Bayesian and frequentist model comparison for nuclear symmetry energy

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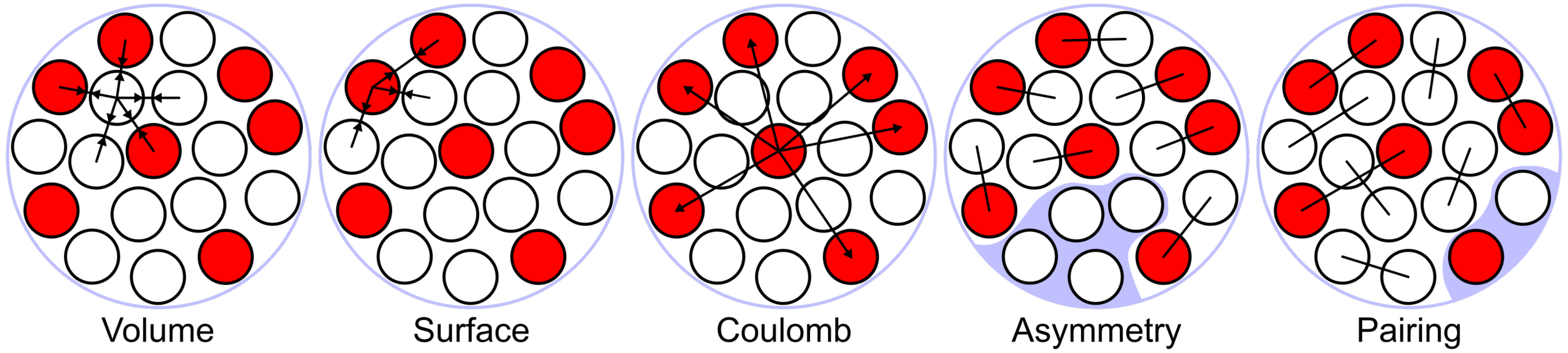
Outline

- ✦ Introduction
- ✦ Bayesian model averaging for $E_{\rm sym}(2\rho_0/3)$ from nuclear masses
  *M. Qiu, B.-J. Cai, L.-W. Chen, C.X. Yuan and **ZZ,** PLB 849, 138435 (2024)*
- ✦ Frequentist model comparison based on covariance analysis
  *J. Zeng, M. Qiu and **ZZ**, in preparation*
- ✦ Summary

Nuclear equation of state and the symmetry energy

- Semi-empirical mass formula (binding energy of a nucleus):

$$E_B = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} \pm \delta(N,Z)\,.$$



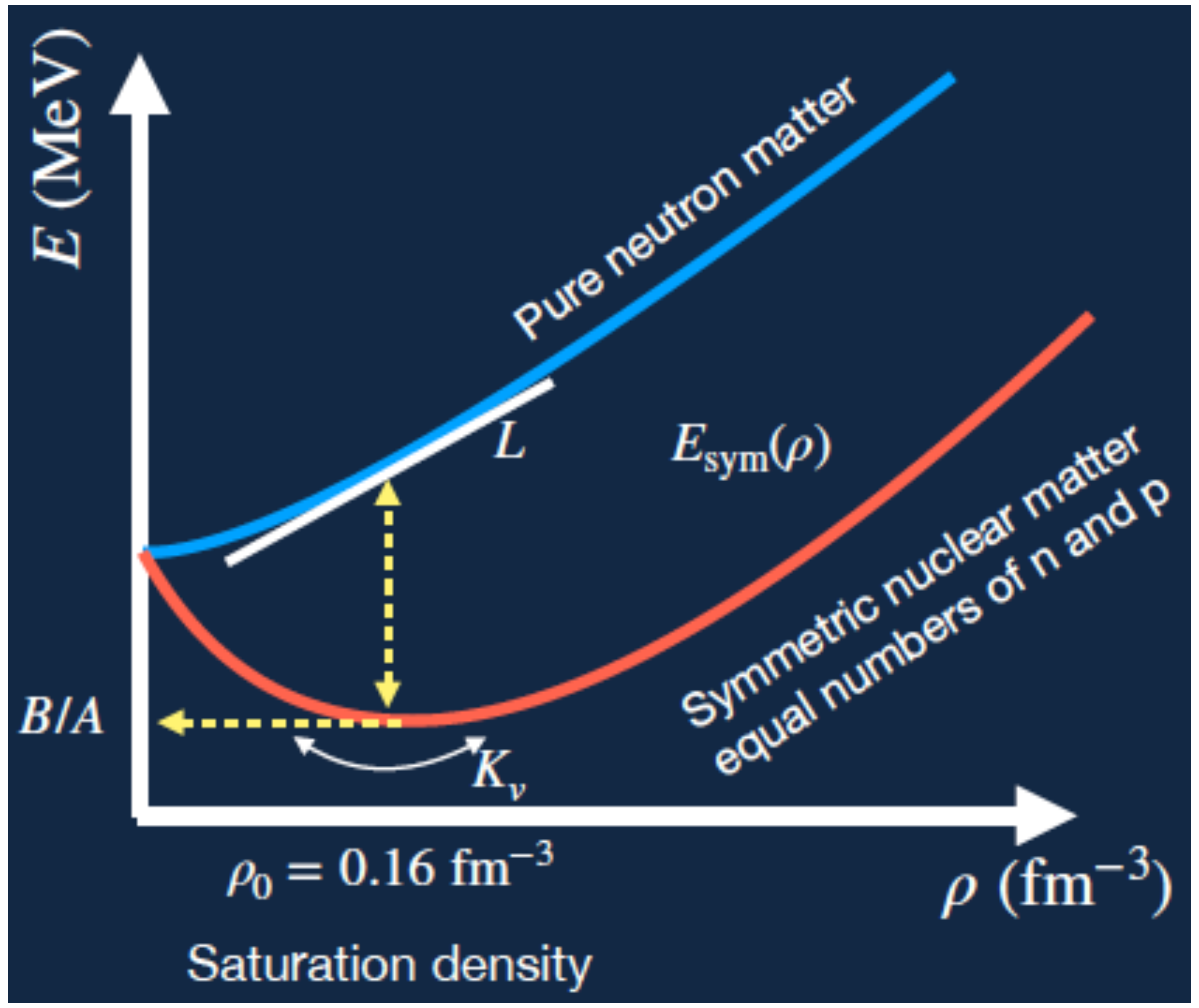
- Nuclear equation of state (E_B/A in uniform nuclear matter):

$$E = E_0(\rho) + E_{sym}(\rho)\delta^2 + O(\delta^4), \quad \delta = (\rho_n - \rho_p)/\rho$$

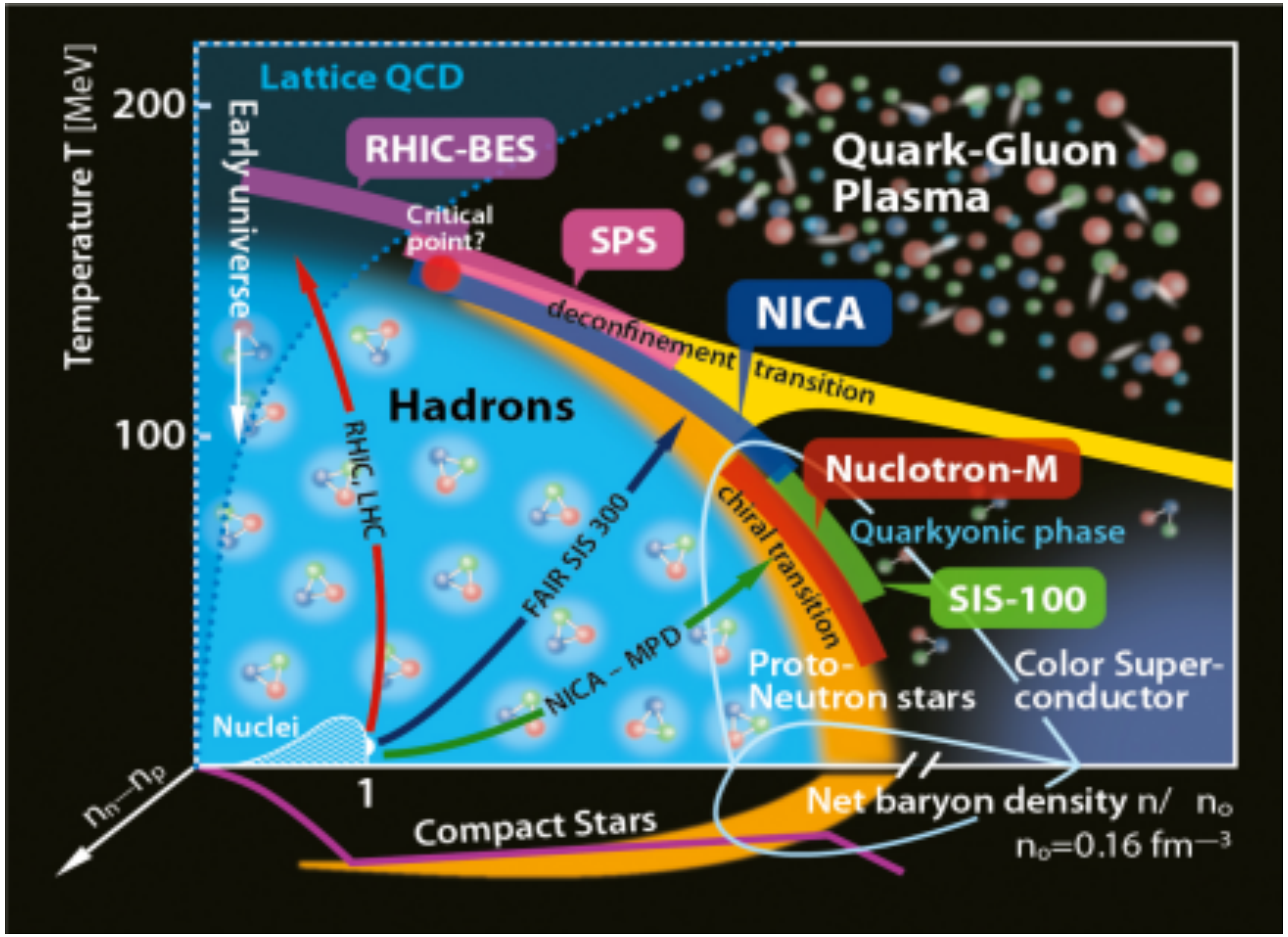
- Symmetry energy:

$$E_{sym}(\rho) \equiv \frac{1}{2}\frac{\partial^2 E}{\partial \delta^2}\bigg|_{\delta=0} = E_{sym}(\rho_0) + L(\rho_0)\chi + \frac{1}{2}K_{sym}\chi^2 + \mathcal{O}(\chi^3)$$

characterizes the isospin dependence of nuclear equation of state.

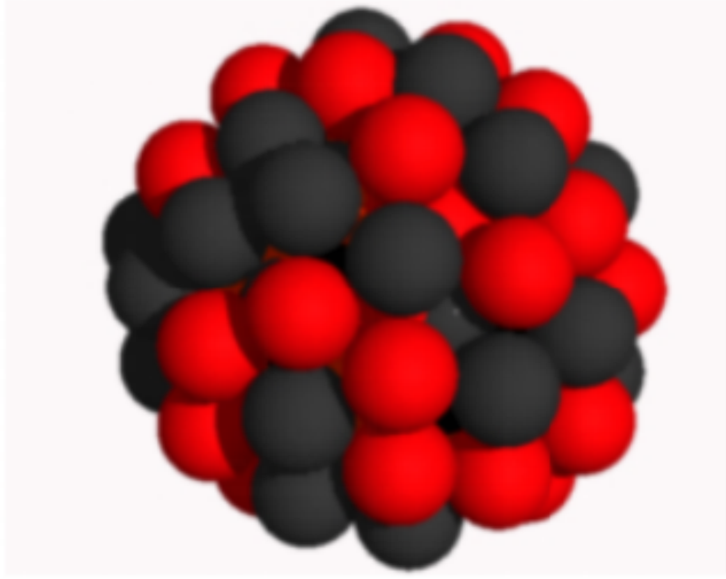


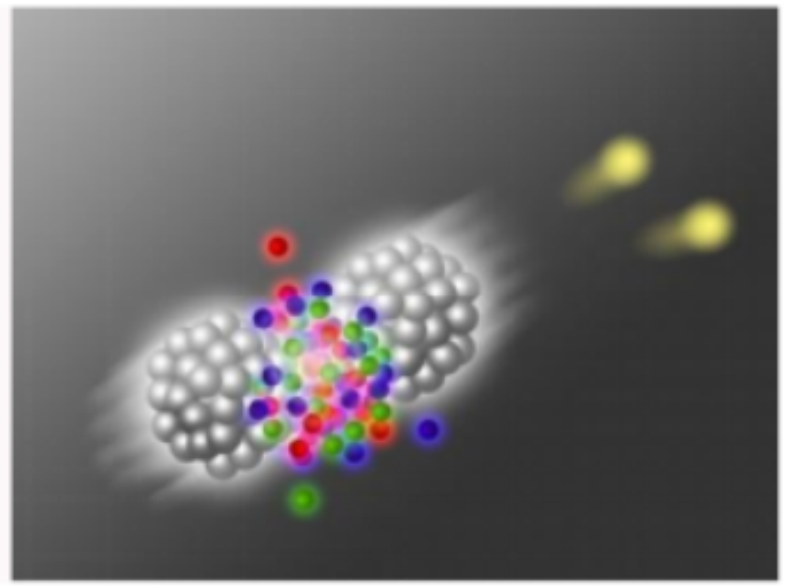
Multifaceted influence of the symmetry energy

