Higher-order ISR corrections to electron-positron annihilation processes

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The 2025 International Workshop
on the High Energy Circular Electron Positron Collider

4 — 10 November 2025

8th November 2025

Outline

1. Motivation
2. Partons in QED
3. Higher-order ISR
4. Factorization scale choice
5. Outlook

Motivation

- Development of physical programs for future high-energy e^+e^- colliders and not only
- Having high-precision theoretical description of basic e^+e^- processes is of crucial importance
- Two-loop QED and EW calculations are in progress, but higher-order QED corrections are also relevant
- The formalism of QED parton distribution functions gives a fast estimate of the bulk of higher-order effects

The talk is based on the series of works [A.A., U.Voznaya, JPG'2023, PRD'2024, arXiv:2511.00437]

Partons in QED

The QED leading (LO) logarithmic corrections

$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^n \frac{s}{m_e^2}$$

were relevant for LEP measurements of Bhabha, $e^+e^- \to \mu^+\mu^-$ etc. for $n \leq 3$ since $\ln(M_Z^2/m_e^2) \approx 24$

- E. Kuraev and V. Fadin 1985
- A. De Rujula, R.Petronzio, A.Savoy-Navarro 1979

NLO contributions

$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^{n-1} \frac{s}{m_e^2}$$

with $n \geq 3$ being required for future e^+e^- experiments

In the collinear approximation we can get them within the NLO QED structure function (PDF) formalism

- F.A. Berends, W.L. van Neerven, G.J. Burgers, NPB'1988
- A.A., K. Melnikov, PRD'2002; A.A. JHEP'2003

News: NNLO QED PDFs

There are two recent works

[1] M. Stahlhofen
NNLO electron structure functions (PDFs) from SCET
arXiv:2508.16964 [hep-ph]

[2] M. Schnubel and R. Szafron
Electron and Photon Structure Functions at Two Loops
arXiv:2509.09618 [hep-ph]

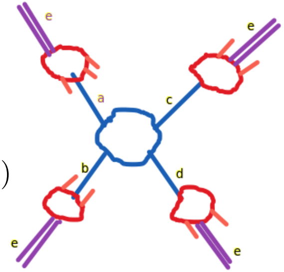
where electron and photon QED PDFs were computed in $\mathcal{O}\left(\alpha^2 L^0\right)$

Comparisons with QCD to be still performed

QED NLO master formula

The NLO Bhabha cross section reads

$$d\sigma = \sum_{a,b,c,d=e,\bar{e},\gamma} \int_{\bar{z}_1}^{1} dz_1 \int_{\bar{z}_2}^{1} dz_2 \mathcal{D}_{ae}^{\text{str}}(z_1)\mathcal{D}_{b\bar{e}}^{\text{str}}(z_2)$$
$$\times \left[d\sigma^{(0)}_{ab\to cd}(z_1,z_2) + d\bar{\sigma}^{(1)}_{ab\to cd}(z_1,z_2) \right]$$
$$\times \int_{\bar{y}_1}^{1} \frac{dy_1}{Y_1} \int_{\bar{y}_2}^{1} \frac{dy_2}{Y_2} \mathcal{D}_{ec}^{\text{frg}}\left(\frac{y_1}{Y_1}\right) \mathcal{D}_{\bar{e}d}^{\text{frg}}\left(\frac{y_2}{Y_2}\right)$$
$$+\mathcal{O}\left(\alpha^n L^{n-2}, \frac{m_e^2}{s}\right)$$



Upgrade to NNLO is straightforward

QED NLO DGLAP evolution equations

$$\mathcal{D}_{ba}\left(x, \frac{\mu_R^2}{\mu_F^2}\right) = \delta_{ab}\delta(1-x) + \sum_{c=e,\gamma,\bar{e}} \int\limits_{\mu_R^2}^{\mu_F^2} \frac{dt}{t} \int\limits_{x}^{1} \frac{dy}{y} P_{bc}(y,t) \mathcal{D}_{ca}\left(\frac{x}{y}, \frac{\mu_R^2}{t}\right)$$

a, b, c are massless partons ($\sim e^{\pm}, \gamma$)

μ_F is the factorization (energy) scale

μ_R is the renormalization (energy) scale

D_{ba} is the parton density function (PDF)

