Analytical calculation of multi-loop Feynman integrals with masses

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Introduction

- Higher-order radiative corrections are more important, with the increasing precision of measurements at the future colliders: CEPC, ILC, HL-LHC, FCC $\cdots$

Introduction

- One-loop Feynman integrals are well known analytically in the time-space dimension $D = 4 - 2\varepsilon$.
  However, how to perform analytically multi-loop Feynman integrals with masses is still a challenge.
- Considering Feynman integrals as the generalized hypergeometric functions, one finds that the $D-$module of a Feynman diagram is isomorphic to Gel'fand-Kapranov-Zelevinsky (GKZ) $D-$module.
- We can construct GKZ hypergeometric system of multi-loop Feynman integrals with masses, to obtain the generalized hypergeometric function solutions.

Literatures

Using GKZ hypergeometric system, we can obtain the hypergeometric function solutions of Feynman integrals with masses in neighborhoods of origin including infinity.

I. Tai-Fu Feng, Chao-Hsi Chang, Jian-Bin Chen, Hai-Bin Zhang

GKZ-hypergeometric systems for Feynman integrals

NPB 953 (2020) 114952 [arXiv: 1912.01726]

II. Tai-Fu Feng, Hai-Bin Zhang, Yan-Qing Dong, Yang Zhou

GKZ-system of the 2-loop self energy with 4 propagators

EPJC 83 (2023)4, 314 [arXiv: 2209.15194]

III. Hai-Bin Zhang, Tai-Fu Feng

GKZ hypergeometric systems of the three vacuum Feynman integrals

JHEP 05 (2023) 075 [arXiv: 2303.02795]

IV. Hai-Bin Zhang, Tai-Fu Feng

GKZ hypergeometric systems of the four vacuum Feynman integrals

JHEP 03 (2025) 013 [arXiv: 2403.13025]

Literatures

Embed in Grassmannians, we can obtain the hypergeometric function solutions of Feynman integrals with masses in neighborhoods of all regular singularities. We generalize Gauss relations among the hypergeometric functions to complete analytic continuation of the solutions.

V. Tai-Fu Feng, Hai-Bin Zhang, Chao-Hsi Chang

Feynman integrals of Grassmannians

PRD 106 (2022) 116025 [arXiv: 2206.04224]

VI. Tai-Fu Feng, Yang Zhou, Hai-Bin Zhang

Gauss relations in Feynman integrals

PRD 111 (2025) 016015 [arXiv: 2407.10287]

VII. Tai-Fu Feng, Yang Zhou, $\cdots$, Hai-Bin Zhang

Feynman Integral of two-loop Dune diagram

[arXiv: 26xx.xxxxx]

Contents

I. Embedding of Feynman Integrals in Grassmannians

- Feynman integrals involving several energy scales can be given by some finite linear combinations of generalized hypergeometric functions.
- Any commonly used functions of one indeterminate of analysis can be expressed as the Gauss function

$$ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\ (c)_n} x^n \,, \tag{1.1} $$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer notation.

- For the given parameters $a,\ b,\ c$, there are 24 hypergeometric series solutions totally of the partial differential equation (PDE) which can be written as the GKZ-system on the Grassmannians $G_{2,4}$.

I. Embedding of Feynman Integrals in Grassmannians

- In α-parameterization, the Feynman integral of one-loop self-energy is

$$
\begin{aligned}
& iA_{1SE}(p^2, m_1^2, m_2^2) \\
= & -\left(\Lambda_{\rm RE}^2\right)^{2-D/2} \int_0^\infty d\alpha_1 d\alpha_2 \int \frac{d^D q}{(2\pi)^D} \exp\Big\{ i \Big[\alpha_1 (q^2 - m_1^2) \\
& + \alpha_2 ((q+p)^2 - m_2^2) \Big] \Big\} \\
= & \frac{i^{2-D/2} \exp\left\{ \frac{i\pi(2-D)}{4} \right\} \Gamma(2 - D/2) \left(\Lambda_{\rm RE}^2\right)^{2-D/2}}{(4\pi)^{D/2}} \\
& \times \int_S \omega_3(t) \delta(t_1 t_2 + t_1 t_3 + t_2 t_3)(t_1 t_2)^{1-D/2} t_3^{D/2-1} \\
& \times \left[t_1 m_1^2 + t_2 m_2^2 + t_3 p^2 \right]^{D/2-2} ,
\end{aligned}
\tag{1.2}
$$