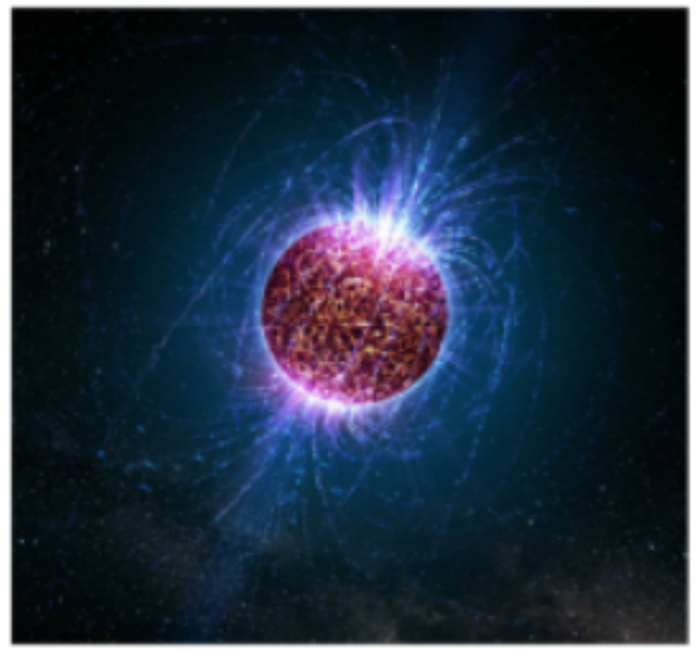


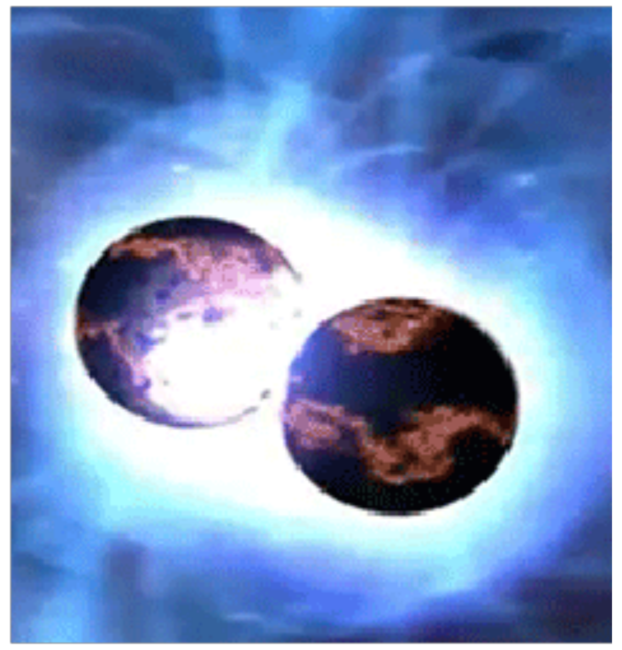
✦Symmetry energy bridges the physics from microscopic nuclei (fm scale) to macroscopic neutron stars (km scale)

$E_{\rm sym}(\rho)$ is important for



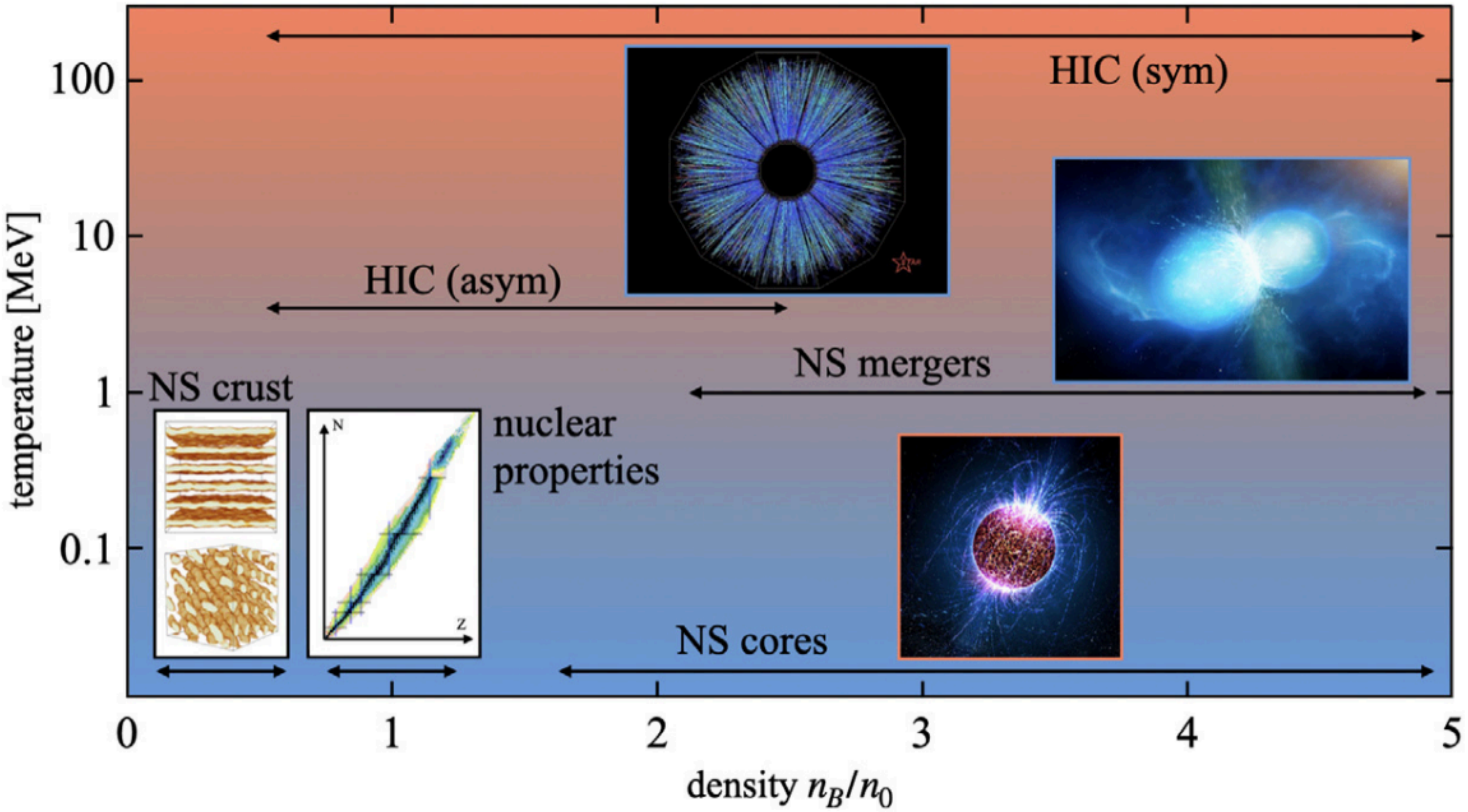




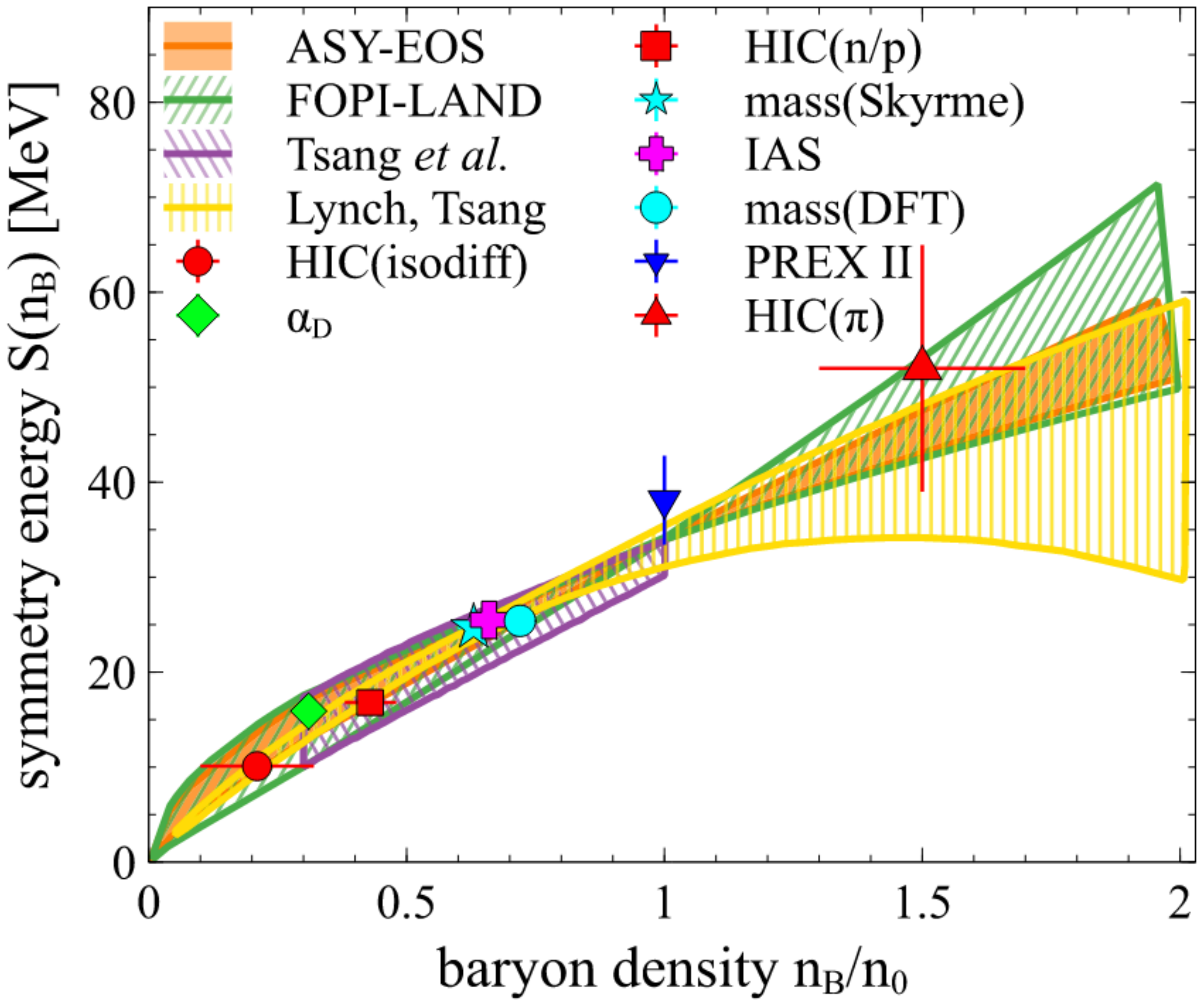


A.W. Steiner, M. Prakash, J.M. Lattimer, P.J. Ellis, Phys. Rep. 411, 325 (2005).
J.M. Lattimer, M. Prakash, Phys. Rep. 442, 109 (2007).
B.-A. Li, L.-W. Chen, C.M. Ko, Phys. Rep. 464, 113 (2008).
M. Oertel, M. Hempel, T. Klähn, S. Typel, Rev. Mod. Phys. 89, 015007 (2017).
X. Roca-Maza, N. Paar, Prog. Part. Nucl. Phys. 101, 96 (2018).
B.-A. Li, B.-J. Cai, W.-J. Xie, N.-B. Zhang, Universe 7, 182 (2021).

How to constrain the symmetry energy?