P_{bc} are splitting function or kernels of the DGLAP equation

QED splitting functions

The perturbative splitting functions are

$$P_{ba}(x, \bar{\alpha}(t)) = \frac{\bar{\alpha}(t)}{2\pi} P^{(0)}_{ba}(x) + \left(\frac{\bar{\alpha}(t)}{2\pi}\right)^2 P^{(1)}_{ba}(x) + \mathcal{O}(\alpha^3)$$

$$\text{e.g.} \quad P^{(0)}_{ee}(x) = \left[\frac{1+x^2}{1-x}\right]_+$$

They come from loop calculations, e.g., $P^{(1)}_{ba}(x)$ comes from 2-loops

The splitting functions can be obtained by reduction of the ones known in QCD to the abelian case of QED

$\bar{\alpha}(t)$ is the QED running coupling constant in the $\overline{\text{MS}}$ scheme

N.B. Factorization in $\bar{\alpha}(t) \times P_{ba}(x)$ is not unique

Running coupling constant

Compare QED-like

$$\bar{\alpha}(t) = \alpha\left\{1 + \frac{\alpha}{2\pi}\left(\frac{2}{3}L - \frac{10}{9}\right) + \left(\frac{\alpha}{2\pi}\right)^2\left(\frac{4}{9}L^2 - \frac{13}{27}L + \ldots\right) + \ldots\right\}$$

and QCD-like

$$\bar{\alpha}(t) = \frac{4\pi}{\beta_0 \ln(t/\Lambda^2)}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln[\ln(t/\Lambda^2)]}{\ln(t/\Lambda^2)} + \ldots\right]$$

Note that "$-10/9$" could have been hidden into Λ

In QED $\beta_0 = -4/3$ and $\beta_1 = -4$

Iterative solution

The NLO “electron in electron” PDF reads [A.A., U.Voznaya, JPG 2023]

$$\begin{aligned}
\mathcal{D}_{ee}(x,\mu_F,m_e) &= \delta(1-x) + \frac{\alpha}{2\pi} L P^{(0)}_{ee}(x) + \frac{\alpha}{2\pi} d^{(1)}_{ee}(x,m_e,m_e) \\
&+ \left(\frac{\alpha}{2\pi}\right)^2 L^2 \left(\frac{1}{2} P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \frac{1}{3} P^{(0)}_{ee}(x) + \frac{1}{2} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e}(x)\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^2 L \left(P^{(0)}_{e\gamma} \otimes d^{(1)}_{\gamma e}(x,m_e,m_e) + P^{(0)}_{ee} \otimes d^{(1)}_{ee}(x,m_e,m_e) - \frac{10}{9} P^{(0)}_{ee}(x) + P^{(1)}_{ee}(x)\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^3 \left(\frac{1}{6} P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \frac{1}{6} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{\gamma e}(x) + \dots\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^2 \left(P^{(0)}_{ee} \otimes P^{(1)}_{ee}(x) + P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes d^{(1)}_{ee}(x,m_e,m_e) + \frac{1}{3} P^{(1)}_{ee}(x) - \frac{10}{9} P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \dots\right) \\
&+ \mathcal{O}(\alpha^2 L^0, \alpha^3 L^1) + \dots
\end{aligned}$$

The large logarithm $L \equiv \ln \frac{\mu_F^2}{\mu_R^2}$, $\alpha \equiv \alpha(\mu_R)$

$\mu_R = m_e$ is well motivated since we a hunting for the large logs

Difference in the LO $\mathcal{O}(\alpha^3 L^3)$ singlet pair contribution from M.Skrzypek'1992

N.B. Naïve re-collection of N_f-dependent terms into $\alpha(\mu_F^2)$ doesn't work

$\mathcal{O}(\alpha)$ matching

The expansion of the master formula for ISR gives

$$d\sigma^{(1)}_{e\bar{e}\to\gamma^*} = \frac{\alpha}{2\pi}\left\{2LP^{(0)}_{ee}\otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*} + 2d^{(1)}_{ee}\otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*}\right\} + d\,\bar{\sigma}^{(1)}_{e\bar{e}\to\gamma^*}$$

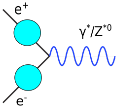
We know the massive $d\sigma^{(1)}$ and massless $d\,\bar{\sigma}^{(1)}$ ($m_e \equiv 0$ with $\overline{\mathrm{MS}}$ subtraction) results in $\mathcal{O}(\alpha)$ $\Rightarrow$

$$d^{(1)}_{ee} = \left[\frac{1+z^2}{1-z}\left(\ln\frac{\mu_R^2}{m_e^2} - 1 - \ln(1-z)\right)\right]_+, \quad P^{(0)}_{ee}(z) = \left[\frac{1+z^2}{1-z}\right]_+, \quad L = \ln\frac{\mu_F^2}{\mu_R^2}$$

