Search for Dark Matter in 2HDMS at LHC and Future Lepton Colliders

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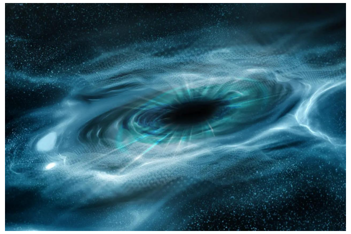
Based on [arXiv:2308.05653]
[arXiv:2504.14529]

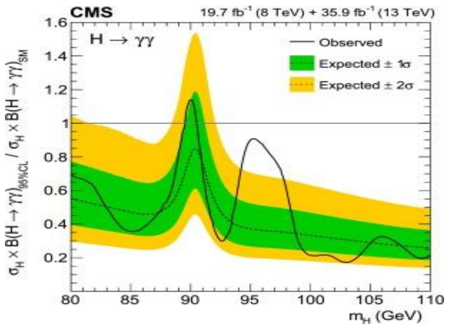
Why 2HDM+complex singlet?

$$V_{SM} = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$$

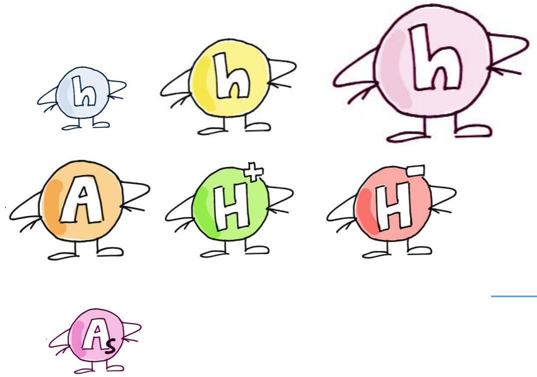
Explain:
- cold dark matter (DM)



- 95 GeV excess
  (at LHC and LEP)



*γγ channel at CMS (~2.9 σ), bb channel at LEP (~2 σ), investigated in S. Heinemeyer, C. Li, et al, 2021, arxiv:2112.11958



$$V_{2HDMS} = V_{2HDM} + V_S$$

Reference model to obtain different **DM mass scenarios**, which lead to various **signal topologies at future colliders**

HiLumi HL-LHC PROJECT

CEPC

ilc

International MUON Collider Collaboration

2HDMS Type II, Higgs Sector Potential

[Notation as in: Baum and Shah, arXiv: 1808.02667]

$$V = V_{2HDM} + V_S$$

$$V_{2HDM} = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + h.c.] + \frac{\lambda_1}{2}(\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2}(\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + \left[\frac{\lambda_5}{2}(\Phi_1^\dagger \Phi_2)^2 + h.c.\right]$$

$$V_S = m_S^2 S^\dagger S + \left[\frac{m_S'^2}{2} S^2 + h.c.\right] + \left[\frac{\lambda_1''}{24} S^4 + h.c.\right] + \left[\frac{\lambda_2''}{6}(S^2 S^\dagger S) + h.c.\right] + \frac{\lambda_3''}{4}(S^\dagger S)^2 + S^\dagger S[\lambda_1' \Phi_1^\dagger \Phi_1 + \lambda_2' \Phi_2^\dagger \Phi_2] + [S^2(\lambda_4' \Phi_1^\dagger \Phi_1 + \lambda_5' \Phi_2^\dagger \Phi_2) + h.c.]$$

for this study: $\lambda_2'' = \lambda_1''$

Type II, Couplings to Fermions

Down-type quarks	Leptons	Up-type quarks
Φ_1	Φ_1	Φ_2

V_{2HDMS} Symmetries

$\Phi_j \xrightarrow{U(1)} e^{i\theta}\Phi_j$ $\Phi_j^\dagger \xrightarrow{U(1)} e^{-i\theta}\Phi_j^\dagger$	avoids charge-parity violation
$\Phi_1 \xrightarrow{Z_2} -\Phi_1$ $\Phi_2 \xrightarrow{Z_2} \Phi_2$ (softly broken by m_{12}^2)	avoids flavour changing neutral currents
$\Phi_j \xrightarrow{Z_2'} \Phi_j$ $S \xrightarrow{Z_2'} -S$	stabilization of DM

2HDMS Type II, Higgs Sector Potential

[Notation as in: Baum and Shah, arXiv: 1808.02667]

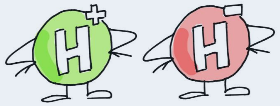
$$\Phi_i = \begin{pmatrix} \phi_i^+ \\ \frac{1}{\sqrt{2}}(v_i + \rho_i + i\eta_i) \end{pmatrix} \qquad \langle \Phi_i \rangle = \begin{pmatrix} 0 \\ \frac{v_i}{\sqrt{2}} \end{pmatrix}$$

$$S = \frac{1}{\sqrt{2}}(v_S + \rho_S + iA_S) \qquad \langle S \rangle = \frac{v_S}{\sqrt{2}}$$

DM Candidate

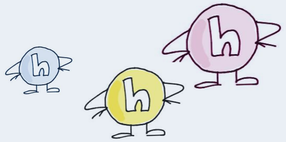
DM Candidate Properties:
massive
electrically neutral
colourless
stable

Higgs Sector particles:



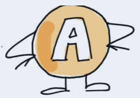
2 charged: $H_{\pm}$,

1 charged GB: $G_{\pm}$,



3 scalars: h_1, h_2, h_3,

SM-like (h_2)



1 pseudo scalar: A,

1 pseudo scalar GB: G_0,

1 DM: A_S

This model is different from 2HDMa, where the pseudoscalar is the DM

2HDMS Type II, Higgs Sector Potential

[Notation as in: Baum and Shah, arXiv: 1808.02667]