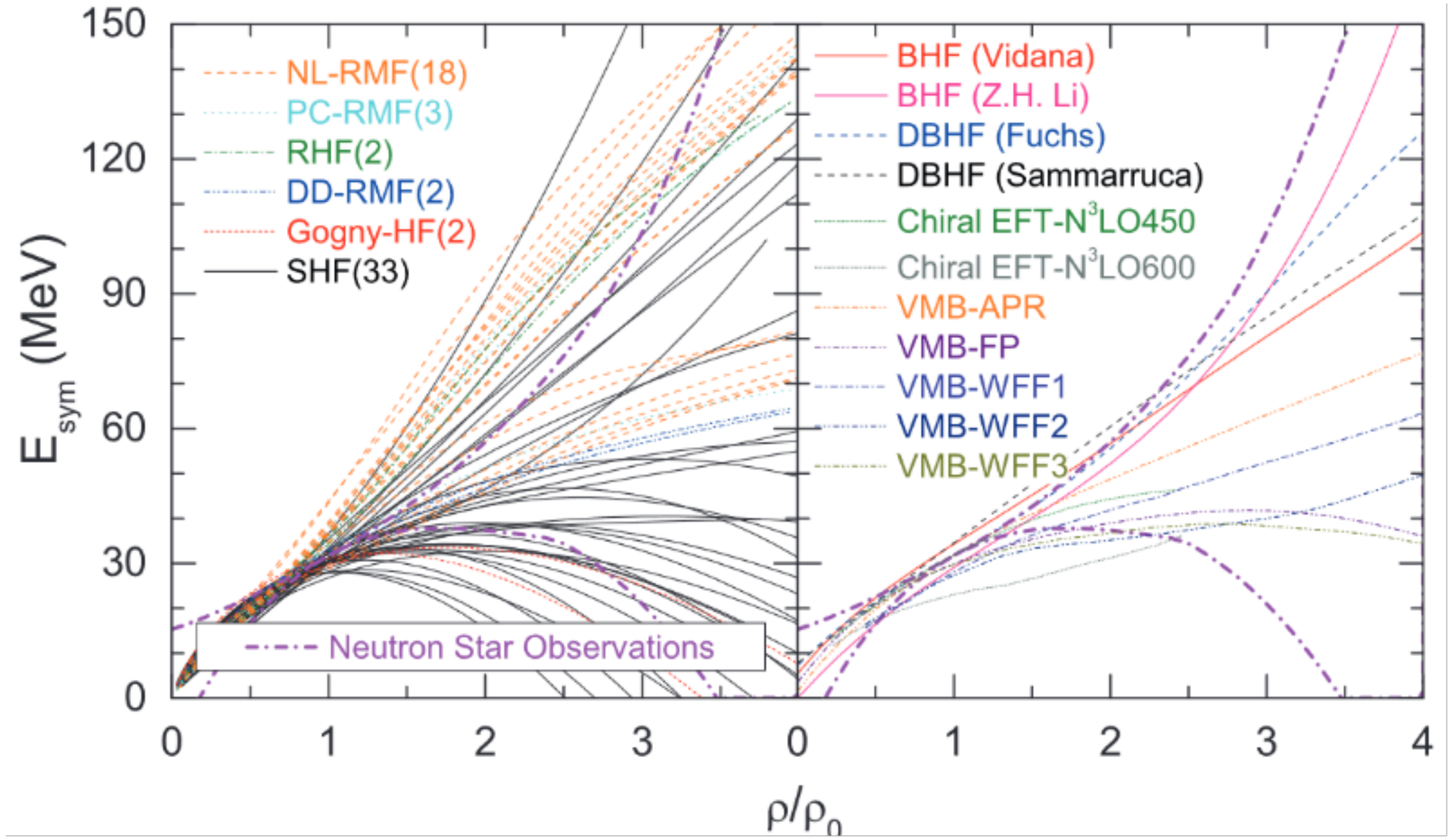
A. Sorensen et al. PPNP 134 (2024) 104080



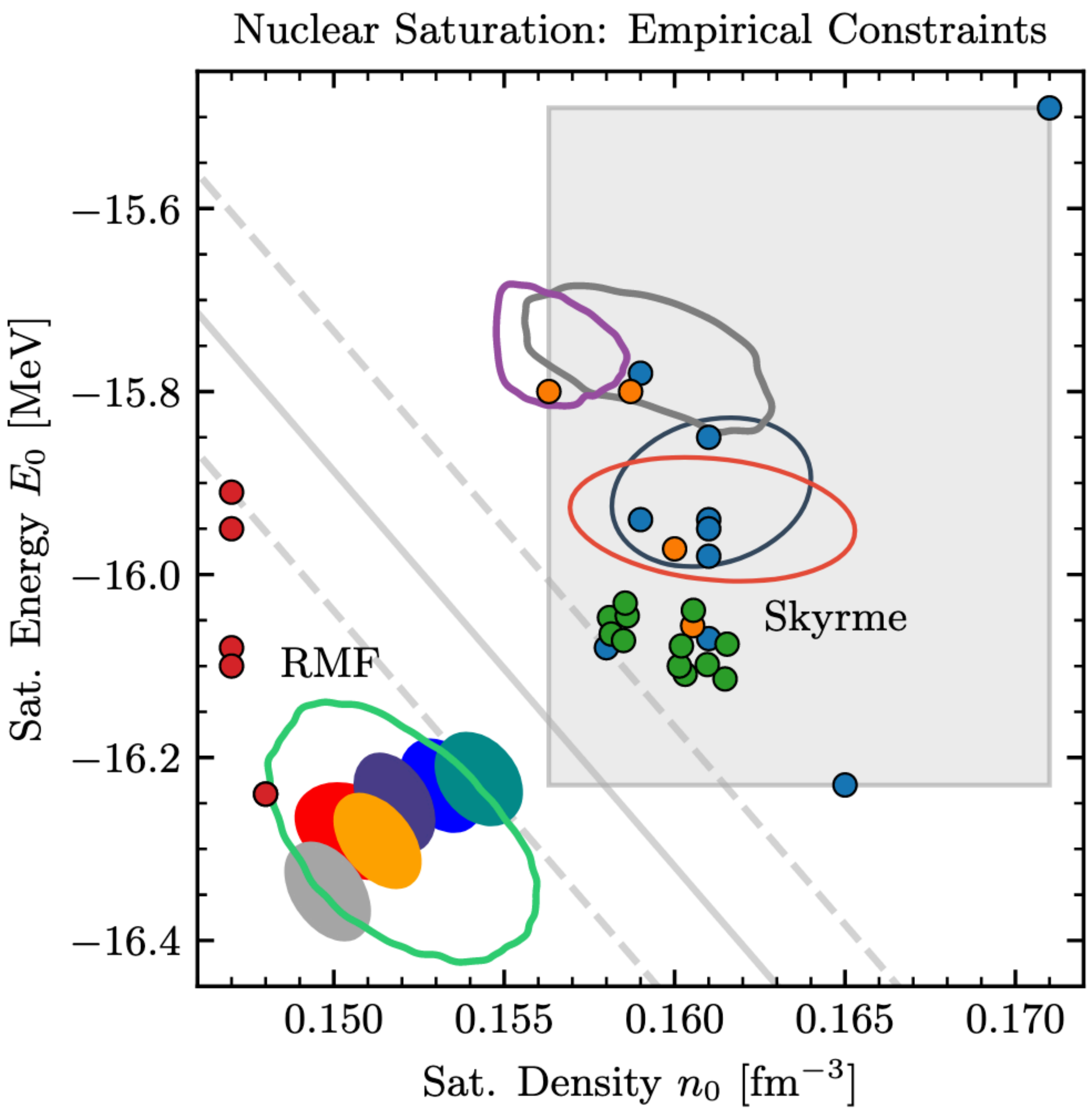
✦ The symmetry energy can be indirectly extracted from various probes!

- Uncertainty propagation/quantification
- Model dependence

Inter-model uncertainties of nuclear EOS



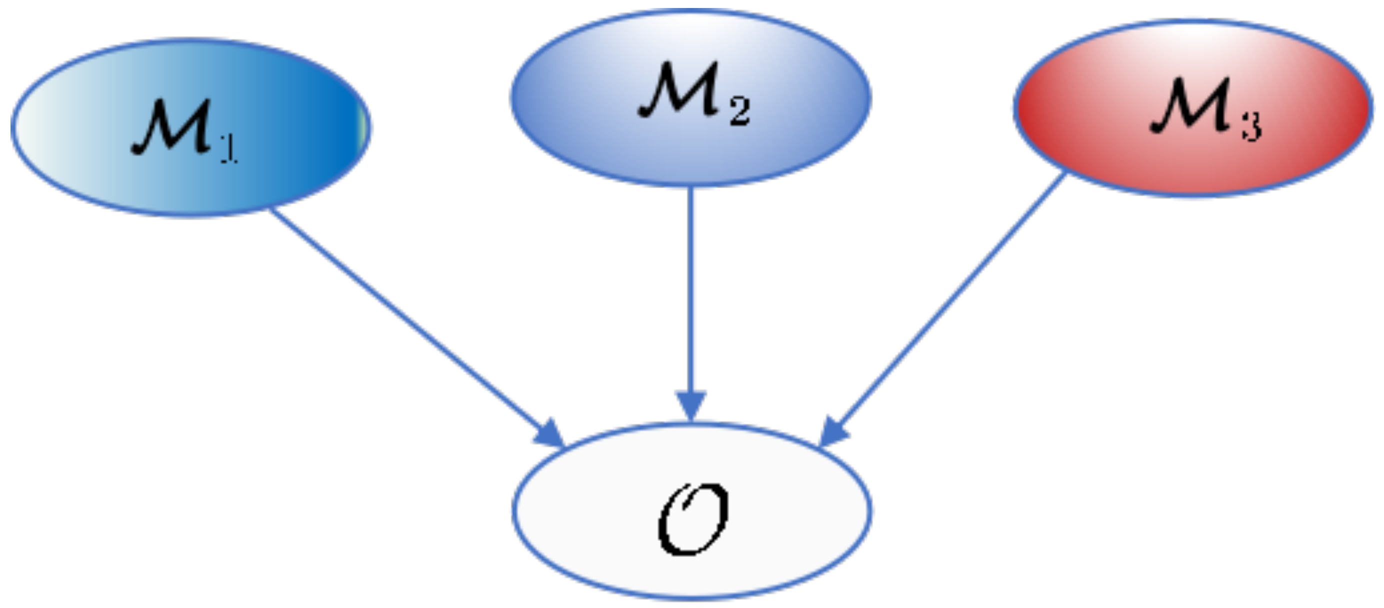
L.W.Chen, Nucl.Phys.Rev.34,20 (2017).
N.B.Zhang, B.A.Li, Eur.Phys.J.A 55,39(2019)



C. Drischler et al., arXiv:2405.02748

✦ How to quantify the inter-model uncertainties?

Model comparison



Which model should we trust?

Determine a weighting factor ω_i for each model.

- **Model selection**: identify the best model(s) with the largest ω_i
- **Model averaging**: averaging model predictions by

$$\mathcal{O}_{MA} = \frac{\sum_i \mathcal{O}_i \omega_i}{\sum_i \omega_i}$$

Model Selection and Multimodel Inference: A Practical Information Theoretic Approach
by Burnham & Anderson

How to determine ω_i?

✦The virtue of a phenomenological model lies in its simplicity and ability to reproduce data.

Bayesian model averaging for $E_{\rm sym}(2\rho_0/3)$ from nuclear masses

*M. Qiu, B.-J. Cai, L.-W. Chen, C.X.Yuan and **ZZ,** PLB 849, 138435 (2024)*

Bayesian inference

Under model $\mathcal{M}$' s assumption

The likelihood function
of observing y given the model $\mathcal{M}$ predictions at Θ
(Usually Gaussian)

The prior probability
of quantities of interest Θ before being confronted with the experimental measurements y
(Usually uniform)

The posterior probability
distribution of quantities of interest Θ given experimental measurements y

$$p(\mathcal{O}|\ \mathbf{y},\mathcal{M}) = \frac{\mathcal{L}(\mathbf{y}|\ \mathcal{O},\mathcal{M})\pi(\mathcal{O}|\ \mathcal{M})}{p(\mathbf{y}|\ \mathcal{M})}$$

The marginal likelihood/Evidence
The probability of model $\mathcal{M}$ giving experimental measurements y

✓infra-model uncertainties

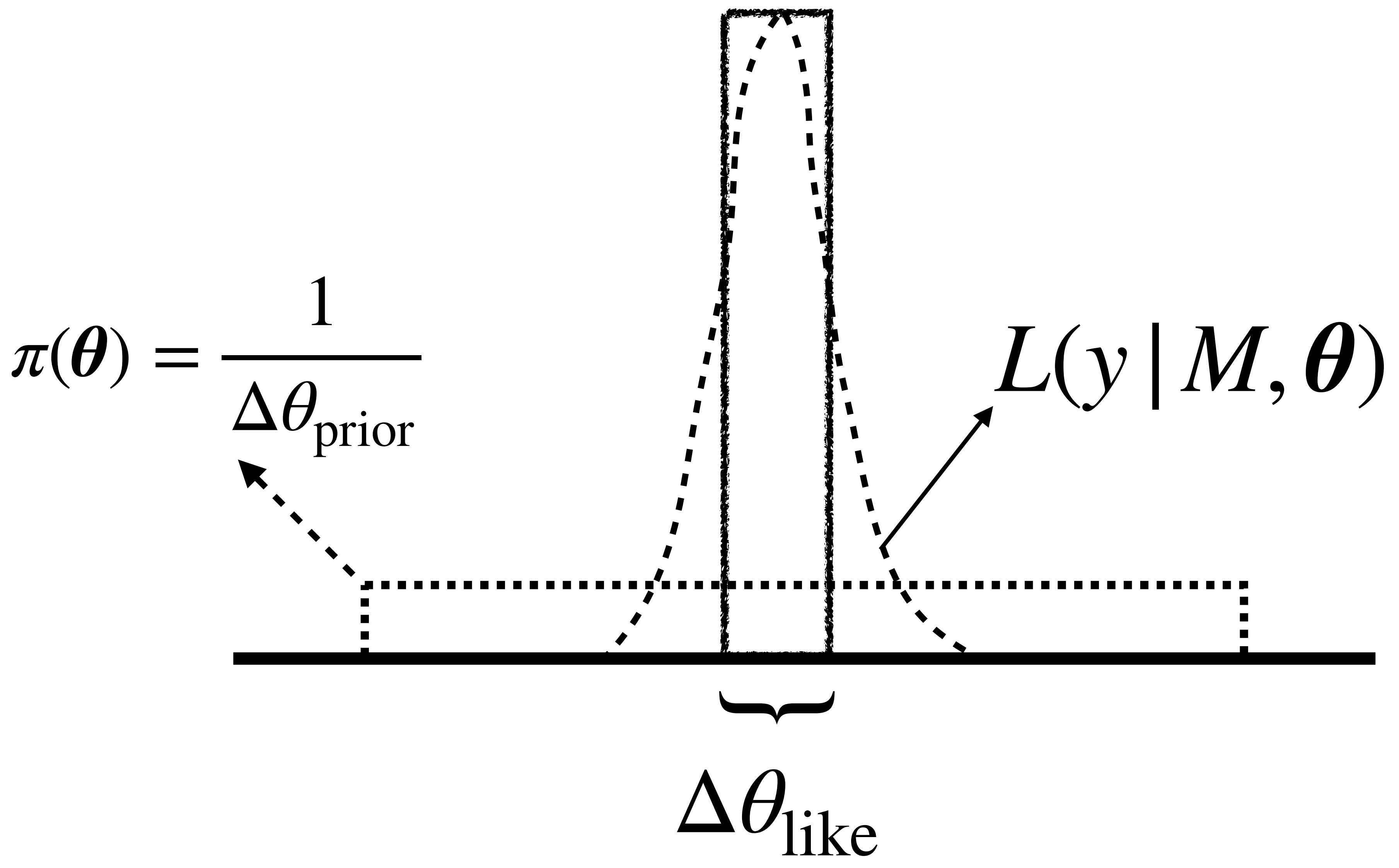
✦ **Evidence** (a larger evidence indicates a better model)

$$Z \equiv p(\boldsymbol{y} \mid M) = \int d\boldsymbol{\theta} L(\boldsymbol{y} \mid M, \theta)\pi(\boldsymbol{\theta})$$

$$\approx L\left(\boldsymbol{y} \mid M, \theta_{\mathrm{ML}}\right)\left(\Delta\theta_{\mathrm{like}}/\Delta\theta_{\mathrm{prior}}\right)^{N_\theta}$$