- The hyperplane S is given by the equation $t_3 + 1 = 0$, and $\omega_3(t) = t_1 dt_2 \wedge dt_3 - t_2 dt_1 \wedge dt_3 + t_3 dt_1 \wedge dt_2$ is the volume element in the projective plane P^2, respectively.

I. Embedding of Feynman Integrals in Grassmannians

$$iA_{1SE}(p^2, m_1^2, m_2^2) \propto \int_S \omega_3(t)\delta(t_1t_2 + t_1t_3 + t_2t_3)(t_1t_2)^{1-D/2}t_3^{D/2-1}\left[t_1m_1^2 + t_2m_2^2 + t_3p^2\right]^{D/2-2}$$

- The integral can be embedded in the subvariety of the Grassmannian $G_{3,5}$

$$\boldsymbol{\xi} = \begin{pmatrix} 1 & 0 & 0 & 1 & r_1 \\ 0 & 1 & 0 & 1 & r_2 \\ 0 & 0 & 1 & 1 & r_3 \end{pmatrix}, \tag{1.3}$$

  with the exponent vector $\beta_{(1S)} = (2 - \frac{D}{2},\ 2 - \frac{D}{2},\ \frac{D}{2},\ -1,\ \frac{D}{2} - 1) \in C^5$, and $r_1 = m_1^2,\ r_2 = m_2^2,\ r_3 = p^2$.
- Row: 1: integration variable t_1, 2: t_2, 3: t_3, respectively.
- Column: 1: the power function $t_1^{1-D/2}$, 2: $t_2^{1-D/2}$, 3: $t_3^{D/2-1}$, 4: δ function, 5: the power polynomial $t_1m_1^2 + t_2m_2^2 + t_3p^2$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

The hypergeometric function on the general stratum of the Grassmannian $G_{3,5}$ with the splitting coordinates in Eq.(1.3) satisfies the GKZ-system as

$$\begin{aligned}
&\Big\{\vartheta_{1,4}+\vartheta_{1,5}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=-\beta_1\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\Big\{\vartheta_{2,4}+\vartheta_{2,5}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=-\beta_2\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\Big\{\vartheta_{3,4}+\vartheta_{3,5}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=-\beta_3\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\Big\{\vartheta_{1,4}+\vartheta_{2,4}+\vartheta_{3,4}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=(\beta_4-1)\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\Big\{\vartheta_{1,5}+\vartheta_{2,5}+\vartheta_{3,5}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=(\beta_5-1)\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,
\end{aligned} \tag{2.1}$$

where the Euler operators $\vartheta_{i,j}=\xi_{i,j}\partial/\partial\xi_{i,j}$, and the exponent vector $\boldsymbol{\beta}=(\beta_1,\cdots,\beta_5)\in C^5$ satisfying $\sum\beta_i=2$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

Corresponding to the Grassmannian $G_{3,5}$ represented by the matroid in Eq.(1.3), the exponent matrix is generally written as

$$\begin{pmatrix} \beta_1 - 1 & 0 & 0 & \alpha_{1,4} & \alpha_{1,5} \\ 0 & \beta_2 - 1 & 0 & \alpha_{2,4} & \alpha_{2,5} \\ 0 & 0 & \beta_3 - 1 & \alpha_{3,4} & \alpha_{3,5} \end{pmatrix} . \tag{2.2}$$

where

$$\begin{aligned} & \sum_{i=1}^{5} \beta_i = 2, \quad \sum_{j=1}^{3} \alpha_{j,4} = \beta_4 - 1, \quad \sum_{j=1}^{3} \alpha_{j,5} = \beta_5 - 1 \\ & \alpha_{j,4} + \alpha_{j,5} = -\beta_j, \quad j = 1, 2, 3. \end{aligned} \tag{2.3}$$

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

Let $\mathcal{N} = \{1, \cdots, 5\}$ denoting the set of indices of the columns in Eq.(1.3). Choosing the affine spanning subset $\mathcal{B}$ of the vector subspace C^3 and the integer lattice, one gets the hypergeometric function accordingly.