Scheme dependence comes from here

Factorization and renormalization scale dependence is also from here

Drell-Yan process in QED: e^+e^- annihilation



$$\frac{d\sigma_{e^+e^-\to\gamma^*,Z}}{ds'} = \frac{1}{s}\sigma^{(0)}(s') \sum_{a,b=e^-,\gamma,e^+} D_{ae^-} \otimes D_{be^+} \otimes \tilde{\sigma}_{ab\to\gamma^*,Z}$$

$a\backslash b$	e^+	γ	e^-
e^-	$D_{e^-e^-}D_{e^+e^+}\sigma_{e^-e^+}$ LO (1)	$D_{\gamma e^-}D_{e^-e^-}\sigma_{e^-\gamma}$ NLO ($\alpha^2 L$)	$D_{e^-e^-}D_{e^-e^+}\sigma_{e^-e^-}$ NNLO ($\alpha^4 L^2$)
γ	$D_{\gamma e^-}D_{e^+e^+}\sigma_{e^+\gamma}$ NLO ($\alpha^2 L$)	$D_{\gamma e^-}D_{\gamma e^+}\sigma_{\gamma\gamma}$ NNLO ($\alpha^4 L^2$)	$D_{\gamma e^-}D_{e^-e^+}\sigma_{e^-\gamma}$ NLO ($\alpha^4 L^3$)
e^+	$D_{e^+e^-}D_{e^+e^+}\sigma_{e^+e^+}$ NNLO ($\alpha^4 L^2$)	$D_{e^+e^-}D_{\gamma e^+}\sigma_{e^+\gamma}$ NLO ($\alpha^4 L^3$)	$D_{e^+e^-}D_{e^-e^+}\sigma_{e^+e^-}$ LO ($\alpha^4 L^4$)

See [J. Ablinger, J. Blümlein, A. De Freitas and K. Schönwald, "Subleading Logarithmic QED Initial State Corrections to $e^+e^- \to \gamma^*/Z^{0^*}$ to $O(\alpha^6 L^5)$," NPB 955 (2020) 115045] and [A.A., U.Voznaya, PRD 2024]

Factorization at NLO

Remind the one-loop correction $\Rightarrow \mu_F^2 = s$

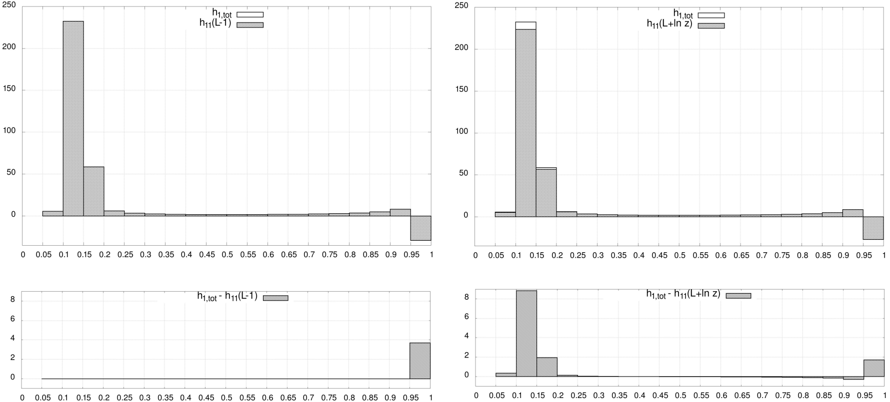
But Berends et al. and Blümlein et al. used $\mu_F^2 = zs$ in PDFs and $\mu_F^2 = ys \neq zs$ in

$$\frac{d\sigma^{(1)}_{e\bar{e}\to\gamma^*}}{d\sigma^{(0)}_{e\bar{e}\to\gamma^*}} = \frac{\alpha}{\pi}\left[\frac{1+y^2}{1-y}\right]_+ \left(\ln\frac{ys}{m_e^2} - 1\right) + \delta(1-y)(\dots)$$

where y is an integration variable in convolution $\overline{\sigma}^{(1)}_{e\bar{e}} \otimes P^{(0)}_{ee}$, sic!

