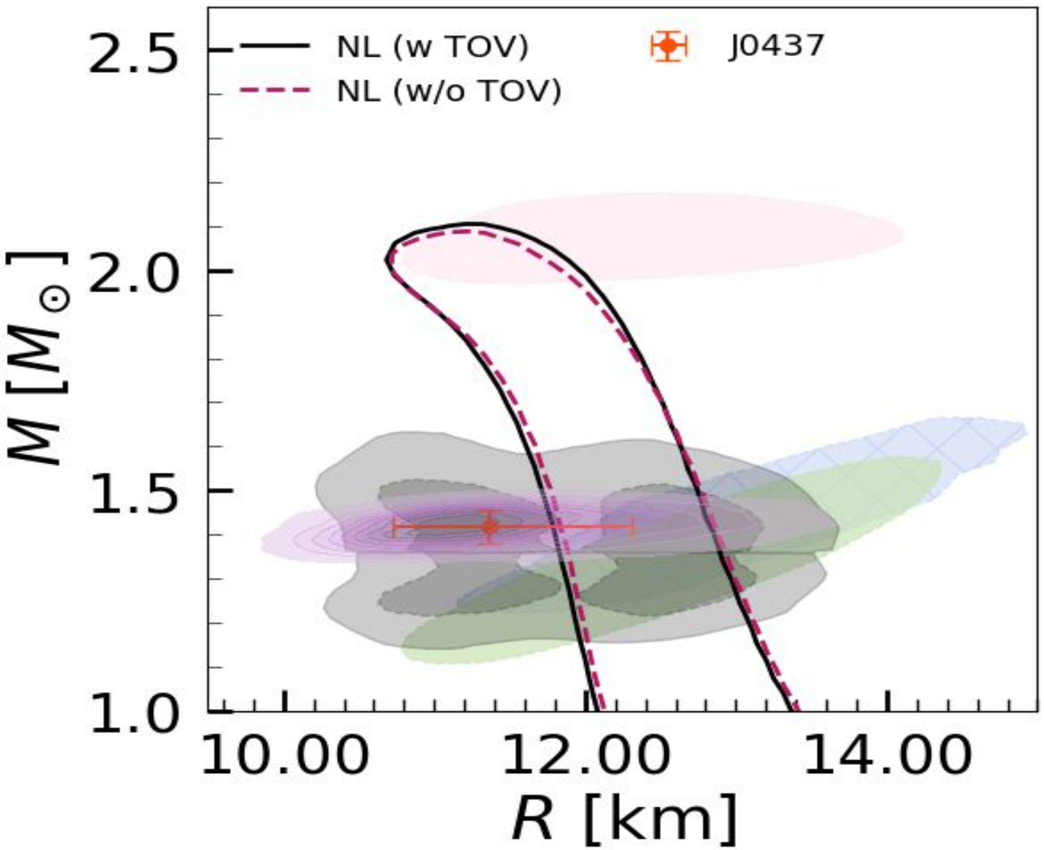
Application of Mmax relation

-In Bayesian analysis, EOSs are filtered with Mmax > 2 Solar mass.

Table 1: Bayesian Inference: the Pymultinest Sampler is employed with 1500 live points on the Deucalion HPC, University of Coimbra, Portugal. The system's large-x86 partition utilizing 4 nodes, each equipped with 128 CPUs. More information can be found at https://docs.macc.fccn.pt/

Model	Total Likelihood evaluated	Total Samples in Posterior	Sampling Time (Sec)	CPU Hr
NL (w TOV)	307018	8291	6828.08	971.10
NL (w/o TOV)	283718	8245	896.49	127.50

Application of Mmax relation



Find EOS parameters in terms of M-R & M-Λ

- ❑ Inputs/Features : Mass, Radius and Tidal deformability as a mass range of 1 to 2 solar mass

- ❑ Outputs/Target : Energy density & Pressure

Direct link between M-R with EOS

$$\frac{\epsilon}{\mathrm{MeV}\cdot\mathrm{fm}^{-3}} = \frac{15500\left(\left(\frac{M}{M_\odot}\right)^3 + 1.0188\left(\frac{M}{M_\odot}\right)^2 + \frac{M}{M_\odot}\right)}{\left(\frac{R}{\mathrm{km}}\right)^2} - \frac{3100\left(\frac{R}{\mathrm{km}}\right)\left(\frac{M}{M_\odot}\right)^2 + 46747.7117}{\left(\frac{R}{\mathrm{km}}\right)^2} \quad (3)$$