Maximum likelihood

No. of parameter, model complexity



Udo von Toussaint, Rev. Mod. Phys. 83, 943 (2011)

A variant of Bayesian evidence

- Limitation of classical evidence:

$$Z_i \equiv \int L\left(\boldsymbol{y} \mid \boldsymbol{\theta}, M_i\right) \pi(\boldsymbol{\theta} \mid M_i) d\boldsymbol{\theta}$$

  - Model calibration and comparison are based on the same data set.
  - All data are equally weighted.

- Pseudo evidence (posterior predictive distribution):

$$Z_i^{\mathrm{pseu}} \equiv \int L\left(\boldsymbol{y}_{\mathrm{ev}} \mid \boldsymbol{\theta}, M_i\right) p\left(\boldsymbol{\theta} \mid \boldsymbol{y}, M_i\right) d\boldsymbol{\theta}$$

  - Model calibration and evaluation use different data sets.
  - Select $\boldsymbol{y}_{\mathrm{ev}}$ that are closely related to the target observables.
  - Posterior from calibration → Prior for evaluation/prediction.

Neufcourt et al., PRL 122, 062502 (2019)
Neufcourt et al., PRC 101, 014319 (2020)
Kejzlar et al., JPG 47, 094001 (2020)
Cirigliano et al., JPG 49, 120502 (2022)

Bayesian model averaging

- Model prior $\pi(M_i)$: our preference on M_i before seeing the data.

- Model likelihood: evidence $Z_i = p(\mathbf{y}\,|\,M_i)$

- Model posterior:

$$p(M_i\,|\,\mathbf{y}) = \frac{p(\mathbf{y}\,|\,M_i)\pi(M_i)}{\sum_j p(\mathbf{y}\,|\,M_j)\pi(M_j)}$$

- Model averaging for observable $\mathcal{O}$:

$$p(\mathcal{O}\,|\,\mathbf{y}) = \sum_i p(\mathcal{O}\,|\,\mathbf{y}, M_i)\,p(M_i\,|\,\mathbf{y})$$

PHYSICAL REVIEW LETTERS **122,** 062502 (2019)

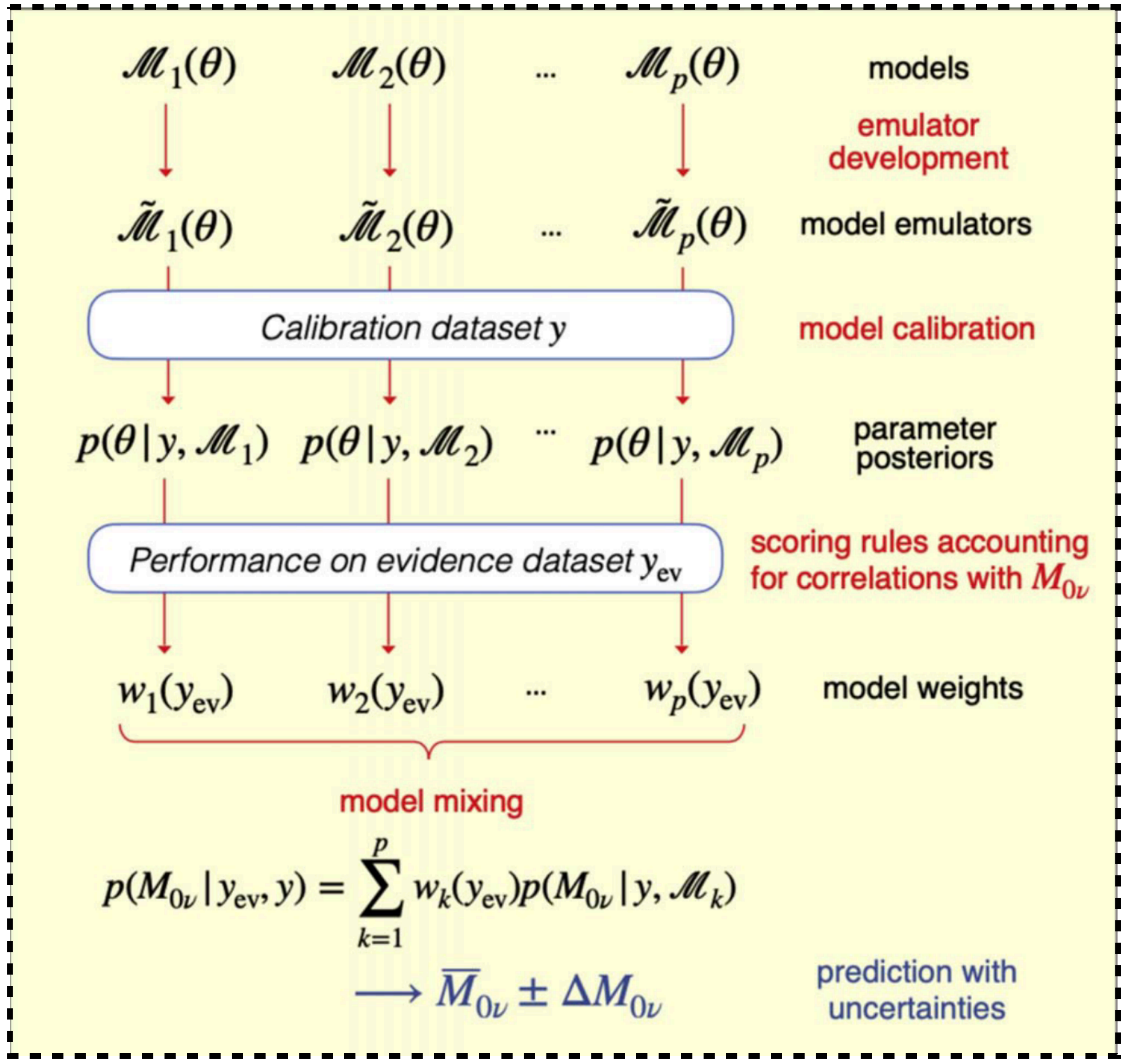
Neutron Drip Line in the Ca Region from Bayesian Model Averaging

Léo Neufcourt,[1,2] Yuchen Cao (曹宇晨),[3] Witold Nazarewicz,[4] Erik Olsen,[2] and Frederi Viens[1]

PHYSICAL REVIEW LETTERS **126,** 242301 (2021)

Editors' Suggestion

Phenomenological Constraints on the Transport Properties of QCD Matter with Data-Driven Model Averaging



V. Cirigliano et al, J. Phys. G 49, 120502 (2022)

Bayesian model selection

✦ Bayes' factor :

$$B_{12} = \frac{Z_1}{Z_2}$$

Jeffreys' Scale (rule of thumb)

- $B_{12} < 1$: support for M_2.
- $1 < B_{12} < 3$: barely worth mentioning.
- $3 < B_{12} < 10$: moderate evidence for M_1.
- $10 < B_{12} < 100$: strong evidence.
- > 100: decisive evidence.

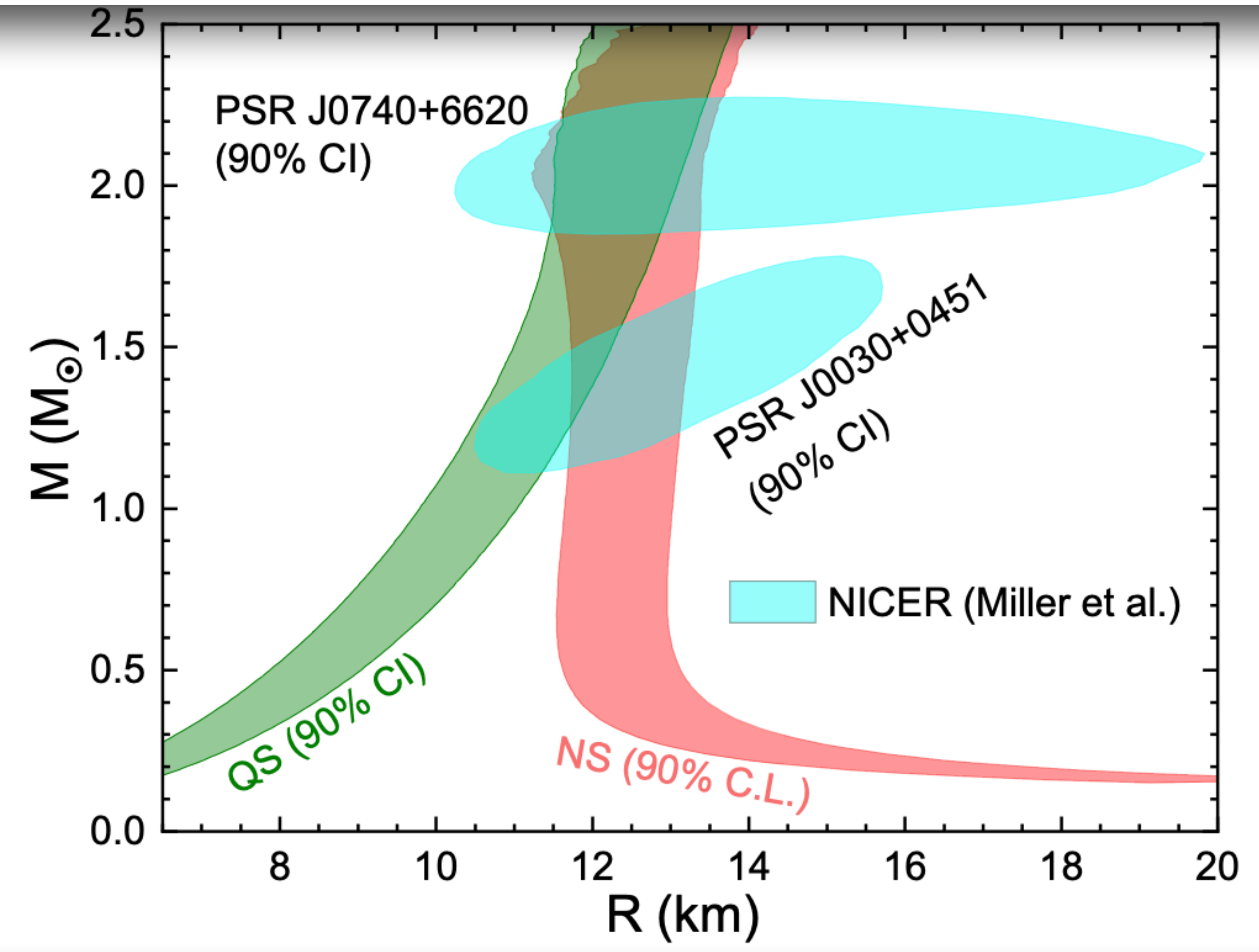
arXiv > astro-ph > arXiv:2308.16783

Astrophysics > High Energy Astrophysical Phenomena

[Submitted on 31 Aug 2023]

Neutron Star vs Quark Star in the Multimessenger Era

Zheng Cao, Lie-Wen Chen



arXiv > nucl-th > arXiv:2507.11394v1

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Nuclear Theory

[Submitted on 15 Jul 2025]

Bayesian Model Selection and Uncertainty Propagation for Beam Energy Scan Heavy-Ion Collisions

Syed Afrid Jahan, Hendrik Roch, Chun Shen

Effective p-n chemical potential difference $\Delta\mu^*_{\rm pn}$

- Effective chemical potential

$$\mu_{\rm n} = \frac{\partial B(N,Z)}{\partial N} \approx \frac{B(N+2,Z) - B(N-2,Z)}{4}$$
$$\mu_{\rm p} = \frac{\partial B(N,Z)}{\partial Z} \approx \frac{B(N,Z+2) - B(N,Z-2)}{4}$$