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = R(\alpha_{1,2,3}) \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}$$

$$v = \sqrt{v_1^2 + v_2^2}$$

$$\tan(\beta) = \frac{v_2}{v_1}$$

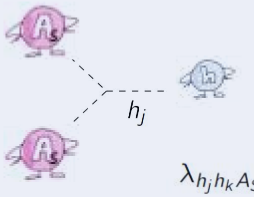
$$\tilde{\mu}^2 = \frac{m_{12}^2}{\sin\beta\cos\beta}$$

$$\lambda'_{14} = \lambda'_1 - 2\lambda'_4$$

$$\lambda'_{25} = \lambda'_2 - 2\lambda'_5$$

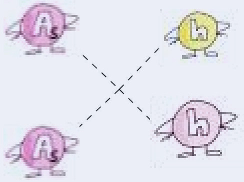
$$\lambda''_{13} = \lambda''_1 - \lambda''_3$$

DM to Higgs portal couplings



$$\lambda_{h_j h_k A_S A_S} = \frac{\partial^4 V}{\partial h_j \partial h_k \partial A_S \partial A_S}$$

$$= -i[(\lambda'_1 - 2\lambda'_4)R_{j1}R_{k1} + (\lambda'_2 - 2\lambda'_5)R_{j2}R_{k2} - \frac{1}{2}(\lambda''_1 - \lambda''_3)R_{j3}R_{k3}]$$



$$\lambda_{h_j h_k A_S A_S} = \frac{\partial^4 V}{\partial h_j \partial h_k \partial A_S \partial A_S}$$

$$= -i[(\lambda'_1 - 2\lambda'_4)R_{j1}R_{k1} + (\lambda'_2 - 2\lambda'_5)R_{j2}R_{k2} - \frac{1}{2}(\lambda''_1 - \lambda''_3)R_{j3}R_{k3}]$$

DM mass:

$$m_{A_S}^2 = \frac{\partial^2 V}{\partial A_S^\dagger \partial A_S}\Big|_{\Phi_1 = \langle\Phi_1\rangle\ \Phi_2 = \langle\Phi_2\rangle,\ S = \langle S\rangle}$$

$$= -(2m_S'^2 + v_S^2(\frac{\lambda''_1}{3} + \frac{\lambda''_1}{3}) + 2(\lambda'_4 v_1^2 + \lambda'_5 v_2^2))$$

2HDMS Basis Change

Interaction Basis Parameters:

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, m_{12}^2, \tan\beta, v_S, m_S^{2\prime},$
$\lambda'_1, \lambda'_2, \lambda'_4, \lambda'_5, \lambda''_1 = \lambda''_2, \lambda''_3$

⬇

Mass Basis Parameters:

$m_{h_1}, m_{h_2}, m_{h_3}, m_A, m_{A_S}, m_{H^\pm}, \alpha_1, \alpha_2, \alpha_3,$
$\tan\beta, v_S, \tilde{\mu}^2, \lambda'_{14}, \lambda'_{25}, \lambda''_{13}$

Benchmark scenarios

- Light 10 GeV < m_{DM} < 100 GeV
- Intermediate 100 GeV < m_{DM} < 1 TeV
- Heavy m_{DM} > 1 TeV

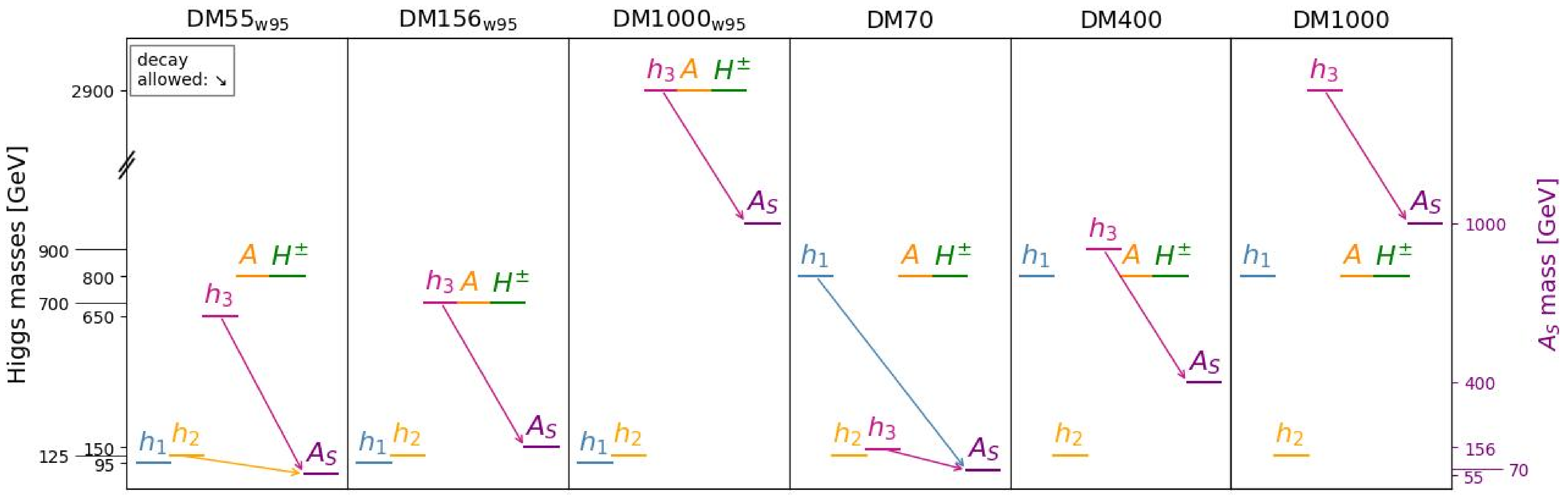
Dark Matter (DM) Phenomenology
(Relic Density,
Indirect Detection,
Direct Detection)

SARAH
SPheno
micrOMEGAs

Collider
Phenomenology
(HL-LHC,
Future Lepton
Colliders)