For example as $\mathcal{B} = \{1, 2, 3\}$, there are 12 choices on the matrix of integer lattice:

$$(0_{3\times 3} \,|\, \pm n_1 E_3^{(i)} \pm n_2 E_3^{(j)}) \tag{2.4}$$

$$E_3^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}, E_3^{(2)} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}, E_3^{(3)} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Corresponding to the integer lattice

$$\left(0_{3\times 3} \middle| n_1 E_3^{(1)} + n_2 E_3^{(2)}\right) = \begin{pmatrix} 0 & 0 & 0 & \boxed{n_2} & -n_2 \\ 0 & 0 & 0 & \boxed{n_1} & -n_1 \\ 0 & 0 & 0 & -n_1 - n_2 & n_1 + n_2 \end{pmatrix}, \qquad (2.5)$$

- the exponents are given by the matrix

$$||\boldsymbol{\alpha}|| = \begin{pmatrix} \beta_1 - 1 & 0 & 0 & \boxed{0} & -\beta_1 \\ 0 & \beta_2 - 1 & 0 & \boxed{0} & -\beta_2 \\ 0 & 0 & \beta_3 - 1 & \beta_4 - 1 & 1 - \beta_3 - \beta_4 \end{pmatrix}, \qquad (2.6)$$

where $\alpha_{1,4} = \alpha_{2,4} = 0$ because $n_{1,2}$ are nonnegative.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The generalized hypergeometric function is

$$\begin{aligned}\Phi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) &= A^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})(r_1)^{-\beta_1}(r_2)^{-\beta_2}(r_3)^{1-\beta_3-\beta_4} \\ &\quad \times\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{r_3}{r_2},\ \frac{r_3}{r_1})\ , \\ \varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) &= \sum_{n_1,n_2} c^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1,n_2)x_1^{n_1}x_2^{n_2}\ ,\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}A^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}) &= \frac{\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(1-\beta_2)\Gamma(2-\beta_3-\beta_4)}\ , \\ c^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1,n_2) &= \frac{(\beta_2)_{n_1}(\beta_1)_{n_2}(1-\beta_4)_{n_1+n_2}}{n_1!n_2!(2-\beta_3-\beta_4)_{n_1+n_2}}\ .\end{aligned} \tag{2.8}$$

with the Pochhammer notation $(a)_n = \Gamma(a+n)/\Gamma(a)$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Corresponding to the integer lattice

$$\left(0_{3\times 3}\middle| n_1 E_3^{(1)} + n_2 E_3^{(3)}\right) = \begin{pmatrix} 0 & 0 & 0 & \boxed{n_2} & -n_2 \\ 0 & 0 & 0 & n_1 - n_2 & -n_1 + n_2 \\ 0 & 0 & 0 & -n_1 & \boxed{n_1} \end{pmatrix}, \tag{2.9}$$

- the exponents are given by the matrix

$$||\boldsymbol{\alpha}|| = \begin{pmatrix} \beta_1 - 1 & 0 & 0 & \boxed{0} & -\beta_1 \\ 0 & \beta_2 - 1 & 0 & \beta_3 + \beta_4 - 1 & \beta_1 + \beta_5 - 1 \\ 0 & 0 & \beta_3 - 1 & -\beta_3 & \boxed{0} \end{pmatrix}, \tag{2.10}$$

where $\alpha_{1,4} = \alpha_{3,5} = 0$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The generalized hypergeometric function is formulated as

$$\Phi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) = A^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta})(r_1)^{-\beta_1}(r_2)^{\beta_1+\beta_5-1} \times\varphi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{r_3}{r_2},\ \frac{r_2}{r_1})\ ,$$

$$\varphi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \sum_{n_1,n_2} c^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1, n_2)x_1^{n_1}x_2^{n_2}\ , \tag{2.11}$$