- Proton-neutron chemical potential differences

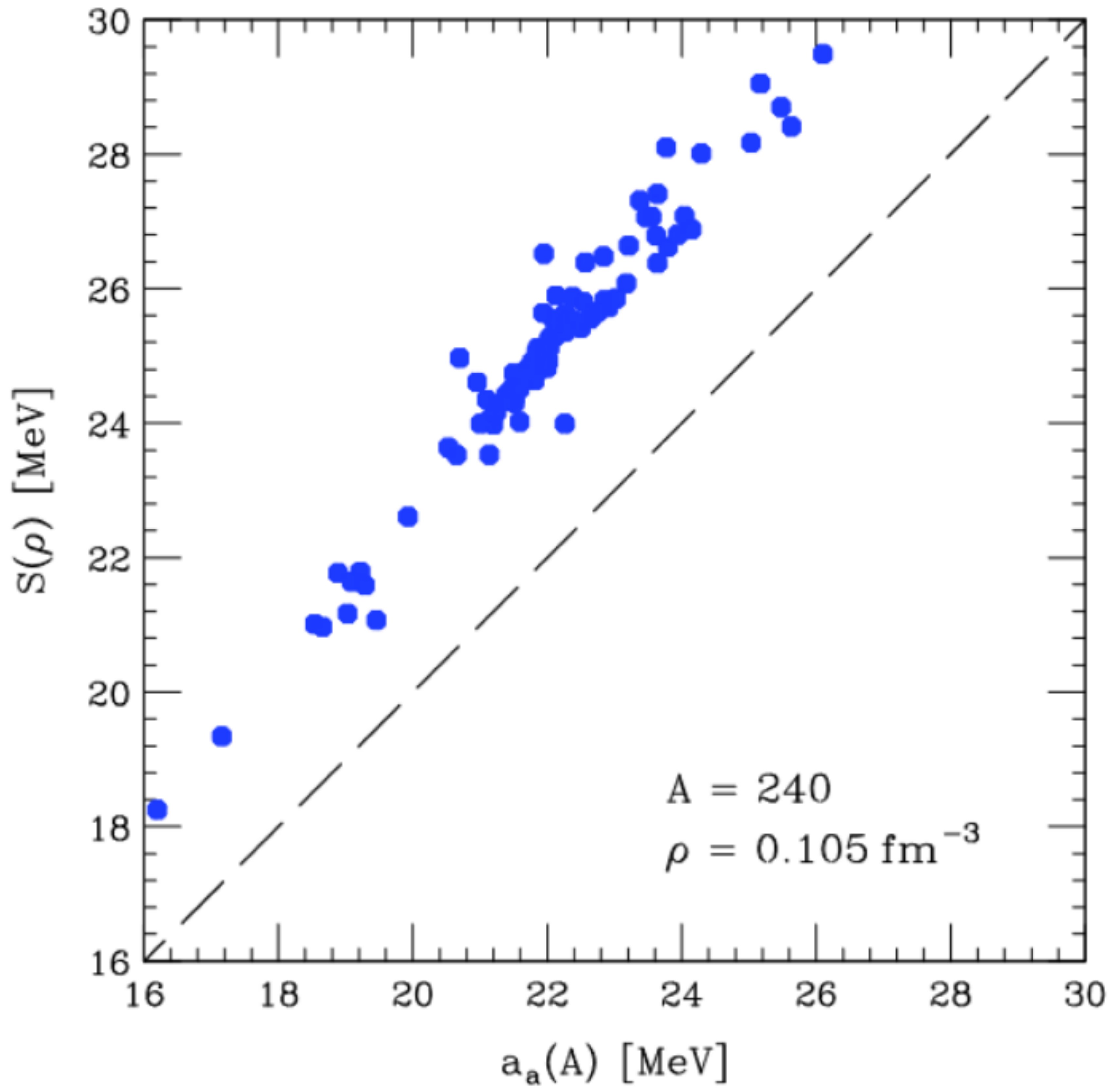
$$\Delta\mu^*_{\rm pn} = \frac{1}{4}[B(N,Z+2) - B(N,Z-2) - B(N+2,Z) + B(N-2,Z)]$$

- Semi empirical mass formul

$$B(N,Z) = a_{\rm v}A - a_{\rm s}A^{2/3} - a_{\rm c}\frac{Z^2}{A^{1/3}} - a_{\rm sym}I^2A + E_{\rm mic}$$

- Expected sensitivity

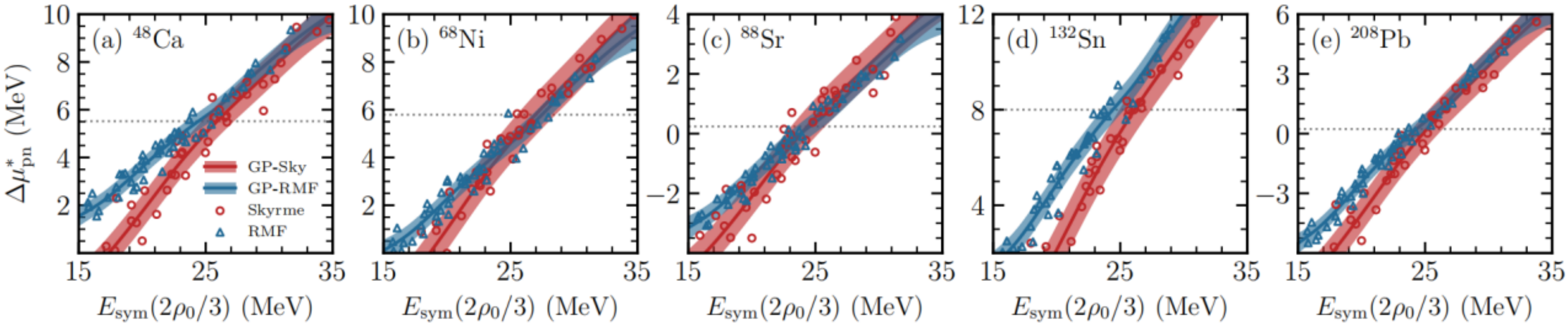
$$\Delta\mu^*_{\rm pn} \simeq a_{\rm c}\left[\frac{1-Z}{(A-2)^{1/3}} - \frac{1+Z}{(A+2)^{1/3}}\right] + a_{\rm sym}\frac{4A^2I}{A^2-4} \simeq -2a_{\rm c}\frac{Z}{A^{1/3}} + 4a_{\rm sym}I$$
$$\Delta\mu^*_{\rm pu} \propto a_{\rm symm} \approx E_{\rm sym}\left(\rho_r\right)$$



Pawel Danielewicz, Jenny Lee, Nuclear Physics A 922 (2014)
M. Centelles, Phys. Rev. Lett. 102, 122502 (2009)
L.-W. Chen, Phys. Rev. C 83, 044308 (2011)
N. Wang, L. Ou, and M. Liu, Phys. Rev. C 87, 034327(2013)

$E_{\mathrm{sym}}(2\rho_0/3)$ and $\Delta\mu^*_{\mathrm{pn}}$

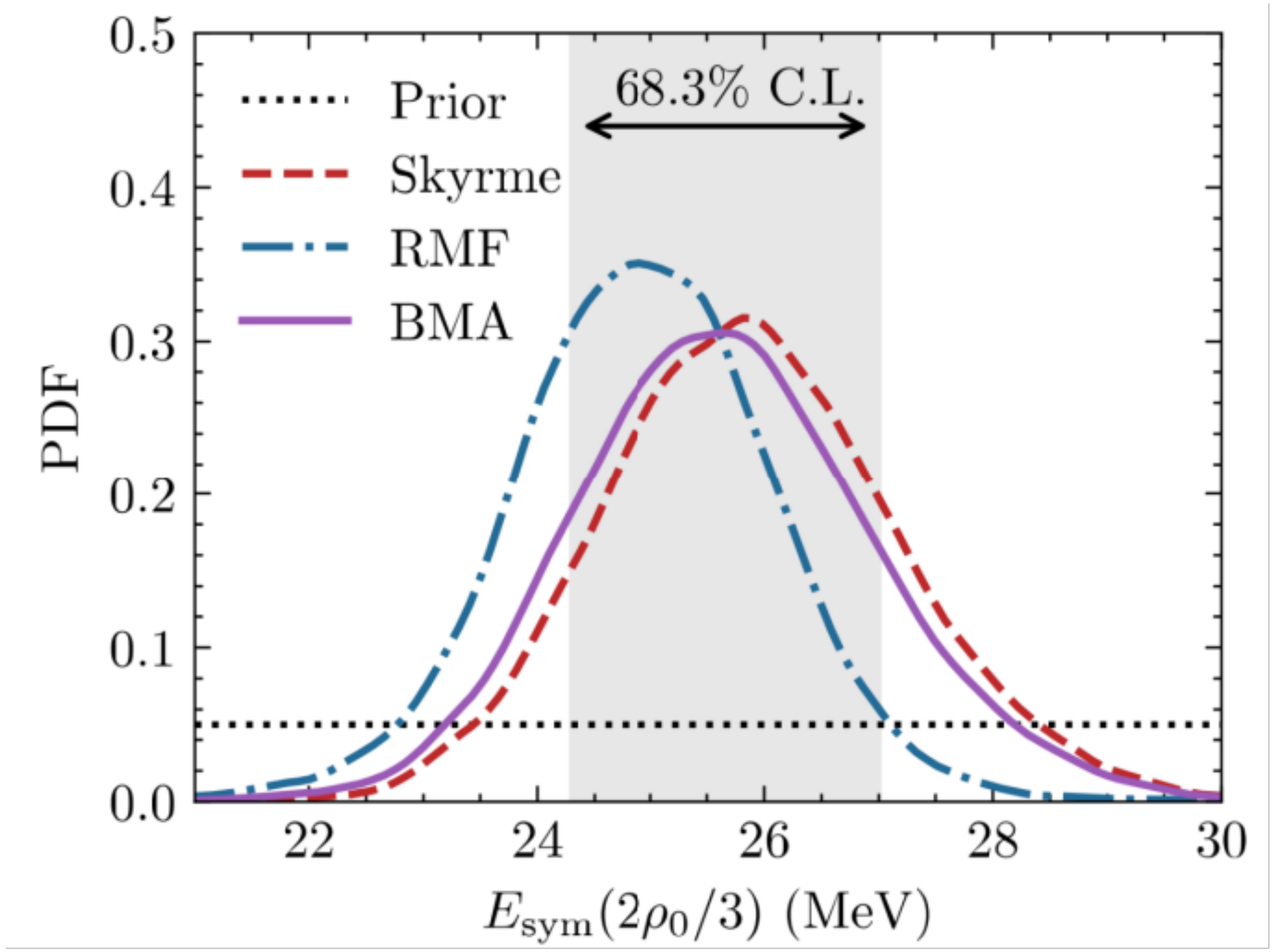
$$\Delta\mu^*_{\mathrm{pn}} = \frac{1}{4}[B(N,Z+2) - B(N,Z-2) - B(N+2,Z) + B(N-2,Z)]$$



*M. Qiu, B.-J. Cai, L.-W. Chen, C.X.Yuan and **ZZ,** PLB 849, 138435 (2024)*

- Strong $\Delta\mu^*_{\mathrm{pn}}$-$E_{\mathrm{sym}}(2\rho_0/3)$ correlations predicted by nuclear energy density functionals.
- Gaussian processes based on predictions of 50 Skyrme EDFs and 50 covariant EDFs.
- Discrepancy between Skyrme EDF and nonlinear RMF model. Model dependence!