Pythia
Delphes
MadAnalysis
WHIZARD
Madgraph

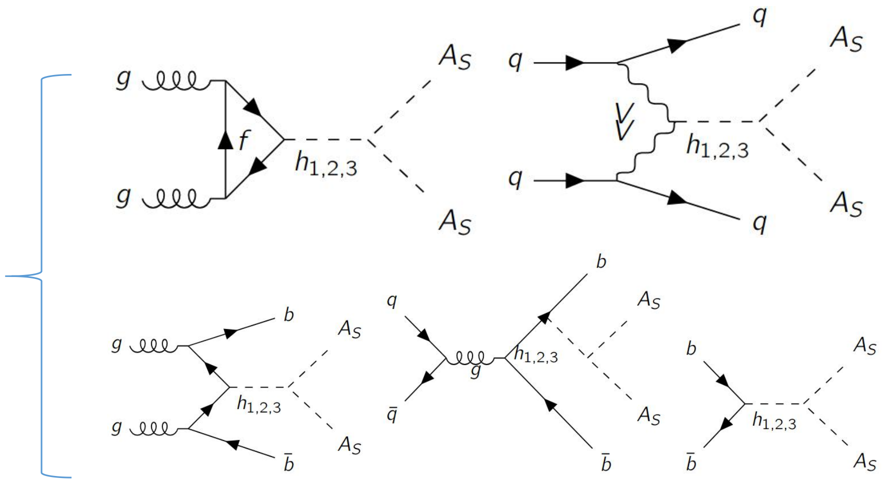
Constraints:
- Theoretical: bfb, unitarity
- Experimental: HiggsBounds, Planck, LUX-ZEPLIN, Fermi-LAT
*including 95 GeV excess



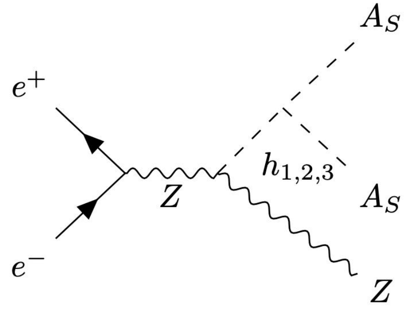
- The DM55 and DM70 couple with h_{125}, which can be interesting to probe at Higgs factory of $e^+ e^-$ collider

Collider Phenomenology

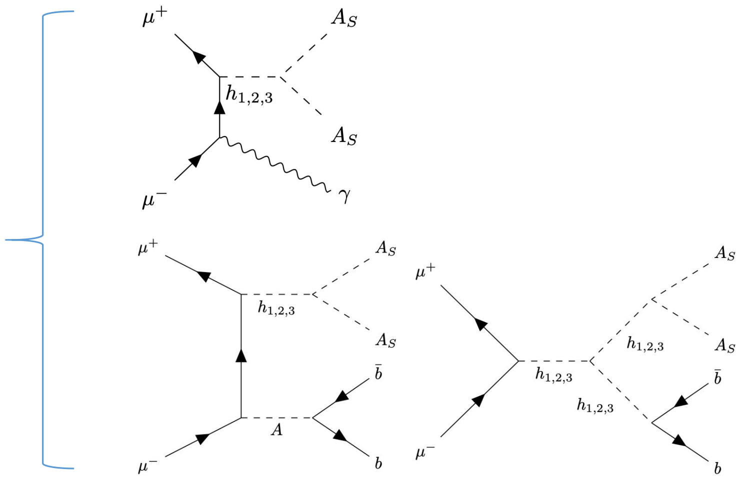
Hadron Collider
(HL-LHC)



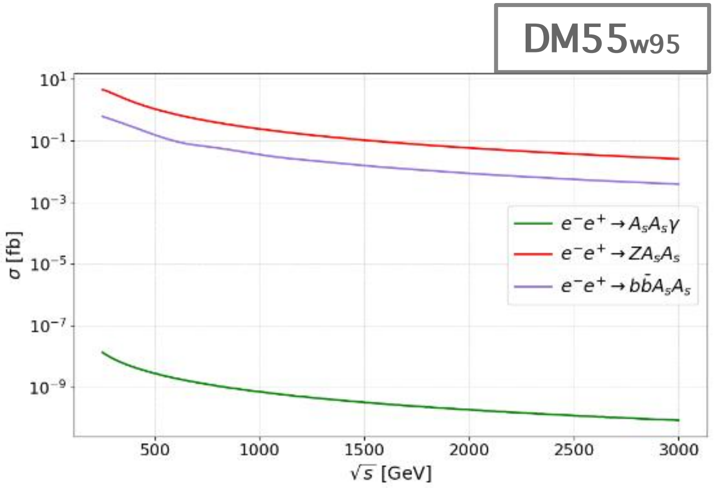
e+e- Colliders
(CEPC, ILC, FCC-ee, CLIC)