Accidentally their “choice” leads to the proper result in $\mathcal{O}(\alpha^2 L^1)$. But it fails in higher orders.

Numercal example $e^+e^- \to \gamma^*, Z \to \mu^+\mu^-$



One-loop ISR corrections [%] at $\sqrt{s} = 240$ GeV vs. $\mu^+\mu^-$ invariant mass

ISR corrections to $e^+e^- \to Z(\gamma^*)$ ($\sqrt{s} = M_Z$)

LO $\mathcal{O}(\alpha^n L^n)$ and NLO $\mathcal{O}(\alpha^n L^{n-1})$ ISR corrections in % at the Z-peak for $z_{\min} = 0.1$

Type / n	1	2	3	4	5
LO γ	−32.7365	4.8843	−0.3776	0.0034	0.0032
NLO γ	2.0017	−0.5952	0.0710	−0.0019	
LO pair	—	−0.3057	0.0875	0.0016	−0.0001
NLO pair	—	0.1585	−0.0460	0.0038	
Σ	−30.7348	4.1419	−0.2651	0.0069	0.0031

Even higher orders seem to be relevant numerically $\Longrightarrow$ exponentiation

Exponentiation of the leading logs is straightforward and known [Gribov-Lipatov, Kuraev-Fadin, . . .]

NLO exponentiation in the MSbar scheme is ambiguous

Impact of new corrections on old LEP results?

ISR corrections to $e^+e^- \to Z(\gamma^*)$ ($\sqrt{s} = 240$ GeV)

LO $\mathcal{O}(\alpha^n L^n)$ and NLO $\mathcal{O}(\alpha^n L^{n-1})$ ISR corrections in % at 240 GeV for $z_{\min} = 0.1$

Type/ n	1	2	3	4	5
LO, γ	324.14638	27.12979	-0.62423	-0.06641	0.00123
NLO, γ	-11.76206	-0.99088	0.61404	0.01487	
LO, pairs	-	25.33367	-0.14006	-0.02686	
NLO, pairs	-	-1.47717	-0.02069	0.04599	
Σ	312.38432	49.99541	-0.17094	-0.03241	0.00123

Preliminary

Factorization scale choice

The final result of calculation in all orders in α and L would not depend on μ_F

But a fixed-order result for an observable does depend on μ_F

Many different methods for choosing μ_F were proposed:

- CSS — Conventional Scale Setting ($\mu_F =$ hard momentum transfer)
- FAC — Fastest Apparent Convergence [G. Grunberg; N. Krasnikov]
- PMS — Principle of Minimal Sensitivity [P.M. Stevenson]
- BLM — Brodsky-Lepage-Mackenzie (absorb β_0-dependent terms)
- PMC — Principle of Maximal Conformality [S.Brodsky et al.]
- ...

Factorization scale choice in $e^+e^- \to \gamma^*, Z \to \mu^+\mu^-$

We propose the FAC-like prescription, i.e., hide the bulk of one-loop corrections into the scale choice

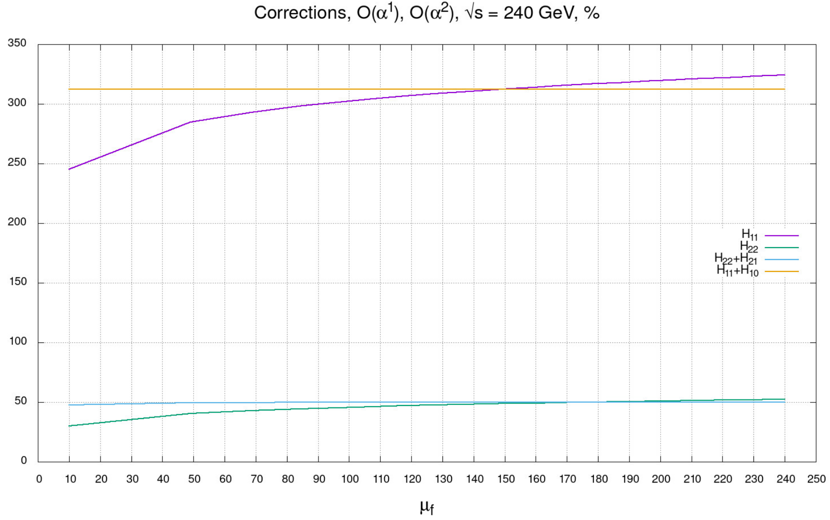
For e^+e^- annihilation

$$\frac{d\sigma^{(1)}_{e\bar{e}\to\gamma^*}}{d\sigma^{(0)}_{e\bar{e}\to\gamma^*}} = \frac{\alpha}{\pi}\left[\frac{1+z^2}{1-z}\right]_+\left(\ln\frac{s}{m_e^2} - 1\right) + \delta(1-z)(\dots) \Rightarrow \mu_F^2 = s \quad \text{or } \mu_F^2 = \frac{s}{e}$$