Bayesian inference of $E_{\mathrm{sym}}(2\rho_0/3)$



*M. Qiu, B.-J. Cai, L.-W. Chen, C.X.Yuan and **ZZ,** PLB 849, 138435 (2024)*

✦ Posterior：

Skyrme EDFs $E_{\mathrm{sym}}(2/3\rho_0) = 25.8^{+1.3}_{-1.2}$ MeV

$\sigma = 0.4^{+0.4}_{-0.2}$ MeV

Nonlinear RMF $E_{\mathrm{sym}}(2/3\rho_0) = 24.9 \pm 1.1$ MeV

$\sigma = 0.8^{+0.6}_{-0.3}$ MeV

✦ Bayesian model averaging

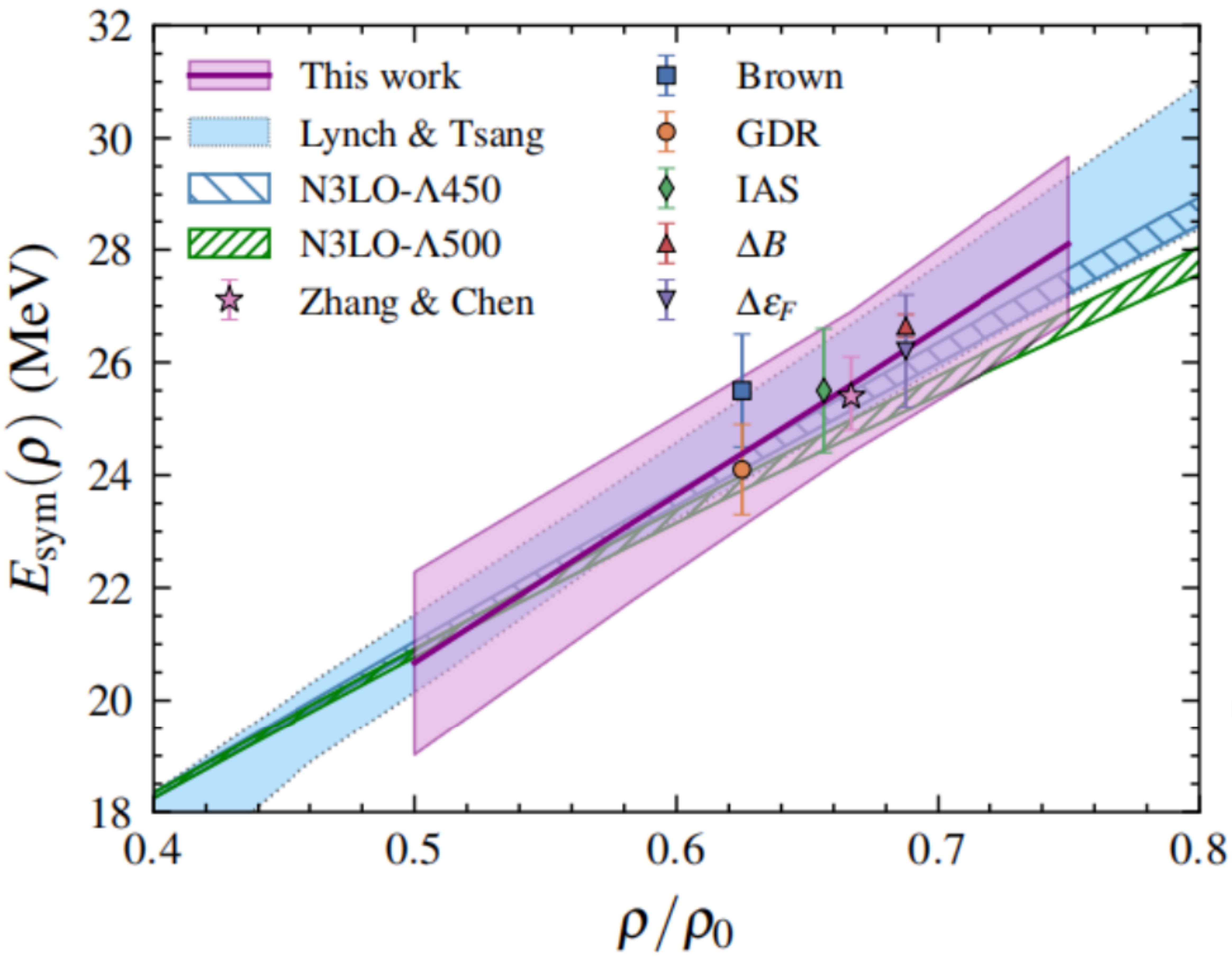
$$p(\mathcal{M}_i|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\mathcal{M}_i)\pi(\mathcal{M}_i)}{\sum_\ell p(\boldsymbol{y}|\mathcal{M}_\ell)\pi(\mathcal{M}_\ell)}$$

BMA = 3.3/4.3 * Skyrme

+ 1/4.3 * RMF

= $25.6^{+1.4}_{-1.3}$ MeV

Symmetry energy around $2\rho_0/3$



- Brown: Doubly magic nuclei
  B.A.Brown, Phys. Rev. Lett. 111, 232502 (2013)
- Lynch & Tsang: various terrestrial and astrophysical constraints
  W.G.Lynch and M.B.Tsang, Phys. Lett. B. 830, 137098 (2022)
- Zhang & Chen: Doubly magic nuclei Fermi energy difference + PREX+ CREX
  Z. Zhang and L.-W. Chen, Phys. Rev. C. 108, 024317 (2023)
- GDR: Giant dipole resonance
  L. Trippa *et al*, Phys. Rev. C 77, 061304 (2008)
- IAS: Isobaric analog states
  Pawel Danielewicz, Jenny Lee, Nuclear Physics A 922 (2014)
- ΔB : Isotope binding energy difference
  Z. Zhang and L.-W. Chen, Phys. Lett. B 726, 234 (2013)
- $\Delta\varepsilon_F$: Fermi energy difference
  N. Wang, L. Ou, and M. Liu, Phys. Rev. C 87, 034327(2013)

*M. Qiu, B.-J. Cai, L.-W. Chen, C.X.Yuan and **ZZ,** PLB 849, 138435 (2024)*

Frequentist model comparison based on covariance analysis

*J. Zeng, M. Qiu and **ZZ**, in preparation*

Frequentist covariance analysis (CA)

- ✦ Assume $\dfrac{\mathcal{O}^{\text{th}}(\boldsymbol{\theta}) - \mathcal{O}^{\text{exp}}}{\Delta\mathcal{O}} \sim \mathcal{N}(0,1)$
- ✦ Likelihood $\mathcal{L} = \dfrac{1}{(2\pi)^{N/2}\prod_i^N \Delta\mathcal{O}_i}\exp(-\dfrac{\chi^2(\boldsymbol{\theta})}{2})$, with $\chi^2 \equiv \sum_i \dfrac{(\mathcal{O}_i^{\text{th}}(\boldsymbol{\theta}) - \mathcal{O}_i^{\text{exp}})^2}{\Delta\mathcal{O}_i^2}$
- ✦ Optimize parameters $\boldsymbol{\theta}$ by minimizing χ^2 or maximizing $\mathcal{L}$
  - Equivalent to Bayesian analysis using uniform prior and Gaussian likelihood

J. D. McDonnell et al., Phys. Rev. Lett. 114, 122501 (2015)

- ✦ Around the optimal $\boldsymbol{\theta}_0$:

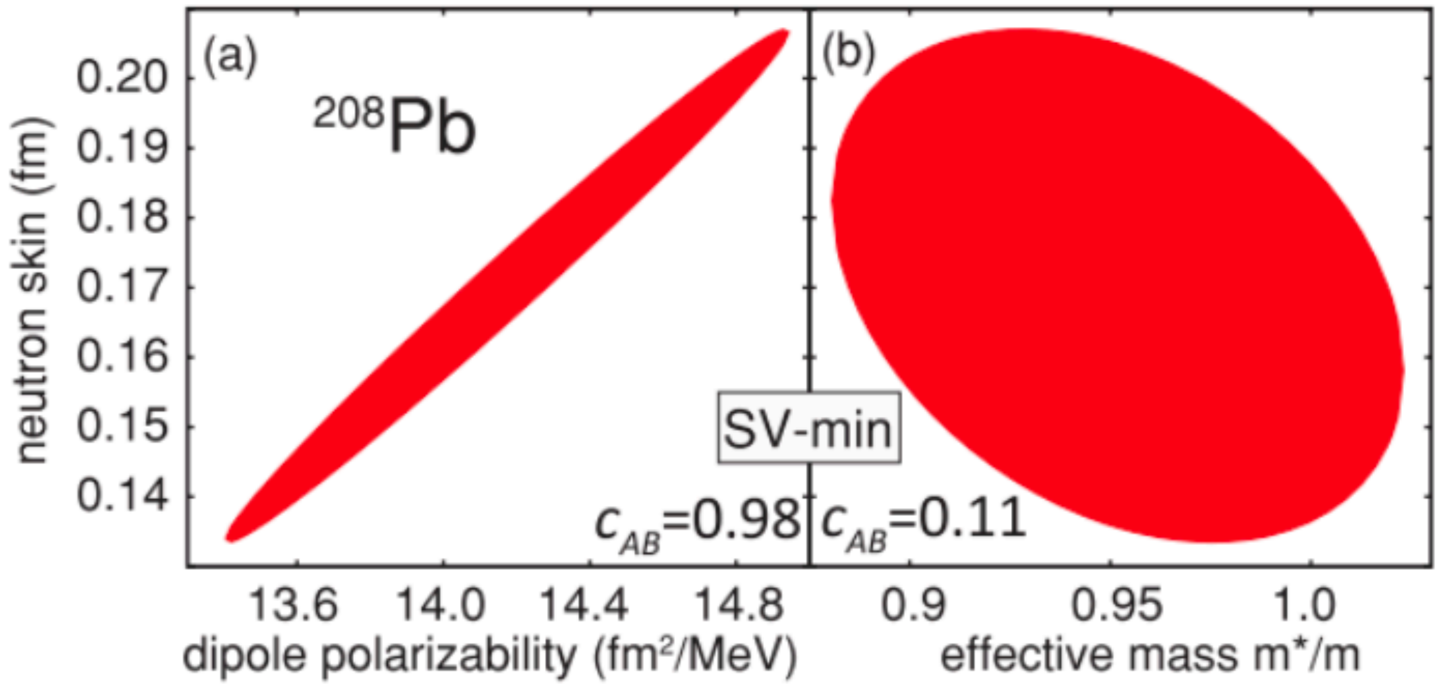
$$\chi^2(\boldsymbol{\theta}) \approx \chi^2(\boldsymbol{\theta}_0) + (\boldsymbol{\theta} - \boldsymbol{\theta}_0)^T \boldsymbol{M}(\boldsymbol{\theta} - \boldsymbol{\theta}_0), \text{ with } \boldsymbol{M} = \frac{1}{2}\frac{\partial^2\chi^2}{\partial\theta_i\partial\theta_j}\bigg|_{\boldsymbol{\theta}_0}$$

- ✦ Multivariate Gaussian distribution:

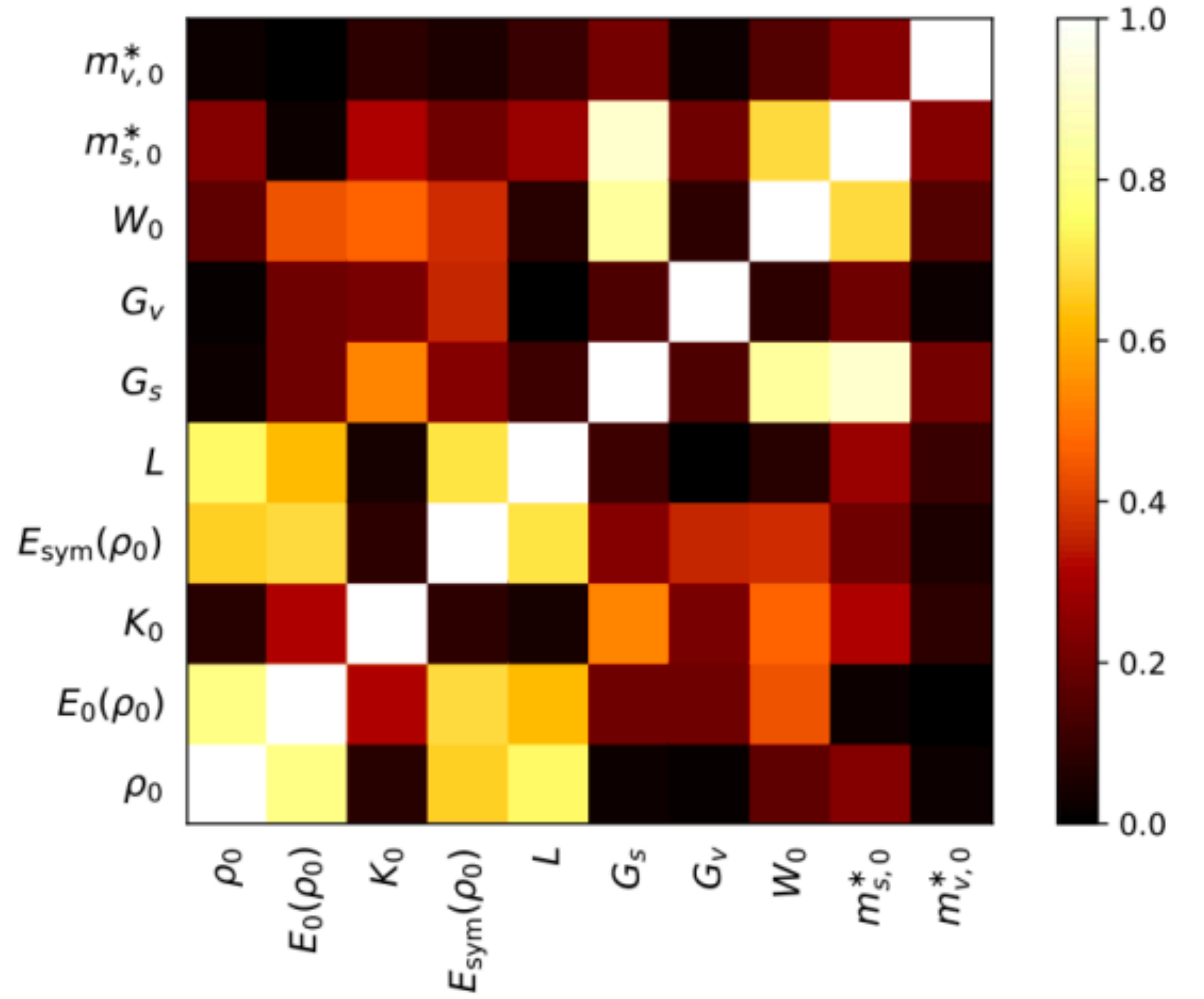
$$p(\boldsymbol{\theta}) \propto \exp\left[-\frac{(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^T \boldsymbol{M}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)}{2}\right].$$