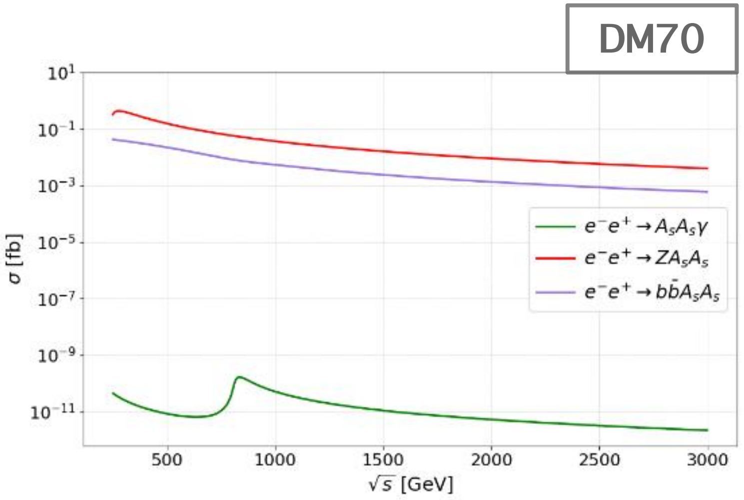


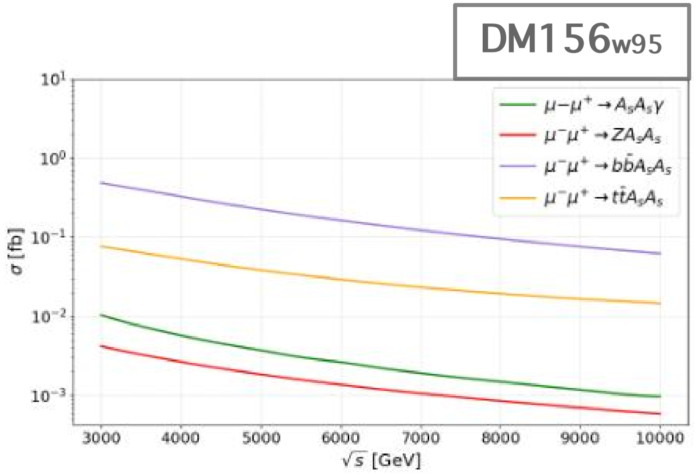
μ+μ- Collider

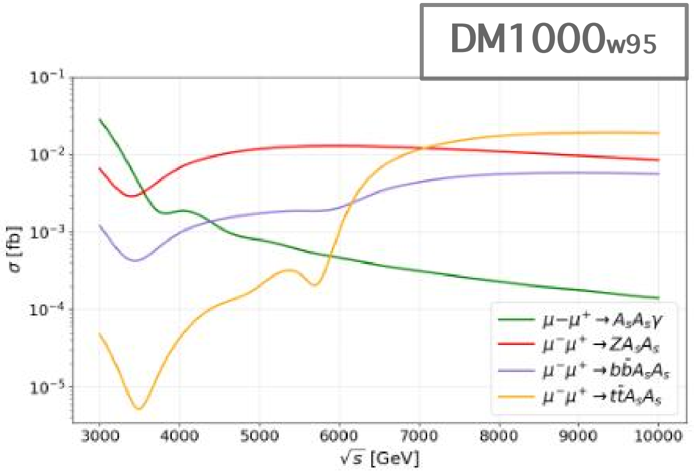


Collider Phenomenology, Lepton colliders







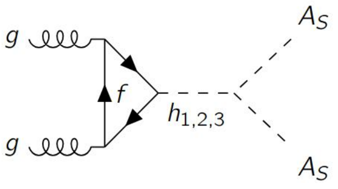


- The mono-Z and b-associated processes of light scenarios are dominant at e^+e^- colliders Higgs factory
- The heavier scenarios would have a better b/t-associated production cross-section at muon collider

Collider Phenomenology, HL-LHC

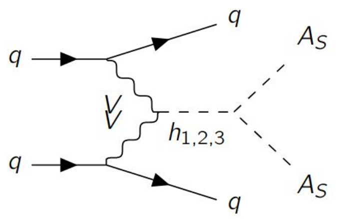
Process	Production cross-section (fb) at $\sqrt{s} = 14$ TeV		
	DM55$_{\mathbf{w95}}$	**DM156**$_{\mathbf{w95}}$	**DM70**
GGF$(h_2 \to A_S A_S)$	533.9	-	19.29×10^3
GGF$(h_3 \to A_S A_S)$	-	0.015	-
VBF$(h_2 \to A_S A_S)$	54.33	-	2.72×10^3
VBF$(h_3 \to A_S A_S)$	-	0.134	0.0022
BBH $((b\bar{b}h_2 \to A_S A_S))$	21.6	-	0.137
BBH $((b\bar{b}h_3 \to A_S A_S))$	-	47.24	-

Process	Production cross-section (fb) at $\sqrt{s} = 14$ TeV		
	DM400	**DM1000**	**DM1000**$_{\mathbf{w95}}$
GGF$(h_3 \to A_S A_S)$	0.013	6.35×10^{-7}	4.5×10^{-6}
VBF$(h_3 \to A_S A_S)$	0.0008	-	-
BBH$(h_3 \to A_S A_S)$	0.007	-	-



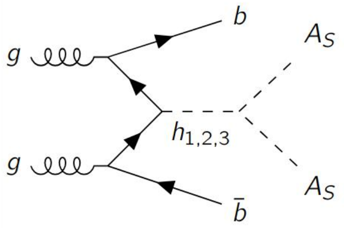
GGF + Mono jet

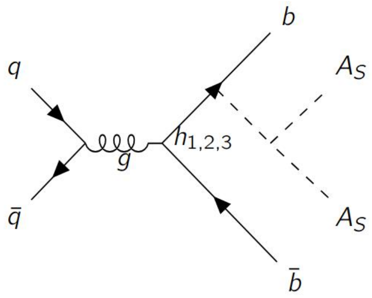
Benchmark	Significance
DM55$_{\mathbf{w95}}$	0.30σ
DM70	0.55σ

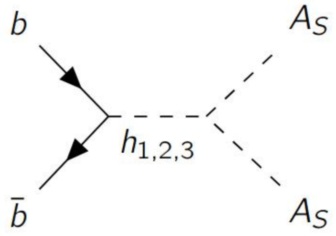


VBF

Benchmark	Significance
DM70	1.94σ







BBH

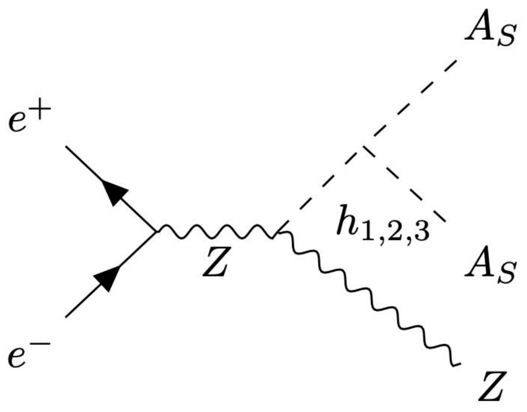
Benchmark	Significance
DM156$_{\mathbf{w95}}$	1.95σ