- Where

$$A^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{\Gamma(\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(1-\beta_3)\Gamma(\beta_1+\beta_5)\Gamma(\beta_3+\beta_4)}\ ,$$

$$c^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1, n_2) = \frac{(-)^{n_1+n_2}(\beta_3)_{n_1}(\beta_1)_{n_2}}{n_1!n_2!(\beta_1+\beta_5)_{-n_1+n_2}(\beta_3+\beta_4)_{n_1-n_2}}\ . \tag{2.12}$$

Note that $1/(a)_{-n} = (-1)^n(1-a)_n$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

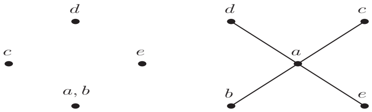

Figure: 1 The geometric configurations of the hypergeometric functions on the projective plane P^2, where the points $a, \cdots, e$ denote the indices of columns of exponent matrix.

- Geometric representation of $\Phi^{(1)}_{\{1,2,3\}}$ is drown in Fig.1(a) where $\{a, b\} = \{3, 4\}$ and $\{c, d, e\} = \{1, 2, 5\}$, which determinant of any 2×2 minor of the submatrix consisted of the third and fourth columns is zero.
- Geometric representation of $\Phi^{(2)}_{\{1,2,3\}}$ is drown in Fig.1(b) where $a = 2,\ \{b, c\} = \{1, 5\}$ and $\{d, e\} = \{3, 4\}$, which determinants of the submatrices $\det(||\alpha||_{\{1,2,5\}}) = \det(||\alpha||_{\{2,3,4\}}) = 0$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- In these hypergeometric functions, $\varphi_{\mathcal{B}}^{(i)}$, $i = 1, 3, 5, 8, 10, 12$ are the first Appell functions, while $\varphi_{\mathcal{B}}^{(j)}$, $j = 2, 4, 6, 7, 9, 11$ are the Horn functions.
- It is easy to find that the convergent regions of $\varphi_{\{1,2,3\}}^{(1)}$, $\varphi_{\{1,2,3\}}^{(2)}$, and $\varphi_{\{1,2,3\}}^{(3)}$ have nonempty intersections in a connected component of definition domain, thus they constitute a fundamental solution system.
- The linear combinations of hypergeometric functions on the different nonempty proper subsets of the parameter space are regarded as analytic continuations of each other.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

$$
\begin{aligned}
\Psi(\boldsymbol{\beta},\ \boldsymbol{\xi}) = & \sum_{i=\{1,2,3\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
& = \sum_{i=\{1,5,6\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
& = \sum_{i=\{3,7,8\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
& = \sum_{i=\{4,5,12\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
& = \sum_{i=\{8,9,10\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
& = \sum_{i=\{10,11,12\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi})\ . \qquad (2.13)
\end{aligned}
$$

Using the Gauss inverse relations below, we can derive the combinatorial coefficients uniquely, then continue the analytic expressions to the whole domain of definition of the Feynman integral by the Gauss-Kummer relations.

III. Generalized Gauss relations

- The Gauss inverse relations include the following analytic continuation together with its various variants

$$
\begin{aligned}
{}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| x\right) = &\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}\, {}_2F_1\left(\begin{array}{c} a, 1+a-c \\ 1+a-b \end{array} \middle| \frac{1}{x}\right) \\
&+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}\, {}_2F_1\left(\begin{array}{c} b, 1+b-c \\ 1-a+b \end{array} \middle| \frac{1}{x}\right) .
\end{aligned}
\tag{3.1}
$$

- The Gauss inverse relations can be obtained through the Mellin-Barnes's contour on the corresponding complex plane, combined with residue theorem.