Remind QCD Drell-Yan processes where we usually take $\mu_F^2 = s' \equiv zs \sim M_Z$, i.e., the energy scale of the hard subprocess (CSS choice)

F.Berends at al. and J. Blümlein et al. used $\mu_F^2 = zs$ without any justification

Factorization scale choice in $e^+e^- \to \mu^+\mu^-$



Factorization (subtraction) scheme choice

NLO exponentiation in the MSbar scheme is ambiguous:

explicit solution for $D_{ee}(x)$ in the $\overline{\mathrm{MS}}$ scheme in the limit $x \to 1$ doesn't match the (pure photonic) exact solution by Gribov and Lipatov '1972

$$\mathcal{D}_{ee}^{(\gamma)}(x)\Big|_{x\to 1} = \frac{\beta}{2}\,\frac{(1-x)^{\beta/2-1}}{\Gamma(1+\beta/2)}\exp\left\{\frac{\beta}{2}\left(\frac{3}{4}-\gamma\right)\right\}$$

where $\beta = 2\alpha/\pi(L-1)$ and γ is the Euler constant

See also [A.V. Kotikov et al., "α_s from DIS data with large x resummation," arXiv:2403.13360]

We suggest a DIS-like scheme with the following modification of the NLO initial condition

$$d_{ee}^{(1)}\Big|_{\overline{\mathrm{MS}}} = \left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_R^2}{m_e^2} - 1 - \ln(1-x)\right)\right]_+ \to \tilde{d}_{ee}^{(1)} = \left[\frac{1+x^2}{1-x}\ln\frac{\mu_R^2}{m_e^2}\right]_+ = 0$$

for $m_R = m_e$ with subsequent modification of $\sigma^{(1)}$ to preserve the NLO matching. Fixed-order results for total cross-sections remain unchanged.

Scale variation test: $\mu_F \to \mu_F/2, \quad \mu_F \times 2$

True (Δ) shifts and the ones estimated (δ) by factorization scale variation by factor 2 in $\mathcal{O}(\alpha^2)$ for $\mu_F = \sqrt{s}$ in the total cross section ($s' \geq s z_{\text{min}}$)

	LO		NLO	
	Δ	δ	Δ	δ
$\sqrt{s} = M_z, \quad z_{min} = 0.1$	0.436	0.524	0.0064	0.0250
$\sqrt{s} = M_z, \quad z_{min} = 0.5$	0.436	0.526	0.0063	0.0250
$\sqrt{s} = 240$ GeV, $z_{min} = 0.1$	2.468	5.569	0.518	0.148
$\sqrt{s} = 240$ GeV, $z_{min} = 0.5$	0.114	0.106	0.0088	0.0061
$\sqrt{s} = 3$ TeV, $z_{min} = 0.1$	0.032	0.096	0.024	0.002
$\sqrt{s} = 3$ TeV $z_{min} = 0.5$	0.1213	0.1145	0.008	0.005

$$\Delta^{\text{LO}} = h_{21}, \qquad \Delta^{\text{NLO}} = h_{20}$$

$$\delta^{\text{LO}} = \frac{|h_{22} - h_{22}(1/2)| + |h_{22} - h_{22}(2)|}{2}$$

$$\delta^{\text{NLO}} = \frac{|h_{22} + h_{21} - (h_{22} + h_{21})(1/2)| + |h_{22} + h_{21} - (h_{22} + h_{21})(2)|}{2}$$

Outlook

- High-precision HEP experiments challenge theory
- New calculations of two-loop and higher-order corrections within QED and full SM are required
- We have progress in NLO and NNLO QED PDFs and fragmentation functions
- QED provides explicit results and serves for cross checks of QCD
- Optimization of factorization scale and scheme choices is important as in QCD as well as in QED
- Explicit higher-order results are given in collinear approximation
- The results can be used

  1) as benchmarks for Monte Carlo codes

  2) on top of complete $\mathcal{O}(\alpha^2)$ results