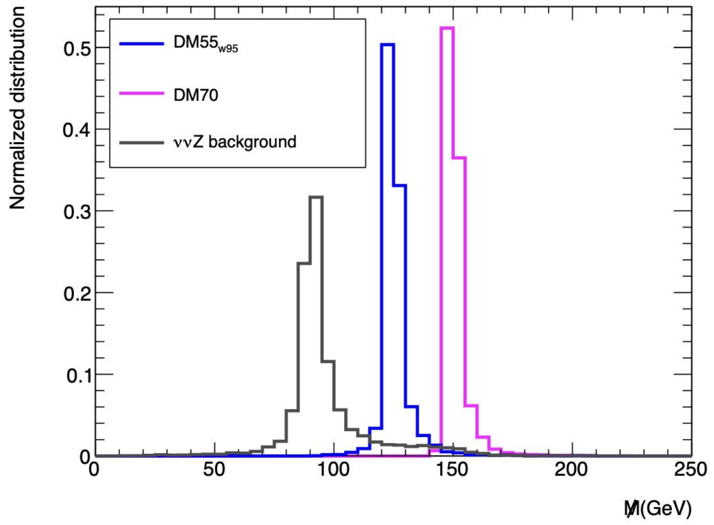
- DM55 has low significance at hardon collider
- DM70 has large cross-section via off-shell h_{125} , while the significance is still lower than 2σ

Collider Phenomenology, e^+e^- Colliders

Z+MET

Benchmark	Production cross-section (fb)		
	at $\sqrt{s} = 250$ GeV	at $\sqrt{s} = 500$ GeV	at $\sqrt{s} = 1$ TeV
DM55$_{\mathbf{w95}}$	4.42	1.1	0.24
DM70	0.33	0.15	0.035
$\nu\bar{\nu}Z$ background	503	491	950





Benchmark	$\sqrt{s}$	Cut	Significance
DM55$_{\mathbf{w95}}$	250 GeV	$\not{M} > 100$ GeV	11σ ($1ab^{-1}$)
DM70	250 GeV	$\not{M} > 130$ GeV	3σ ($3ab^{-1}$)
DM55$_{\mathbf{w95}}$	500 GeV	$\not{M} > 100$ GeV and $\not{M} < 150$ GeV	3.6σ ($1ab^{-1}$)
DM70	500 GeV	$\not{M} > 140$ GeV and $\not{M} < 190$ GeV	1.5σ ($3ab^{-1}$)
DM55$_{\mathbf{w95}}$	1 TeV	$\not{M} > 120$ GeV and $\not{M} < 250$ GeV	2.4σ ($3ab^{-1}$)
DM70	1 TeV	$\not{M} > 120$ GeV and $\not{M} < 250$ GeV	0.36σ ($3ab^{-1}$)

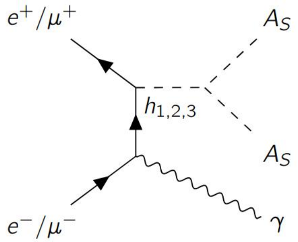
- e^+e^- colliders can be particularly sensitive to the light DM scenarios with h_{125} as mediator

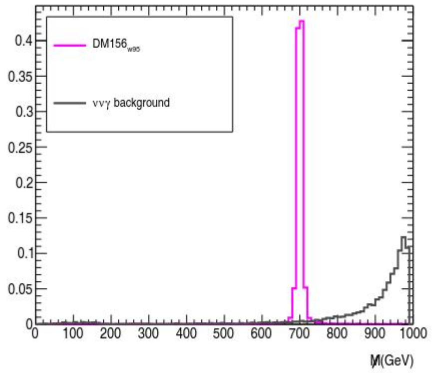
Collider Phenomenology, $\mu^+\mu^-$ Collider

Mono photon

Benchmark	Production cross-section (fb)at $\sqrt{s} = 1$ TeV
$\mathbf{DM156_{w95}}$	0.23
$\nu\nu\gamma$ background	2.45

$\mathbf{DM156_{w95}}$	690 GeV $< \not{M} <$ 710 GeV	3σ $(3ab^{-1})$, 5.3σ $(10ab^{-1})$

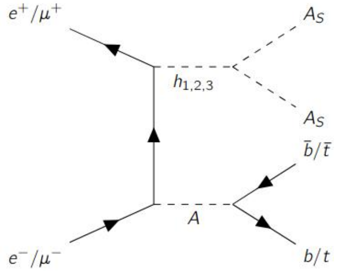


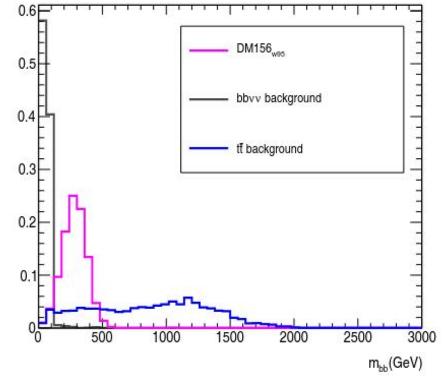


bb+MET

Benchmark	Production cross-section (fb)	
	at $\sqrt{s} = 3$ TeV	at $\sqrt{s} = 10$ TeV
$\mathbf{DM156_{w95}}$	0.48	0.063
$b\bar{b}\nu\nu$ background	758	1.3
$t\bar{t}$ background	20	1.7

Benchmark	Cut	Significance
$\mathbf{DM156_{w95}}$	100 GeV $< m_{bb} <$ 500 GeV	6.3σ $(3ab^{-1})$