Covariance of A and B

$$Cov(A,\, B) = \sum_{ij}\left(\frac{\partial A}{\partial p_i}\right)_0 \mathcal{C}_{ij}\left(\frac{\partial B}{\partial p_j}\right)_0 \quad C = M^{-1}$$



Reinhard and Nazarewicz, PRC81, 051303(R) (2010)
Dobaczewski et al, JPG41, 074001 (2014)



Zhang et al. PLB 777,73 (2018)

Estimate (pseudo-)evidence with covariance analysis

✦**Classical evidence:**

$$Z = \int L(\boldsymbol{\theta} \,|\, \mathcal{O}^{\exp})\pi(\boldsymbol{\theta})d\boldsymbol{\theta} \approx \prod_i^{N_p} (\Delta\theta_i^{\text{prior}})^{-1} \int L(\boldsymbol{\theta})d\boldsymbol{\theta}$$

$$L(\boldsymbol{\theta} \,|\, \mathcal{O}^{\exp}) \approx \frac{1}{(2\pi)^{\frac{N_d}{2}} \prod_i^{N_d} \Delta\mathcal{O}_i} \exp\left(-\frac{\chi_0^2}{2} - \frac{(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}} \boldsymbol{M} (\boldsymbol{\theta} - \boldsymbol{\theta}_0)}{2} \right),$$

$$\ln Z \approx -\frac{\chi_0^2}{2} - \frac{1}{2}\ln|\boldsymbol{M}| - \frac{N_d - N_p}{2}\ln(2\pi) - \sum_i^{N_p} \ln \Delta\theta_i^{\text{prior}} - \sum_i^{N_d} \ln \Delta\mathcal{O}_i \,.$$

- $\chi_0^2 \equiv \chi^2(\boldsymbol{\theta}_0)$
- N_d: data number
- N_p: parameter number

✦**Pseudo-evidence:**

$$Z_i^{\text{pseu}} = \int L\left(\mathcal{O}_{\text{ev}}^{\exp} \,|\, \boldsymbol{\theta}\right) p\left(\boldsymbol{\theta} \,|\, \mathcal{O}^{\exp}\right) d\boldsymbol{\theta}$$

$$p(\boldsymbol{\theta} \,|\, \mathcal{O}^{\exp}) \approx \frac{|\boldsymbol{M}|^{1/2}}{(2\pi)^{N_p/2}} \exp\left[-\frac{(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^T \boldsymbol{M} (\boldsymbol{\theta} - \boldsymbol{\theta}_0)}{2} \right], \quad L\left(\mathcal{O}_{\text{ev}}^{\exp} \,|\, \boldsymbol{\theta}\right) = \propto \exp\left[-\sum_i^n \frac{(\mathcal{O}_{\text{ev,i}}^{th} - \mathcal{O}_{\text{ev,i}}^{exp})^2}{2\Delta\mathcal{O}_{\text{ev,i}}^2} \right]$$

$$Z_i^{\text{pseu}} \approx \frac{\exp\left[-\frac{1}{2}(\mathcal{O}_{\text{ev}}^{th}(\boldsymbol{\theta}_0) - \mathcal{O}_{\text{ev}}^{exp})^T \boldsymbol{W}^{-1} (\mathcal{O}_{\text{ev}}^{th}(\boldsymbol{\theta}_0) - \mathcal{O}_{\text{ev}}^{exp}) \right]}{\sqrt{(2\pi)^n |\boldsymbol{W}|}}$$

$$\boldsymbol{W} = \boldsymbol{C} + \Sigma, \quad C_{ij} = \sum_{k,l} \left.\frac{\partial \mathcal{O}_{\text{ev,i}}}{\partial \theta_k}\right|_{\boldsymbol{\theta}_0} \boldsymbol{M}_{kl}^{-1} \left.\frac{\partial \mathcal{O}_{\text{ev,j}}}{\partial \theta_l}\right|_{\boldsymbol{\theta}_0},$$

Σ is a diagonal matrix with $\Sigma_{ii} = (\Delta\mathcal{O}_{\text{ev,i}})^2$

✦More efficient for predictions. Typically, $3n \times N_p$ model calculations for n new observables.

✦Covariance analysis has been applied in the construction of many nuclear EDFs (e.g. UNEDF0, FSU2, DDPC-REX,…). The corresponding matrices M for these models are available.

Liquid drop model

$$B(N,Z) = a_v A - a_s A^{2/3} - a_C \frac{Z^2}{A^{1/3}} + a_p \frac{\delta(N,Z)}{A^{1/3}} - a_{sym}(A)\frac{(N-Z)^2}{A} + E_W \, .$$

- $\delta(N,Z) = [(-1)^Z + (-1)^N]$

M. Liu, N. Wang, Z. X. Li and F. S. Zhang, PRC 82, 064306 (2010)
N. Wang, Z. Y. Liang, M. Liu and X. Z. Wu, PRC 82, 044304 (2010)

- Wigner term: $E_W = C_0 \exp(-W|I|/C_0)$ with $W = 42$ MeV and $C_0 = 10$ MeV

W. D. Myers and W. J. Swiatecki, NPA 612, 249 (1997).

✦ Symmetry energy term

- **LD1**: leptodermous expansion in powers of $A^{-1/3}$

$$a_{sym}(A) = \alpha - \beta A^{-1/3}$$

P. G. Reinhard, M. Bender, W. Nazarewicz, and T. Vertse, PRC 73, 014309 (2006)

- **LD2**: droplet model

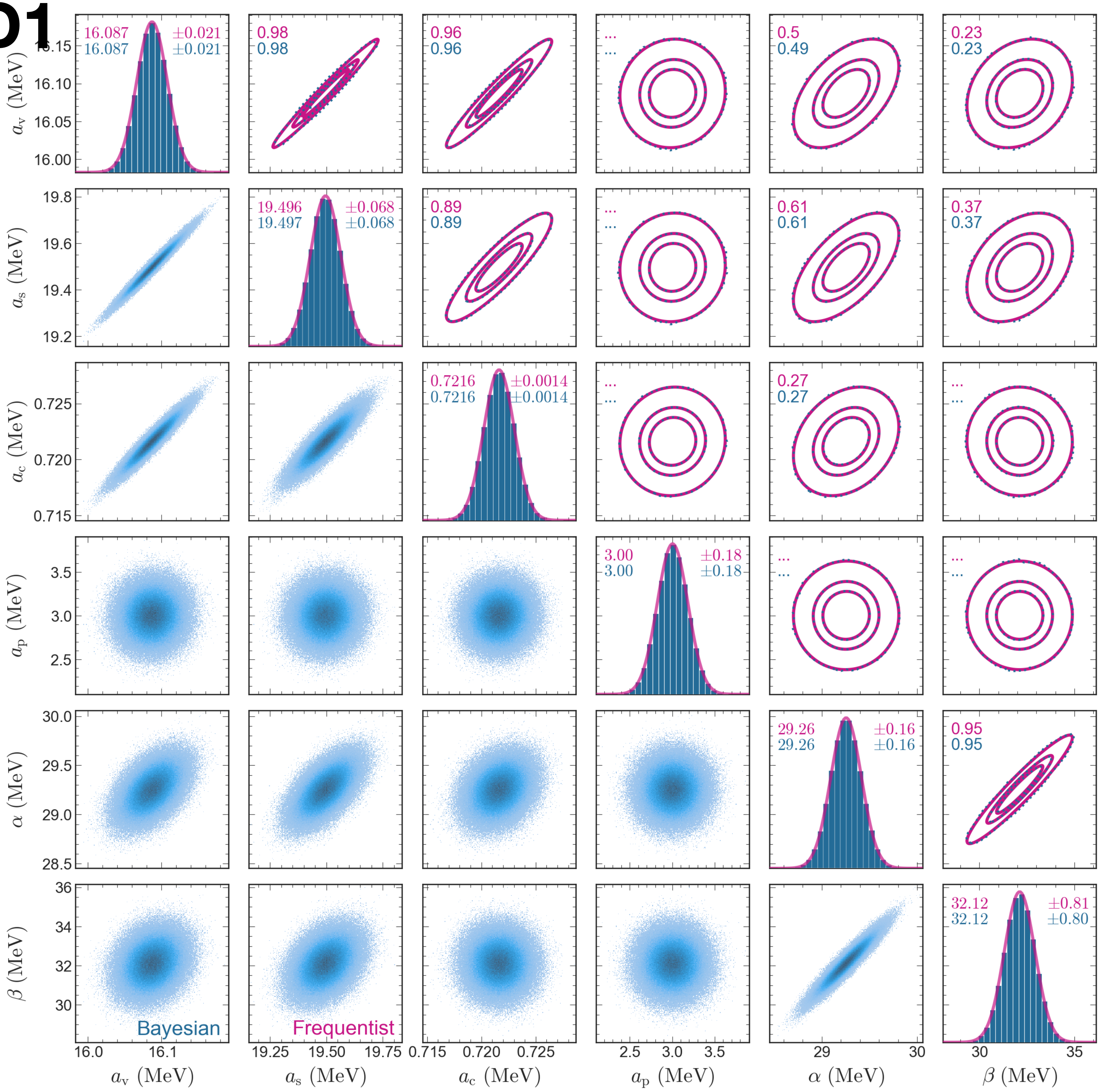
$$a_{sym}(A) = \frac{\alpha}{1 + \kappa A^{-1/3}}$$

W. Myers and W. Swiatecki, Annals of Physics 84, 186 (1974).
P. Danielewicz, Nucl. Phys. A727, 233 (2003)

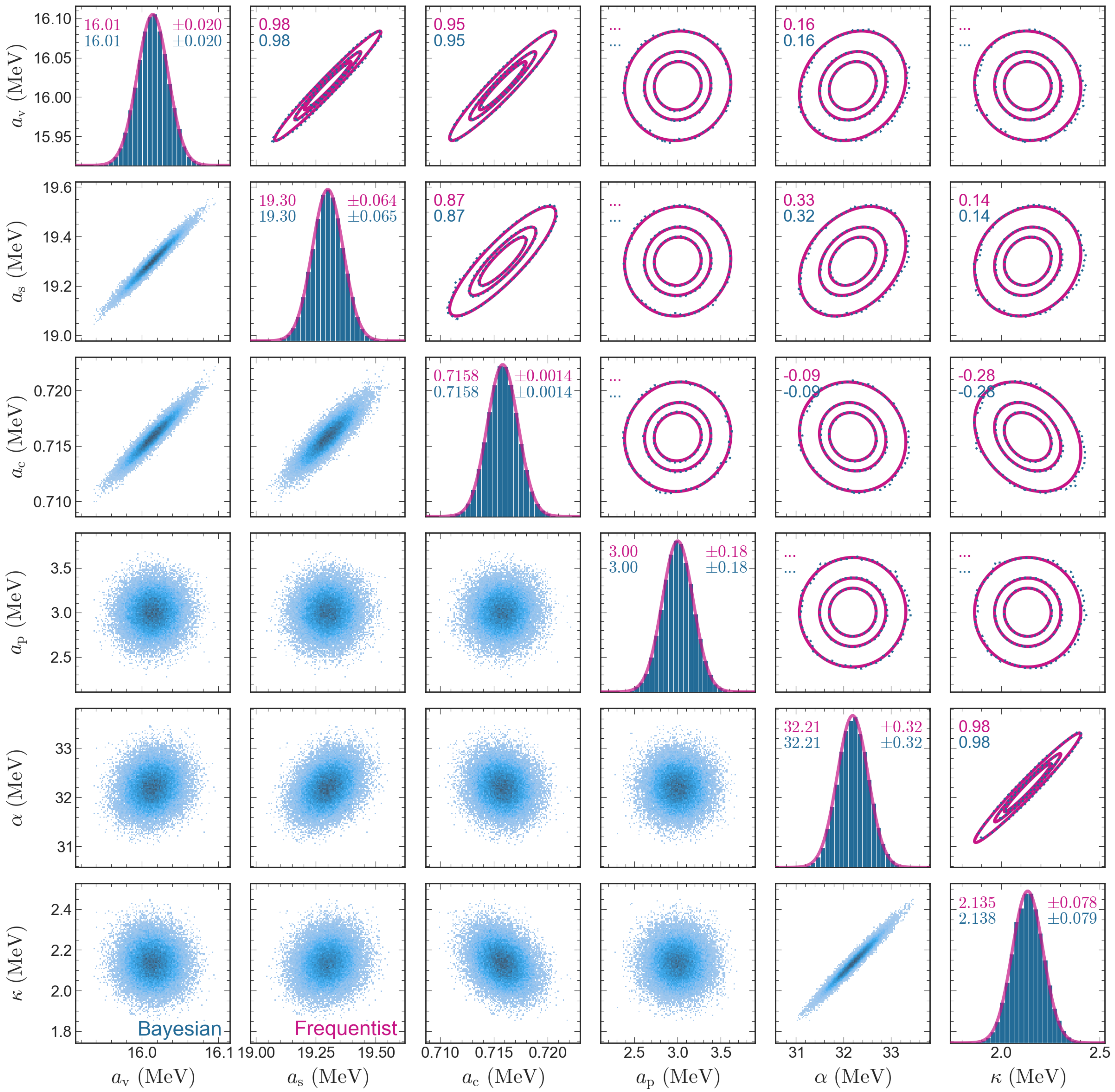
$a_{sym}(A) \to E_{sym}(\rho_0)$ as $A \to \infty$; $E_{sym}(\rho_0) = \alpha$