III. Generalized Gauss relations

We can give generalized Gauss inverse relations with two variables

$$\begin{aligned} & \varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) \\ = & \frac{\Gamma(1-\beta_1-\beta_4)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_2+\beta_5)\Gamma(1-\beta_4)}(-x_2)^{-\beta_1}\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ \frac{1}{x_2}) \\ & +\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_1)\Gamma(1-\beta_3)}(-x_2)^{\beta_4-1}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_2},\ \frac{x_1}{x_2})\ . \end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned} & \varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) \\ = & \frac{\Gamma(1-\beta_2-\beta_4)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_1+\beta_5)\Gamma(1-\beta_4)}(-x_1)^{-\beta_2}\varphi^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_1},\ x_2) \\ & +\frac{\Gamma(\beta_2+\beta_4-1)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_2)\Gamma(1-\beta_3)}(-x_1)^{\beta_4-1}\varphi^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_1},\ \frac{x_2}{x_1})\ . \end{aligned} \tag{3.3}$$

III. Generalized Gauss relations

- Gauss-Kummer relations are derived through Kummer's classification, which can be written as

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, x\right) = (1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix}\,\middle|\, x\right) \\
&= (1-x)^{-a}\,{}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix}\,\middle|\, \frac{x}{x-1}\right) \\
&= (1-x)^{-b}\,{}_2F_1\left(\begin{matrix} c-a,\ b \\ c \end{matrix}\,\middle|\, \frac{x}{x-1}\right) \qquad (3.4)
\end{aligned}
$$

  and its various variants.
- For the GKZ-system on the Grassmannian, the generalized hypergeometric solutions corresponding to the same geometric representation are proportional to each other in the intersection of their convergent regions.

III. Generalized Gauss relations

Corresponding to the geometric representation shown in Fig.1(a) with $\{a, b\} = \{3, 5\}$, $\{c, d, e\} = \{1, 2, 4\}$, we obtain the generalized Gauss-Kummer relations as

$$\begin{aligned}
&\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x,\ y) \\
&= (1-y)^{-\beta_1}(1-x)^{-\beta_2}\varphi^{(1)}_{\{1,2,5\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ \frac{y}{y-1}) \\
&= (1-x)^{\beta_5-1}\varphi^{(5)}_{\{1,3,4\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ \frac{x-y}{x-1}) \\
&= (1-y)^{\beta_5-1}\varphi^{(5)}_{\{2,3,4\}}(\boldsymbol{\beta},\ \frac{y}{y-1},\ \frac{y-x}{y-1}) \\
&= (1-x)^{1-\beta_2-\beta_3}(1-y)^{-\beta_1}\varphi^{(10)}_{\{1,4,5\}}(\boldsymbol{\beta},\ x,\ \frac{x-y}{1-y}) \\
&= (1-y)^{1-\beta_1-\beta_3}(1-x)^{-\beta_2}\varphi^{(10)}_{\{2,4,5\}}(\boldsymbol{\beta},\ y,\ \frac{y-x}{1-x})\ ,
\end{aligned} \tag{3.5}$$

with $x = r_2/r_3,\ y = r_1/r_3$.

IV. The analytic expressions for 1-loop self energy

In this scheme, we can have the generally strategy:

- Embedding the Feynman integral on Grassmannian $G_{k,n}$, we construct hypergeometric solutions from GKZ-systems under all possible affine spanning.
- We derive the Gauss inverse relations under the same affine spanning, and the Gauss-Kummer relations from different affine spanning.
- Feynman integral can be written as a finite linear combinations of the hypergeometric solutions.
- The combination coefficients are obtained by Gauss relations, then the analytic expressions of the Feynman integral are continued to its whole domain of definition.