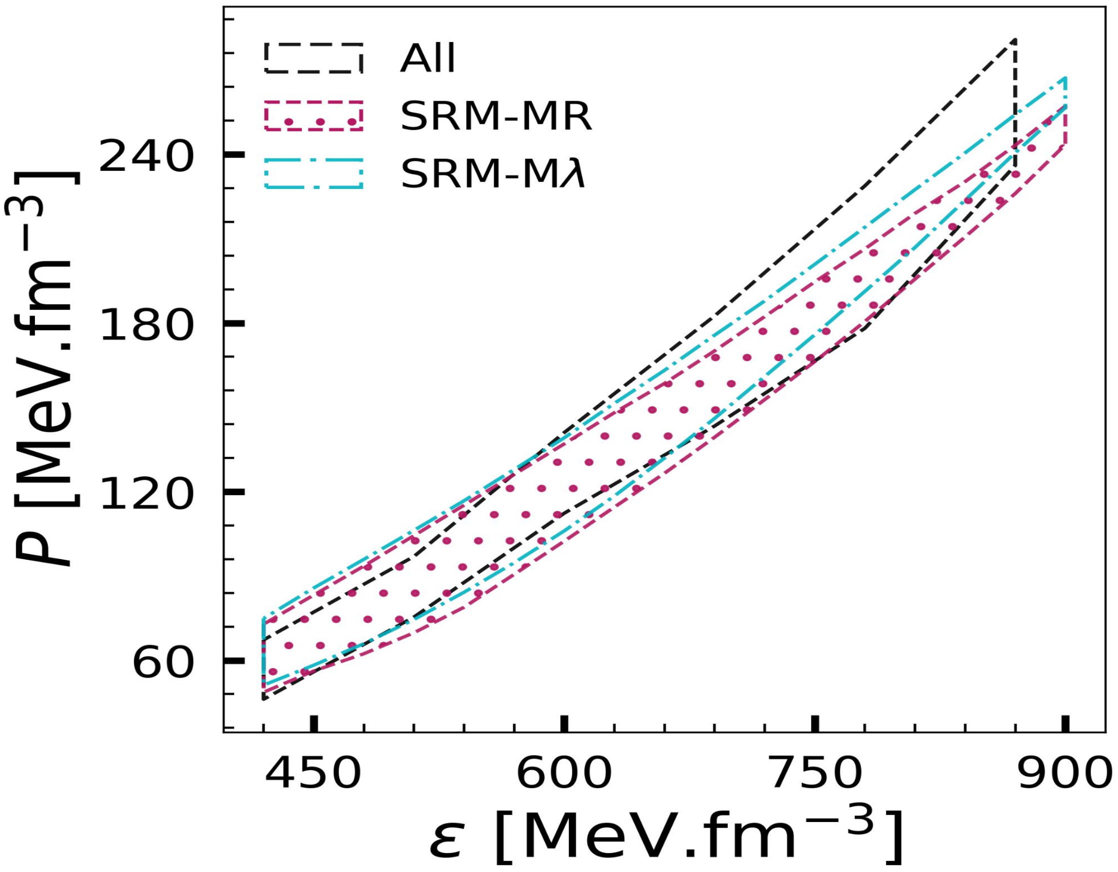
$$\frac{P}{\mathrm{MeV}\cdot\mathrm{fm}^{-3}} = \frac{5\times 10^{5.54094}\times 10^{\left(\frac{M}{M_\odot}\right)}}{\left(\frac{R}{\mathrm{km}}\right)^{5.54094}} + 18.55325 \quad (4)$$

Direct link between M-Λ with EOS

$$\frac{\epsilon}{\mathrm{MeV}\cdot\mathrm{fm}^{-3}} = 155\left(\left(\frac{M}{M_\odot}\right)^2 + \left(\frac{M}{M_\odot}\right) + 30.4241\right)^{1.06601\left(\frac{\Lambda}{km^5}\right)^{-0.03318}}$$

$$\frac{P}{\mathrm{MeV}\cdot\mathrm{fm}^{-3}} = \frac{873161.2165\left(\frac{M}{M_\odot}\right)}{\left(\frac{\Lambda}{km^5}\right) + 2057.0829} \qquad (6)$$

Application of Our relations



Individual Relative Error (RE)

Table 2: The optimal equations obtained through feature and target variable minimization. In this context, "**Y**" specifically denotes the ϵ, and P, while the features "**X**" pertain to the neutron star properties, i.e. mass M, radius R and tidal deformability λ, in a mass range of 1-2 $M_{\odot}$. We provide the relative error values, "RE" for all the datasets.

Y	**X**	**Eqs.**	**Datasets [RE (%)]**							
			NL	**NL-hyp**	**CompOSE**	**CSE**	**PWP**	**DDB**	**SR**	**All**
ϵ	[M,R]	3	3.4	5.8	6.2	10.2	11.0	4.4	4.5	5.5
P	[M,R]	4	5.3	5.3	6.0	7.8	9.3	5.2	5.5	5.7
ϵ	[M,λ]	5	3.8	7.0	4.6	7.4	7.5	4.0	5.2	5.1
P	[M,λ]	6	6.5	7.9	8.4	11.9	8.9	6.7	7.4	7.5

Summary

- ➢ We found the maximum mass of NS constraints which helps to reduce the computational time during Bayesian Analysis.