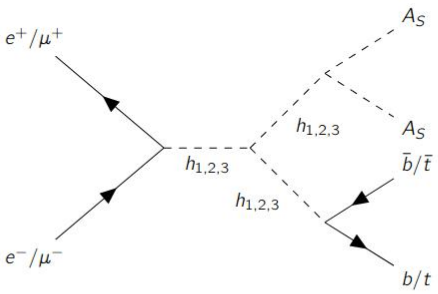


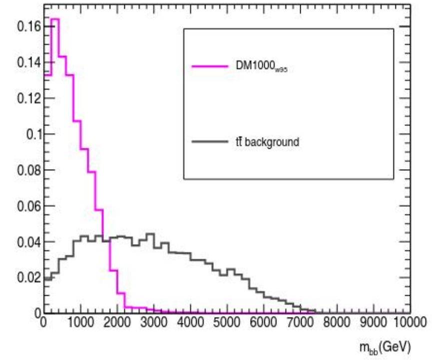


tt+MET

Benchmark	Production cross-section (fb) at $\sqrt{s} = 10$ TeV
$\mathbf{DM1000_{w95}}$	0.027
$t\bar{t}$+MET background	1.66

Benchmark	Cut	Significance
$\mathbf{DM1000_{w95}}$	$m_{bb} < 2$ TeV	2.9σ $(10ab^{-1})$





Collider Phenomenology, Challenging Scenarios

$\mu^+ \rightarrow \leftarrow \mu^-$

DM400

Final state	Production cross-section (fb) at muon collider	
	at $\sqrt{s} = 3$ TeV	at $\sqrt{s} = 10$ TeV
γ+MET	5.3×10^{-7}	4.9×10^{-8}
Z+MET	1.1×10^{-5}	1.5×10^{-6}
$b\bar{b}$+MET	2.7×10^{-3}	4.5×10^{-3}
$t\bar{t}$+MET	3.7×10^{-3}	8.9×10^{-3}

DM1000

Final state	Production cross-section (fb) at muon collider	
	at $\sqrt{s} = 3$ TeV	at $\sqrt{s} = 10$ TeV
γ+MET	3.5×10^{-9}	1.3×10^{-10}
Z+MET	4.4×10^{-8}	2.2×10^{-6}
$b\bar{b}$+MET	3.7×10^{-8}	2.0×10^{-5}
$t\bar{t}$+MET	7.8×10^{-9}	3.7×10^{-5}

LHC

Process	Production cross-section (fb) at $\sqrt{s} = 14$ TeV	
	DM400	**DM1000**
GGF	0.016	1.27×10^{-4}
VBF	0.001	4.7×10^{-6}
BBH	0.008	1.96×10^{-6}

FCC-hh

Process	Production cross-section (fb) at $\sqrt{s} = 100$ TeV	
	DM400	**DM1000**
GGF	1.456	0.117
VBF	0.039	1.182
BBH	0.264	0.029

- The scenario with the singlet Higgs as the DM mediator would be difficult for lepton colliders

Conclusions

Process	Benchmark	Significance
GGF	**DM55**$_{w95}$	0.30σ
GGF	**DM70**	0.55σ
VBF	**DM70**	1.94σ
BBH	**DM156**$_{w95}$	1.95σ

- DM search at future colliders:
    - 6 BPs (light, intermediate, heavy)
    - Hardon, electron, muon colliders

$e^+ \rightarrow \leftarrow e^-$

Final State	Benchmark	$\sqrt{s}$	Significance
Z+MET	**DM55**$_{w95}$	250 GeV	11σ ($1\,ab^{-1}$)
Z+MET	**DM70**	250 GeV	3σ ($3\,ab^{-1}$)
Z+MET	**DM55**$_{w95}$	500 GeV	3.6σ ($1\,ab^{-1}$)
Z+MET	**DM70**	500 GeV	1.5σ ($3\,ab^{-1}$)
Z+MET	**DM55**$_{w95}$	1 TeV	2.4σ ($3\,ab^{-1}$)
Z+MET	**DM70**	1 TeV	0.36σ ($3\,ab^{-1}$)

- Best prospects for light BPs at e^+e^- colliders Higgs factory (e.g. CEPC), via on-shell or off-shell h_{125}

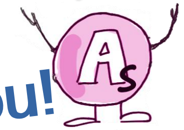
- Best prospects for intermediate and heavy BPs at $\mu^+\mu^-$ colliders, via heavy doublet Higgs

$\mu^+ \rightarrow \leftarrow \mu^-$

Final State	Benchmark	$\sqrt{s}$	Significance
$b\bar{b}$+MET	**DM156**$_{w95}$	3 TeV	6.3σ ($3\,ab^{-1}$)
γ+MET	**DM156**$_{w95}$	1 TeV	3σ ($3ab^{-1}$), 5.3σ ($10ab^{-1}$)
$t\bar{t}$+MET	**DM1000**$_{w95}$	10 TeV	2.9σ ($10\,ab^{-1}$)

- Challenging scenarios probably require much higher energy (e.g. FCC-hh)

Thank you!



Backup

m_{h_1}	m_{h_2}	m_{h_3}	m_A	$m_{H^\pm}$	χ^2
95.4 GeV	125.09 GeV	650 GeV	800 GeV	800 GeV	1.26
m_{A_S}	$\lambda_1' - 2\lambda_4'$	$\lambda_2' - 2\lambda_5'$	$\lambda_1'' - \lambda_3''$	$\tan\beta$	
55.596 GeV	0.0020912	0.00074611	−0.025735	2	
v_S	$\tilde{\mu}$	α_1	α_2	α_3	
300 GeV	650 GeV	−1.932	1.272	1.484	

Table 23: The benchmark point $\mathbf{DM55_{w95}}$ in the mass basis.

m_{h_1}	m_{h_2}	m_{h_3}	m_A	$m_{H^\pm}$	χ^2
95.4 GeV	125.09 GeV	700 GeV	700 GeV	700 GeV	0.422
m_{A_S}	$\lambda_1' - 2\lambda_4'$	$\lambda_2' - 2\lambda_5'$	$\lambda_1'' - \lambda_3''$	$\tan\beta$	
156 GeV	12.753	−0.31351	−2.6747	6.6	
v_S	$\tilde{\mu}$	α_1	α_2	α_3	
239.86 GeV	700 GeV	1.4661	1.1920	−1.5989	