IV. The analytic expressions for 1-loop self energy

- In example of 1-loop self energy, its Feynman integral can be written as linear combinations of the hypergeometric solutions in the different definition domains below.
- $|p^2| < m_2^2 < m_1^2$:

$$
\begin{aligned}
A_{1SE} = & C^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{-\beta_2}(p^2)^{1-\beta_3-\beta_4}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{p^2}{m_1^2}) \\
& + C^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{\beta_1+\beta_5-1}\varphi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_2^2}{m_1^2}) \\
& + C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_5-1}\varphi^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_1^2}, \frac{m_2^2}{m_1^2}) \, . \qquad (4.1)
\end{aligned}
$$

- $|p^2| < m_1^2 < m_2^2$:

$$
\begin{aligned}
A_{1SE} = & C^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{-\beta_2}(p^2)^{1-\beta_3-\beta_4}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{p^2}{m_1^2}) \\
& + C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{\beta_5-1}\varphi^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}) \\
& + C^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_2+\beta_5-1}(m_2^2)^{-\beta_2}\varphi^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_1^2}, \frac{m_1^2}{m_2^2}) \qquad (4.2)
\end{aligned}
$$

IV. The analytic expressions for 1-loop self energy

- $m_2^2 < |p^2| < m_1^2$:

$$
\begin{aligned}
A_{1SE} = {} & C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_5-1}\varphi^{(3)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{p^2}{m_1^2},\ \frac{m_2^2}{m_1^2}\Big) \\
& + C^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(p^2)^{\beta_1+\beta_5-1}\varphi^{(7)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{m_2^2}{p^2},\ \frac{p^2}{m_1^2}\Big) \\
& + C^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{1-\beta_2-\beta_4}(p^2)^{-\beta_3}\varphi^{(8)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{m_2^2}{p^2},\ \frac{m_2^2}{m_1^2}\Big)
\end{aligned}
\tag{4.3}
$$

- $m_2^2 < m_1^2 < |p^2|$:

$$
\begin{aligned}
A_{1SE} = {} & C^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{1-\beta_2-\beta_4}(p^2)^{-\beta_3}\varphi^{(8)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{m_2^2}{p^2},\ \frac{m_2^2}{m_1^2}\Big) \\
& + C^{(9)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_3+\beta_5-1}(p^2)^{-\beta_3}\varphi^{(9)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{m_1^2}{p^2},\ \frac{m_2^2}{m_1^2}\Big) \\
& + C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})(p^2)^{\beta_5-1}\varphi^{(10)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{m_2^2}{p^2},\ \frac{m_1^2}{p^2}\Big)
\end{aligned}
\tag{4.4}
$$

IV. The analytic expressions for 1-loop self energy

- $m_1^2 < m_2^2 < |p^2|$:

$$\begin{aligned} A_{1SE} =& C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})(p^2)^{\beta_5-1}\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_1^2}{p^2}) \\ &+C^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{\beta_3+\beta_5-1}(p^2)^{-\beta_3}\varphi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_1^2}{m_2^2}) \\ &+C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{1-\beta_1-\beta_4}(m_2^2)^{-\beta_2}(p^2)^{-\beta_3}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_1^2}{p^2}, \frac{m_1^2}{m_2^2}) \end{aligned} \quad (4.5)$$

- $m_1^2 < |p^2| < m_2^2$:

$$\begin{aligned} A_{1SE} =& C^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{-\beta_2}(p^2)^{\beta_2+\beta_5-1}\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_1^2}{p^2}) \\ &+C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{\beta_5-1}\varphi^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}) \\ &+C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{1-\beta_1-\beta_4}(m_2^2)^{-\beta_2}(p^2)^{-\beta_3}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_1^2}{p^2}, \frac{m_1^2}{m_2^2}) \end{aligned} \quad (4.6)$$

IV. The analytic expressions for 1-loop self energy

- The boundary conditions:

$$iA_{1SE}(p^2,0,0) = \frac{i\Gamma(2-\frac{D}{2})\Gamma^2(\frac{D}{2}-1)}{(4\pi)^{D/2}\Gamma(D-2)}\Big(\frac{-p^2}{\Lambda^2_{\rm RE}}\Big)^{\frac{D}{2}-1},$$
$$iA_{1SE}(0,m^2,0) = iA_{1SE}(0,0,m^2) = \frac{i\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2}-1)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\Big(\frac{m^2}{\Lambda^2_{\rm RE}}\Big)^{\frac{D}{2}-1}. \quad (4.7)$$

Here $\Lambda_{\rm RE}$ is the renormalization scale.