- ➢ We successfully reconstructed the EOS from radius and tidal deformability of NS within a mass range 1-2 solar masses.

Acknowledgement

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Dr. Helena Pais (University of Coimbra)

n/3 expansion

$$E(\rho,0) = \sum_{n=2}^{6}(a'_{n-2})\left(\frac{\rho}{\rho_0}\right)^{\frac{n}{3}}, \quad (7)$$

$$E_{\rm sym}(\rho) = \sum_{n=2}^{6}(b'_{n-2})\left(\frac{\rho}{\rho_0}\right)^{\frac{n}{3}}, \quad (8)$$

$$E(\rho,\delta) = \sum_{n=2}^{6}(a'_{n-2}+b'_{n-2}\delta^2)\left(\frac{\rho}{\rho_0}\right)^{\frac{n}{3}}. \quad (9)$$

$$\begin{pmatrix} \varepsilon_0 \\ 0 \\ K_0 \\ Q_0 \\ Z_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ -2 & 0 & 4 & 10 & 18 \\ 8 & 0 & -8 & -10 & 0 \\ -56 & 0 & 40 & 40 & 0 \end{pmatrix} \begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \\ a'_3 \\ a'_4 \end{pmatrix}, \quad (10)$$

$$\begin{pmatrix} J_0 \\ L_0 \\ K_{\rm sym,0} \\ Q_{\rm sym,0} \\ Z_{\rm sym,0} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ -2 & 0 & 4 & 10 & 18 \\ 8 & 0 & -8 & -10 & 0 \\ -56 & 0 & 40 & 40 & 0 \end{pmatrix} \begin{pmatrix} b'_0 \\ b'_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{pmatrix}. \quad (11)$$

$$\begin{aligned}
a'_0 &= \frac{1}{24}(360\varepsilon_0 + 20K_0 + Z_0), \\
a'_1 &= \frac{1}{24}(-960\varepsilon_0 - 56K_0 - 4Q_0 - 4Z_0), \\
a'_2 &= \frac{1}{24}(1080\varepsilon_0 + 60K_0 + 12Q_0 + 6Z_0), \\
a'_3 &= \frac{1}{24}(-576\varepsilon_0 - 32K_0 - 12Q_0 - 4Z_0), \\
a'_4 &= \frac{1}{24}(120\varepsilon_0 + 8K_0 + 4Q_0 + Z_0),
\end{aligned}$$

$$\begin{aligned}
b'_0 &= \frac{1}{24}(360J_0 - 120L_0 + 20K_{\rm sym,0} + Z_{\rm sym,0}), \\
b'_1 &= \frac{1}{24}(-960J_0 + 328L_0 - 56K_{\rm sym,0} - 4Q_{\rm sym,0} \\
&\quad -4Z_{\rm sym,0}), \\
b'_2 &= \frac{1}{24}(1080J_0 - 360L_0 + 60K_{\rm sym,0} + 12Q_{\rm sym,0} \\
&\quad +6Z_{\rm sym,0}), \\
b'_3 &= \frac{1}{24}(-576J_0 + 192L_0 - 32K_{\rm sym,0} - 12Q_{\rm sym,0} \\
&\quad -4Z_{\rm sym,0}), \\
b'_4 &= \frac{1}{24}(120J_0 - 40L_0 + 8K_{\rm sym,0} + 4Q_{\rm sym,0} \\
&\quad +Z_{\rm sym,0}). \quad (1
\end{aligned}$$

Speed of Sound EOS

- The velocity of sound for $\rho > \rho_{c_s}(1.5 - 2\rho_0)$ is given as,

$$\frac{c_s^2}{c^2} = \frac{1}{3} - c_1 exp\left[-\frac{(\rho - c_2)^2}{n_b^2}\right] + h_p exp\left[-\frac{(\rho - n_p)^2}{w_p^2}\right] \left[1 + erf(s_p \frac{\rho - n_p}{w_p})\right].$$

- If energy density ($\epsilon(\rho_{c_s})$), the pressure ($\mathrm{P}(\rho_{c_s})$) and the derivative of energy density ($\epsilon'(\rho_{c_s})$) are known

$$\begin{aligned} \rho_{i+1} &= \rho_i + \Delta\rho, \\ \epsilon_{i+1} &= \epsilon_i + \Delta\epsilon \\ &= \epsilon_i + \Delta\rho\frac{\epsilon_i + P_i}{\rho_i}, \\ P_{i+1} &= P_i + c_s^2(\rho_i)\Delta\epsilon. \end{aligned}$$

Ref. Tews et al. APJ. 149 (2018)