Table 24: The benchmark point $\mathbf{DM156_{w95}}$ in the mass basis.

m_{h_1}	m_{h_2}	m_{h_3}	m_A	$m_{H^\pm}$	χ^2
95.4 GeV	125.09 GeV	2950 GeV	2950 GeV	2950 GeV	2.13
m_{A_S}	$\lambda_1' - 2\lambda_4'$	$\lambda_2' - 2\lambda_5'$	$\lambda_1'' - \lambda_3''$	$\tan\beta$	
1000 GeV	21.231	0	−1.4153	5	
v_S	$\tilde{\mu}$	α_1	α_2	α_3	
10005 GeV	2949.29 GeV	−1.769	1.250	1.569	

Table 25: The benchmark point $\mathbf{DM1000_{w95}}$ in the mass basis.

m_{h_1}	m_{h_2}	m_{h_3}	m_A	$m_{H^\pm}$
800 GeV	125.09 GeV	150 GeV	800 GeV	800 GeV
m_{A_S}	$\lambda_1' - 2\lambda_4'$	$\lambda_2' - 2\lambda_5'$	$\lambda_1'' - \lambda_3''$	$\tan\beta$
70 GeV	−0.10783	0.063127	−0.47818	1.3728
v_S	$\tilde{\mu}$	α_1	α_2	α_3
219.05 GeV	751.54 GeV	−0.60016	0.042445	−0.054807

Table 26: The benchmark point **DM70** in the mass basis.

m_{h_1}	m_{h_2}	m_{h_3}	m_A	$m_{H^\pm}$
800 GeV	125.09 GeV	900 GeV	800 GeV	800 GeV
m_{A_S}	$\lambda_1' - 2\lambda_4'$	$\lambda_2' - 2\lambda_5'$	$\lambda_1'' - \lambda_3''$	$\tan\beta$
400 GeV	0.077784	0.036923	−0.42725	2.1309
v_S	$\tilde{\mu}$	α_1	α_2	α_3
587.17 GeV	755.39 GeV	−0.41245	−0.0086501	−0.0055431

Table 27: The benchmark point **DM400** in the mass basis.

m_{h_1}	m_{h_2}	m_{h_3}	m_A	$m_{H^\pm}$
800 GeV	125.09 GeV	2900 GeV	800 GeV	800 GeV
m_{A_S}	$\lambda_1' - 2\lambda_4'$	$\lambda_2' - 2\lambda_5'$	$\lambda_1'' - \lambda_3''$	$\tan\beta$
1000 GeV	0.32873	0.21320	−0.41541	1.3414
v_S	$\tilde{\mu}$	α_1	α_2	α_3
2271.3 GeV	768.14 GeV	−0.54917	0.036530	−0.056095

Table 28: The benchmark point **DM1000** in the mass basis.

Backup, Collider Pheno, HL-LHC

$$\not{M}^2 = (p_{in} - p_{out})^2$$

$$\sigma^{w_i} = \frac{\sigma^{4f} + w_i \sigma^{5f}}{1 + w_i}$$

$$w_i = ln(\frac{m_{h_i}}{m_b}) - 2$$

Benchmark	Cross-section after cuts (fb)
DM156$_{w95}$	0.357
SM Background	
$b\bar{b}Z$	18.3
$b\bar{b}\nu\bar{\nu}$	13.46
$t\bar{t}$	66.46
$Z+j$	2.04
hZ	0.012
Total Background	100.27

Table 8: The cross-sections for the signal and backgrounds after applying the cuts **E1-E4** as discussed in the text for signal-background distinction for BBH for HL-LHC at an integrated luminosity of 3000 fb^{-1}.

Gluon Fusion

We consider the final state mono-jet + MET from the gluon fusion production channel. For the collider analyses, we use the following cuts [70]:

- **C1**: The final state consists of up to four jets with $p_T > 30$ GeV and $|\eta| < 2.8$.
- **C2**: We demand a large $\not{E}_T > 250$ GeV.
- **C3**: The hardest leading jet has $p_T > 250$ GeV with $|\eta| < 2.4$.
- **C4**: We demand $\Delta\Phi(j, \not{E}_T) > 0.4$ for all jets and $\Delta\Phi(j, \not{E}_T) > 0.6$ for the leading jet.
- **C5**: A lepton-veto is imposed for electrons with $p_T > 20$ GeV and $|\eta| < 2.47$ and muons with $p_T > 10$ GeV and $|\eta| < 2.5$.

The SM background of 7.07 pb is obtained from the mono-jet + $\not{E}_T$ search studied in Ref. [71].

Vector Boson Fusion

We consider the final state two forward-jets + MET from the vector boson fusion production channel. For the collider analyses, we use the following cuts [72]:

- **D1**: The final state consists of at least two jets with $p_T(j_1) > 80$ and $p_T(j_2) >$ 40 GeV and $\Delta\Phi(j_i, \not{E}_T) > 0.5$.
- **D2**: We demand $\eta(j_1 j_2) < 0$ and $\Delta\Phi j_1 j_2 < 1.5$.
- **D3**: We demand $|\Delta\eta|_{jj} > 3.0$.
- **D4**: The invariant mass of the two forward jets is required to be large, i.e, $M_{jj} > 600$ GeV.
- **D5**: We demand $\not{E}_T > 200$ GeV.
- **D6**: Furthermore, a lepton veto is imposed for electrons with $p_T > 20$ GeV or muons with $p_T > 10$ GeV.

$b\bar{b}$ Higgs associated production

- **E1**: The final state consists of two b jets and no photons or leptons. We demand $\Delta R(b_1, b_2) > 0.4$, $p_T(b_1) > 150$ GeV and $p_T(b_2) > 100$ GeV.
- **E2**: We demand a large missing transverse momenta (MET) $\not{E}_T > 200$ GeV to reduce SM background.
- **E3**: We demand the invariant mass of the $b\bar{b}$ pair (as seen in Fig. 5) is outside the Z (76 GeV $< M(b\bar{b}) <$ 105 GeV) or SM Higgs mass window (115 GeV $< M(b\bar{b}) <$ 135 GeV) to remove background contributions from on-shell Z or Higgs bosons.
- **E4**: Further, we demand $M(b\bar{b}) > 200$ GeV to reduce SM background contributions.