- Using the boundary conditions in Eq.(4.7), we have

$$C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) = C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{\Gamma(\frac{D}{2}-1)\Gamma(2-\frac{D}{2})}{(4\pi)^{D/2}\Gamma(\frac{D}{2})},$$
$$C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{(-1)^{D/2-2}\Gamma^2(\frac{D}{2}-1)\Gamma(2-\frac{D}{2})}{(4\pi)^{D/2}\Gamma(D-2)}. \quad (4.8)$$

IV. The analytic expressions for 1-loop self energy

- Other coefficients are linear combinations of the above coefficients through the Gauss inverse relations.
- For example, performing the inverse transformation of suitable variables in Eq.(4.2) and Eq.(4.3), one gets

$$
\begin{aligned}
C^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_2} \frac{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5-1)\Gamma(\beta_2+\beta_5)} C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_2+\beta_5} \frac{\Gamma(2-\beta_2-\beta_5)\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_2)\Gamma(\beta_2+\beta_5)} C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) , \\
C^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_1+\beta_5} \frac{\Gamma(2-\beta_1-\beta_5)\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(\beta_2)\Gamma(\beta_1+\beta_5)} C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_1} \frac{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5)\Gamma(\beta_2+\beta_5-1)} C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) . \qquad (4.9)
\end{aligned}
$$

- Taking the time-space dimension $D = 4 - 2\varepsilon$ in dimensional regularization, one finds that the ultraviolet divergence in Eq(4.1)$\sim$Eq.(4.6) is $1/\varepsilon$.

V. The 2-loop Massive Dune diagram

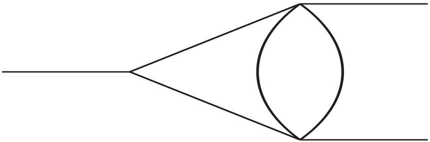

Figure: 1 The 2-loop Massive Dune diagram.

- The Feynman integral of the 2-loop massive Dune diagram is embedded in Grassmannian $G_{5,8}$.
- Tai-Fu Feng, Yang Zhou, $\cdots$, Hai-Bin Zhang

  Feynman Integral of two-loop Dune diagram, [arXiv: 26xx.xxxxx]

V. The 2-loop Massive Dune diagram

The splitting coordinates are reduced to the matroid ξ_{DU}

$$\xi_{DU} = \left(\begin{array}{c|c} I_5 & Z_{DU} \end{array} \right)_{5\times 8} \tag{5.1}$$

with the exponent vector
$\beta_{DU} = (0,0,0,0,0,-D,D-4,-1) \in \mathbf{C}^8$, and

$$Z_{DU} = \begin{pmatrix} 1 & p_1^2 & m_1^2 \\ 1 & p_1^2 & m_2^2 \\ 1 & p_2^2 & m_3^2 \\ 1 & p_3^2 & m_4^2 \\ \zeta & \Lambda^2 & \Lambda^2 \end{pmatrix} . \tag{5.2}$$

V. The 2-loop Massive Dune diagram

- For Grassmannian $G_{5,8}$, there are 56 affine spanning. In each affine spanning there are 1905 linearly independent hypergeometric functions. In total there are 106680 hypergeometric functions.
- The matroid ξ_{DU} represent a collection of eight points in the projective space P^4 which has nine geometric configurations. Those 106680 hypergeometric functions above are attributed to nine types of hypergeometric functions, which are transformed into each other by the various Gauss relations.

VI. Summary

- In this approach, one topological diagram corresponds to one set of hypergeometric solutions. We make the classification among those hypergeometric solutions by the geometric configurations.
- GKZ-systems of Grassmannians give the analytic expressions of multi-loop Feynman integrals with masses in whole domain of definition, combined with generalized Gauss relations.
- Next we are considering how to better implement the programmatization of this method, which can be applicable to the high-order corrections of physical quantities in future high-precision colliders, like CEPC.

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THANKS!