Bayesian vs Frequentist parameter estimation

LD1

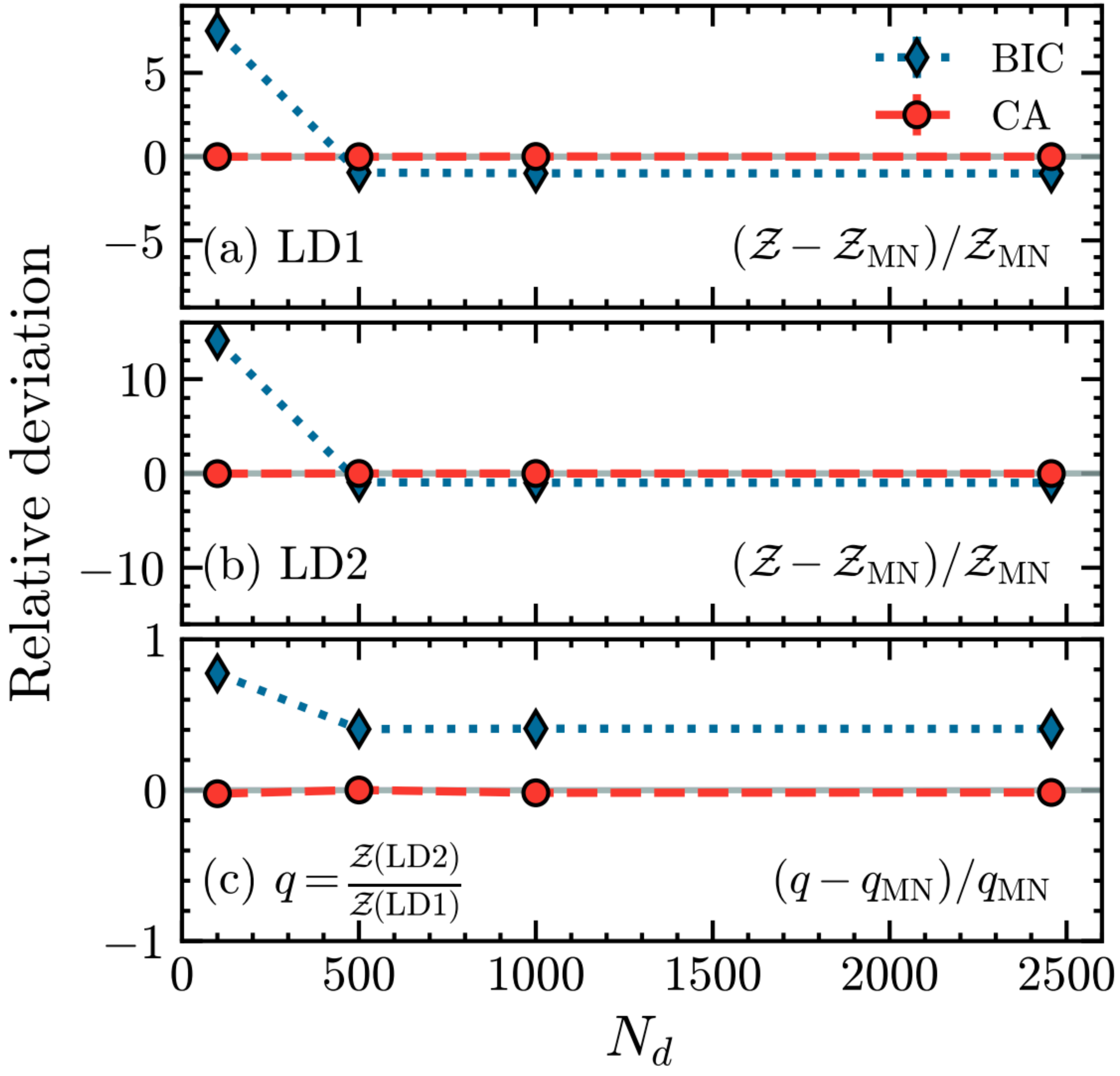


LD2



2457 data from AME2020, *Wang et al. CPC 45, 030003 (2021)*

Estimation of classical evidence



✦N_d data from AME 2020

✦MultiNest (MN) sampling results as benchmark *Buchner et al. 2014, A&A*

✦Bayesian information criterion (BIC)

$$\mathrm{BIC} = -2\ln L(\boldsymbol{\theta}_0) + N_p \ln N_d$$

G. Schwarz, The annals of statistics 6, 461 (1978)

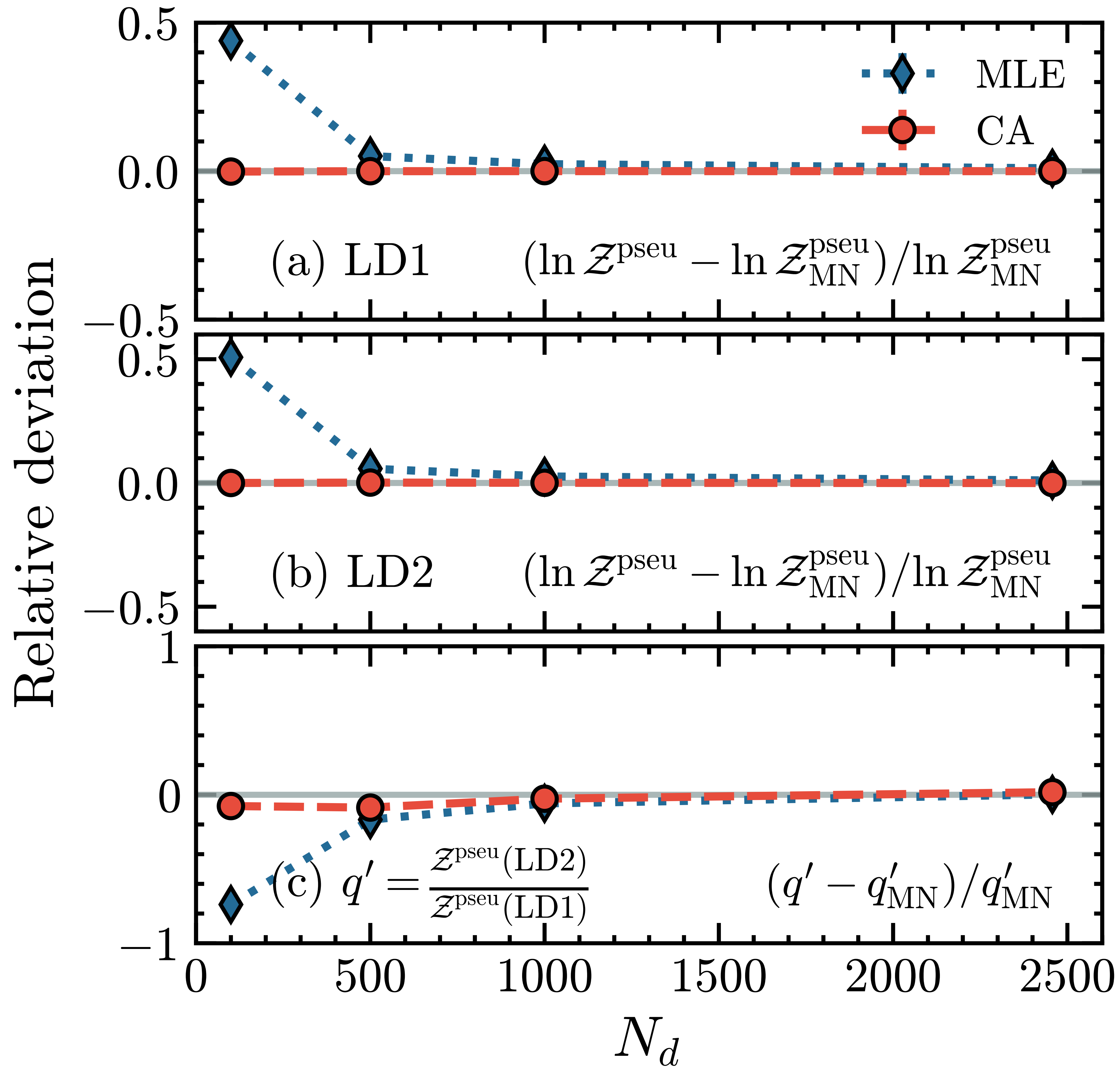
$$\omega \propto \exp(-\mathrm{BIC}/2)$$

✦Bayes factor:

$$q = \frac{Z(\mathrm{LD2})}{Z(\mathrm{LD1})}$$

✦Covariance analysis (CA) results well agree with MN results.

Estimation of pseudo evidence



- $\mathcal{O}_{\text{ev}}$: ^{131,132,133,134,135}Sn (5 data)

- MN results

$$Z^{\text{pseu}}_{\text{MN}} = \frac{Z_{\text{MN}}\left[\mathcal{L}(\mathcal{O}^{\text{exp}}_{\text{ev}} + \mathcal{O}^{\text{exp}})\right]}{Z_{\text{MN}}\left[\mathcal{L}(\mathcal{O}^{\text{exp}})\right]}$$

- A local criterion using maximum likelihood estimate (MLE) $\boldsymbol{\theta}_0$

$$Z^{\text{pseu}}_{\text{MLE}} = L(\mathcal{O}^{\text{ev}} | \boldsymbol{\theta}_0)$$

- For large $N_d(2457)$, the three methods lead to almost the same results.

- CA results are in better agreement with MN results for limited N_d.

Quantitative results

N_d	LD1			LD2			$\ln q_{\mathrm{MN}}$	$\ln q_{\mathrm{CA}}$	$\ln q_{\mathrm{BIC}}$
	$\ln \mathcal{Z}_{\mathrm{MN}}$	$\ln \mathcal{Z}_{\mathrm{CA}}$	$-0.5\mathrm{BIC}$	$\ln \mathcal{Z}_{\mathrm{MN}}$	$\ln \mathcal{Z}_{\mathrm{CA}}$	$-0.5\mathrm{BIC}$			
100	-232.9743(56)	-232.9711	-230.8297	-232.3611(67)	-232.3747	-229.6359	0.6132(87)	0.5964	1.1938
500	-1171.8567(59)	-1171.8564	-1174.8607	-1164.7391(59)	-1164.7451	-1167.4107	7.1176(83)	7.1113	7.4500
1000	-2387.5220(56)	-2387.5215	-2392.5745	-2374.8013(58)	-2374.7990	-2379.4921	12.7207(81)	12.7225	13.0824
2457	-5916.1730(65)	-5916.1713	-5923.9607	-5892.1474(64)	-5892.1463	-5899.5795	24.0256(91)	24.0249	24.3811

N_d	LD1			LD2			$\ln q'_{\mathrm{MN}}$	$\ln q'_{\mathrm{CA}}$	$\ln q'_{MLE}$
	$\ln \mathcal{Z}^{\mathrm{pseu}}_{\mathrm{MN}}$	$\ln \mathcal{Z}^{\mathrm{pseu}}_{\mathrm{CA}}$	$\ln \mathcal{Z}^{\mathrm{pseu}}_{\mathrm{MLE}}$	$\ln \mathcal{Z}^{\mathrm{pseu}}_{\mathrm{MN}}$	$\ln \mathcal{Z}^{\mathrm{pseu}}_{\mathrm{CA}}$	$\ln \mathcal{Z}^{\mathrm{pseu}}_{\mathrm{MLE}}$			
100	-59.4516(79)	-59.3644	-85.5644	-54.0410(101)	-54.0635	-81.5281	5.3820(128)	5.3010	4.0363
500	-56.1159(80)	-56.1113	-58.9688	-52.4357(87)	-52.5114	-55.4622	3.6802(118)	3.5999	3.5066
1000	-49.2871(82)	-49.2949	-50.4551	-46.5428(80)	-46.5716	-47.7765	2.7443(115)	2.7232	2.6901
2457	-50.1317(88)	-50.1273	-50.6143	-47.6510(85)	-47.6298	-48.1319	2.4807(122)	2.4974	2.4824

- ✦ LD2 model with $a_{\mathrm{sym}}(A) = \dfrac{\alpha}{1 + \kappa A^{-1/3}}$ is preferred by AME2020 data.
- ✦ $\alpha = 32.21 \pm 0.32\mathrm{MeV}$ and $\kappa = 2.135 \pm 0.078$.

Summary and outlook

- Bayesian evidence is a proper criterion for model comparison.
  - $E_{\rm sym}(2\rho_0/3) = 25.6^{+1.4}_{-1.3}$ from BMA analysis of nuclear masses.
- Frequentist covariance analysis can effectively estimate Bayesian (pseudo) evidence.
  - AME 2020 data prefer $a_{\rm sym}(A) = \dfrac{\alpha}{1 + \kappa A^{-1/3}}$ with $\alpha = 32.21 \pm 0.32\ \mathrm{MeV}$ and $\kappa = 2.135 \pm 0.078$.
- Ongoing work: incorporating more observables into the multi-model analysis.

Thanks for your attention

Sources of uncertainty

✦ **Statistical error**

- Uncertainty propagation:

- Numerical uncertainty (e.g., Monte Carlo)

✦ **Systematic error** from imperfect modeling.

- Inter-model uncertainties and model dependence.
- Could be estimated by compare different models.

Dobaczewski, Nazarewicz and Reinhard, JPG41 (2014) 074001

Large statistical error:

Large systematic error:

Bayesian vs Frequentist (100 nuclei)