Backup

$$0 = \frac{\partial V}{\partial \Phi_1}\Big|_{\substack{\Phi_1=\langle\Phi_1\rangle \\ \Phi_2=\langle\Phi_2\rangle \\ S=\langle S\rangle}} = \frac{1}{\sqrt{2}}[m_{11}^2 v_1 - m_{12}^2 v_2 + \frac{\lambda_1}{2}v_1^3 + \frac{\lambda_{345}}{2}v_1 v_2^2 + (\frac{\lambda_1'}{2}v_1 + \lambda_4' v_1)v_S^2] \tag{A.1a}$$

$$0 = \frac{\partial V}{\partial \Phi_2}\Big|_{\substack{\Phi_1=\langle\Phi_1\rangle \\ \Phi_2=\langle\Phi_2\rangle \\ S=\langle S\rangle}} = \frac{1}{\sqrt{2}}[m_{22}^2 v_2 - m_{12}^2 v_1 + \frac{\lambda_2}{2}v_2^3 + \frac{\lambda_{345}}{2}v_1^2 v_2 + (\frac{\lambda_2'}{2}v_2 + \lambda_5' v_2)v_S^2] \tag{A.1b}$$

$$0 = \frac{\partial V}{\partial S}\Big|_{\substack{\Phi_1=\langle\Phi_1\rangle \\ \Phi_2=\langle\Phi_2\rangle \\ S=\langle S\rangle}} = \frac{1}{\sqrt{2}}[m_S^2 v_S + m_S'^2 v_S + \frac{\lambda_1''}{12}v_S^3 + \frac{\lambda_2''}{3}v_S^3 + \frac{\lambda_3''}{4}v_S^3 + \frac{v_S}{2}(\lambda_1' v_1^2 + \lambda_2' v_2^2) + v_S(\lambda_4' v_1^2 + \lambda_5' v_2^2)]. \tag{A.1c}$$

$$R = \begin{pmatrix} c_{\alpha_1}c_{\alpha_2} & s_{\alpha_1}c_{\alpha_2} & s_{\alpha_2} \\ -s_{\alpha_1}c_{\alpha_3} - c_{\alpha_1}s_{\alpha_2}s_{\alpha_3} & c_{\alpha_1}c_{\alpha_3} - s_{\alpha_1}s_{\alpha_2}s_{\alpha_3} & c_{\alpha_2}s_{\alpha_3} \\ s_{\alpha_1}s_{\alpha_3} - c_{\alpha_1}s_{\alpha_2}c_{\alpha_3} & -c_{\alpha_1}s_{\alpha_3} - s_{\alpha_1}s_{\alpha_2}c_{\alpha_3} & c_{\alpha_2}c_{\alpha_3} \end{pmatrix}$$

$$\begin{aligned}
m_{12}^2 &= \tilde{\mu}^2 \cdot \sin\beta\cos\beta \\
\lambda_1 &= \frac{1}{v^2\cos^2\beta}(\Sigma_{i=1}^3 m_i^2 R_{i1}^2 - \tilde{\mu}^2\sin^2\beta), \\
\lambda_2 &= \frac{1}{v^2\sin^2\beta}(\Sigma_{i=1}^3 m_i^2 R_{i2}^2 - \tilde{\mu}^2\cos^2\beta), \\
\lambda_3 &= \frac{1}{v^2}(\frac{1}{\sin\beta\cos\beta}\Sigma_{i=1}^3 m_i^2 R_{i1}R_{i2} - \tilde{\mu}^2 + 2m_{H^\pm}^2), \\
\lambda_4 &= \frac{1}{v^2}(m_A^2 + \tilde{\mu}^2 - 2m_{H^\pm}^2), \\
\lambda_5 &= \frac{1}{v^2}(-m_A^2 + \tilde{\mu}^2), \\
\lambda_1' &= \frac{1}{2}(\frac{1}{vv_S\cos\beta}\Sigma_{i=1}^3 m_i^2 R_{i1}R_{i3} + \lambda_{14}'), \\
\lambda_2' &= \frac{1}{2}(\frac{1}{vv_S\sin\beta}\Sigma_{i=1}^3 m_i^2 R_{i2}R_{i3} + \lambda_{25}'), \\
\lambda_4' &= \frac{1}{4}(\frac{1}{vv_S\cos\beta}\Sigma_{i=1}^3 m_i^2 R_{i1}R_{i3} - \lambda_{14}'), \\
\lambda_5' &= \frac{1}{4}(\frac{1}{vv_S\sin\beta}\Sigma_{i=1}^3 m_i^2 R_{i2}R_{i3} - \lambda_{25}'), \\
\lambda_1'' &= \frac{3}{4v_S^2}(\Sigma_{i=1}^3 m_i^2 R_{i3}^2 + \frac{v_S^2}{2}\lambda_{13}''), \\
\lambda_2'' &= \lambda_1'', \\
\lambda_3'' &= \frac{3}{4v_S^2}(\Sigma_{i=1}^3 m_i^2 R_{i3}^2 + \frac{5v_S^2}{6}\lambda_{13}''), \\
m_S'^2 &= -(\frac{1}{2}m_{A_S}^2 + \frac{1}{4}\Sigma_{i=1}^3 m_i^2(R_{i3}^2 + R_{i1}R_{i3}\frac{v\cos\beta}{v_S} + R_{i2}R_{i3}\frac{v\sin\beta}{v_S}) \\
&\quad - \frac{v^2}{4}(\lambda_{14}'\cos^2\beta + \lambda_{25}'\sin^2\beta) + \frac{v_S^2}{8}\lambda_{13}'')
\end{aligned}$$